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MACS2

# The Wave Equation and Geometric Optics

This course has three intertwined goals.

**I.** Given an introdution to the behavior of solutions of D'Alembert's wave equation

$$0 = \Box u := (\partial_t^2 u - \Delta_x)u, \qquad \Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\,,$$

which is one of three most important partial differential equations (the others are Laplace's equation and the heat equation).

**II.** Illustrate by example the use of three of the most important tools in the study of partial differential equations;

**1.** the Fourier transform,

**2.** the energy method,

**3.** aymptotic analysis (an example of multiscale analysis).

**III.** With high frequency asymptotic solutions show how a wave theory can lead to a particle like theory in the short wavelength limit. Explain the physical theory of geometric optics from a wave theory governed by a partial differential equation.

The physical theory called *Geometric Optics* is at the heart of the design of telescopes, microscopes, cameras, ... etc. The three basic laws of this theory are;

**1.** Light progagates in straight lines.

**2.** Reflects at boundaries following the usual law that the angles of incidence and reflection are equal.

**3.** Refracts at interfaces of materials with different indices of refraction according to Snell's Law (sometimes incorrectly called Descarte's law in France).

Goal **III** of this course is to show how these laws are explained by a wave theory of light. A key ingredient is short wavelength asymptotic analysis. The passage from wave optics to the ray theory is an example of the *wave particle duality.*

## Outline

**Lecture 1.** Present some basic facts about (D'Alembert's) wave equation, in particular in dimesnsions $d > 1$. This is the model for wave propagation in higher dimensions.

**Lectures 2-3.** Use the Fourier trasnform to derive short wavelength asymptotic solutions. Inroduce the boundary value problems and transmission problems relevant to the laws of reflection and refraction. Use the short wavelength solutions to explain the three laws of geometric optics.

**References**

The material in the first lecture is in my book *Partial Differential Equations*, and in others as well.

The second and third lecture closely follow §1.4, 1.6, and 1.7 from my book in preparation. They are available on my web page (`www.math.lsa.umich.edu/~rauch`). Follow the link to Course Materials then this course.

**Required work**

Each student is required to solve two problems. There are problems in §1.4, §1.6 and §1.7, as well as additional problems written for this course and posted on my web page under the rubric Course Materials. You may choose any two problems. You may work on the problems in groups. You must write up your own solutions. I expect the written solutions from different students to look different. Solutions can be in French or English.

Solutions can be delivered to me at the fac before the end of February, or given to the MACS secretary before the end of March. Or, a .pdf (either TeXed or scanned) emailed to me (rauch@umich.edu) also before the end of March.

**Outline of first lecture**

**1.** D'Alembert's formula for the case $d = 1$.

**2.** Fourier transform, Fourier Inversion Formula, Parseval Identity.

**3.** Plane wave solutions.

**4.** Solution of the Cauchy problem by Fourier transformation.

**5.** Conservation of energy in Fourier and local identity.

**6.** Finite speed of propagation.

**7.** Spherical solutions.