the Klein Gordon equation at least, both answers can be determined from considerations of group velocities.

§1.4. Fourier synthesis and rectilinear propagation.

For equations with constant coefficients, solutions of the initial value problem are expressed as Fourier integrals. Injecting short wavelength initial data and performing an asymptotic analysis yields the approximations of geometric optics. This is how such approximations were first justified in the nineteenth century. It is also the motivating example for the more general theory. The short wavelength approximations explain the *rectilinear propagation of waves* in homogeneous media. This is the first of the three basic physical laws of geometric optics. It explains, among other things, the formation of shadows. The short wavelength solutions are also the building blocks in the analysis of the laws of reflection and refraction.

Consider the initial value problem

$$\Box u \ := \ u_{tt} - \Delta\, u \ := \ \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \ = \ 0\,, \qquad u(0,x) = f\,, \quad u_t(0,x) = g\,. \tag{1.4.1}$$

Fourier transformation with respect to the x variables yields

$$\partial_t^2 \hat u(t,\xi) \ + \ |\xi|^2\, \hat u(t,\xi) \ = \ 0\,, \qquad \hat u(0,\xi) = \hat f\,, \quad \partial_t \hat u(0,\xi) = \hat g\,.$$

Solve the ordinary differential equations in t to find

$$\hat u(t,\xi) \ = \ \hat f(\xi)\, \cos t|\xi| + \hat g(\xi)\, \frac{\sin t|\xi|}{|\xi|}\,.$$

Write

$$\cos t|\xi| \ = \ \frac{e^{it|\xi|} + e^{-t|\xi|}}{2}, \qquad \sin t|\xi| \ = \ \frac{e^{it|\xi|} - e^{-t|\xi|}}{2i},$$

to find

$$\hat u(t,\xi) \ = \ a_+(\xi)\, e^{i(x\xi - t|\xi|)} \ - \ a_-(\xi)\, e^{i(x\xi + t|\xi|)}\,, \tag{1.4.2}$$

with,

$$2\, a_+ \ := \ \hat f \ + \ \frac{\hat g}{i|\xi|}\,, \qquad 2\, a_- \ := \ \hat f \ - \ \frac{\hat g}{i|\xi|}\,. \tag{1.4.3}$$

The right hand side of (1.4.2) is an expression in terms of the plane waves $e^{i(x\xi \mp t|\xi|)}$ with amplitudes $a_\pm(\xi)$ and dispersion relations $\tau = \mp|\xi|$. The group velocities associated to $a_\pm$ are

$$\mathbf{v} \ = \ -\nabla_\xi \tau \ = \ -\nabla_\xi(\mp|\xi|) \ = \ \pm\frac{\xi}{|\xi|}\,.$$

The solution is the sum of two terms,

$$u^\epsilon_\pm(t,x) \ := \ \frac{1}{(2\pi)^{d/2}} \int a_\pm(\xi)\ e^{i(x\xi \mp t|\xi|)}\, d\xi\,.$$

Using $\mathcal{F}\big(\partial u/\partial x_j\big) = i\xi_j \hat u$, and Parseval's Theorem shows that the conserved energy for the wave equation is equal to

$$\frac{1}{2} \int |u_t(t,x)|^2 \ + \ |\nabla_x u(t,x)|^2 \ dx \ = \ \int |\xi|^2 \big(|a_+(\xi)|^2 \ + \ |a_-(\xi)|^2\big)\ d\xi\,.$$

There are conservations of all orders. Each of the following quantities is independent of time,

$$\frac{1}{2}\,\|\nabla_{t,x}u(t)\|^2_{H^s(\mathbb{R}^d)} \;=\; \int \langle\xi\rangle^{2s}\;|\xi|^2\;\big(|a_+(\xi)|^2\;+\;|a_-(\xi)|^2\big)\,d\xi\,.$$

Consider initial data a wave packet with wavelength of order ϵ and phase equal to x_1/ϵ,

$$u^\epsilon(0,x)\;=\;\gamma(x)\,e^{ix_1/\epsilon}\,,\qquad u^\epsilon_t(0,x)\;=\;0\,,\qquad \gamma\;\in\;\cap_s H^s(\mathbb{R}^d)\,. \tag{1.4.4}$$

The choice $u_t = 0$ postpones dealing with the factor $1/|\xi|$ in (1.4.3). The initial value is an envelope or profile γ multiplied by a rapidly oscillating exponential.

Applying (1.4.3) with $g = 0$ and with

$$\hat f(\xi)\;=\;\hat u(0,\xi)\;=\;\mathcal{F}\big(\gamma(x)\,e^{ix_1/\epsilon}\big)\;=\;\hat\gamma(\xi-\mathbf{e}_1/\epsilon)\,,$$

yields $u = u_+ + u_-$ with,

$$u^\epsilon_\pm(t,x)\;:=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\xi-\mathbf{e}_1/\epsilon)\;e^{i(x\xi\mp t|\xi|)}\,d\xi\,.$$

Analyse u^ϵ_+. The other term is analogous. For ease of reading, the subscript plus is omitted. Introduce

$$\zeta\;:=\;\xi\;-\;\mathbf{e}_1/\epsilon,\qquad \xi\;=\;\mathbf{e}_1\;+\;\epsilon\zeta,$$

to find,

$$u^\epsilon(t,x)\;=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\;e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\;e^{-it|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\,d\zeta\,. \tag{1.4.5}$$

The approximation of geometric optics comes from injecting the first order Taylor approximation,

$$\big|\mathbf{e}_1+\epsilon\zeta\big|\;\approx\;1+\epsilon\zeta_1\,,$$

yielding,

$$u^\epsilon_{\text{approx}}\;:=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\;e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\;e^{-it(1+\epsilon\zeta_1)/\epsilon}\,d\zeta\,.$$

Collecting the rapidly oscillating terms $e^{i(x_1-t)/\epsilon}$ which do not depend on ζ gives,

$$u_{\text{approx}}\;=\;e^{i(x_1-t)/\epsilon}\;a(t,x),\qquad a(t,x)\;:=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x\zeta-t\zeta_1)}\,d\zeta\,. \tag{1.4.6}$$

Write $x - t\zeta_1 = (x - t\mathbf{e}_1).\zeta$ to find,

$$a(t,x)\;=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x-t\mathbf{e}_1)\zeta}\,d\zeta\;=\;\frac{1}{2}\,\gamma(x-t\mathbf{e}_1)\,.$$

The approximation is a wave translating rigidly with velocity equal to $\mathbf{e}_1$. The waveform γ is arbitrary. The approximate solution resembles the collumnated light from a flashlight. If the support of γ is small the approximate solution resembles a light ray.

The amplitude a satisfies the transport equation

$$\frac{\partial a}{\partial t}+\frac{\partial a}{\partial x_1}\;=\;0$$

so is constant on the **rays** $x = \underline{x} + t\mathbf{e}_1$. The construction of a family of short wavelength approximate solutions of D'Alembert's wave equations requires only the solutions of a simple transport equation.

The dispersion relation of the family of plane waves,

$$e^{i(x.\xi+\tau t)} \;=\; e^{i(x.\xi-|\xi|t)},$$

is $\tau = -|\xi|$. The velocity of transport, $\mathbf{v} = (1, 0, \ldots, 0)$, is the group velocity $\mathbf{v} = -\nabla_\xi \tau(\underline{\xi}) = \underline{\xi}/|\underline{\xi}|$ at $\underline{\xi} = (1, 0, \ldots, 0)$. For the opposite choice of sign the dispersion relation is $\tau = |\xi|$, the group velocity is $-\mathbf{e}_1$, and the rays are the lines $x = \underline{x} - t\mathbf{e}_1$.

Had we taken data with oscillatory factor $e^{ix.\xi/\epsilon}$ then the propagation would be at the group velocity $\pm\xi/|\xi|$. The approximate solution would be

$$\frac{1}{2}\left(e^{i(x.\xi-t|\xi|)/\epsilon}\,\gamma\Big(x - t\frac{\xi}{|\xi|}\Big) \;+\; e^{i(x.\xi+t|\xi|)/\epsilon}\,\gamma\Big(x + t\frac{\xi}{|\xi|}\Big)\right).$$

The approximate solution (1.4.6) is a function $H(x - t\mathbf{e}_1)$ with $H(x) = e^{ix_1/\epsilon}\, h(x)$. When h has compact support or more generally tends to zero as $|x| \to \infty$ the approximate solution is localized and has velocity equal to $\mathbf{e}_1$. The next result shows that when $d > 1$, no exact solution can have this form. In particular the distribution $\delta(x - \mathbf{e}_1 t)$ which is the most intuitive notion of a light ray is **not** a solution of the wave equation or Maxwell's equation.

Proposition 1.4.1. *If $d > 1$, $s \in \mathbb{R}$, $K \in H^s(\mathbb{R}^d)$ and $u = K(x - \mathbf{e}_1 t)$ satisfies $\Box u = 0$, then $K = 0$.*

Exercise 1.4.1. *Prove Proposition 1.4.1.* **Hint.** Prove and use a Lemma. **Lemma.** *If $k \le d$, $s \in \mathbb{R}$, and, $w \in H^s(\mathbb{R}^d)$ satisfies $0 = \sum_k^d \partial^2 w/\partial^2 x_j$, then $w = 0$.*

Next, analyse the error in (1.4.6). The first step is to extract the rapidly oscillating factor in (1.4.5) to define an exact amplitude $a^\epsilon_{\text{exact}}$,

$$u^\epsilon(t,x) = e^{i(x_1-t)/\epsilon}\; a_{\text{exact}}(\epsilon, t, x)\,,$$

$$a_{\text{exact}}(\epsilon, t, x) \;:=\; \frac{1}{(2\pi)^{d/2}2}\int \hat\gamma(\zeta)\; e^{ix.\zeta}\; e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon}\; d\zeta\,. \tag{1.4.7}$$

Proposition 1.4.2. *The exact and approximate solutions of $\Box u^\epsilon = 0$ with Cauchy data (1.4.4) are given by*

$$u^\epsilon = \sum_\pm e^{i(x_1\mp t)/\epsilon}\; a^\pm_{\text{exact}}(\epsilon, t, x)\,, \qquad u^\epsilon_{\text{approx}} = \sum_\pm e^{i(x_1\mp t)/\epsilon}\; \frac{\gamma(x \mp \mathbf{e}_1 t)}{2}\,,$$

as in (1.4.7) and (1.4.6). The error is $O(\epsilon)$ on bounded time intervals. Precisely, there is a constant $C > 0$ so that for all s, ϵ, t,

$$\Big\| a^\pm_{\text{exact}}(\epsilon, t, x) \;-\; \frac{\gamma(x \mp \mathbf{e}_1 t)}{2}\Big\|_{H^s(\mathbb{R}^N)} \;\le\; C\,\epsilon\,|t|\; \big\|\gamma\big\|_{H^{s+2}(\mathbb{R}^d)}\,.$$

Proof. It suffices to estimate the error with the plus sign. The definitions yield

$$a^+_{\text{exact}}(\epsilon,t,x) \;-\; \gamma(x-\mathbf{e}_1 t)/2 \;=\; C\int \hat\gamma(\zeta)\;e^{ix.\zeta}\;\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon)} - e^{-it\zeta_1}\big)\;d\zeta\,.$$

The definition of the $H^s(\mathbb{R}^d)$ norm yields

$$\Big\|a^+_{\text{exact}}(\epsilon,t,x) \;-\; \gamma(x-\mathbf{e}_1 t)/2\Big\|_{H^s(\mathbb{R}^N)} \;=\; \Big\|\langle\zeta\rangle^s\;\hat\gamma(\zeta)\;\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^N)}.$$

Taylor expansion yields for $|\beta|\le 1/2$,

$$|\,\mathbf{e}_1+\beta\,| \;=\; 1+\beta_1+r(\beta)\,, \qquad |r(\beta)|\le C\,|\beta|^2\,.$$

Increasing C if needed, the same inequality is true for $|\beta|\ge 1/2$ as well.

Applied to $\beta=\epsilon\zeta$ this yields,

$$\Big|\,t\big(\big|\mathbf{e}_1+\epsilon\zeta\big|-1\big)/\epsilon - \zeta_1 x_1\Big| \;\le\; C\,\epsilon\,|t|\,|\zeta|^2\,,$$

so

$$\Big|e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;-\; e^{-it\zeta_1}\Big| \;\le\; C\,\epsilon\,|t|\,|\zeta|^2\,.$$

Therefore

$$\Big\|\langle\zeta\rangle^s\;\hat\gamma(\zeta)\;\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^d)} \;\le\; C\,\epsilon\,|t|\,\Big\|\langle\zeta\rangle^s|\zeta|^2\;\hat\gamma(\zeta)\Big\|_{L^2}. \tag{1.4.8}$$

Combining (1.4.7-1.4.8) yields the estimate of the Proposition. ∎

The approximation retains some accuracy so long as $t=o(1/\epsilon)$.

The approximation has the following geometric interpretation. One has a superposition of plane waves $e^{i(x\xi+t|\xi|)}$ with $\xi=(1/\epsilon,0,\ldots,0)+O(1)$. Replacing ξ by $(1/\epsilon,0,\ldots,0)$ and $|\xi|$ by $1/\epsilon$ in the plane waves yields the approximation (1.4.6).

The wave vectors, ξ, make an angle $O(\epsilon)$ with $\mathbf{e}_1$. The corresponding rays have velocities which differ by $O(\epsilon)$ so the rays remain close for times small compared with $1/\epsilon$. For longer times the fact that the group velocities are not parallel is important. The wave begins to spread out. Parallel group velocities is a reasonable approximation for times $t=o(1/\epsilon)$.

The example reveals several scales of time. For times $t<<\epsilon$, u and its gradient are well approximated by their initial values. For times $\epsilon<<t<<1$ $u\approx e^{i(x-t)/\epsilon}a(0,x)$. The solution begins to oscillate in time. For $t=O(1)$ the approximation $u\approx a(t,x)\,e^{i(x-t)/\epsilon}$ is appropriate. For times $t=O(1/\epsilon)$ the approximation ceases to be accurate. The more refined approximations valid on this longer time scale are called *diffractive geometric optics*. The reader is referred to [Donnat, Joly Métiver, and Rauch] for an introduction in the spirit of Chapters 7-8.

It is typical of the approximations of geometric optics, that

$$\Box\big(u_{\text{approx}}-u_{\text{exact}}\big) \;=\; \Box u_{\text{approx}} = O(1)\,,$$

is not small. The error $u_{\text{approx}}-u_{\text{exact}}=O(\epsilon)$ is smaller by a factor of ϵ. The residual $\Box u_{\text{approx}}$ is rapidly oscillatory, so applying $\Box^{-1}$ gains the factor ϵ.

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase. A nice description including the phase shift on crossing a focal point can be found in [Hörmander 1983, §12.2].

Next the approximation is pushed to higher accuracy with the result that the residuals can be reduced to $O(\epsilon^N)$ for any N. Taylor expansion to higher order yields,

$$|\mathbf{e}_1 + \eta| \;=\; 1 + \eta_1 + \sum_{|\alpha|\geq 2} c_\alpha \eta^\alpha\,, \qquad |\eta| < 1, \tag{1.4.9}$$

so

$$\big(|\mathbf{e}_1 + \epsilon\zeta| - 1\big)/\epsilon \sim \zeta_1 + \sum_{|\alpha|\geq 2} \epsilon^{|\alpha|-1}\, c_\alpha\, \zeta^\alpha,$$

$$e^{it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;\sim\; e^{it\zeta_1}\, e^{\sum_{|\alpha|\geq 2} it\epsilon^{|\alpha|-1}\, c_\alpha \zeta^\alpha} \;\sim\; e^{it\zeta_1}\Big(1 + \sum_{j\geq 1} \epsilon^j\, h_j(t,\zeta)\Big)\,.$$

Here, $h_j(t,\zeta)$ is a polynomial in t,ζ. Injecting in the formula for $a_{\text{exact}}(\epsilon,t,x)$ yields an expansion

$$a_{\text{exact}}(\epsilon,t,x) \;\sim\; a_0(t,x) + \epsilon\, a_1(t,x) + \epsilon^2 a_2(t,x) + \cdots\,, \qquad a_0(t,x) = \gamma(x - \mathbf{e}_1 t)/2, \tag{1.4.10}$$

$$a_j \;=\; \frac{1}{(2\pi)^{-d/2}\,2}\int \hat\gamma(\zeta)\, e^{i(x\zeta - t\zeta_1)}\;\, h_j(t,\zeta)\; d\zeta \;=\; \frac{1}{2}\big(h_j(t,\partial/i)\gamma\big)(x - \mathbf{e}_1 t)\,. \tag{1.4.11}$$

The series is asymptotic as $\epsilon \to 0$ in the sense of Taylor series. For any s, N, truncating the series after N terms yields an approximate amplitude which differs from a_{exact} by $O(\epsilon^{N+1})$ in L^2 uniformly on compact time intervals. The H^s error for $s \geq 0$ is $O(\epsilon^{N+1-s})$.

Exercise 1.4.2. *Compute the precise form of the first corrector a_1.*

Formula (1.4.11) implies that if the Cauchy data are supported in a set $\mathcal{O}$, then the amplitudes a_j are all supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x) \;:\; x = \underline{x} + t\mathbf{e}_1, \quad \underline{x} \in \mathcal{O}\Big\}\,. \tag{1.4.12}$$

Warning. Though the a_j are supported in this tube, it is not true that $a^\epsilon_{\text{exact}}$ is supported in the tube. The map $\epsilon \mapsto a_{\text{exact}}(\epsilon,t,x)$ is not analytic. If it were, the Taylor series would converge to the exact solution which would then have support in the tube. When $d \geq 2$, the function $u = 0$ is the only solution of D'Alembert's equation with support in a tube of rays with cross section of finite d dimensional Lebesgue measure. This follows from the fact that for finite energy solutions, the energy in the tube tends to zero. †

To analyse the oscillatory initial value problem with $u(0) = 0$, $u_t(0) = \beta(x)\, e^{ix_1/\epsilon}$ requires one more idea to handle the contributions from $\xi \approx 0$ in the expression

$$u(t,x) \;=\; (2\pi)^{-d/2}\int \frac{\sin t|\xi|}{|\xi|}\; \hat\beta\Big(\xi - \frac{\mathbf{e}_1}{\epsilon}\Big)\; e^{ix\xi}\; d\xi\,.$$

† This is proved by approximation by regular solutions. For Cauchy data in $C_0^\infty(\mathbb{R}^d)$, the energy in the tube is $O(t^{(1-d)})$. This can be proved using the fundamental solution. Alternatively, if the Fourier transform of the Cauchy data belongs to $C_0^\infty(\mathbb{R}^d_\xi \setminus 0)$ one has the same estimate using the inequality of stationary phase from Appendix 3.II (see Lemma 3.4.2).

Choose $\chi \in C_0^\infty(\mathbb{R}^d_\xi)$ with $\chi = 1$ on a neighborhood of $\xi = 0$. The cutoff integrand is equal to

$$\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi - \mathbf{e}_1/\epsilon)\ e^{ix\xi}\,, \qquad k_s(\xi)\ :=\ \langle\xi\rangle^s\ \hat\beta(\xi)\ \in\ L^2(\mathbb{R}^d_\xi)\,.$$

A simple upper bound is,

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\Big\|_{L^\infty(\mathbb{R}^d)}\ \le\ C_s\,|t|\,\epsilon^s\,, \qquad 0<\epsilon\le 1\,.$$

It follows that

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi - \mathbf{e}_1/\epsilon)\ \Big\|_{L^2(\mathbb{R}^d)}\ \le\ C_s\,|t|\,\epsilon^s \big\|\beta\big\|_{H^s(\mathbb{R}^d)}\,.$$

The small frequency contribution is negligable in the limit $\epsilon \to 0$. It is removed with a cutoff as above and then the analysis away from $\xi = 0$ proceeds by decomposition into plane wave as in the case with $u_t(0) = 0$. It yields left and right moving waves with the same phases as before.

Exercise 1.4.3. *Solve the Cauchy problem for the anisotropic wave equation,* $u_{tt} = u_{xx} + 4u_{yy}$ *with initial data given by*

$$u^\epsilon(0,x)\ =\ \gamma(x)\,e^{ix.\xi/\epsilon}\,, \qquad u^\epsilon_t(0,x)\ =\ 0\,, \qquad \gamma\ \in\ \cap_s H^s(\mathbb{R}^d)\,.$$

Find the leading term in the approximate solution to u_+. *In particular, find the velocity of propagation as a function of* ξ. **Discussion.** The velocity is equal to the group velocity from §1.3.

§1.5. A cautionary example in geometric optics.

A typical science text discussion of a mathematics problem involves simplifying the underlying equations. The usual criterion applied is to ignore terms which are small compared to other terms in the equation. It is striking that in many of the problems treated under the rubric of geometric optics, such an approach can lead to completely inaccurate results. It is an example of an area where more careful mathematical consideration is not only useful but necessary.

Consider the initial value problems

$$\partial_t u^\epsilon + \partial_x u^\epsilon + u^\epsilon\ =\ 0\,, \qquad u^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,,$$

in the limit $\epsilon \to 0$. The function a is assumed to be smooth and to vanish rapidly as $|x| \to \infty$ so the initial value has the form of wave packet. The initial value problem is uniquely solvable and the solution depends continuously on the data. The exact solution of the general problem

$$\partial_t u + \partial_x u + u\ =\ 0\,, \qquad u\big|_{t=0} = f(x)\,,$$

is $u(t,x) = e^{-t}\,f(x-t)$ so the exact solution u^ϵ is

$$u^\epsilon(t,x)\ =\ e^{-t}\,a(x-t)\ \cos((x-t)/\epsilon)\,.$$

In the limit as $\epsilon \to 0$ one finds that both $\partial_t u^\epsilon$ and $\partial_x u^\epsilon$ are $O(1/\epsilon)$ while $u^\epsilon = O(1)$ is negligibly small in comparison. Dropping this small term leads to the simplified equation for an approximation v^ϵ,

$$\partial_t v^\epsilon + \partial_x v^\epsilon\ =\ 0\,, \qquad v^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,.$$

The exact solution is

$$v^\epsilon(t,x) \;=\; a(x-t)\cos\big((x-t)/\epsilon\big)\,,$$

which misses the exponential decay. It is **not** a good approximation. The two large terms compensate so that the small term is not negligible compared to their sum.

§1.6. The law of reflection.

Consider the wave equation $\Box u = 0$ in the half space $\mathbb{R}^d_- := \{x_1 \le 0\}$. At $\{x_1 = 0\}$ a boundary condition is required. The condition encodes the physics of the interaction with the boundary.

Since the differential equation is of second order one might guess that two boundary conditions are needed as for the Cauchy problem. An analogy with the Dirichlet problem for the Laplace equation suggests that one condition is required.

A more revealing analysis concerns the case of dimension $d = 1$. D'Alembert's formula shows that at all points of space time the solution consists of the sum of two waves one moving toward the boundary and the other toward the interior. The waves approaching the boundary will propagate to the edge of the domain. At the boundary one does not know what values to give to the waves which move into the domain. The boundary condition must give the value of the incoming wave in terms of the outgoing wave. That is one boundary condition.

Factoring

$$\partial_t^2 - \partial_x^2 \;=\; (\partial_t - \partial_x)(\partial_t + \partial_x) \;=\; (\partial_t + \partial_x)(\partial_t - \partial_x),$$

shows that $(\partial_t - \partial_x)(u_t + u_x) = 0$ so $u_t + u_x$ is transported to the left. Similarly, $u_t - u_x$ moves to the right. Thus from the initial conditions, $u_t - u_x$ is determined everywhere in $x \le 0$ including the boundary $x = 0$. The boundary condition at $\{x = 0\}$ must determine $u_t + u_x$. The conclusion is that half of the information needed to find all the first derivatives is already available and one needs only one boundary condition.

For the Dirichlet condition,

$$u(t,x)\Big|_{x_1=0} = 0\,. \tag{1.6.1}$$

Differentiating (1.6.1) with respect to t shows that $u_t(t,0) = 0$, so at $t = 0$ $(u_t + u_x) = -(u_t - u_x)$ showing that at the boundary, the incoming wave is equal to -1 times the outgoing wave.

In the case $d \ge 1$ consider the Cauchy data,

$$u(0,x) = f\,, \qquad u_t(0,x) = g\,, \quad \text{for} \quad x_1 \le 0\,. \tag{1.6.2}$$

If the data are supported in a compact subset of $\mathbb{R}^d_-$ then, for small time the support of the solution does not meet the boundary. When waves hit the boundary they are reflected. The goal of this section is to describe this reflection process.

Uniqueness of solutions and finite speed of propagation for (1.6.1)-(1.6.2) are both consequences of a local energy identity. A function is a solution if and only if the real and imaginary parts are solutions. Thus it suffices to treat the real case for which

$$u_t\,\Box u \;=\; \partial_t e - \sum_{j\ge 1} \partial_j(u_t \partial_j u)\,, \qquad e := \frac{u_t^2 + |\nabla_x u|^2}{2}\,.$$

Denote by Γ a backward light cone

$$\Gamma \;:=\; \Big\{(t,x) \;:\; |x - \underline{x}|^2 < \underline{t} - t\Big\}$$

and by $\tilde\Gamma$ the part in $\{x_1<0\}$,

$$\tilde\Gamma \ := \ \Gamma \ \cap \ \big\{x_1<0\big\}\,.$$

For any $0\le s<\underline{t}$ the section at time s is denoted

$$\tilde\Gamma(s) \ := \ \tilde\Gamma \ \cap \ \big\{t=s\big\}\,.$$

Both uniqueness and finite speed follow from the following energy estimate.

Proposition 1.6.1. *If u is a smooth solution of (1.6.1)-(1.6.2), then for $0<t<\underline{t}$,*

$$\phi(t) \ := \ \int_{\tilde\Gamma(t)} e(t,x)\ dx$$

is a nonincreasing function of t.

Proof. Translating the time if necessary it suffices to show that for $s>0$, $\phi(s)\le\phi(0)$.
In the identity

$$0 \ = \ \int_{\tilde\Gamma\cap\{0\le t\le s\}} u_t\ \Box u\ dt\,dx\,.$$

Integrate by parts to find integrals over four distinct parts of the boundary. The tops and bottoms contribute $\phi(t)$ and $-\phi(0)$ respectively. The intersection of $\tilde\Gamma(s)$ with $x_1=0$ yields

$$\int_{\tilde\Gamma(s)\cap\{x_1=0\}} u_t\ \partial_1 u\ dt\,dx_2\ \ldots\ dx_d\,.$$

The Dirichlet condition implies that $u_t=0$ on this boundary so the integral vanishes.
The contribution of the sides $|x-\underline{x}|=\underline{t}-t$ yield an integral of

$$n_0\,e \ + \ \sum_{j=1}^{d} n_j\,u_t\,\partial_j u\,,$$

where $(n_0,n_1,n_2,\ldots,n_d)$ is the outward unit normal. Then

$$n_0 \ = \ \Big(\sum_{j=1}^{d} n_j^2\Big)^{1/2} \ = \ \frac{1}{\sqrt2}\,, \qquad \Big|\sum_{j=1}^{d} n_j\,u_t\,\partial_j u\Big| \ \le \ \frac{1}{\sqrt2}\ |u_t||\nabla_x u| \ \le \ \frac{1}{\sqrt2}\,e\,.$$

Thus the integrand from the contributions of sides is nonnegative, so the integral over the sides is nonnegative.
Combining yields

$$0 \ = \ \int_{\tilde\Gamma\cap\{0\le t\le s\}} u_t\ \Box u\ dt\,dx \ \ge \ \phi(t)-\phi(0)\,,$$

and the estimate follows. ∎

§1.6.1. The method of images.

Introduce the notations,

$$x=(x_1,x'),\quad x':=(x_2,\ldots,x_d),\qquad \xi=(\xi_1,\xi'),\quad \xi':=(\xi_2,\ldots,\xi_d).$$

Definitions. *A function f on $\mathbb{R}^{1+d}$ is* **even** *(resp.* **odd***) in x_1 when*

$$f(t,x_1,x') \;=\; f(t,-x_1,x') \qquad \text{resp.} \quad f(t,-x_1,x') \;=\; -f(t,x_1,x').$$

Define the **reflection operator** R *by*

$$(Rf)(t,x_1,x') \;:=\; f(t,-x_1,x')\,.$$

The even (resp. odd) parts of a function f are defined by

$$\frac{f+Rf}{2}, \qquad \text{resp.} \qquad \frac{f-Rf}{2}.$$

Proposition 1.6.2. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ is a solution of $\Box u = 0$ that is odd in x_1, then its restiction to $\{x_1 \le 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Dirichlet boundary condition (1.6.1) .*
ii. *Conversely, if $u \in C^\infty(\{x_1 \le 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.1) then the odd extension of u to $\mathbb{R}^{1+d}$ is a smooth odd solution of $\Box u = 0$.*

Proof. i. Setting $x_1 = 0$ in the identity $u(t,x_1,x') = -u(t,-x_1,x')$ shows that (1.6.1) is satisfies.
ii. First prove by induction on n that

$$\forall n \ge 0, \qquad \frac{\partial^{2n} u}{\partial^{2n} x_1}\bigg|_{x_1=0} \;=\; 0\,. \tag{1.6.3}$$

The case $n = 0$ is (1.6.1).
Since the derivatives ∂_t and ∂_j for $j > 1$ are parallel to the boundary along which $u = 0$, it follows that u_{tt} and $\partial_j^2 u$ with $j > 1$ vanish at $x_1 = 0$. The equation $\Box u = 0$ implies

$$\frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

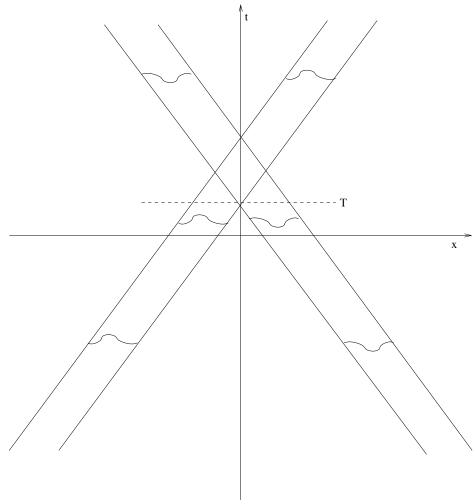
The right hand side vanishes on $\{x_1 = 0\}$ proving the case $n = 1$.
If the case $k \ge 1$ is known, apply the case k to the odd solution $\partial_1^2 u$ to prove the case $k+1$. This completes the proof of (1.6.3).
Denote by $\tilde u$, the odd extension of u. It is not hard to prove using Taylor's theorem that (1.6.3) is a necessary and sufficient condition for $\tilde u \in C^\infty(\mathbb{R}^{1+d})$. The equation $\Box \tilde u = 0$ for $x_1 \ge 0$ follows from the equation in $x_1 \le 0$ since $\Box \tilde u$ is odd. ∎

Example. Suppose that $d = 1$ and that $f \in C_0^\infty(]-\infty, 0[)$ so that $u = f(x-t)$ is a solution of (1.6.1), (1.6.2) representing a wave which approaches the boundary $\{x = 0\}$ from the left. To describe the reflection use images as follows. The solution in $\{x < 0\}$ is the restriction to $x < 0$ of an odd solution of the wave equation. For $x < 0$ that solution is equal to the given function in $x < 0$ and to minus its reflection in $\{x > 0\}$,

$$u \;=\; f(x-t) \;-\; f(-x-t)\,.$$

The formula on the right is an odd solution of the wave equation which is equal to u in $t < 0$ so is therefore the solution for all time. The solution u is the restriction to $x < 0$.

An example is sketched in the figure. In $\mathbb{R}^{1+1}$ one has an odd solution of the wave equation.

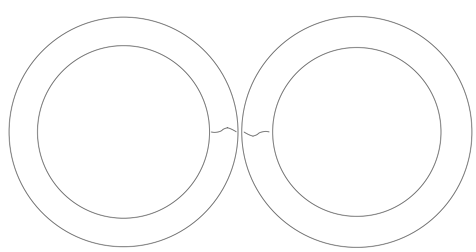


Reflection in dimension $d = 1$

There is a righward moving wave with postitve profile and a leftward moving wave with negative profile equal to -1 times the reflection of the first.

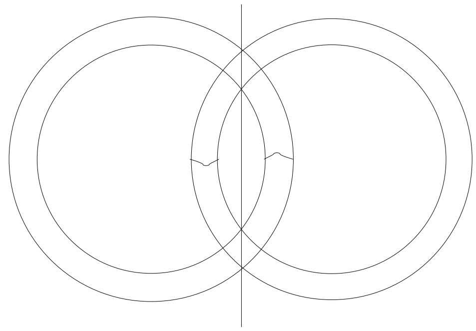
Viewed from $x < 0$, there is a wave with positive profile which arrives at the boundary at time T. At that time a leftward moving wave seems to emerge from the boundary. It is the reflection of the wave arriving at the boundary. If the wave arrives at the boundary with amplitude a on an incoming ray, the reflected wave on the reflected ray has amplitude $-a$. The *coefficient of reflection* is equal to -1. This is the same result found in the first paragraphs of §1.6.

Example. Suppose that $d = 3$ and in $t < 0$ one has a spherically symmetric wave approaching the boundary. Until it reaches the boundary the boundary condition does not play a role. The reflection is computed by extending the incoming wave to an odd solution consisting of the given solution and its negative in mirror image. The moment when the original wave reaches the boundary from the left, its image arrives from the right.



Spherical wave arrives at the boundary

In the figure the wave on the left has positive profile and that on the right a negative profile.



Spherical wave with reflection

In the figure above the middle line represents the boundary. Viewed from $x<0$, the wave on the left disappears into the boundary and a reflected spherical wave emerges with profile flipped. The profiles of spherical waves in three space preserve their shape but decrease in amplitude as they spread.

§**1.6.2. The plane wave derivation.**

In many texts you will find a derivation which goes as follows. Begin with the plane wave solutions

$$e^{i(x.\xi+t\tau)}\,, \qquad \xi\in\mathbb{R}^d, \quad \tau=\mp|\xi|\,.$$

Since u is everywhere of modulus one, no solution of this sort can satisfy the Dirichlet boundary condition.

Seek a solution of the initial boundary value problem which is a sum of two plane waves,

$$e^{i(x.\xi-t|\xi|)}\ +\ A\,e^{i(x.\eta+t\sigma)}\,, \qquad A\in\mathbb{C}\,.$$

In order that the solutions satisfy the wave equation one must have $\sigma^2=|\eta|^2$. In order that the plane waves sum to zero at $x_1=0$ it is necessary and sufficient that $\eta'=\xi'$, $\sigma=-|\xi|$, and $A=-1$. Since $\sigma^2=|\eta|^2$ it follows that $|\eta|=|\xi|$ so

$$\eta=(\pm\xi_1,\xi_2,\dots,\xi_d)\,.$$

The sign + yields the solution $u=0$. Denote

$$\tilde{x}\ :=\ (-x_1,x_2,\dots,x_d), \qquad \tilde{\xi}\ :=\ (-\xi_1,\xi_2,\dots,\xi_d).$$

The sign minus yields the interesting solution.

$$e^{i(x.\xi-t|\xi|)}\ -\ e^{i(x.\tilde{\xi}-t|\tilde{\xi}|)}$$

which is twice the odd part of $e^{i(x.\xi-t|\xi|)}$.

The textbook interpretation of the solution with $\tau=-|\xi|$ and $\xi_1>0$ is that $e^{i(x.\xi-t|\xi|)}$ is a plane wave approaching the boundary $x_1=0$, and $e^{i(x.\tilde{\xi}-t|\tilde{\xi}|)}$ moves away from the boundary. The first is an incident wave and the second is a reflected wave. The factor $A=-1$ is the reflection coefficient. The direction of motions are given group velocity computed from the dispersion relation.

Both waves are of infinite extent and of modulus one everywhere in space time. They have finite energy density but infinite energy. They both meet the boundary at all times. It is questionable to think of either one as incoming or reflected. The next subsection shows that there are localized waves which are clearly incoming and reflected waves with the property that when they interact with the boundary the local behavior resembles the plane waves.

For more general mixed initial boundary value problems, there are other wave forms which need to be included. The key is that solutions of the form $e^{i(x.\xi+t\tau)}$ are acceptable in $x_1<0$ for ξ',τ real and $\operatorname{Im}\xi_1\le 0$. When $\operatorname{Im}\xi_1<0$ the associated waves are localized near the boundary. The Rayleigh waves in elasticity are a classic example. They carry the devastating energy of earth quakes. Waves of this sort which do not propagate are needed to analyse total reflection which is described at the end of §1.7. The reader is referred to [Benzoni-Gavage - Serre], [Chazarain-Piriou], [Taylor 1981], [Hormander 1982 v.II], [Sakamoto], for more information.

§1.6.3. Reflected high frequency wave packets.

Consider solutions which for small time are equal to high frequency solutions from §1.3,

$$u^\epsilon \;=\; e^{i(x.\xi-t|\xi|)/\epsilon}\,a(\epsilon,t,x)\,,\qquad a(\epsilon,t,x)\;\sim\;a_0(t,x)+\epsilon\,a_1(t,x)+\cdots\;, \tag{1.6.5}$$

with

$$\xi=(\xi_1,\xi_2,\dots,\xi_d)\,,\qquad \xi_1>0\,.$$

Then $a_0(t,x)\;=\;h(x-t\xi/|\xi|)$ is constant on the rays $\underline{x}+t\xi/|\xi|$. If the Cauchy data are supported in a set $\mathcal{O}\subset\subset\{x_1<0\}$ then the amplitudes a_j are supported in the tube of rays

$$\mathcal{T}\;:=\;\Big\{(t,x)\;:\;x=\underline{x}+t\xi/|\xi|,\quad \underline{x}\in\mathcal{O}\Big\}\,, \tag{1.6.6}$$

Finite speed shows that the wave as well as the geometric optics approximation stays strictly to the left of the boundary for small $t>0$.

The method of images computes the reflection. Define v^ϵ to be the reversed mirror image solution,

$$v^\epsilon(t,x_1,x_2,\dots,x_d)\;:=\;-u^\epsilon(t,-x_1,x_2,\dots,x_d)\,.$$

The solution of the Dirichlet problem is then equal to the restriction of $u^\epsilon+v^\epsilon$ to $\{x_1\le 0\}$.

Then

$$\tilde v^\epsilon\;=\;-\,e^{i(\tilde x.\xi-t)/\epsilon}\,h(\tilde x-t\xi)\;+\;\text{h.o.t}\;=\;-\,e^{i(\tilde x.\xi-t)/\epsilon}\,\tilde h(x-t\tilde\xi)\;+\;\text{h.o.t}\,.$$

To leading order, $u^\epsilon+v^\epsilon$ is equal to

$$e^{i(x.\xi-t)/\epsilon}\,h(x-t\xi)\;-\;e^{i(\tilde x.\xi-t)/\epsilon}\,\tilde h(x-t\tilde\xi)\,. \tag{1.6.7}$$

The wave represented by u^ϵ has leading term which moves with velocity $\xi/|\xi|$. The wave corresponding to v^ϵ has leading term with velocity $\tilde\xi/|\tilde\xi|$. which comes from $\xi/|\xi|$ by reversing the first component. At the boundary $x_1=0$, the tangential components of $\xi/|\xi|$ and $\tilde\xi/|\tilde\xi|$ are equal and their normal components are opposite. The directions are related by the standard law that the angle of incidence equals the angle of reflection. The amplitude of the reflected wave v^ϵ on the reflected ray is equal to -1 time the amplitude of the incoming wave u^ϵ on the incoming wave. This is summarized by the statement that the reflection coefficient is equal to -1.

Suppose that $\underline{t},\underline{x}$ is a point on the boundary and $\mathcal{O}$ in a neighborhood of size large compared to the wavelength ϵ and small compared to the scale on which h varies. Then, on $\mathcal{O}$, the solution is approximately equal to

$$e^{i(x.\xi-t)/\epsilon}\,h(\underline{x}-\underline{t}\xi/|\xi|)\;-\;e^{i(\tilde x.\xi-t)/\epsilon}\,\tilde h(\underline{x}-\underline{t}\tilde\xi/|\tilde\xi|)\,.$$

This recovers the reflected plane waves of §1.6.2. An observer on such an intermediate scale sees the structure of the plane waves. Thus, even though the plane waves are completely nonlocal, the asymptotic solutions of geometric optics shows that they predict the local behavior at points of reflection.

The method of images also solves the Neumann boundary value problem in a half space using *even* mirror reflection in $x_1 = 0$. It shows that for the Neumann condition, the reflection coefficient is equal to 1.

Proposition 1.6.2. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ is an even solution of $\Box u = 0$, then its restiction to $\{x_1 \leq 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Neumann boundary condition*

$$\partial_1 u|_{x_1=0} = 0\,, \tag{1.6.8}$$

ii. *Conversely, if $u \in C^\infty(\{x_1 \leq 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.8) then the even extension of u to $\mathbb{R}^{1+d}$ is a smooth odd solution of $\Box u = 0$.*

The analogue of (1.6.3) in this case is

$$\forall n \geq 0, \qquad \frac{\partial^{2n+1} u}{\partial x_1^{2n+1}}\bigg|_{x_1=0} \;=\; 0\,. \tag{1.6.9}$$

Exercise 1.6.1. *Prove the Proposition.*

Exercise 1.6.2. *Prove uniqueness of solutions by the energy method.* **Hint.** Use the local energy identity.

Exercise 1.6.3 *Verify the assertion concerning the reflection coefficient by following the examples above. That is, consider the case of dimension $d = 1$, the case of spherical waves with $d = 3$ and the behavior in the future of a solution which near $t = 0$ is a high frequency asymptotic solution approaching the boundary.*

§1.7. Snell's law of refraction.

Refraction is the bending of waves as they pass through media whose propagation speeds vary from point to point. The simplest situation is when media with different speeds occupy half spaces, for example $x_1 < 0$ and $x_1 > 0$. The classical physical situations are when light passes from air to water or from air to glass. It is observed that the angles of incidence and refraction are so that for fixed materials the ratio $\sin\theta_i/\sin\theta_r$ is independent of the incidence angle. Fermat observed that this would hold if the speed of light were different in the two media and light light path was a path of least time. In that case, the quotient of sines equal to the ratio of the speeds, c_i/c_r. In this section we derive this behavior for a model problem quite close to the natural Maxwell equations.

The simplified model with the same geometry is,

$$u_{tt} - \Delta\, u = 0 \quad \text{in} \quad x_1 < 0\,, \qquad u_{tt} - c^2\,\Delta\, u = 0 \quad \text{in} \quad x_1 > 0\,, \qquad 0 < c < 1\,. \tag{1.7.1}$$

In $x_1 < 0$ the speed is equal to 1 which is greater than the speed c in $x > 0$. To see that c is the speed of the latter equation one can factor the one dimensional operator $\partial_t^2 - c^2\partial_x^2 = (\partial_t = c\partial_x)(\partial_t - c\partial_x)$ or use the formula for group velocity with dispersion relation $\tau^2 = |\xi|^2$.

A transmission condition is required at $x_1 = 0$ to encode the interaction of waves with the interface. In the one dimesional case, there are waves which approach the boundary from both sides. The waves which move from the boundary into the interior must be determined from the waves which arrive from the interior. There are two arriving waves and two departing waves. One needs two boundary conditions.

We analyse the transmission condition that imposes continuity of u and $\partial_1 u$ across $\{x_1 = 0\}$. Seek solutions of (1.7.1) satsifying the transmission condition,

$$u(t, 0^-, x') \;=\; u(t, 0^+, x')\,, \qquad \partial_1 u(t, 0^-, x') \;=\; \partial_1 u(t, 0^+, x')\,. \tag{1.7.2}$$

Denote by square brackets the jump

$$\big[u\big](t, x') \;:=\; u(t, 0^+, x') \;-\; u(t, 0^-, x')\,.$$

The transmission condition is then

$$\big[u\big] \;=\; 0\,, \qquad \big[\partial_1 u\big] \;=\; 0\,.$$

For solutions which are smooth on both sides of the boundary $\{x_1 = 0\}$, the transmission condition (1.7.2) and be differentiated in t or $x_2, \dots, x_d$ to find

$$\Big[\partial^\beta_{t,x'} u\Big] \;=\; 0\,, \qquad \Big[\partial^\beta_{t,x'} \partial_1 u\Big] \;=\; 0\,. \tag{1.7.3}$$

The partial differential equations then imply that in $x_1 < 0$ and $x_1 > 0$ respectively one has

$$\frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}\,, \qquad \frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}\,,$$

Therefore at the boundary

$$\left[\frac{\partial^2 u}{\partial x_1^2}\right] \;=\; \left(1 - \frac{1}{c^2}\right) \frac{\partial^2 u}{\partial t^2}\,.$$

The second derivative $\partial_1^2 u$ is expected to be discontinuous at $\{x_1 = 0\}$.

The physical conditions for Maxwell's Equations at an air-water or air-glass interface can be analysed in the same way. In that case, the dielectric constant is discontinuous at the interface.

Define

$$\gamma(x) \;:=\; \begin{cases} 1 & \text{when} \quad x_1 > 0 \\ c^{-2} & \text{when} \quad x_1 < 0\,, \end{cases} \qquad e(t,x) \;:=\; \frac{\gamma\, u_t^2 + |\nabla_x u|^2}{2}\,,$$

From (1.7.1) it follows that solutions suitably small at infinity satisfy

$$\partial_t \int_{x_1 < 0} e\; dx \;=\; \int u_t(t, 0^-, x')\; \partial_1 u(t, 0^+, x')\; dx'\,,$$
$$\partial_t \int_{x_1 > 0} e\; dx \;=\; -\int u_t(t, 0^+, x')\; \partial_1 u(t, 0^+, x')\; dx'\,.$$

The transmission condition guarantees that the terms on the right compensate exactly so

$$\partial_t \int_{\mathbb{R}^3} e\; dx \;=\; 0\,.$$

This suffices to prove uniqueness of solutions. A localized argument as in §1.6.1, shows that signals travel at most at speed one.

Exercise 1.7.1. *Prove this finite speed result.*

A function $u(t,x)$ is called **piecewise smooth** if its restriction to $x_1 < 0$ (resp. $x_1 > 0$) has a C^∞ extension to $x_1 \le 0$ (resp. $x_1 \ge 0$). The Cauchy data of piecewise smooth solutions must be piecewise smooth (with the analogous definition for functions of x only). They must, in addition, satisfy conditions analogous to (1.6.3).

Propostion 1.7.1. *If u is a piecewise smooth solutions u of the transmission problem, then the partial derivatives satisfy the sequence of compatibility conditions, for all $j \ge 0$,*

$$\Delta^j \{u, u_t\}(t, 0^-, x_2, x_3) \;=\; (c^2\Delta)^j \{u, u_t\}(t, 0^+, x_2, x_3)\,,$$

$$\Delta^j \partial_1 \{u, , u_t\}(t, 0^-, x_2, x_3) \;=\; (c^2\Delta)^j \partial_1 \{u, u_t\}(t, 0^+, x_2, x_3)\,.$$

ii. *Conversely, if the piecewise smooth f, g satisfy for all $j \ge 0$,*

$$\Delta^j \{f, g\}(0^-, x_2, x_3) \;=\; (c^2\Delta)^j \{f, g\}(0^+, x_2, x_3)\,, \tag{1.7.4}$$

$$\Delta^j \partial_1 \{f, g\}(0^-, x_2, x_3) \;=\; (c^2\Delta)^j \partial_1 \{f, g\}(0^+, x_2, x_3)\,, \tag{1.7.5}$$

then there is a piecewise smooth solution with these Cauchy data.

Proof. **i.** If u is a piecewise smooth solution then so is $\partial_t^j u$ for any j. Use (1.7.2) for pure time derivatives,

$$\big[\partial_t^j u\big] \;=\; 0\,, \qquad \big[\partial_t^j \partial_1 u\big] \;=\; 0\,. \tag{1.7.6}$$

The case $j = 1$ yields the necessary condition

$$\big[g\big] \;=\; 0\,, \qquad \big[\partial_1 g\big] \;=\; 0\,.$$

For the higher orders, compute with $k \ge 1$,

$$\partial_t^{2k} u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1 < 0 \\ (c^2\Delta)^k u & \text{when} \quad x_1 > 0, \end{cases}$$

$$\partial_t^{2k-1} u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1 < 0 \\ (c^2\Delta)^k u & \text{when} \quad x_1 > 0. \end{cases}$$

Thus, the transmission conditions (1.7.6) proves **i.**

The proof of **ii.** is technical, interesting, and omitted. One can construct solutions using finite differences almost as in §2.2. The shortest existence proof to state uses the spectral theorem for self adjoint operators.* The general regularity theory for such transmission problems can be obtained

* For those with sufficient background, the Hilbert space is $\mathcal{H} := L^2(\mathbb{R}^d\,;\, \gamma\, dx)$.

$$D(\mathcal{A}) \;:=\; \Big\{ w \in H^2(\mathbb{R}^d_+) \cap H^2(\mathbb{R}^d_-) \;:\; [w] = [\partial_1 w] = 0 \Big\}\,,$$

by folding them to a boundary value problem and using the results of [Rauch-Massey, Sakamoto]. ∎

Next consider the mathematical problem whose solution explains Snell's law. The idea is to send a wave in $x_1 < 0$ toward the boundary and ask how it behaves in the future. Suppose

$$\xi \in \mathbb{R}^d, \qquad |\xi| = 1\,, \qquad \xi_1 > 0\,,$$

and consider a short wavelength asymptotic solution in $\{x_1 < 0\}$ as in §1.6.3,

$$I^\epsilon \ \sim\ e^{i(x.\xi - t)/\epsilon}\, a(\epsilon, t, x)\,, \qquad a(\epsilon,t,x)\ \sim\ a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\,, \tag{1.7.5}$$

where for $t < 0$ the support of the a_j is contained in a tube of rays with compact cross section and moving with speed ξ. One can take a to vanish outside the tube. Since the incoming waves are smooth and initially vanish identically on a neighborhood of the interface $\{x_1 = 0\}$, the compatibilities are satisfied and there is a family of piecewise smooth solutions u^ϵ defined on $\mathbb{R}^{1+d}$. The tools prepared yield an infinitely accurate description of the family of solutions u^ϵ.

To solve the problem, seek an asymptotic solution which at $\{t = 0\}$ is equal to this incoming wave. A first idea is to find a transmitted wave which continues the incoming wave into $\{x_1\} > 0$.

Seek the transmitted wave in $x_1 > 0$ in the form

$$T^\epsilon \ \sim\ e^{i(x.\eta + t\tau)/\epsilon}\ d(\epsilon, t, x)\,, \qquad d(\epsilon,t,x)\ \sim\ d_0(t,x) + \epsilon\, d_1(t,x) + \cdots\,,$$

In order that this be an approximate solution moving away from the interface one must have

$$\tau^2 = c^2|\eta|^2, \qquad |\eta| = 1/c\,.$$

The incoming wave, when restricted to the interface $x_1 = 0$ oscillates with phase $(x'.\xi' - t)/\epsilon$. At the interface, the proposed transmitted wave oscillates with phase $(x'.\eta' - t\tau)/\epsilon$. In order that there be any chance at all of satisfying the transmission condtions one must take

$$\eta' = \xi', \qquad \tau = -1,$$

so that the two expressions oscillate together.

$$\mathcal{A}w := \Delta w \quad \text{in} \quad x_1 < 0, \qquad \mathcal{A}w := c^2\Delta \quad \text{in} \quad x_1 > 0\,.$$

Then,

$$(\mathcal{A}u, v)_{\mathcal{H}} \ =\ (u, \mathcal{A}v)_{\mathcal{H}} \ =\ -\int \nabla u.\nabla v\ dx\,,$$

so $-\mathcal{A} \geq 0$. The elliptic regularity theorem implies that $\mathcal{A}$ is self adjoint. The regularity theorem is proved, for example, by the methods in [Rauch 1992, Chapter 10]. The solution of the initial value problem is

$$u \ =\ \cos t\sqrt{-\mathcal{A}}\ f \ +\ \frac{\sin t\sqrt{-\mathcal{A}}}{\sqrt{-\mathcal{A}}}\ g\,.$$

For piecwise H^∞ data, the sequence of compatibilities is equivalent to the data belonging to $\cap_j D(\mathcal{A}^j)$.

The equation $\tau^2 = c^2|\eta|^2$ implies

$$\eta_1^2 \;=\; \frac{\tau^2}{c^2} - |\eta'|^2 \;=\; \frac{1}{c^2} - |\xi'|^2 \,.$$

Impose $\eta_1 > 0$ so the transmitted wave moves into the region $x_1 > 0$ to find

$$\eta_1 = \Big(\frac{1}{c^2} - |\xi'|^2\Big)^{1/2} \;>\; \xi_1.$$

Thus,

$$T^\epsilon \;\sim\; e^{i(x.\eta - t)/\epsilon}\, d(\epsilon, t, x)\,, \qquad \eta = \bigg(\Big(\frac{1}{c^2} - |\xi'|^2\Big)^{1/2}\,,\ \xi'\bigg). \tag{1.7.6}$$

From section 1.6.3 we know that the leading amplitude d_0 must be constant on the rays $t \mapsto (t, \underline{x} + c\,t\,\eta/|\eta|)$. To determine d_0 it suffices to know the values $d_0(t, 0^+, x')$ at the interface. One could choose d_0 to guarantee the continuity of u or of $\partial_1 u$, but not both. One cannot construct an good approximated solution consisting of just an incident and transmitted wave.
Add to the recipe a reflected wave. Seek a reflected wave in $x_1 \geq 0$ in the form

$$R^\epsilon \;\sim\; e^{i(x.\zeta + t\sigma)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots \,.$$

In order that the reflected wave oscillate with the same phase as the incident wave in the boundary $x_1 = 0$, one must have $\zeta' = \xi'$ and $\sigma = -1$. To satisfy the wave equation in $x_1 < 0$ requires $\sigma^2 = |\zeta|^2$. Together these imply $\zeta_1^2 = \xi_1^2$. To have propagation away from the boundary requires $\zeta_1 = -\xi_1$ so $\zeta = \tilde{\xi}$. Therefore,

$$R^\epsilon \;\sim\; e^{i(x.\tilde{\xi} - t)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots \,. \tag{1.7.7}$$

Summarizing seek

$$v^\epsilon \;=\; \begin{cases} I^\epsilon + R^\epsilon & \text{in} \quad x_1 < 0 \\ \quad T^\epsilon & \text{in} \quad x_1 > 0 \end{cases} .$$

The continuity required at $x_1 = 0$ forces

$$e^{i(x'.\xi' - t)/\epsilon}\, \big(a(\epsilon, t, 0, x') + b(\epsilon, t, 0, x')\big) \;=\; e^{i(x'.\xi' - t)/\epsilon}\, d(\epsilon, t, 0, x')\,. \tag{1.7.8}$$

The continuity of u and $\partial_1 u$ hold if and only if at $x_1 = 0$ one has

$$a \;+\; b \;=\; d\,, \qquad \text{and,} \qquad \frac{i\xi_1}{\epsilon} a + \partial_1 a \;-\; \frac{i\xi_1}{\epsilon} b + \partial_1 b = \frac{i\eta_1}{\epsilon} d + \partial_1 d\,. \tag{1.7.9}$$

The first of these relations yields

$$\Big(a_j \;+\; b_j \;-\; d_j\Big)_{x_1 = 0} = 0, \qquad j = 0, 1, 2, \dots\,, \tag{1.7.10}$$

The second relation in (1.7.9) is expanded in powers of ϵ. The coefficients of ϵ^j must match for all all $j \geq -1$. The leading order is ϵ^{-1} and yields

$$\big(a_0 - b_0 - (\eta_1/\xi_1) d_0\big)_{x_1 = 0} = 0\,. \tag{1.7.11}$$

Since a_0 is known, the $j = 0$ equation from (1.7.10) together with (1.7.11) yield a system of two linear equations for the two unknown b_0, d_0

$$\begin{pmatrix} -1 & 1 \\ 1 & \eta_1/\xi_1 \end{pmatrix} \begin{pmatrix} b_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 \end{pmatrix} .$$

Since the matrix is invertible, this determines the values of b_0 and d_0 at $x_1 = 0$.

The amplitude b_0 (resp. d_0) is constant on rays with velocity $\tilde{\xi}$ (resp. $c\eta/|\eta|$). Thus the leading amplitudes are determined throughout the half spaces on which they are defined.

Once these leading terms are known the ϵ^0 term from the second equation in (1.7.9) shows that on $x_1 = 0$,

$$a_1 - b_1 - d_1 = \text{known} .$$

Note that a_1 is also known so that together with the case $j = 2$ from (1.7.10) this suffices to determine b_1, d_1 on $x_1 = 0$. Each satisfies a transport equation along rays which is the analogue of (1.4.12). Thus from the initial values just computed on $x_1 = 0$ they are determined everywhere. The higher order correctors are detemined analogously.

Once the b_j, d_j are determined, one can choose b, c as functions of ϵ with the known Taylor expansions at $x = 0$. They can be chosen to have supports in the appropriate tubes of rays and to satisfy the transmission conditions (1.7.9) exactly.

The function u^ϵ is then an infinitely accurate approximate solution in the sense that it satisfies the transmission and initial conditions exactly while the residuals

$$v^\epsilon_{tt} - \Delta\, v^\epsilon := r^\epsilon \quad \text{in} \quad x_1 < 0\,, \qquad v^\epsilon_{tt} - c^2\, \Delta\, v^\epsilon := \rho^\epsilon,$$

satisfy for all N, s, T there is a C so that

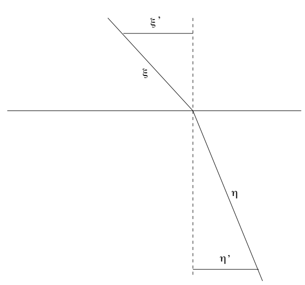
$$\big\| r^\epsilon \big\|_{H^s([-T,T]\times\{x_1<0\})} + \big\| \rho^\epsilon \big\|_{H^s([-T,T]\times\{x_1>0\})} \le C\,\epsilon^N .$$

From the analysis of the transmission problem it follows that with new constants,

$$\big\| u^\epsilon - v^\epsilon \big\|_{H^s([-T,T]\times\{x_1>0\})} \le C\,\epsilon^N .$$

The proposed problem of describing the family of solutions u^ϵ is solved.

The angles of incidence and refraction, θ_i and θ_r, given by the directions of propagation of the incident and transmitted waves. From the figure



one finds,

$$\sin\theta_i = \frac{|\xi'|}{|\xi|}, \qquad \text{and}, \qquad \sin\theta_r = \frac{|\eta'|}{|\eta|} = \frac{|\xi'|}{|\xi|/c} .$$

Therefore

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{1}{c},$$

is independent of θ_i. The high frequency asymptotic solutions explain Snell's law. This is the last of the three basic laws of geometric optics. The law depends only on the phases. The phases are determined by the requirement that the restriction of the phases to $x_1 = 0$ equal the restriction of the incoming phase. They do not depend on the transmission condition that we chose. It is for this reason that the conclusion is the same for the correct transmission problem for Maxwell's equations.

On a neighborhood $(\underline{t}, \underline{x}) \in \{x_1 = 0\}$ which is small compared to the scale on which a, b, c vary and large compared to ϵ, the solution resembles three interacting plane waves. In science texts one usually computes for which such triples the transmission condition is satisfied in order to find Snell's law. The asymmptotic solutions of geometric optics show how to overcome the criticism that the plane waves have modulus independent of (t, x) so cannot reasonably be viewed as either incoming or outgoing.

For a more complete discussion of reflection and refraction see [Taylor 1981, Benzoni-Gavage and Serre]. In particular these treat the phenomenon of *total reflection* which can anticipated as follows. From Snell's law one sees that $\sin\theta_r < 1/c$ and approaches that value as θ_i approaches $\pi/2$. The refracted rays lie in the cone $\theta_r < \arcsin(1/c)$. Reversing time shows that light rays from below approaching the surface at angles smaller than this critical angle traverse the surface tracing backward the old incident rays. For angles larger than $\arcsin(1/c)$ there is no continuation as a ray above the surface possible. One can show by constructing infinitely accurate approximate solutions that there is total reflection. Below the surface there is a reflected ray with the usual law of reflection. The role of a third wave is played by a boundary layer of thickness $\sim \epsilon$ above which the solution is $O(\epsilon^\infty)$.