Smooth Localized Parametric Resonance for Wave Equations

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§1. Introduction.

The classic example of parametric resonance for ordinary differential equations is a vibrating spring with time varying spring constant

$$\frac{d^2x}{dt^2} \ + \ k^2(t)\, x \ = \ 0\,, \qquad 0 < k \in C^\infty(\mathbb{R}), \quad \text{periodic}\,.$$

This example is studied using Floquet theory. A classic example is $k(t) = 1 + \varepsilon \cos \omega t$, where one finds *regions of instability* in the ε, ω plane for which there are solutions which grow exponentially in time. The instability regions are open and with closures touching $\varepsilon = 0$ at critical frequencies (see [A], [MW]). For any $\underline{t}$, the constant coefficient problems with k frozen at $k(\underline{t})$ is conservative and have no such growth.

In this paper we construct analogous examples for partial differential equations. That is variable coefficient wave equations with smooth time periodic coefficients so that the problems with coefficients frozen in time are conservative, but for which the time variation leads to exponential growth. The novelty is that the coefficients are constant outside a compact spatial domain. In such a situation one might expect waves to scatter to infinity and thereby escape the effect of the perturbations.

There are known examples of the form

$$u_{tt} \ - \ \partial_x\big(a(t)\partial_x u\big) \ = \ 0\,,$$

with $0 < a$ smooth and periodic (see [CS] and [CJS]). In that case, the analysis is by explicit solution using the Fourier transform in x to transform to ordinary differential equations. The method works as well for

$$u_{tt} - u_{xx} + p(t) u \ = \ 0\,,$$

with $0 < p$ periodic. It is harder to construct examples with localized perturbations.

There are well analysed examples in domains with periodically moving boundaries. The practical problem of assessing the impact of large wind mills on telecommunications falls into this domain. In the case of moving boundaries, growth can occur along multiply

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reflected rays analogous to a ping pong ball accelerated by multiple hits by a racquet. The definitive analysis is in [PR]. In the one dimensional case the analysis is easier and has a close link with the asymptotic behavior of iterates of maps from the circle to itself see [C2], [C3], and especially [CK]. In the mathematical physics literature one finds the related one dimensional examples [DDG] and [DD] and especially [DP].

In this paper we construct a compactly supported smooth perturbation, periodic in time, of the 1+2 dimensional wave equation which leads by parametric resonance to exponentially growing solutions.

The perturbations are in the leading order terms, so rays bend by refraction. The growth is created by producing a periodic ray which is in a sense amplifying.

We do not know how to construct analogous examples with perturbation term $p(t,x)\,u$ with $0\le p$ smooth compactly supported and periodic. In those cases the rays of geometric optics are straight lines and escape the region of perturbed coefficients in finite time and our strategy appears not to be effective. There are interesting numerical studies suggesting that such examples might exist and that they should be hard to find (see [C1], [C4]).

For the same reason, escape of rays, we do not know how to construct examples in dimension $d=1$. Our construction easily yields examples in all $d\ge 2$, though we present only the case $d=2$.

For $x\in\mathbb{R}^d$, consider the equation

$$u_{tt}\ -\ \operatorname{div}(a(t,x)\,\operatorname{grad} u)\ =\ 0\,, \tag{1.1}$$

with scalar strictly positive smooth a periodic in t with

$$a\ =\ 1,\qquad \text{when}\quad |x|\ \geq\ 2\,. \tag{1.2}$$

Denote by $\mathcal{H}$ the Hilbert space which is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|\phi\|^2_{\mathcal{H}}\ :=\ \int_{\mathbb{R}^d} |\operatorname{grad} u|^2\ dx\,. \tag{1.3}$$

For compactly supported smooth solutions of (1.1), the mappings $\mathcal{R}(t,s)$ defined by

$$C_0^\infty\times C_0^\infty\ \ni\ u_t(s,\cdot),u(s,\cdot)\quad\mapsto\quad u_t(t,\cdot),u(t,\cdot)\ \in\ C_0^\infty\times C_0^\infty \tag{1.4}$$

extend uniquely to bounded maps of $L^2\times\mathcal{H}$ to itself. The next proposition, proved in §2, gives a simple upper bound on the possible growth.

Proposition 1.1. *With*

$$M\ :=\ \Big\|\frac{a_t}{2\,a}\Big\|_{L^\infty(\mathbb{R}^{1+d})}\,,$$

one has the upper bound

$$\big\|\mathcal{R}(t,s)\big\|_{\operatorname{Hom}(L^2\times\mathcal{H})}\ \leq\ \left(\frac{\sup_{\mathbb{R}^{1+d}} a}{\inf_{\mathbb{R}^{1+d}} a}\right)^{1/2} e^{\frac{M}{2}|t-s|}\,.$$

For a satisfying,

$$0 \ < \ \frac{1}{C} \ \leq \ a \ \leq \ C\,, \tag{1.5}$$

this shows that the rate of exponential growth is at most a multiple of the $\|a_t\|_{L^\infty}$. Our main result shows that such rates of growth can be achieved.

Theorem 1.2. *For $d = 2$ and any $C > 1$ there is a constant $c > 0$ so that for all M there exist coefficients a satisfying (1.5) with $\|a_t\|_{L^\infty} > M$ and, solutions u of the associated wave equation (1.1) so that*

$$\liminf_{t\to\infty} \quad e^{-c\,\|a_t\|_{L^\infty}\, t}\ \big\|u_t(t), u(t)\big\|_{L^2\times\mathcal{H}} \ > \ 0\,.$$

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§2. The energy bound.

Proof of Proposition 1.1. For a smooth solution of (1.1) with compactly supported initial data the standard energy is defined by

$$e(t) \ := \ \int_{\mathbb{R}^d} \frac{|u_t|^2 \ + \ a(t,x)|\mathrm{grad_x} u|^2}{2}\ dx\,.$$

The basic energy estimate reads

$$\partial_t e \ = \ \int \frac{a_t}{2}\ |\mathrm{grad}\, u|^2\ dx \ = \ \int \frac{a_t}{2\,a}\ a\,|\mathrm{grad}\, u|^2\ dx \ \leq \ M\, e(t)\,.$$

Therefore,

$$e(t) \ \leq \ e^{M|t-s|}\, e(s)\,.$$

But using (1.2) one has $\inf_{\mathbb{R}^{1+d}} a \leq 1 \leq \sup_{\mathbb{R}^{1+d}} a$, so in addition one has

$$\frac{2}{\sup_{\mathbb{R}^{1+d}} a}\ e(t) \ \leq \ \big\|u_t, u\big\|^2_{L^2\times\mathcal{H}} \ \leq \ \frac{2}{\inf_{\mathbb{R}^{1+d}} a}\ e(t)\,.$$

Combining these two yields

$$\begin{aligned}\big\|u_t(t), u(t)\big\|^2_{L^2\times\mathcal{H}} \ &\leq \ \frac{2}{\inf a}\ e(t) \ \leq \ \frac{2}{\inf a}\ e^{M|t-s|}\ e(s)\\ &\leq \ \frac{2}{\inf a}\ e^{M|t-s|}\ \frac{\sup a}{2}\ \big\|u_t(s), u(s)\big\|^2_{L^2\times\mathcal{H}}\,,\end{aligned}$$

proving the desired bound. ∎

Remarks. i. We construct examples with periodic a showing that solutions can grow essentially as fast as the proposition allows. **ii.** The estimate is dimensionally correct.

§3. Trapped circular rays in $\mathbb{R}^2_x$.

The *bicharacteristics* of the wave equation (1.1) are defined to be the integral curves of the hamiltonian vector field with hamiltonian

$$H(t,x,\tau,\xi) := \frac{\tau^2 - a(t,x)\,|\xi|^2}{2}$$

along which the conserved hamiltonian vanishes. These curves in (t,x,τ,ξ) space belong to the characteristic variety of (1.1).

Hamilton's equations read

$$\frac{d\,(t,x)}{ds} = -\frac{\partial H}{\partial\,(\tau,\xi)}, \qquad \frac{d\,(\tau,\xi)}{ds} = \frac{\partial H}{\partial\,(t,x)}.$$

Therefore,

$$\frac{dt}{ds} = -\tau, \qquad \frac{dx}{ds} = \xi a, \qquad \frac{d\tau}{ds} = -\frac{|\xi|^2}{2}\frac{\partial a}{\partial t}, \qquad \frac{d\xi}{ds} = -\frac{|\xi|^2}{2}\nabla_x a\,.$$

Proposition 3.1. *If $d \geq 2$ and on $\{x \in \mathbb{R}^d : |x| = R\}$ the coefficient a satisfies*

$$2\,x.\nabla_x a = a\,,$$

then the set

$$\Big\{(t,x,\tau,\xi) \;:\; |x| = R, \quad \text{and} \quad x.\xi = 0\Big\} \tag{3.1}$$

is invariant under the hamiltonian flow.

Proof. It suffices to show that the hamiltonian field is tangent to the variety (3.1). Thus, it suffices to show that for orbits through a point of the variety one has

$$\frac{d\,|x|^2}{ds} = \frac{d\,x.\xi}{ds} = 0\,.$$

Compute

$$\frac{d\,|x|^2}{ds} = 2x.\frac{dx}{ds} = 2x.\xi\,a = 0\,,$$

since $x.\xi = 0$.

Similarly,

$$\frac{d\,x.\xi}{ds} \;=\; x\,.\,\frac{d\xi}{ds} + \frac{dx}{ds}\,.\,\xi \;=\; x.\Big(-\,\frac{|\xi|^2}{2}\,\nabla_x a\Big) + (\xi a).\xi \;=\; \frac{|\xi|^2}{2}\Big(2\,a \;-\; x.\nabla_x a\Big) \;=\; 0,$$

since $2\,a = x.\nabla_a$ when $|x| = R$. ∎

In dimension $d = 2$ this shows that if the circle, $\{|x| = R\}$, satisfies the hypotheses then the bicharacteristics starting on the circle with $\xi \perp x$ stay on the circle.

In the same vein, the set $\{\xi = 0\}$ is invariant. Thus if $\xi(0) \neq 0$, then $\xi(s) \neq 0$ for all s. Thus for such ξ, dx/dt is nonvanishing so the bicharacteristics lie on points over $|x| = R$ and their projections have nonvanishing speed.

In the two dimensional case, the space of t, x, τ, ξ space has dimension 6. The homogeneity with respect to τ, ξ means the dynamics is essentially five dimensional. The intersection of the set (3.1) with an energy surface is of codimension 3, therefore two dimensional. The six dimensional dynamics is reduced to dynamics in dimension 2.

To express that reduction use the standard identification $\mathbb{C} \ni u + iv \mapsto (u, v) \in \mathbb{R}^2$ of $\mathbb{R}^2$ with the complex plane. Writing

$$x \;=\; R\,e^{i\phi(s)}\,, \qquad \xi \;=\; \alpha(s)\,e^{i(\phi(s)+\pi/2)} \;=\; i\,\alpha(s)\,e^{i\phi(s)}\,,$$

the motion is parameterized by $\phi \in S^1$ and $\alpha \in \mathbb{R} \setminus 0$. For simplicity, consider only $\alpha > 0$. Use the Hamilton equations to find

$$\frac{dx}{ds} \;=\; R\,i\phi'\,e^{i\phi} \;=\; \xi\,a \;=\; \alpha\,i\,a\,e^{i\phi}\,.$$

Therefore

$$\phi' \;=\; \frac{\alpha\,a}{R}\,.$$

Similarly,

$$\frac{d\xi}{ds} \;=\; \alpha'\,i\,e^{i\phi} \;-\; \alpha\,\phi'\,e^{i\phi} \;=\; -\,\frac{|\xi|^2}{2}\,\nabla_x a \;=\; -\,\frac{\alpha^2}{2}\,\nabla_x a\,.$$

Taking the scalar product with the vector of length R tangent to $|x| = R$ in the sense of increasing θ one finds

$$R\,\alpha' \;=\; -\,\frac{\alpha^2}{2}\,\frac{\partial a}{\partial\theta}\,.$$

Summarizing, the dynamics of ϕ, α, τ, t is defined by the equations

$$\frac{d\phi}{ds} \;=\; \frac{\alpha\,a}{R}\,, \qquad \frac{d\alpha}{ds} \;=\; -\,\frac{\alpha^2}{2\,R}\,\frac{\partial a}{\partial\theta}\,, \qquad \frac{dt}{ds} \;=\; -\tau\,, \qquad \frac{d\tau}{ds} \;=\; -\frac{\alpha^2}{2}\,\frac{\partial a}{\partial t}\,. \tag{3.2}$$

The solutions of interest lie in the characteristic variety, that is, they satisfy

$$\tau^2 - a\,\alpha^2 \;=\; 0\,. \tag{3.3}$$

There are two roots

$$\tau \;=\; \pm\sqrt{a}\,\alpha\,. \tag{3.4}$$

We find a two dimensional dynamical system by eliminating s. First

$$\frac{d\phi}{dt} \;=\; \frac{d\phi/ds}{dt/ds} \;=\; \frac{\alpha\,a/R}{-\tau} \;=\; \frac{\alpha\,a/R}{\mp\alpha\sqrt{a}} \;=\; \mp\,\frac{\sqrt{a}(t,R\,e^{i\phi(t)})}{R}\,, \tag{3.5}$$

an ordinary differential equation for ϕ alone. Similarly,

$$\frac{d\alpha}{dt} \;=\; \frac{d\alpha/ds}{dt/ds} \;=\; \frac{-\alpha^2\,a_\theta/2R}{\mp\alpha\,\sqrt{a}} \;=\; \pm\,\frac{\alpha\,a_\theta}{2\,R\,\sqrt{a}}\,. \tag{3.6}$$

To recover the bicharacteristic from $\phi(t),\alpha(t)$, start with (3.4) to find $\tau(t)$. Since $ds/dt = 1/\tau(t)$ one recovers $s(t)$. Inversion yields $t(s)$ finishing the identification. Note that there are two such curves one for each of the sign choices in (3.4).

For $d=2$ and bicharacteristics turning above $|x|=R$, (3.5) shows that

$$0 \;<\; \frac{\inf\sqrt{a}}{R} \;\le\; \Big|\frac{d\phi}{dt}\Big| \;\le\; \frac{\sup\sqrt{a}}{R} \;<\; \infty.$$

Therefore the projected bicharacteristic turns steadily, winding around and around.

§4. The Rauch-Taylor lower bound.

For each t,

$$-\mathrm{div_x}\, a(t,x)\,\mathrm{grad}_x$$

is a positive, and not strictly positive, self adjoint operator on $L^2(\mathbb{R}^d)$ with domain equal to the Sobolev space $H^2(\mathbb{R}^d)$. Denote by $B(t)$ its positive square root. For each t the map $B(t)$ is an an bijection from $\mathcal{H}$ to $L^2(\mathbb{R}^d)$ with norm and norm of B^{-1} bounded uniformly in time.

Write the wave equation in the form

$$u_{tt} \;-\; \big(\mathrm{div_x}\, a(t,x)\,\mathrm{grad}_x\big)u \;=\; u_{tt} + B(t)^2 u \;=\; 0\,, \qquad 0 \le B = B^*\,. \tag{4.1}$$

Introduce

$$\mathcal{U}(t) \;:=\; (v(t), w(t)) \;:=\; (u_t\,,\, B(t)\,u)\,. \tag{4.2}$$

Then

$$v_t = u_{tt} = -B^2\,u = -Bw\,, \qquad w_t = B\,u_t + B_t u = B\,v + B_t B^{-1} B\,u = B\,v + B_t B^{-1} w\,.$$

This yields the system

$$\mathcal{U}_t \;+\; \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}\mathcal{U} \;+\; \begin{pmatrix} 0 & 0 \\ 0 & -B_t\,B^{-1} \end{pmatrix}\mathcal{U} \;=\; 0\,. \tag{4.3}$$

This a a perturbation of the equation

$$\mathcal{V}_t \;+\; \begin{pmatrix} 0 & B(t) \\ -B(t) & 0 \end{pmatrix} \mathcal{V} \;=\; 0.$$

The latter generates unitary maps on $L^2(\mathbb{R}^d)$. Since $B_t B^{-1}$ is a smooth family of bounded operators on L^2, (4.3) generates a family, $S(t,s)$ of continuous linear maps on $L^2(\mathbb{R}^d)$.

That is, $\mathcal{U}(t) := S(t,s)f$ is the solution of (4.3) which satisfies the initial condition $\mathcal{U}(s) = f$. The operator $S(t,s)$ maps

$$u_t(s,\cdot)\,,\; B(s)u(s\,,\,\cdot) \quad\mapsto\quad u_t(t,\cdot)\,,\; B(t)u(t\,,\,\cdot)$$

Comparing the definition of $\mathcal{R}$ from (1.4) and S yields,

$$\mathcal{R}(t,s) \;=\; \begin{bmatrix} I & 0 \\ 0 & B(t)^{-1} \end{bmatrix} \; S(t,s) \; \begin{bmatrix} I & 0 \\ 0 & B(s) \end{bmatrix}\,. \tag{4.4}$$

The matrix on the right is an isomorphism from $L^2 \times \mathcal{H}$ to $L^2 \times L^2$ and the matrix on the left an isomorphism in the opposite direction. Their norms are bounded independent of t, s.

The operator $B(t)$ is a classical pseudodifferential operators of degree 1 (see [S], [T]). The principal symbol of B is equal to

$$b(t,x,\xi) \;:=\; \sqrt{a}\;|\xi| \;>\; 0\,.$$

The symbol of B is smooth in $\xi \neq 0$, 2π periodic in time, and classical

$$B(t,x,\xi) \;\sim\; b(t,x,\xi) + \sum_{j=1}^{\infty} b_{-j}(t,x,\xi)\,,$$

with b_{-j} homogeneous of degree $-j$ and smooth on $\mathbb{R}_t \times \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0)$. For $|x| \geq 2$, $B(t,x,\xi)$ is identically equal to $|\xi|$. Each symbol $b_j(t,x,\xi)$ is independent of x for $|x| \geq 2$.

The paper of Rauch-Taylor [RT] is devoted to pseudodifferential systems like (4.3) on compact manifolds. Their basic computations apply directly to (4.3). Their analysis gives lower bounds for the norm of $\chi S(t,s)\chi$ modulo compact operators when $\chi \in C_0^\infty(\mathbb{R}^d_x)$. In [RT] a central role is played by the eigenprojectors $p_\pm(t,x,\xi)$ of the principal symbol

$$\begin{pmatrix} 0 & \sigma(B)(t,x,\xi) \\ -\sigma(B)(t,x,\xi) & 0 \end{pmatrix}\,.$$

For our problem that symbol is a scalar function of t, x, ξ times a fixed matrix. Therefore our $p_\pm$ are independent of (t,x,ξ) and are equal to

$$p_\pm \;=\; \frac{1}{2}\begin{pmatrix} -1 & \pm i \\ \mp i & 1 \end{pmatrix}\,.$$

The scalar functions $\gamma_\pm(t,x,\xi)$ defined by

$$p_\pm \begin{pmatrix} 0 & 0 \\ 0 & -\sigma(B_t\, B^{-1}) \end{pmatrix} p_\pm \;=\; \gamma_\pm\, p_\pm\,, \tag{4.5}$$

are the key growth and decay coefficient in the transport equation (4.7) below. To compute $\gamma_\pm$, look at the $(2,2)$ (i.e. second row second column) elements of both sides. Since

$$\begin{pmatrix} 0 & 0 \\ 0 & -\sigma(B_t\, B^{-1}) \end{pmatrix} p_\pm \;=\; \begin{pmatrix} 0 & 0 \\ * & -\sigma(B_t B^{-1})/2 \end{pmatrix},$$

the $(2,2)$ element of the left hand side of (4.5) is equal to $\sigma(B_t B^{-1})/4$. This is equal to $\gamma_\pm/2$ yielding the formula

$$\gamma_\pm \;=\; \frac{b_t(t,x,\xi)}{2\, b(t,x,\xi)} \;=\; \frac{\partial_t \sqrt{a}}{2\sqrt{a}} \;=\; \frac{a_t}{4\, a}\,. \tag{4.6}$$

The fundamental transport equation of [RT] is

$$\frac{d\beta}{ds} \;+\; \gamma(t(s),x(s),\xi(s))\,\beta \;=\; 0\,. \tag{4.7}$$

with $(t(s),x(s),\tau(s),\xi(s))$ a bicharacteristic. The quantity β measures amplification of amplitudes associated to solutions microlocally localized along the bicharacteristic. The coefficient of β in (4.7) is an exact time derivative of a bounded function. It need not be an exact s derivative. This is what allows the possibility of growth.

The transport equation is analysed by introducing the integrating factor, $\Gamma(s)$ defined by

$$\frac{d\Gamma}{ds} \;=\; \gamma(t(s),x(s),\xi(s))\,, \qquad \Gamma(0) \;=\; 0\,. \tag{4.8}$$

Equivalently

$$\Gamma \;=\; \int_0^s \gamma(t(s),x(s),\xi(s))\; ds\,.$$

Then

$$\beta(s) \;=\; e^{-\Gamma(s)}\;\beta(0)\,.$$

The goal is to find examples where $\Gamma \to -\infty$ as $t(s) \to \infty$. Rauch-Taylor give the following estimate.

Theorem 4.1. *Suppose that $(t(s),x(s),\tau(s),\xi(s))$ is a bicharacteristic and that $\chi \in C_0^\infty(\mathbb{R}^d)$ is identically equal to one on a neighborhood of $x([0,\underline{s}])$ and $\underline{t} := t(\underline{s})$. Then with $\mathcal{K}$ denoting the ideal of compact operators in* $\mathrm{Hom}(L^2(\mathbb{R}^d))$,

$$\|\chi S(\underline{t},0)\chi\|_{\mathrm{Hom}(L^2(\mathbb{R}^d))/\mathcal{K}} \;\geq\; e^{-\Gamma(\underline{s})}\,.$$

In the case of compact manifolds, the supremum of the right hand side over bicharacteristics is an exact formula for the norm of S modulo compacts.

Change the integration variable in the definition of Γ to t to find,

$$\Gamma \;=\; \int_0^s \gamma\; ds \;=\; \int_0^{t(s)} \frac{a_t}{4\,a}\,\frac{ds}{dt}\; dt \;=\; \int_0^{t(s)} \frac{a_t}{4a}\,\frac{1}{\mp\alpha\sqrt{a}}\; dt \;=\; \int_0^{t(s)} \frac{\mp a_t}{4\,a^{3/2}}\,\frac{1}{\alpha}\; dt\,. \qquad (4.9)$$

The sign is from (3.4). We construct a so that Γ is negative.

§5. Amplifying circular bicharacteristics.

The method is to construct, at the same time, the function $a(t,x)$, 2π periodic in time, and a periodic null bicharacteristic which lies over the curve $(t, e^{i\phi(t)}) \in \mathbb{R}^{1+2}_{t,x}$.

The projection will make one complete circle each 2π units of time and during that interval the function Γ will decrease by a fixed nonzero amount. Then as t tends to ∞, Γ approaches $-\infty$ at an essentially linear rate. From this we will be able to deduce the existence of solutions of the wave equation which grow exponentially in time.

Choose a so that $\{|x| = 1\} \cap \{x.\xi = 0\}$ is invariant. The proposition of §3 shows that it is sufficient that

$$a_r(t,r,\theta) \;=\; a(t,r,\theta)/2\,, \quad \text{when} \quad r = 1\,. \qquad (5.1)$$

Here we use standard polar coordinates r, θ.

Choose $\phi(t)$ with $d\phi/dt$ strictly positive and 2π periodic. In addition suppose that

$$\phi(2\pi) - \phi(0) \;=\; 2\pi\,,$$

so that $e^{i\phi(t)}$ winds once about the origin as t increases from 0 to 2π.

Equation (3.5) shows that in order that $(t, 1, \phi(t))$ be the projection in polar coordinates of a null bicharacteristic it is necessary and sufficient that

$$a(t, 1, \phi(t)) \;=\; (\phi'(t))^2\,. \qquad (5.2)$$

We are interested in $a(t,r,\theta)$ for $r \approx 1$. It is 2π periodic as a function of t and of θ. Formula (5.2) shows that if a bicharacteristic is to project to $(t, 1, \phi(t))$ then the values of a are determined on the curve $(t, 1, \phi(t))$. Differentiating (5.2) yields,

$$a_t + \phi'\, a_\theta \;=\; 2\,\phi'\;\phi'', \qquad \text{on} \quad \big\{\big(t, 1, \phi(t)\big)\;:\; t \in \mathbb{R}\big\}\,. \qquad (5.3)$$

Equations (5.1) and (5.2) are two linear constraints on the differential of a on $\{t, 1, \phi(t)\}$. There is an additional derivative which can be assigned arbitrarily along this set. We take it to be

$$a_\theta(t, 1, \phi(t)) \;:=\; p(t)\,. \qquad (5.4).$$

Choosing $p(t)$ appropriately, allows us to control Γ.

Lemma 5.1. *If $p(t)$ is a smooth 2π-periodic function, then there is a strictly positive function $a(t,r,\theta)$ identically equal to one outside a small neighborhood of $r=1$, $2\pi\times 2\pi$ periodic in (t,θ) so that (5.1), (5.2), (5.3), and, (5.4) are satisfied.*

Proof. The explicit formula

$$a(t,r,\theta) \;=\; e^{(r-1)/2}\,(\phi')^2\,\Big(1 \;+\; \sin(\theta-\phi(t))\,p(t)\Big)\,,$$

has the desired properties on a neighborhood of $r=1$.

Use a partition of unity consisting of two functions, one of which is supported near $\{r=1\}$ and identically equal to 1 on a smaller neighborhood of $\{r=1\}$, to finish the construction. ∎

Lemma 5.2. *Suppose that $R=1$ and that p is a smooth 2π-periodic function satisfying*

$$\int_0^{2\pi} \frac{p(t)}{\phi'(t)}\;dt \;=\; 0\,. \tag{5.5}$$

Then, the null bicharacteristic which project to $(t,\phi(t))$, when reparameterized as a function of t

$$\Big(t\,,\,x(s(t))\,,\,\xi(s(t))\,,\,\tau(s(t))\Big)$$

is 2π periodic.

Proof. Advancing time by 2π increases ϕ by 2π so, $x(s(t))=e^{i\phi(t)}$ is 2π periodic.

From (3.6),

$$\frac{d\ln\alpha}{dt} \;=\; \pm\frac{a_\theta}{2\sqrt{a}} \;=\; \pm\frac{p(t)}{2\,\phi'(t)}\,.$$

As the right hand side is periodic with integral zero, it follows that $\ln\alpha$ and therefore α is 2π periodic. This shows that $\xi(s(t))=i\alpha e^{i\phi}$ is 2π periodic.

Then (3.4) together with the fact that α cannot vanish shows that τ is 2π periodic. ∎

Next examine the behavior of Γ when the bicharacteristic runs over a period. Define $\overline{s}$ by $t(\overline{s})=2\pi$. The preceding lemma implies that as a function s the bicharacteristic is $\overline{s}$ periodic so,

$$\forall n\in\mathbb{N},\quad t(n\overline{s}) \;=\; 2\pi n\,.$$

Use (4.9) with $s=\overline{s}$, and, replace a_t and $a^{1/2}$ using (5.2), (5.3) and (5.4) respectively to find

$$\Gamma \;=\; \mp\int_0^{2\pi} \frac{2\,\phi'\,\phi''-\phi'\,p(t)}{4\,(\phi')^3\,\alpha}\;dt\,.$$

The sign comes from (3.4).

Seek $p(\theta)$ satisfying (5.5) so that Γ is negative. If Γ is nonnegative for all $p(t)$ satisfying (5.5) it is necessary that for all such p,

$$\int_0^{2\pi} \frac{\phi'\, p(t)}{(\phi')^3\,\alpha}\, dt \;=\; \int_0^{2\pi} \frac{p(t)}{(\phi')^2\,\alpha}\, dt \;=\; 0\,.$$

Equivalently

$$p \;\perp\; \frac{1}{\phi'(t)} \quad\Longrightarrow\quad p\;\perp\; \frac{1}{\alpha(t)(\phi'(t))^2}\,.$$

For this to be so it is necessary (and sufficient) that $1/\phi'$ be proportional to $1/((\phi')^2\alpha)$. Equivalently,

$$\phi'\,\alpha \;=\; \text{constant.} \tag{5.6}$$

Proposition 5.3. *If q satisfies*

$$q\;\neq\; 2\,\phi'' \qquad \text{and} \qquad q\;\neq\; -2\,\phi''\,, \tag{5.7}$$

and

$$\int_0^{2\pi} \frac{q(t)}{\phi'(t)}\, dt \;=\; 0\,,$$

then there are constants c_1, c_2 with $c_2 \neq 0$ so that with $p = \lambda q$ one has,

$$\Gamma \;=\; c_1 \mp c_2\,\lambda\,. \tag{5.8}$$

The sign is determined from (3.4). In particular, choosing λ appropriately, Γ can be made as negative as one likes.

Proof. The formula for Γ is first degree polynomial in λ. It suffices to show the coefficient of λ is nonzero. The coefficient is zero only when (5.6) is satisfied.
Differentiate (5.6) to find

$$\phi''\,\alpha \;+\; \phi'\,\alpha' \;=\; 0\,.$$

Then,

$$\big(\ln\alpha\big)' \;=\; \frac{\alpha'}{\alpha} \;=\; -\frac{\phi''}{\phi'}\,.$$

On the other hand, using (3.6) yields

$$\frac{d\ln\alpha}{dt} \;=\; \frac{1}{\alpha}\,\frac{d\alpha}{dt} \;=\; \frac{\pm\,a_\theta}{2\,\sqrt{a}} \;=\; \frac{\pm p(t)}{2\,|\phi'|} \;=\; \frac{\pm\text{sign}(\phi')\;p(t)}{2\,\phi'}\,.$$

These two expressions are equal only when $p(t) = 2\phi''$ or $p(t) = -2\phi''$. Thus (5.6) can hold only when hypothesis (5.7) is violated. ∎

§6. The main theorems.

Theorem 6.1. *Suppose that ϕ, $p(t)$, χ and a are chosen as above, with $\Gamma < 0$. Then for any $\varepsilon > 0$, there is a solution $u \in C(\mathbb{R}\,;\,\mathcal{H}) \cap C^1(\mathbb{R}\,;\,L^2(\mathbb{R}^d))$ of (1.1) and an infinite sequence $n_1 < n_2 < \cdots$ of integers so that*

$$\Big\|\chi\Big(u_t(2\pi n_j)\,,\,u(2\pi n_j)\Big)\Big\|_{L^2(\mathbb{R}^d)\times\mathcal{H}} \;\geq\; e^{(|\Gamma|-\varepsilon)n_j}\,. \tag{6.1}$$

The example is constructed with initial data supported in $\operatorname{supp}\chi$.

Proof. The assertion is equivalent to finding an $f \in L^2 \times \mathcal{H}$ so that

$$\|\chi\,\mathcal{R}(2\pi n_j, 0)f\|_{L^2\times\mathcal{H}} \;\geq\; e^{(|\Gamma|-\varepsilon)n_j}\,.$$

For time $t = 2\pi n$ one makes n circuits of the amplifying bicharacteristic and finds $\Gamma|_{t=2\pi n} = n\,\Gamma|_{2\pi}$. The estimate of Rauch-Taylor yields

$$\big\|\chi\,S(2\pi n, 0)\chi\big\|_{\mathrm{Hom}(L^2)/\mathcal{K}} \;\geq\; c_3\,e^{|\Gamma|n}\,. \tag{6.2}$$

The calculus of pseudodifferential operators implies that the commutators

$$\big[\chi\,,\,B(t)^{-1}\big]\,, \qquad \text{and} \qquad \big[\chi\,,\,B(s)\big]\,,$$

are compact in

$$\mathrm{Hom}(L^2;\mathcal{H}) \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{H};L^2)$$

respectively. Using this, identity (4.4) implies

$$\chi\,\mathcal{R}(t,s)\chi \;\equiv\; \begin{bmatrix} I & 0 \\ 0 & B(t)^{-1} \end{bmatrix}\;\chi\,S(t,s)\,\chi\;\begin{bmatrix} I & 0 \\ 0 & B(s) \end{bmatrix} \quad \mathrm{mod}\;\;\mathcal{K}\,.$$

Together with (6.2) this yields,

$$\big\|\chi\,\mathcal{R}(2\pi n, 0)\chi\big\|_{\mathrm{Hom}(L^2)/\mathcal{K}} \;\geq\; c_4\,e^{|\Gamma|n}\,.$$

Since,

$$\big\|\chi\,\mathcal{R}(2\pi n, 0)\chi\big\|_{\mathrm{Hom}(L^2)} \;\geq\; \big\|\chi\,\mathcal{R}(2\pi n, 0)\chi\big\|_{\mathrm{Hom}(L^2)/\mathcal{K}}\,,$$

the sequence of operators

$$e^{(-|\Gamma|+\varepsilon)n}\;\chi\,\mathcal{R}(2\pi n, 0)\,\chi$$

is not uniformly bounded. The uniform boundedness principal implies that there is a Cauchy datum g so that

$$\Big\{e^{(-|\Gamma|+\varepsilon)n}\;\chi\,\mathcal{R}(2\pi n, 0)\,\chi\,g \;:\; n = 1, 2, \ldots\;\Big\}$$

is not bounded in $L^2 \times \mathcal{H}$.

Choose $n_1 < n_2 < \cdots$ so that

$$\left\| e^{(-|\Gamma|+\varepsilon)n_j} \; \chi \, \mathcal{R}(2\pi n_j, 0) \, \chi \, g \right\|_{L^2} \;\geq\; 1,$$

and monotonically increase to infinity as $j \to \infty$. Setting

$$f \;:=\; \chi \, g$$

yields (6.1) and the compact support. ∎

Proof of Theorem 1.2. The rate of exponential growth is given by Γ which is essentially linear in λ. Clearly one can carry out the construction so that (1.5) is satisfied and with $\sup a_t \leq \text{const.}\, \lambda$ for $\lambda >> 1$. In this case the rate of exponential growth is $\sim \Gamma \geq c\lambda$. ∎

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