Diffractive Short Pulse Asymptotics for Nonlinear Wave Equations

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Abstract

We present an algorithm for constructing approximate solutions to nonlinear wave propagation problems in which diffractive effects and nonlinear effects come into play on the same time scale. The approximate solutions describe the propagation of short pulses. In a separate paper the equations used to construct the approximate solutions are derived using the method of multiple scales and the approximate solutions are proved accurate in the short wavelength limit. We present numerical studies which even in the linear case indicate significant qualitative differences between these approximations and those derived using the slowly varying envelope ansatz.

1   Introduction

Wave propagation problems from water waves to electromagnetic waves have been studied through high frequency asymptotics, analyzing the solutions in the limit as the wavelength approaches zero. Solutions are often assumed to be wavetrains, expressed as a slowly varying amplitude multiplying a rapidly oscillating phase such as $A(t,x)e^{i(k \cdot x - \omega t)/\varepsilon}$. To see diffractive effects, a slow time scale $T = \varepsilon t$ is introduced, leading to solutions of the form $A(\varepsilon t, t, x)e^{i(k \cdot x - \omega t)/\varepsilon}$. This approach, applied to nonlinear Maxwell's equations, yields the Nonlinear Schrödinger equation (NLS) as a description of the evolution of the amplitude function $A(T, t, x)$. Note that the variations in the amplitude A happen on space-time scales $\mathcal{O}(1)$ and $\mathcal{O}(1/\varepsilon)$, much slower than the variations in the phase $e^{ix/\varepsilon}$ which happen on space-time scale $\mathcal{O}(\varepsilon)$. The equations determining the amplitude $A(T, t, x)$ are simpler than the original system and A does not vary on the short scale ε so the equations can be solved numerically without using an $\mathcal{O}(\varepsilon)$ mesh.

The development of ultrafast lasers in the last decade raises questions about the slowly varying amplitude assumption. These lasers typically pass a point in times measured in femtoseconds and contain only a few wavelengths. Currently
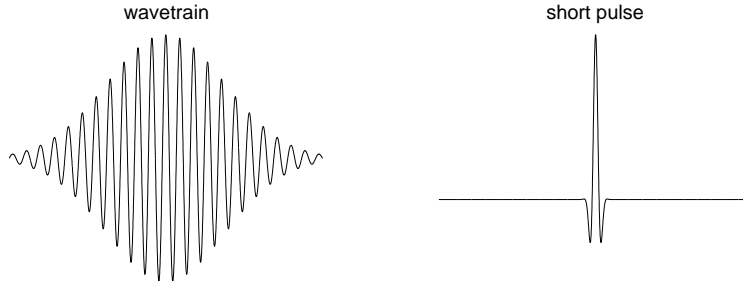
Figure 1: Example of a Wavetrain and a Short Pulse

scientists are working to develop lasers which produce "Half Cycle Pulses" [1]. The output of these lasers last only half a wavelength. Figure 1 gives examples of a wavetrain, lasting many wavelengths, and a short pulse.

Notice that the wavetrain oscillates rapidly, and can be considered to have integral zero. In other words, the spectrum of the wavetrain has no DC component. This description does not apply to the short pulse solution. In fact, ultrafast laser pulses are created using a broad spectrum including low frequency components. This difference lies at the heart of the different profile equations which describe the propagation of wavetrains or short pulses, as discussed at the end of Section 3.

Two major approaches have been used to address this problem. The full Maxwell's equations can be studied as in [2]-[6]. This approach yields important insight into the differences between short pulse and wavetrain solutions, but is costly. Few physical experiments yield explicit solutions and computations on the full system must use a mesh which resolves the solution on the small $\mathcal{O}(\varepsilon)$ scale.

Another line of inquiry, taken in [7]-[12] uses asymptotics to derive new approximate solutions valid for short pulse initial data. Sometimes weaker assumptions about the pulse length or the evolution of the amplitude are substituted for the slowly varying envelope assumption.

In this Letter we present an approximate solution derived by asymptotic analysis. Note that we make few assumptions, clearly stated in Section 3 and derive approximate solutions for an entire class of nonlinear wave propagation problems. This gives a result more general than those in [7] - [12] while avoiding the intensive computations of [2] - [6].

2 Asymptotics for a Linear Example

First we consider the linear case, because an explicit solution can be found using Fourier analysis. Consider solutions of the wave equation for $t, x \in \mathbf{R}^{1+2}$:

$$\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u = 0.$$

Denote by ε a small parameter, representing the wavelength of light. Consider the wavetrain with initial data having slowly varying amplitude, given by

$$u^\varepsilon(0, x_1, x_2) = f(x_1, x_2)e^{ix_1/\varepsilon} \qquad \partial_t u^\varepsilon(0, x_1, x_2) = 0.$$

2

The solution is a sum of the terms

$$u^\varepsilon_\pm(t, x) = \frac{1}{4\pi} \int \hat{f}(\xi_1 - 1/\varepsilon, \xi_2) e^{i(x \cdot \xi \mp t|\xi|)} d\xi.$$

Make the change of variables $(\zeta_1, \zeta_2) = (\xi_1 - 1/\varepsilon, \xi_2)$, and expand the exponent in powers of ε to find

$$\varepsilon|\xi| = \sqrt{(1 + \varepsilon\zeta_1)^2 + (\varepsilon\zeta_2)^2} \approx 1 + \varepsilon\zeta_1 + \frac{\varepsilon^2}{2}\zeta_2^2 + \mathcal{O}(|\varepsilon\zeta|^3) \qquad (1)$$

Using the leading term $1 + \varepsilon\zeta_1$ in the integral yields

$$u^\varepsilon_\pm(t, x) \approx \frac{1}{4\pi} e^{i(x_1 \mp t)/\varepsilon} \int \hat{f}(\zeta) e^{i\zeta_1(x_1 \mp t)} e^{i\zeta_2 x_2} d\zeta = f(x_1 \mp t, x_2) e^{i(x_1 \mp t)/\varepsilon} + \mathcal{O}(\varepsilon t).$$

In other words, for times $t \ll 1/\varepsilon$, the solution $u^\varepsilon_\pm$ travels with speed ± 1 in the positive x_1 direction without changing shape.

This approximation has a clear flaw. Amplitudes remain constant along rays $x_1 - t = c$, however exact solutions to the wave equation on $\mathbf{R}^{1+2}_{t,x}$ decay like $t^{-1/2}$ as $t \to \infty$. Therefore for long times, a more accurate asymptotic description is needed which includes diffractive effects. This can be found by including the next term in (1):

$$u^\varepsilon_+(t, x) \approx \frac{1}{4\pi} e^{i(x_1 - t)/\varepsilon} \int \hat{f}(\zeta) e^{i\varepsilon t\zeta_2^2/2} e^{i\zeta_1(x_1 - t)} e^{i\zeta_2 x_2} d\zeta.$$

To analyze this formula, introduce the slow time variable $T = \varepsilon t$. Then $u^\varepsilon_+(t, x) \approx A(\varepsilon t, x_1 - t, x_2) e^{i(x_1 - t)/\varepsilon}$ where

$$A(T, x) = \frac{1}{4\pi} \int \hat{f}(\zeta) e^{iT\zeta_2^2/2} e^{i\zeta_1 x_1} e^{i\zeta_2 x_2} d\zeta \qquad (2)$$

Note that $A(T, x_1, x_2)$ satisfies the Schrödinger equation:

$$\partial_T A = -\frac{i}{2} \partial^2_{x_2} A \qquad A(0, x_1, x_2) = f(x_1, x_2) \qquad (3)$$

This is how the Schrödinger equation arises to describe the evolution of amplitudes of wavetrain solutions.

Differences arise when short pulse initial data are studied. Consider solutions with new initial data:

$$u^\varepsilon(0, x_1, x_2) = f(x_2, x_1/\varepsilon) \qquad \partial_t u^\varepsilon(0, x_1, x_2) = 0$$

where $\lim_{|\varphi| \to \infty} f(x_2, \varphi) = 0$. These initial data represent a pulse of width $\mathcal{O}(\varepsilon)$ in x_1 and width $\mathcal{O}(1)$ in x_2. Such initial data better describe the output of an ultrafast laser. The width in the x_2 direction could correspond to the width of the laser beam.

The short pulse solution is a sum of the terms

$$u^\varepsilon_\pm(t, x) = \frac{1}{4\pi} \int \varepsilon \hat{f}(\xi_2, \varepsilon\xi_1) e^{i(x \cdot \xi \mp t|\xi|)} d\xi.$$

Make the change of variables $\rho = \varepsilon\xi_1$ and expand the exponent in powers of ε to find

$$\varepsilon|\xi| = |\rho|\sqrt{1 + (\varepsilon\xi_2/\rho)^2} \approx |\rho| \left(1 + \frac{(\varepsilon\xi_2/\rho)^2}{2} + \mathcal{O}((\varepsilon\xi_2/\rho)^3)\right) \qquad (4)$$

3

Using the leading term $|\rho|$ in the integral, and regrouping $u_{\pm}^{\varepsilon}$ yields a sum of terms of the form

$$u_{\pm}^{\varepsilon}(t,x) \approx \frac{1}{4\pi} \int \hat{f}(\xi_2, \rho) e^{i\xi_2 x_2} e^{i\rho(x_1 \mp t)/\varepsilon} d\xi_2 d\rho = f(x_2, (x_1 \mp t)/\varepsilon) + \mathcal{O}(\varepsilon t).$$

This has the same form, valid for times $t \ll 1/\varepsilon$, as found for wavetrain initial data, and faces the same problems. Again we include additional terms from (4) to find an asymptotic description valid for longer times. The additional terms yield

$$u_{\pm}^{\varepsilon}(t,x) \approx \frac{1}{4\pi} \int \hat{f}(\xi_2, \rho) e^{\mp i\varepsilon t \xi_2^2 / 2\rho} e^{i\xi_2 x_2} e^{i\rho(x_1 \mp t)/\varepsilon} d\xi_2 d\rho.$$

Introduce the slow time variable $T = \varepsilon t$. Then $u_{+}^{\varepsilon}(t,x) \approx B(\varepsilon t, x_2, (x_1 - t)/\varepsilon)$ where

$$B(T, x_2, \varphi) = \frac{1}{4\pi} \int \hat{f}(\xi_2, \rho) e^{-iT\xi_2^2 / 2\rho} e^{i\xi_2 x_2} e^{i\rho\varphi} d\xi_2 d\rho$$

Note that $B(T, x_2, \varphi)$ satisfies a *different* linear equation.

$$\partial_T \partial_\phi B = -\frac{1}{2} \partial_{x_2}^2 B \qquad B(0, x_2, \varphi) = f(x_2, \varphi) \tag{5}$$

Differences emerge even in the linear case because of the different form of the initial data. Section 4 shows some qualitative differences between the wavetrain approximation given by (3) and the short pulse approximation given by (5).

3   Nonlinear Problem

An approximate solution can be constructed for a general class of semilinear problems. Let $y = (t, x) \in \mathbf{R}_y^{d+1}$ and consider the behavior for $t \sim 1/\varepsilon$ of solutions to a system of equations

$$L(\partial_y)u^\varepsilon + \Phi(u^\varepsilon) = 0 \qquad u^\varepsilon(0, x) = \varepsilon^p f(x, (k \cdot x)/\varepsilon) \tag{6}$$

where $k \in \mathbf{R}^d$, which satisfies the following assumptions.

Assumption 0. Short pulse initial data. The function $f(x, \varphi)$ is in $H^s(\mathbf{R}_{x,\varphi}^{d+1})$ for all $s > 0$ and decays rapidly in φ.

Assumption 1. Symmetric hyperbolicity. The operator $L(\partial_y)$ can be written as $\partial_t + \sum_{j=1}^d A_j \partial_{x_j}$ where the coefficients A_j are $N \times N$ are hermitian symmetric matrices. This condition holds for most wave equations.

Assumption 2. Order J nonlinearity. The nonlinear function $\Phi(u)$ is of order $J \geq 2$ in the sense that for all $|\alpha| \leq J - 1$, $\partial_u^\alpha \Phi(0) = 0$. We denote by $\Phi_J(u)$ the homogeneous polynomial of degree J which is the first nonzero term of the Taylor expansion of $\Phi(u)$ about 0. For optics applications, typically $J = 2$ or $J = 3$.

We then seek an approximate solution of the form

$$\mathrm{U}^\varepsilon(y) = \varepsilon^p U_0(T, y, \varphi) \Big|_{T = \varepsilon t, \varphi = (k \cdot x - \omega t)/\varepsilon} \tag{7}$$

Assumption 3. Smooth characteristic variety. ω and k satisfy the dispersion relation smoothly. In other words, the matrix $L(\omega, k)$ is singular, equivalently

(ω, k) is characteristic, and the characteristic variety is smooth there. Fix $k = k_0$ and write $\omega = \omega(k_0)$.

Assumption 4. Magnitude of the solution. The exponent p is chosen so that $p = 1/(1 - J)$. This insures that nonlinear effects and diffractive effects both have the same order of magnitude and come into play on the same time scale.

Definition 1. Define the orthogonal projection operator $\pi = \pi(k_0)$ which projects into the kernel of $L(\omega, k_0)$. Define a partial inverse $Q(k_0)$ by

$$Q\pi = 0 \qquad QL(\omega, k_0)w = (I - \pi)w$$

for all vectors $w \in \mathbf{R}^d$.

Assumption 5. Polarization. Both U_0 and the initial data f satisfy the polarization condition $\pi U_0(T, y, \varphi) = U_0(T, y, \varphi)$ and $\pi f(x, \varphi) = f(x, \varphi)$.

Definition 2. Define the group velocity v by

$$v := (v_1, \ldots, v_d) \qquad \text{where} \quad v_j = -\frac{\partial \omega}{\partial k_j}\bigg|_{k=k_0} \tag{8}$$

Definition 3. Define the second order differential operator $R(\partial_x)$ by

$$R(\partial_x) := -\frac{1}{2} \sum_{l,m=1}^{d} \frac{\partial^2 \omega}{\partial k_l \partial k_m}\bigg|_{k=k_0} \frac{\partial^2}{\partial x_l \partial x_m} \tag{9}$$

The approximate solution $\mathrm{U}^\varepsilon(y)$ in (7) is determined by $\varepsilon^p U_0(T, y, \varphi)$ which can be written

$$U_0(T, y, \varphi) = F(T, x - vt, \varphi) \tag{10}$$

where $F(T, x, \varphi)$ satisfies

$$\partial_T \partial_\varphi F + R(\partial_x)F + \partial_\varphi \pi \Phi_J(F) = 0 \qquad F(0, x, \varphi) = f(x, \varphi) \tag{11}$$

Equation (11) is a bit unusual, and the surface $T = 0$ is characteristic. Theorem 1, proved in [13] and [14] shows that this initial value problem has a unique local solution $F \in \cap_s C\left([0, T^*)\,;\, H^s(\mathbf{R}_{x,\varphi}^{d+1})\right)$. By imposing an additional boundary condition and seeking a solution in $L^2(\mathbf{R}_{x,\varphi}^{d+1})$, a unique solution is selected. If the initial data have derivatives in L^2, the solution will be in appropriate Sobolev spaces. Precisely:

Theorem 1 *Given $f(x, \varphi)$ in $\cap_{s>0} H^s(\mathbf{R}_{x,\varphi}^{d+1})$, there exists a maximal (possibly infinite) time $T^* > 0$ and a unique function $F(T, x, \varphi)$ in $\cap_{s>0} C([0, T^*) : H^s(\mathbf{R}_{x,\varphi}^{d+1}))$ which solves*

$$\partial_T \partial_\varphi F = -R(\partial_x)F - \partial_\varphi \pi \Phi_J(F)$$
$$F(0, x, \varphi) = f(x, \varphi).$$

Equations (10) and (11) can be used to construct an approximate solution $\mathrm{U}^\varepsilon(y) := \varepsilon^p U_0(\varepsilon t, y, (k_0 \cdot x - \omega t)/\varepsilon)$. Theorem 2, proved in [13] and [14] shows that $\mathrm{U}^\varepsilon(y)$ provides a good approximation to the exact solution of (6). Standard Sobolev norms cannot be used to show convergence because derivatives of the approximate solution typically grow like $1/\varepsilon$ and hence are not bounded independent of ε. Define a set of directions in $\mathbf{R}_x^d$, $\partial_1', \ldots, \partial_{d-1}'$ to be a basis for the set of constant coefficient vector fields that vanish on $k_0 \cdot x$. Define $\partial^\perp$ to be one of the remaining directions, chosen so that $\partial^\perp(k_0 \cdot x) = 1$.

Theorem 2 *The percentage error of the approximate solution goes to zero as* $\varepsilon \to 0$. *For all* $T < T^*$ *and all* α, β

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T/\varepsilon} \frac{1}{\sqrt{\varepsilon}} \left\| (\varepsilon \partial^\perp)^\alpha (\partial')^\beta \left(u^\varepsilon(y) - \mathrm{U}^\varepsilon(y) \right) \right\|_{L^\infty(\mathbf{R}_x^d)} = 0$$

and

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T/\varepsilon} \frac{1}{\varepsilon} \left\| (\varepsilon \partial^\perp)^\alpha (\partial')^\beta \left(u^\varepsilon(y) - \mathrm{U}^\varepsilon(y) \right) \right\|_{L^2(\mathbf{R}_x^d)} = 0$$

Example: The wave equation on $\mathbf{R}_{t,x}^{1+2}$. In this case, the differential operator $L(\partial_y)$ is given by

$$L(\partial_y)u^\varepsilon := \partial_t u^\varepsilon + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_{x_1} u^\varepsilon + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{x_2} u^\varepsilon$$

Choose $k_0 = (1, 0)$. Then $\omega = \omega(k_0) = 1$, the group velocity $v = (1, 0)$, and the differential operator $R(\partial_x) = \frac{1}{2}\partial_{x_2}^2$. Let $\Phi(u)$ be a cubic nonlinearity. This determines the exponent $p = 1/2$. The approximate solution $\mathrm{U}^\varepsilon(y)$ is then given by $\mathrm{U}^\varepsilon(y) = \sqrt{\varepsilon} U_0(\varepsilon t, y, (x_1 - t)/\varepsilon)$ where $U_0(T, y, \varphi) = F(T, x_1 - t, x_2, \varphi)$ and $F(t, x, \varphi)$ satisfies

$$\partial_T \partial_\varphi F = -\frac{1}{2}\partial_{x_2}^2 F - \partial_\varphi \pi \Phi(F) \qquad F(0, x, \varphi) = f(x, \varphi) \qquad (12)$$

Difference from Wavetrain Approximation. The slowly varying amplitude equation is related to equation (11). The SVEA approximation constructs a function $U_0(T, y, \theta)$ which is 2π periodic in θ and satisfies $\int_0^{2\pi} U_0 d\theta = 0$. The function $U_0(T, y, \theta)$ also satisfies equation (11), and because it has discrete spectrum with mean zero, $\partial_\theta^{-1} U_0(T, y, \theta)$ is bounded. For models with monochromatic solutions, ∂_θ^{-1} becomes $-\frac{i}{k}$ leading to an equation such as the Nonlinear Schrödinger Equation (NLS),

$$\partial_T U = \frac{i}{2k}\partial_{x_2}^2 U + U|U|^2. \qquad (13)$$

4 Numerics

At this point one would like to compare directly the predictions of the short pulse approximation and the wavetrain approximation in a problem like the example at the end of Section 3. Difficulties arise when trying to compute a solution to an equation like (13) with initial data which decay on the $\mathcal{O}(\varepsilon)$ scale. Treating the short pulse as a slowly varying amplitude requires introduction of the parameter ε into the function. This requires that computations for the NLS be performed on an extremely fine mesh to capture behavior on the $\mathcal{O}(\varepsilon)$ scale. The short pulse approximation is adapted to such initial data and the parameter ε does not appear in the computations.

Because the wavetrain approximation could not be computed reliably with short pulse initial data, we compare the two approximations in a specific linear example where choice of ε is not required. The choice of initial data yields an explicit solution for the wavetrain approximation. Even in this linear case, clear differences occur between the wavetrain and short pulse approximations.

Choose the initial datum to be $u^\varepsilon(0, x_1, x_2) = e^{-(x_1/\varepsilon)^2} e^{-x_2^2} \sin(x_1/\varepsilon)$. The wavetrain approximation considers the "slowly varying amplitude" $A(0, x_1, x_2)$ to be $e^{-(x_1/\varepsilon)^2} e^{-x_2^2}$. Of course this is preposterous, because it decays on a length scale $\mathcal{O}(\varepsilon)$. Because of the specific form of the initial datum, an explicit solution for $A^\varepsilon(T, x_1, x_2)$ can be found at $T = 1 = \varepsilon t$, namely

$$A^\varepsilon(1, x_1, x_2) = e^{-(x_1/\varepsilon)^2} \Re \left(\frac{e^{-x_2^2/5} e^{i2x_2^2/5}}{\sqrt{1 + 2i}} \right) \sin\left(x_1/\varepsilon\right) \qquad (14)$$

With the same initial datum, the short pulse asymptotics constructs an approximate solution $B(\varepsilon t, x_2, x_1/\varepsilon)$ where $B(T, x_2, \varphi)$ satisfies (5). This equation can be solved numerically using a spectral method. The ODE on the Fourier side is solved using a Crank-Nicholson method.

Figure 2 shows a the solution of (5) with the same initial datum, at time $t = 1/\varepsilon$. Note the emergence of the additional pulse at the trailing edge, and the splitting of the pulse in the transverse, or x_2, direction.
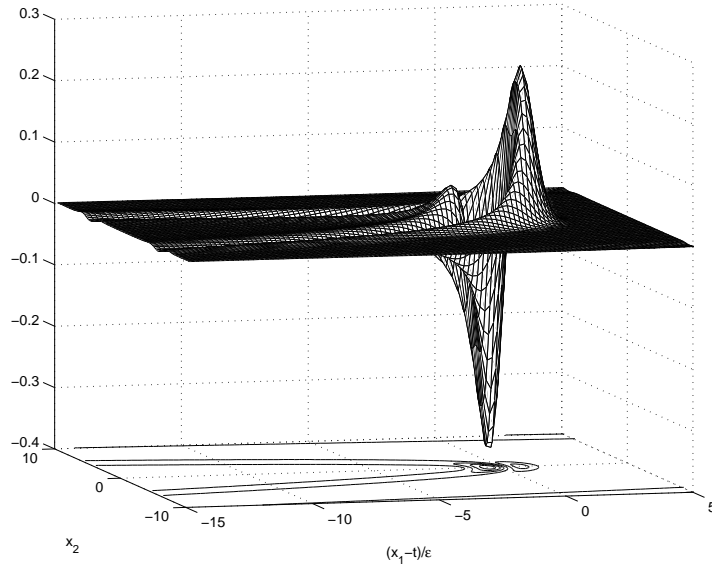

Figure 2: Solution to (5) at time $\varepsilon t = 1$

The numerical short pulse solution can be compared to the explicit formula (14) resulting from an unfounded slowly varying amplitude assumption. Striking qualitative differences between the two approximations emerge.

Figure 3 shows a cross section for x_2 fixed at 0. For $x_1 - t > 0$, the two approximations agree fairly closely. On the other side, the short pulse approximation has a significantly greater amplitude. This shift of the energy to the trailing edge of the pulse confirms the numerical studies of Rothenberg [7] and reveals qualitative errors resulting from inappropriately using the wavetrain approximation.

Figure 4 shows two transverse cross sections, for $x_1 - t$ fixed. In the trailing edge, for $x_1 - t < 0$, the short pulse approximation gives a narrower solution. This explains how the trailing edge of the pulse can have a larger intensity than
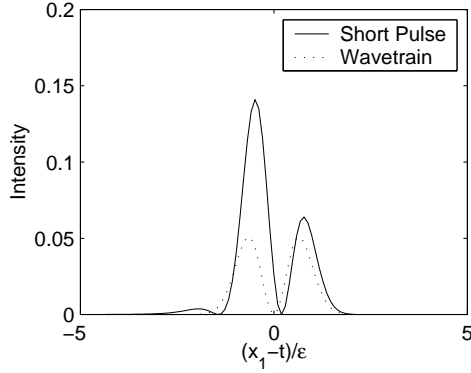
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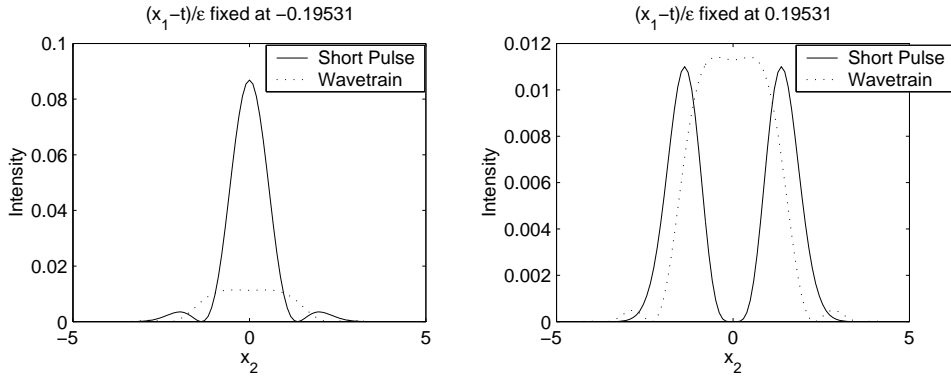Figure 3: Comparison of the two approximations for x_2 fixed at 0.

Figure 4: Comparison of the two approximations for $x_1 - t$ fixed.

the wavetrain approximation, as it is concentrated over a smaller area. For $x_1 - t > 0$, the short pulse approximation splits into two pulses, an effect not seen in the wavetrain approximation.

5   Conclusions

In both nonlinear and linear cases, asymptotic analysis reveals a different profile equation than the commonly used one arising from a slowly varying amplitude assumption, that must be used to describe approximate short pulse solutions to wave propagation and other hyperbolic systems of equations. In Section 4, we see that this gives rise to a qualitatively different description of the pulse. These differences, emerging even in the linear case, underline the importance of using short pulse approximation when describing the the propagation of short pulses. The short pulse approximation also has computational advantages, eliminating the need for a fine mesh which can resolve on a spatial scale $\mathcal{O}(\varepsilon)$.

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