Complements and Exercises for Boundary Value Problems

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1 Lesson 1

In using the principal of dense convergence one needs handy dense subsets. I used linear combinations of characteristic functions of rectangles.

Exercise 1.1 *Show that the linear combinations of characteristic functions of rectangles are dense in $L^2(\mathbb{R}^d)$ by verifying that the only element of $L^2(\mathbb{R}^d)$ that annihilates them all is the zero element.*

Exercise 1.2 *Show that $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ by a similar argument.*

The following are classic applications of the principal of dense convergence.

Exercise 1.3 *For $h \in \mathbb{R}^d$ define the translation operator τ_h on $L^p(\mathbb{R}^d)$ by*

$$(\tau_h f)(x) \;:=\; f(x-h)\,.$$

Prove that for all $f \in L^p$ and $1 \le p < \infty$ one has

$$\lim_{h\to 0} \|\tau_h f \;-\; f\|_{L^p} \;=\; 0\,.$$

Exercise 1.4 *Prove that for all $1 < p < \infty$ and q the dual index that for all $f \in L^p$ and $g \in L^q$*

$$\lim_{h\to\infty} \int \tau_h f \; g \; dx \;=\; 0\,.$$

Exercise 1.5 *Denote χ_{B_R} the characteristic function of the ball of radius R. Prove that for all $f \in L^p$ and $1 \le p < \infty$*

$$\chi_{B_R} f \;\to\; 0 \qquad \text{in} \quad L^p(\mathbb{R}^d)\,.$$

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Exercise 1.6 *Prove the Riemann-Lebesgue Lemma asserting that for all* $f \in L^1(\mathbb{R}^d)$ *the Fourier transform* $\widehat{f}$ *satisfies*

$$\lim_{\xi \to \infty} \widehat{f}(\xi) \;=\; 0\,.$$

Exercise 1.7 *Show that* $\tau_h f - f$ *as* $h \to 0$, $\chi_{B_R} f$ *as* $R \to \infty$, *and* $j_\varepsilon * f - f$ *as* $\varepsilon \to 0$, *do not converge to zero in* $L^\infty(\mathbb{R}^d)$ *for arbitrary* $f \in L^\infty$. **Discussion.** They do converge to zero on the weak star dense subset of continuous functions that tend to zero at infinity.

Exercise 1.8 *Suppose that if* B_j *are Banach spaces and* $T_n : B_1 \to B_2$ *is a sequence of continous linear maps with*

$$\lim_{n \to \infty} T_n b \;=\; 0$$

for all $b \in B_1$. *Prove that for any compact subset* $\Gamma \subset B_1$ *the convergence is uniform on* Γ.

Exercise 1.9 *Prove that a subset* $\Gamma \in L^p(\mathbb{R}^d)$ *is compact if and only if the family of operators* $\tau_h - I$ *and* χ_{B_R} *tend uniformly to zero on* Γ.

Exercise 1.10 *Suppose that* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *is* C^∞ *and the set of critical points of* φ *has Lebesgue measure zero. Prove that for all* $f \in L^1(\mathbb{R}^d)$

$$\lim_{\varepsilon \to 0} \int e^{i\varphi(x)/\varepsilon}\; f(x)\; dx \;=\; 0\,.$$

Discussion. 1. This sufficient condition is clearly necessary. 2. As with the Riemann-Lebesgue Lemma, the result does not extend from $f(x)\,dx$ to finite measures μ. 3. The dense set to take is the set of smooth functions compactly supported in the open subset of nonstationary points of φ. 4. If you have never seen the prinicipal of (non)stationary phase, look in my PDE book for example. Or most of E. Stein's books.

The hypotheses $A_j, B \in L^\infty$ were used in showing that L maps $H^1 \to L^2$.

Exercise 1.11 *Verify that* $L(x,\partial)u = f$ *in* $H^{-1}(\mathbb{R}^d)$ *holds if and only if for all* $\varphi \in C_0^\infty(\mathbb{R}^d)$ *one has*

$$\big(f, \varphi\big) \;=\; \big(u, L^\dagger(x,\partial)\varphi\big)$$

Exercise 1.12 *Show that* $u, v \in L^2(\mathbb{R}^d)$ *then the convolution* $u*v$ *is a continuous function tending to zero as* $x \to \infty$ *and satisfying*

$$\|u * v\|_{L^\infty} \;\leq\; \|u\|_{L^2}\; \|v\|_{L^2}\,.$$

Formulate and prove an $L^p \times L^q$ *version.*

2 Lesson 2

2.1 Friedrichs' lemma

I want simple examples showing that the Lipschitz hypothesis in Friedrichs' Lemma cannot be much relaxed. For example, C^α with $\alpha < 1$ does not suffice. If you find them I'll make them into an exercise!

Exercise 2.1 *1. If $A_j, B, \partial A, \partial B, \partial^2 A \in L^\infty(\mathbb{R}^d)$ show that $[L, J_\varepsilon]$ is a bounded from $H^1(\mathbb{R}^d)$ to itself uniformly for $0 < \varepsilon < 1$.*
2. Show that for $u \in H^1(\mathbb{R}^d)$, $[L, J_\varepsilon]u \to 0$ in $H^1(\mathbb{R}^d)$ as $\varepsilon \to 0$.

Exercise 2.2 *Under the same hypotheses as above prove the analogous results with $H^{-1}(\mathbb{R}^d)$ in place of $H^1(\mathbb{R}^d)$.* **Hint.** Use a duality argument. **Discussion.** Interpolation with the preceding exercise yields the same result for $H^s(\mathbb{R}^d)$ with $|s| \le 1$.

Exercise 2.3 *With the hypotheses of the preceding exercises prove that if $u, v \in H^{1/2}(\mathbb{R}^d)$ then $Lu, L^\dagger v \in H^{-1/2}(\mathbb{R}^d)$ and Greens' identity*

$$\big(Lu\,,\,v\big) \;=\; \big(u\,,\,L^\dagger v\big)$$

holds, Here parentheses indicate the pairing $H^{1/2} \times H^{-1/2} \to \mathbb{C}$ that is conjugate linear in the second variable. That is, the unique continuation of the L^2 scalar product from $C_0^\infty \times C_0^\infty$ to $H^{1/2} \times H^{-1/2}$. **Discussion.** One can also interpolate the hypotheses. The $H^{1/2}$ result only requires $C^{3/2}$ regularity of A.

3 Lesson 3

3.1 Boundary value problems for ordinary differential equations

Consider boundary value problems for the constant coefficient system of ordinary differential operators

$$L \;=\; A\,\frac{d}{dx} \;+\; B$$

on the half line $\Omega := \{x < 0\}$.

Hypothesis 3.1 *The matrix A is invertivle and the spectrum of $A^{-1}B$ is disjoint from the imaginary axis.*

The spectral assumption shows that solutions of $Lu = 0$ either grow exponentially or decay exponentially as $x \to \pm\infty$. There are no bounded solutions at either infinity.

3.1.1 Greens' function on $\mathbb{R}$

Denote by $\mathbb{E}_+ \subset \mathbb{C}^N$ the set of vectors w so that $e^{xA^{-1}B}w \to 0$ as $x \to -\infty$. They are indicated with a plus because they are associated to the eigenvalues of $A^{-1}B$ that have positive real part. Similarly define $\mathbb{E}_-$ as initial data of solutions decaying as $x \to +\infty$. Then hypthesis 3.1 implies that $\mathbb{E}_+ \oplus \mathbb{E}_- = \mathbb{C}^N$ with corresponding projectors $\Pi_\pm$ satisfying $\Pi_+ + \Pi_- = I$

Exercise 3.1 *Verify.*

The subspaces $\mathbb{E}_\pm$ are invariant under $e^{xA^{-1}B}$. The matrix valued function $e^{xA^{-1}B}\ \chi_{x>0}\, \Pi_-$ jumps from 0 on $]-\infty, 0[$ to Π_- at $0+$ so in the sense of distributions

$$L\Big(e^{xA^{-1}B}\ \chi_{x>0}\, \Pi_-\Big) \ =\ \Pi_-\ \delta\,.$$

A fundamental solution exponentially small at infinty is found by adding two such expressions to find

$$\mathcal{E} \ :=\ e^{xA^{-1}B}\ \chi_{x>0}\, \Pi_- \ -\ e^{xA^{-1}B}\ \chi_{x<0}\, \Pi_+\,, \qquad L\,\mathcal{E}\ =\ I\,\delta\,.$$

For $f \in L^2$, $L(\mathcal{E} * f) = f$. Write

$$(\mathcal{E} * f)' \ =\ L(\mathcal{E} * f) \ -\ B\mathcal{E} \ -\ f\,,$$

a sum of three terms in L^2. Therefore $(\mathcal{E} * f)' \in L^2(\mathbb{R}^d)$, so

$$\mathcal{E} * f \ =\ e^{xA^{-1}B}\ \chi_{x>0} \ *\ \Pi_- f \ -\ e^{xA^{-1}B}\ \chi_{x<0} \ *\ \Pi_+ f \ \in\ H^1(\mathbb{R}) \tag{3.1}$$

3.1.2 Boundary value problems in $x < 0$

For $f \in L^2(\mathbb{R})$ supportedf in $x \le 0$ the summand in the middle of (3.1) vanishes on $[0, \infty[$. The first term takes values in $\mathbb{E}_-$. The next exercise asks you to show that the values of the first term in (3.1) at $x = 0$ yield arbitrary elements in $\mathbb{E}_-$.

Exercise 3.2 *Show that for any vector $w \in \mathbb{E}_-$ there is an $f \in C_0^\infty(]-\infty, 0[)$ so that*

$$\Big(e^{xA^{-1}B}\ \chi_{x>0}\, \Pi_- \ *\ f\Big)\Big|_{x=0} \ =\ w\,.$$

Hint. *The set of values attained is a subspace of $\mathbb{E}_-$ hence closed. Take $f = \varphi(x)w$ with φ supported very close to 0.* **Discussion.** Therefore, if $f \in L^2(\mathbb{R})$ has support in $x \le 0$, the values at $x = 0$ of the solutions of $Lu = f$ that tend to zero as $x \to -\infty$ are exactly the vectors in $\mathbb{E}_-$.

Proposition 3.1 *When Hypothesis 3.1 is satisfied and $\mathcal{N} \subset \mathbb{C}^N$ is a subspace, the following two conditions are equivalent.*
i. *For every $f \in L^2(]-\infty, 0])$, the boundary value problem*

$$Lu = f\,, \qquad u(0) \in \mathcal{N}$$

has a unique solution $u \in H^1(]-\infty,0])$.
ii. *The complementarity conditon*

$$\mathcal{N} \ \oplus\ \mathbb{E}_+ \ =\ \mathbb{C}^N \tag{3.2}$$

is satisfied.

Proof. $u \in L^2(]-\infty,0[)$ satisfies

$$L\,u \ = 0, \qquad u(0) \in \mathcal{N}$$

if and only if $u(0) \in \mathbb{E}_+ \cap \mathcal{N}$. Therefore uniquenss of solutions is equivalent to

$$\mathcal{N} \ \cap\ \mathbb{E}_+ \ =\ \{0\}\,. \tag{3.3}$$

is necessary for uniqueness.
Extend f by zero in $x \geq 0$. In $x \leq 0$ the general solution of $Lu = f$ that tends to zero as $x \to -\infty$ is equal to

$$u \ =\ \mathcal{E} * f \ +\ e^{xA^{-1}B}\, e_+\,, \qquad e_+ \in \mathbb{E}_+\,.$$

To have existence for arbitrary f, the boundary condition $u(0) \in \mathcal{N}$ together with the preceding exercise demands that for arbitrary $w \in \mathbb{E}_-$ one find an element $e_+ \in \mathbb{E}_+$ so that $w + e_+ \in \mathcal{N}$. Existence of solutions is equivalent to is

$$\forall w \in \mathbb{E}_-,\ \ \exists e_+ \in \mathbb{E}_+, \qquad w\ +\ e_+\ \in\ \mathcal{N}\,. \tag{3.4}$$

Thus **i** is equivalent to (3.3) and (3.4) holding simultaneously.
They hold simultaneously if and only if

$$\forall w \in \mathbb{E}_-,\ \ \exists!\, e_+ \in \mathbb{E}_+, \qquad w\ +\ e_+\ \in\ \mathcal{N}\,.$$

This defines a one to one map $w \mapsto w + e_+$ from $\mathbb{E}_- \to \mathcal{N}$. Therefore $\dim \mathcal{N} \geq \dim \mathbb{E}_-$. Therefore $\dim \mathcal{N} + \dim \mathbb{E}_+ \geq N$. Together with (3.3) this is equivalent to (3.2). ■

Thus one must have $\dim \mathcal{N} = \dim \mathbb{E}_-$. In particular the number of boundary conditons required is $\dim \mathbb{E}_+$. That is equal to the number, counting multiplicity, of eigenvalues of $A^{-1}B$ that have strictly postive real parts.

3.2 Complementarity, elliptic and otherwise

In the elliptic case one can reverse the process, constructing good approximate solutions of boundary value problems by changing coordinates, freezing coefficients, analysing as above and then summing in ξ'. The key is that as $\xi' \to \infty$ $M(\xi')$ is of size ξ', the lower order term is negligible, M is essentially homogeneous in ξ' and one obtains uniform estimates from homogeneity and the compact set of $|\xi'| = 1$.

In the non elliptic case, general arguments of this style don't work. Control for $|\xi'$ large must come from elsewhere. The count of boundary conditions needed is general and very instructive.

3.3 $\mathcal{H}(L,\Omega)$

Definition 3.1 *Suppose that* $A_j, B, \nabla A_j \ \in\ L^\infty(\Omega)$.

$$\mathcal{H}(L,\Omega) \ :=\ \Big\{u\in L^2(\Omega)\ :\ Lu\ \in\ L^2(\Omega)\Big\}\,, \qquad \|u\|^2_{\mathcal{H}(L,\Omega)}\ :=\ \|u\|^2_{L^2(\Omega)}\ +\ \|Lu\|^2_{L^2(\Omega)}$$

Proposition 3.2 i. $\mathcal{H}(L,\Omega)$ *is complete, therefore Hilbert.*
ii. $H^1(\Omega)$ *is dense in* $\mathcal{H}(L,\Omega)$.

Exercise 3.3 *Prove that* $C_0^\infty(\Omega)$ *is dense in* $\mathcal{H}(L,\Omega)$. **Hint.** Use the Proposition and density of $C^\infty_{(0)}(\Omega)$ in $H^1(\Omega)$. **Discussion.** $C_0^\infty(\Omega)$ is **not** dense when $\partial\Omega$ is nonempty and $\sum A_j\nu_j$ is invertible at the boundary.

Proposition 3.3 First Trace Theorem. *Suppose that the boundary is noncharacteristic. The the map*

$$C^\infty_{(0)}(\Omega)\ \ni\ u\ \mapsto\ u|_{\partial\Omega}\ \in\ C_0^\infty(\partial\Omega)$$

extends unquely to a continuous linear map

$$\mathcal{H}(L,\Omega)\ \ni\ u\ \mapsto\ u|_{\partial\Omega}\ \in\ H^{-1/2}(\Omega)\,.$$

Proposition 3.4 Second Trace Theorem. *Suppose that the boundary is noncharacteristic. The the map*

$$C^\infty_{(0)}(\Omega)\times C^\infty_{(0)}(\Omega)\ \ni\ u\ \mapsto\ \big\langle \sum A_j\nu_j u\,,\, v\big\rangle\ \in\ C_0(\partial\Omega)$$

extends unquely to a continuous linear map

$$\mathcal{H}(L,\Omega)\times\mathcal{H}(L^\dagger,\Omega))\ \mapsto\ \Big\langle \sum A_j\nu_j\, u\,,\, v\Big\rangle\ \in\ \mathrm{Lip}(\partial\Omega)'\,.$$

And Green's identity

$$\int_\Omega \big\langle Lu, v\big\rangle\ dx\ =\ \int_\Omega \big\langle u, L^\dagger v\big\rangle\ dx\ +\ \int_{\partial\Omega}\Big\langle \sum A_j\nu_j\, u\,,\, v\Big\rangle\ dS(x)$$

holds for $u, v\in\mathcal{H}(L,\Omega)\times\mathcal{H}(L^\dagger,\Omega)$. *The boundary integral has the sense of the value of an element in* $\mathrm{Lip}(\Omega)'$ *at the test function* 1.

Remark 3.1 The traces of u,v at the boundary are $H^{-1/2}$. The product in the second trace theorem has no right to exist from that regularity. This is an example of the phenomenon sometimes called compensated compactness. There are quadratric forms that exist because of the differential equations satisfied by u and v and not just from regularity.

4 Lesson 4

Exercise 4.1 *The first trace theorem is sharp. Show that for any $s > -1/2$ there is an L even with constant coefficients and a $u\mathcal{H}_L$ on a half space so that the trace of u on the boundary is not H^s.*

Exercise 4.2 *Show that it is impossible for the trace of $\langle \sum A_j \nu_j u, v\rangle$ to be equal to the derivative of the Dirac delta.*

Open Question. I don't know if the Lip$'$ regularity for the trace of $\langle \sum A_j \nu_j u, v\rangle$ is comparably sharp. Is there an example where the trace is a distribution of order -1 and of no higher order?

Exercise 4.3 *Show that the adjoint of $L^\dagger$ is the original operator L*

Exercise 4.4 *Show that the adoint boundary space of the adjoint boundary space is the original boundary space.*

Exercise 4.5 *Show that in the $C(I; L^2)$ lemma, continuity conclusion can not be strengthened to C^α with $\alpha > 0$.*