A Second Example as in §1.4. J. RAUCH

A second example, together with a third proposed in the problems, shows that the geometric optics approximation works in a variety of contexts. Consider the wave equation with friction,

$$u_{tt} - \Delta u + u_t \;=\; 0. \tag{1}$$

With e denoting the wave equation energy density, the identity $e_t - \text{div}\,(u_t \nabla u) = u_t \Box u$ shows that

$$e_t - \text{div}\,(u_t \nabla u) = -u_t^2\,.$$

This the energy dissipation law for solutions of (1),

$$\partial_t \int e(t,x)\, dx \;=\; -\int u_t^2\, dx \le 0\,.$$

There is a similar dissipation law for complex solutions derived by multiplying by $\overline{u}_t$ and taking real part.

Fourier Transformation in x yields the ordinary differential equation in time,

$$\hat{u}_{tt}(t,\xi) \;+\; |\xi|^2\, \hat{u}(t,\xi) \;+\; \hat{u}_t(t,\xi) \;=\; 0\,.$$

For each ξ this is the differential equation satisfied by a spring with friction. Solve the ordinary differential equation by inserting the $e^{i\tau t}$ to find the equation for τ,

$$\tau^2 - i\tau - |\xi|^2 = 0, \qquad \text{with roots,} \qquad \tau_\pm(\xi) \;=\; \frac{i \pm \sqrt{-1 + 4|\xi|^2}}{2}\,.$$

This leads to decaying plane wave solutions

$$e^{i(\tau_\pm t + \xi x)}$$

the by superposition to exact solutions

$$u_\pm \;:=\; (2\pi)^{-d/2} \int \hat{\gamma}\Big(\xi - \frac{\mathbf{e}_1}{\epsilon}\Big)\; e^{i(\tau_\pm(\xi)t - x\xi)}\; d\xi\,. \tag{2}$$

Consider u_+ and drop the subscript.

Introduce $\zeta = \xi - \mathbf{e}_1/\epsilon$ and change variable of integration to ζ. Compute

$$2\,\tau_+(\xi) \;=\; i + \sqrt{4\Big(\frac{\mathbf{e}_1 + \epsilon\zeta}{\epsilon}\Big)^2 - 1}\,.$$

Expansion in powers of ϵ yields,

$$\tau_+ t = \frac{t}{\epsilon} \;+\; \frac{it}{2} \;+\; \zeta_1 t \;+\; O(\epsilon)\,, \qquad x\xi = \frac{x_1}{\epsilon} \;+\; x\zeta\,.$$

The root τ_+ is not real. The leading term is real. The imaginary part leads to decay in time for the plane waves and the exact solution.

Plug into (2) and keep only the leading terms to find,

$$u_{\text{approx}} \;=\; e^{i(t-x_1)/\epsilon}\, e^{-t/2}\, (2\pi)^{d/2} \int \hat{\gamma}(\zeta)\, e^{i\zeta_1 t}\, e^{-ix\zeta} d\zeta \;=\; e^{i(t-x_1)/\epsilon}\, e^{-t/2}\, \gamma(x - x_1 t)\,.$$

The geometric optics approximation is like the example in the notes but with exponential decay at the rate $e^{-t/2}$.