Boundary Value Problems for Partial Differential Equations

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Outline

1. Friedrichs' dream, boundary value problems without reference to type.

2. Rectangular domains of computation leads to problems in domains with corners.

3. Standard PDE theory

- Characteristic polynomial.
- Elliptic operators
- Hyperbolic operators
- Plane waves

4. A motivating example from bounded operators

5 A little real analysis

- Principal of dense convergence
- Mollifiers $j_\epsilon * \to I$

6. Differential operators and adjoints

- Integration by parts formula
- Definition of symmetric postive systems

7. Estimates and existence theorem for symmetric postitive systems on $\mathbb{R}^d$

8. Examples

- $d = 1$
    - failure of ellipticity
    - singular solutions
- Transport for $d > 1$
    - failure of ellipticity
    - singular solutions
- Maxwell's equations
    - laplace transformed equations
    - the exponential trick

9. Friedrichs' Lemma

10. Applications

- Integration by parts formula
- Uniqueness
- Weak = Strong
- $L^\dagger(x,\partial)(C_0^\infty(\mathbb{R}^d))$ is dense

11. Symmetric hyperbolic systems

ODE on $\mathbb{C}^N$ decreasing euclidean norm

Symmetric postive operators generate contractive evolutions on $L^2(\mathbb{R}^d)$.

Exponential trick reduces to a symmetric positive system on $\mathbb{R}^{1+d}$

$A_0 \partial_t$ with strictly positive A_0

Electromagnetism example: calcite

12. Linearized compressible inviscid flow

Symmetric form not invariant by $L \mapsto M(x)L$ nor by $u \mapsto M(x)\widetilde{u}$

13. Determining the number of boundary conditions needed for a boundary value problem

Constant transport in a half space

The complementarity condition, ODE case (see online handout)

Spectral condition for symmetric positives

Tangential Fourier transform argument, Lopatinski complementarity.

14. $H^1(\Omega)$ is dense in $\mathcal{H}_L$

15. Boundary traces

First trace theorem, $u \in \mathcal{H}_L \Rightarrow u\big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$

$v \in L^2(I; H^s)$ and $v' \in L^2(I; H^{-s})$ imply $v \in C(I; L^2)$

Second trace theorem

$u, v \in \mathcal{H}_L, \times \mathcal{H}_{L^\dagger} \;\; \Rightarrow \;\; \langle \sum A_j \nu_j u \, , \, v \rangle\big|_{\partial\Omega} \in \mathrm{Lip}(\Omega)'$

Greens' identity holds

Connection with compensated compactness

16. Adjoint boundary space

Equivalent definitions of weak solutions

Equivalent definitions of strong solutions

17. Weak and strong solutions, existence and uniqueness

Inequality implying existence of weak solutions

Inequlaity implying uniqueness of strong solutions

18. Proof of weak=strong

Follows Lax-Phillips 1960. Tangential smoothing + Friedrichs Lemma

19. Positive boundary conditions (Online handout posted)

postive, strictly postive, conservative

Maximal postive boundary spaces

Algebra of positive boundary conditions

20 The fundamental existence and uniqueness Theorems

A priori estimates for positive boundary conditions

21. ODE example of Friedrichs 1958, pages 334-335.

22. Differentiability

For $\Omega = \mathbb{R}^d$

For bounded domains with noncharacteristic boundary

Normal form at boundary. Change x, change u, multiply.

Tangential derivative.

Then noncharacteristic argument.

23. Generalization to boundaries characteristic of constant multiplicity

Example of Maxwell equations.

24. $\Omega \in \mathbb{R}^2$ equal to a equadrant

Coupled transport examples

Elliptic version

25. Elliptic strictly dissipative generators are OK at corners.