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Symmetric Positive Linear Differential Equations*

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Introduction

Since the solutions of elliptic and hyperbolic differential equations have very many differing properties, and since quite different types of data must be imposed to determine such solutions it would seem unnatural to attempt a unified treatment of these equations. Still such a unified treatment—up to a certain point—will be given in this paper.

While most of the treatments of these equations naturally employ completely different tools, some of them employ variants of the same: positive definite quadratic forms, the so-called "energy integrals". We shall show that this tool can be adapted to a large class of differential equations which include the classical elliptic and hyperbolic equations of the second order.

The main motivation for this approach was not the desire for a unified treatment of elliptic and hyperbolic equations, but the desire to handle equations which are partly elliptic, partly hyperbolic, such as the Tricomi equation[1]

$$\left(y \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \phi = 0,$$

with $-\partial^2/\partial y^2$ in place of $+\partial^2/\partial y^2$.

As Tricomi has shown, an appropriate way of giving data for the solution of this equation consists in prescribing the function ϕ on an arc which connects two points $\mathscr{P}_+$ and $\mathscr{P}_-$ on the line $y = 0$ through a curve $\mathscr{A}$ in the half-plane $y < 0$, where the equation is elliptic, and in addition on a characteristic arc $\mathscr{C}_-$ issuing from the point $\mathscr{P}_-$ and ending in the half-plane $y > 0$ at its intersection with the characteristic arc $\mathscr{C}_+$ issuing from $\mathscr{P}_+$. No

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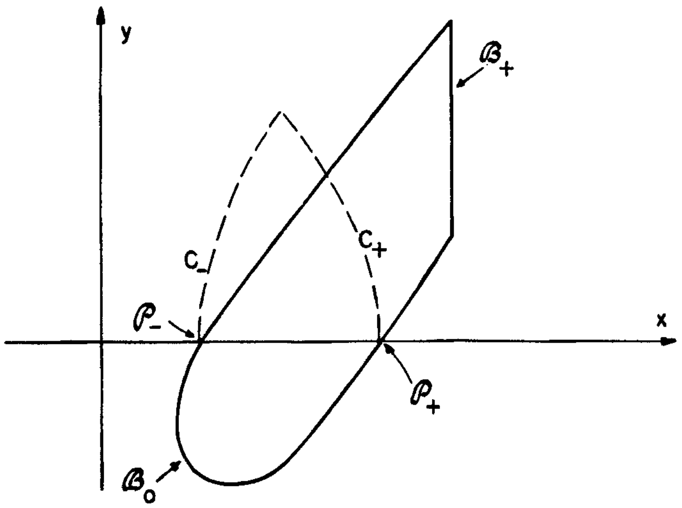
[1]Tricomi's equation and related equations play a basic role in the theory of transonic flow. In fact, the present work originated from an attempt at a numerical approach to a transonic flow problem. For an exposition of mathematical problems of transonic flow theory and an extensive bibliography, see L. Bers [15].

boundary condition is to be imposed on $\mathscr{C}_+$. Frankl has suggested to modify this problem by replacing the arc $\mathscr{C}_-$ by an arc on which $y\, dy^2 < dx^2$. A further modification consists in letting the boundary $\mathscr{B}$ of the region consist of two parts, $\mathscr{B}_0$ and $\mathscr{B}_+$ on which $y\, dy^2 < dx^2$ and $> dx^2$, respectively. The value of ϕ is to be given on the segment $\mathscr{B}_0$, while nothing is to be prescribed on $\mathscr{B}_+$. It is conjectured that this problem is well posed. We shall prove that this is the case for a somewhat special region of this type, see Section 18.



It is remarkable that for the solution of the Tricomi equation fewer boundary data are prescribed than for the solution of a second order elliptic equation. A similar discrepancy in the number of boundary data is found if one compares the Tricomi equation with a hyperbolic equation. Suppose the boundary of the domain contains an "initial surface" and an "end surface." On the end surface, the analogue of the segment $\mathscr{B}_+$, nothing should be prescribed but two data must be given on the initial surface.

One may perhaps say that the condition that the solution of the Tricomi equation should cross the line $y = 0$ smoothly takes the place of the additional boundary condition which must be imposed for solutions of elliptic and hyperbolic equations.

This remark may be illustrated by discussing the analogous situation arising in connection with the ordinary differential equation

$$2\alpha' x \frac{du}{dx} + \gamma u = f(x)$$

in which $\alpha' \neq 0$ and $\gamma > \alpha'$ are constants and $u = u(x)$ is a single function of the single variable x. We ask which boundary conditions must be imposed at $x_- < 0$ and $x_+ > 0$ in order that this equation have a unique solution. Because of the singularity of the equation at the point $x = 0$, one must permit the solution to have a singularity at this point; but we require that this singularity be so weak that $\int_{x_-}^{x_+} u^2(x)dx$ is finite.

If the equation did not have a singularity, the value of u could be prescribed either at one end point, x_-, or at the other, x_+. In the present case, we maintain that a unique solution exists if values at both end points, $x = x_\pm$, are prescribed in case $\alpha' < 0$ and if no boundary values are prescribed in case $\alpha' > 0$. For, as one reads off from the explicit expression of the solutions

$$u(x) = u_0(x) + c_+ u_+(x) + c_- u_-(x),$$
$$u_0(x) = \int_{x_0}^{x} f(x') \left|\frac{x'}{x}\right|^{\gamma/2\alpha'} \frac{dx'}{2\alpha' x'}$$

with $x_0 = x_\pm$ for $\pm x_\pm > 0$ when $\alpha' < 0$, while $x_0 = 0$ when $\alpha' > 0$,

$$u_\pm(x) = \begin{cases} (\pm x)^{-\gamma/2\alpha'} & \text{for } \pm x > 0, \\ 0 & \text{for } \pm x < 0, \end{cases}$$

at least one square-integrable solution of the non-homogeneous equation exists, and no square-integrable solution of the homogeneous solution in case $\alpha' < 0$. In this case, therefore, no additional condition may be imposed. In case $\alpha' > 0$, on the other hand, there are two square-integrable solutions of the homogeneous solution, vanishing identically for $x \gtrless 0$, respectively. To determine the solution uniquely, two conditions must be imposed, one for $x > 0$, one for $x < 0$.

The loss or gain of a boundary condition for the simple equation just considered, which was read off from the explicit representation of the solution, can also be deduced from the general criteria to be formulated in this paper. From the same criteria, and in a closely analogous manner, we shall be able to derive a loss (or gain) of a boundary condition for the Tricomi equation and other partly elliptic, partly hyperbolic equations.

The elliptic or hyperbolic character of an equation is defined by algebraic conditions on its coefficients. The equations we shall study will be characterized by different algebraic conditions. We shall show that to each equation of this type a class of proper boundary conditions can be assigned. These conditions—and in particular the number of data involved in these conditions—depend only on the algebraic nature of the coefficients at the boundary. We shall also show that the standard boundary problems for elliptic, hyperbolic, or parabolic equations, and also the indicated boundary problems of the Tricomi equations—after some adjustments—are special cases of the general boundary problem investigated in this paper.

Instead of working with one equation of higher than first order we prefer to work with a system of equations of the first order. We denote the independent variable by $x = \{x^1, \cdots, x^m\}$. The "value" of a function $u = u(x)$ at a point x will be a set of numbers $u = \{u_1, \cdots, u_k\}$. The differential operator acting on such a function will produce a set of functions $Ku = \{(Ku)_1, \cdots, (Ku)_k\}$, where

$$(Ku)_\lambda = \sum_{\nu=1}^{k} \left(2 \sum_{\rho=1}^{m} \alpha^\rho_{\lambda\nu} \frac{\partial u_\nu}{\partial x^\rho} + \gamma_{\lambda\nu} u_\nu \right)$$

and where the coefficients $\alpha^\rho_{\lambda\nu}$ and γ_ν are functions of $\{x^1, \cdots, x^m\}$. Introducing the matrices

$$\alpha^\rho = \{\alpha^\rho_{\lambda\nu}\}, \qquad \gamma = \{\gamma_{\lambda\nu}\}$$

we may write the operator K as

$$K = 2\alpha^\rho \frac{\partial}{\partial x^\rho} + \gamma$$

where summation with respect to ρ from $\rho = 1$ to m is implied. We impose two requirements on the matrices α^ρ and γ. The first is

I. *The matrices α^ρ are symmetric.*

The second requirement involves the matrix

$$\kappa = \gamma - \frac{\partial}{\partial x^\rho} \alpha^\rho$$

and reads

II. *The symmetric part $\frac{1}{2}(\kappa+\kappa')$ of the matrix κ is positive definite.*

Operators satisfying these conditions will be called "symmetric positive".

The function $u(x)$ should be defined in a region $\mathscr{R}$ of the m-dimensional space whose boundary $\mathscr{B}$ has a piecewise smooth normal $n = \{n_1, \cdots, n_m\}$. The boundary conditions which we shall admit depend only on the nature of the matrix

$$\beta = n_\rho \alpha^\rho$$

on the boundary.

To formulate the boundary condition we assume that the matrix β is the sum

$$\beta = \beta_+ + \beta_-$$

of two matrices $\beta_\pm$ having the following two properties:

III$_0$. *The matrix*

$$\mu = \beta_+ - \beta_-$$

has a non-negative symmetric part $\frac{1}{2}(\mu+\mu')$.

III_1. *Every u can be written as $u = u_+ + u_-$ with*

$$\beta_+ u_- = \beta_- u_+ = 0.$$

The condition

$$\beta_- u = 0$$

will then be called an "admissible" boundary condition. (This admissibility requirement is equivalent with the requirement of "maximality" proposed by P. D. Lax, see Section 5.)

Our boundary problem is the problem of finding the solution of the differential equation

$$Ku = f,$$

where $f = \{f_1, \cdots, f_k\}$ is a given function of x, under the boundary condition $\beta_- u = 0$, which for convenience is assumed to be homogeneous.

The requirements I, II, III_0 are designed in such a way that the uniqueness of the solution of this problem follows immediately, see Section 3. In fact, the uniqueness proof given in Section 3 is nothing but the natural generalization of the classical uniqueness proofs for solutions of elliptic or hyperbolic equations of the second order with the aid of energy integrals.

It is to be observed that requirement I does not restrict the type of the equation since it does not involve any inequality, while condition II which does imply inequalities does not involve the terms of highest order of the equation.

It should be mentioned that operators satisfying conditions I and II have also been studied by Ralph Phillips [6] in his theory of "dissipative" hyperbolic systems[2]

$$\frac{\partial u}{\partial t} + Ku = 0.$$

He determines the most general boundary conditions to be imposed on the function u such that this equation has a strong solution assuming given values for $t = 0$. This problem is reduced to that of the equation $Ku + \lambda u = f$ with $\mathscr{Re}\,\lambda > 0$, which is essentially of the class here considered.

Using our results, it could be shown that the conditions of the form $\beta_- u = 0$ satisfying $\mathrm{III}_{0,1}$ are special cases of the conditions formulated by Phillips. Whether or not this can be shown without using our results is an open question.*

We proceed to show that the Cauchy problem of the simplest hyperbolic equation is a special case of the problem $Ku = f$ in $\mathscr{R}$, $\beta_- u = 0$ on $\mathscr{B}$ formu-

[2]Phillips calls the operator $-K$ (with boundary condition $\beta_- u = 0$) "dissipative"; accordingly the operator K (with $\beta_- u = 0$) might be called "accretive" instead of "symmetric positive".

**Added in proof:* See, however, footnote on page 22.

lated here. We consider the equation $\partial^2\phi/\partial t^2-\partial^2\phi/\partial x^2 = h(x, t)$ and impose the conditions $\phi = \partial\phi/\partial t = 0$ on the initial segment $\mathscr{B}_-$, $x_- \leqq x \leqq x_+$, $t = 0$, prescribing nothing on the end segment $\mathscr{B}_+$, $t = t(x)$; the function $t(x)$ is such that $t(x) > 0$ for $x_- < x < x_+$, $t(x) = 0$ for $x = x_\pm$, $|t_x| < 1$ for $x_- \leqq x \leqq x_+$. Setting $u_1 = e^{-\lambda t}\partial\phi/\partial t$, $u_2 = e^{-\lambda t}\partial\phi/\partial x$, we write the equation in the form

$$\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\left(\frac{\partial}{\partial t}+\lambda\right)+\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\frac{\partial}{\partial x}\right]=\begin{pmatrix}e^{-\lambda t}h\\0\end{pmatrix}$$

and the initial condition as $u_1 = u_2 = 0$. Evidently, the equation is symmetric positive for $\lambda > 0$. The matrix β reduces to

$$\beta = -\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \text{on } \mathscr{B}_-,$$

$$\beta = \frac{1}{\sqrt{1+t_x^2}}\begin{pmatrix}1 & t_x\\t_x & 1\end{pmatrix} \qquad \text{on } \mathscr{B}_+.$$

Setting

$$\beta_- = \beta \qquad \text{on } \mathscr{B}_-, \qquad\qquad \beta_- = 0 \qquad \text{on } \mathscr{B}_+,$$

we see that conditions III are satisfied and that the conditions $u_1 = u_2 = 0$ on $\mathscr{B}_-$, nothing on $\mathscr{B}_+$, are evidently admissible in our sense.

If the domain consists of the rectangle $x_- \leqq x \leqq x_+$, $0 \leqq t \leqq t_+$, the conditions on the initial and end segment $\mathscr{B}_+$ and $\mathscr{B}_-$ are the same as before. On the sides $x = x_\pm$, we have $\beta = \mp\left(\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right)$. We maintain that the condition $u_1 = 0$, corresponding to $\phi = 0$, is admissible there. To see this, we set $\beta_- = \mp\left(\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right)$. Since the symmetric part of $\mu = \mp\left(\begin{smallmatrix}0 & 1\\-1 & 0\end{smallmatrix}\right)$ vanishes, condition III_0 is satisfied and clearly III_1 holds.

For the parabolic equation $\partial\phi/\partial t-\partial^2\phi/\partial x^2 = h(x, t)$ we set $u_1 = e^{-\lambda t}\phi$, $u_2 = e^{-\lambda t}\partial\phi/\partial x$ and write the equation as the system

$$\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\frac{\partial}{\partial t}+\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}\lambda & 0\\0 & 1\end{pmatrix}\right]\begin{matrix}u_1\\u_2\end{matrix}=\begin{pmatrix}e^{-\lambda t}h\\0\end{pmatrix},$$

which is evidently symmetric positive. As domain we consider again $x_- \leqq x \leqq x_+$, $0 \leqq t \leqq t_+$. We maintain that the conditions $u_1 = 0$ on $\mathscr{B}_-$ and nothing on $\mathscr{B}_+$, are admissible.

We note that $\beta = \pm\left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right)$ on $\mathscr{B}_\pm$ and set $\beta_- = \beta$ on $\mathscr{B}_-$, $\beta_- = 0$ on $\mathscr{B}_+$ so that $\mu = \left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right)$ on $\mathscr{B}_+$ and $\mathscr{B}_-$. Hence conditions III are satisfied. Clearly III_1 holds. On the sides $x = x_\pm$ the situation is the same as in the previous example.

To handle the non-homogeneous Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = h(x, y)$$

in a region $\mathscr{R}$ of the x, y-plane, we set $u_0 = \phi$, $u_1 = \partial\phi/\partial x$, $u_2 = \partial\phi/\partial y$. Employing a pair of functions $p_1 = \varepsilon p_1^0$, $p_2 = \varepsilon p_2^0$ with the property that

$$\frac{\partial p_1^0}{\partial x} + \frac{\partial p_2^0}{\partial y} > 0 \quad \text{in } \mathscr{R}+\mathscr{B},$$

we write the equation in the form

$$\begin{pmatrix} -p_1\dfrac{\partial}{\partial x} - p_2\dfrac{\partial}{\partial y} & p_1 - \dfrac{\partial}{\partial x} & p_2 - \dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial x} & 1 & 0 \\ -\dfrac{\partial}{\partial y} & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}.$$

Evidently, the matrix

$$\kappa = \begin{pmatrix} \dfrac{1}{2}\dfrac{\partial p_1}{\partial x_1} + \dfrac{1}{2}\dfrac{\partial p_2}{\partial x_2} & p_1 & p_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is positive definite provided the constant ε is taken sufficiently small.

The matrix β at the boundary $\mathscr{B}$ is

$$\beta = \begin{pmatrix} -(n_x p_1 + n_y p_2) & -n_x & -n_y \\ -n_x & 0 & 0 \\ -n_y & 0 & 0 \end{pmatrix}.$$

To show that the condition $u_0 = 0$ is admissible, we choose two positive functions $q^\pm$ on $\mathscr{B}$ such that

$$-n_x p_1 + n_y p_2 = q^+ - q^-$$

there; then we set

$$\beta_- = -\begin{pmatrix} q^- & 0 & 0 \\ n_x & 0 & 0 \\ n_y & 0 & 0 \end{pmatrix}$$

so that condition $\beta_- u = 0$ is the same as $u_0 = 0$. The matrix

$$\mu = \begin{pmatrix} q^+ + q^- & -n_x & -n_y \\ n_x & 0 & 0 \\ n_y & 0 & 0 \end{pmatrix}$$

evidently satisfies condition III_0.

What has been discussed here for the simplest differential equations for functions of two independent variables can easily be extended to equations for functions of any number of variables, so as to cover general elliptic, parabolic, and hyperbolic equations of the second order and those equations of higher order which can be associated with a variational principle. Also, symmetric hyperbolic systems of the first order are covered (see Section 4).

It would seem possible to bring all those problems of linear differential equations into the framework here explained which can be handled by locally positive definite quadratic forms, i.e. by forms expressed as integrals with a positive definite integrand.

On the other hand, it seems that those problems, in which forms are used which are positive definite but not locally so, cannot be subsumed under our theory. Thus Gårding's theory of general elliptic equations, Leray's theory of general hyperbolic equations, and the various related theories which have been developed during recent years can apparently not be tackled by our method.

Still there is a vast number of equations—of which only a few have been treated so far—which are made accessible by our approach. This applies in particular to the Tricomi equation and related, partially elliptic, partially hyperbolic, equations, at least for a certain class of boundaries and a certain class of boundary condition.

Introducing the functions

$$u_1 = e^{-\lambda x} \frac{\partial \phi}{\partial x}, \qquad u_2 = e^{-\lambda x} \frac{\partial \phi}{\partial y},$$

we write the non-homogeneous Tricomi equation in the form

$$Lu = \left[\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{\partial}{\partial x} + \lambda \right) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial y} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{-\lambda x} h \\ 0 \end{pmatrix}.$$

This equation is symmetric, but not positive, because of the change of sign of y. We multiply the operator L by the matrix

$$Z = \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix},$$

obtaining

$$K = \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix} L = \begin{pmatrix} y & y \\ y & 1 \end{pmatrix} \left(\frac{\partial}{\partial x} + \lambda \right) - \begin{pmatrix} y & 1 \\ 1 & 1 \end{pmatrix} \frac{\partial}{\partial y}.$$

Evidently, the matrix

$$\kappa = \begin{pmatrix} 1+\lambda y & \lambda y \\ \lambda y & \lambda \end{pmatrix}$$

is positive definite for sufficiently small λ.

We maintain that the boundary condition $u_1 n_y - u_2 n_x = 0$ on $\mathscr{B}_0$ is admissible provided that $n_x + n_y < 0$ and $y < 1$ on the arc $\mathscr{B}_0$. We have

$$\beta = \begin{pmatrix} y(n_x - n_y) & yn_x - n_y \\ yn_x - n_y & n_x - n_y \end{pmatrix}$$

on $\mathscr{B}_0$. Setting

$$\beta_- = \frac{1-y}{n_x+n_y} \begin{pmatrix} n_y^2 & -n_x n_y \\ -n_x n_y & n_x^2 \end{pmatrix}, \qquad \beta_+ = \frac{yn_x^2 - n_y^2}{n_x + n_y} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we find that the matrix $\mu = \beta_+ - \beta_-$ is non-negative since $n_x + n_y < 0$, $n_y^2 - yn_x^2 > 0$, $y < 1$, by assumption. Hence III_0 holds. One easily verifies III_1. Evidently, the boundary condition is $\beta_- u = 0$, as desired.

We choose the arc $\mathscr{B}_+$ in such a way that on it $yn_x^2 - n_y^2 > 0$. Assuming also $y < 1$ there, we have $n_x \pm n_y > 0$ and hence β is positive definite on $\mathscr{B}_+$. Therefore, we must take $\beta_- = 0$, i.e., impose no boundary condition on $\mathscr{B}_+$.

In the simple example of a single ordinary differential equation described earlier we found that a boundary condition is lost or gained according to the sign of the derivative α' of the critical coefficient $\alpha(x)$ at the place where this coefficient changes sign. An analogous situation obtains for the Tricomi equation. If we employ the factor $e^{\lambda x}$ instead of $e^{-\lambda x}$ and the multiplier $Z = \left(\begin{smallmatrix} 1 & -y \\ -1 & 1 \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} 1 & y \\ 1 & 1 \end{smallmatrix}\right)$, we again obtain a symmetric positive equation. This time, however, the matrix β is negative definite on the arc $\mathscr{B}_+$. Therefore, two boundary conditions, $\phi = 0$ and $\partial\phi/\partial x = 0$ must be imposed there. On the arc $\mathscr{B}_0$, on the other hand, the condition $\phi = 0$ is no longer admissible but the condition $\partial\phi/\partial x = 0$ is. In the present problem the arc $\mathscr{B}_+$ plays the role of an initial segment for a hyperbolic equation, while in the previous problem it played the role of an end segment.

In Sections 2 and 5 we shall assign to each properly posed boundary problem a properly posed adjoint problem with—in general—different boundary conditions, in such a way that if no condition is imposed on an arc in one problem, everything is prescribed there in the other problem.

In the two problems for the Tricomi equation described we have restricted the domain by the requirements $n_x + n_y < 0$ on $\mathscr{B}_0$ and $y < 1$, which were adopted just for convenience. It is possible to relax these requirements, but not completely. Although it is true that for a large class of domains it is possible to determine an appropriate multiplier matrix Z, (see Protter [10], Morawetz [13]) this is not possible. This remarkable fact was proved by C. Morawetz [14], unless the domain satisfies certain stringent conditions. For more general domains one must supplement the equations

$$y \frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y} = h, \qquad \frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x} = 0,$$

from which the equation $Lu = f$ was built up, by the equations $\partial\phi/\partial x - \phi_x = 0$, $\partial\phi_y/\partial y - \phi_y = 0$, and then select three appropriate linear combinations of these four equations, see Ou and Ding [12] and Protter, unpublished.

This remark illuminates the role of our approach to the theory of linear differential equations: it offers a framework into which some classes of differential equation problems can be fitted. This fitting, or "adaptation", is primarily an algebraic proposition; the main difficulty in the treatment of problems in the manner here proposed consists in finding such an "adaptation". Once it is found, the analytical questions of existence, uniqueness and differentiability are once and for all answered by the general theory.

It is naturally hoped that, in addition to the Tricomi equation, other non-standard differential equation problems can be fitted into our framework. That this is not always possible is evident from the fact, discovered by H. Lewy [7], that differential equations exist which have no solution unless the data are analytic. Lewy's equation $Lu = f$ with

$$L = -\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} + 2i(x_1 + ix_2)\frac{\partial}{\partial y_1}$$

applies to a complex-valued function $u = u_1 + iu_2$ of three real variables x_1, x_2, y_1; the right member $f = f(y_1)$ is assumed real. Lewy proves that the equation $Lu = f$ has no solution in $\mathfrak{C}_1$ unless f is analytic. A slight extension of his argument yields that this equation does not even have a square integrable solution which admits the operator L in the strong sense.

On the other hand, it is possible to enclose the equation $Lu = F$ in a set of equations, differing from it arbitrarily little only in the term of lowest order, which do possess such solutions. One need only take

$$Lu \pm \kappa\bar{u} = f$$

with κ a positive real constant, since the operators $\pm\overline{Lu} + \kappa u$ are symmetric positive with respect to the inner product $\mathscr{R}e\, \bar{v}\cdot u = v_1u_1 + v_2u_2$; hence our theory is applicable.

A similar imbedding is possible for the *ultrahyperbolic equation*

$$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} - \frac{\partial^2\phi}{\partial x_3^2} - \frac{\partial^2\phi}{\partial x_4^2} = 0.$$

Introducing the function $u = \{u_\rho\} = \{\partial\phi/\partial x_\rho\}$ we may write this equation as a symmetric system $Lu = 0$ with

$$Lu = \left\{\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} - \frac{\partial u_4}{\partial x_4},\ \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1},\ -\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1},\ -\frac{\partial u_1}{\partial x_4} + \frac{\partial u_4}{\partial x_1}\right\}.$$

We imbed this equation in the set of equations

$$Lu \pm 2\kappa\{0, u_2, 0, 0\} = 0$$

with a positive constant κ. For these equations boundary conditions may be imposed so that the problem is properly set. For, the equations

$$\pm Lv + \kappa v = 0$$

obtained from the previous equation by setting $u = e^{\pm\kappa x_1} v$, are symmetric positive.

Which boundary conditions are admissible for these equations depends on whether the $+$ or $-$ sign is taken in front of L; specifically, these conditions are complementary to each other. This fact alone, however, should not be taken as a suggestion that for $\kappa = 0$ no boundary conditions produce a properly posed problem. After all, also the Cauchy-Riemann equations $(\partial/\partial x_1 + i\partial/\partial x_2)u = 0$ can be imbedded in a set of symmetric positive equations,

$$\pm \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}\right) u + \kappa \bar{u} = 0,$$

such that the boundary conditions associated with these equations change when x passes through zero. Still, solutions of the Cauchy-Riemann equations can be derived from solutions of a symmetric positive equation (see Section 19).

It was said above that the notion of a "symmetric positive" operator was so designed that the uniqueness of the solution of the equation $Ku = f$ under an admissible boundary condition $\beta_- u = 0$ is immediately evident. The same applies to an appropriately defined adjoint equation $K^* v = g$ under the "adjoint" condition $\beta'_+ v = 0$. From the uniqueness of this adjoint problem one may derive the existence of a weak solution of the original problem, i.e. of a square integrable function u for which

$$\int_{\mathscr{R}} (K^* v) \cdot u \, dR = \int_{\mathscr{R}} v \cdot f \, dR$$

for all functions v in $\mathfrak{C}_1$ with $\beta'_+ v = 0$. Here $\mathscr{R}$ is the region where u and v are defined, and $dR = dx^1 \cdots dx^n$. For the Tricomi problem the existence of such a weak solution was first proved by C. Morawetz [14].

A weak solution as such is not necessarily unique. Uniqueness is insured, however, if the weak solution is a strong one, i.e. if the pair $[u, f]$ lies in the closure of the graph $[\dot{u}, K\dot{u}]$ for $\dot{u}$ in $\mathfrak{C}_1$ with $\beta_- \dot{u} = 0$. For the standard elliptic and hyperbolic problems this can be shown to be the case in a direct way, e.g. with the aid of smoothing operators. But it is not obvious whether or not this tool can be adapted to the general problem here treated.* There-

**Added in proof:* See, however, footnote on page 22.

fore we were compelled to employ a detour via a differentiability theorem.

The *differentiability theorem* states that the solution—of the non-homogeneous problem with homogeneous boundary conditions—admits differentiation (in the strong sense) if the right member does. For elliptic equations, differentiability is a local property; it can be established in many different ways. For Cauchy's problem of hyperbolic equations the Cauchy-Kowalewski solution of the problem with analytic data can be employed. For the general problem considered here, a different method is needed which does not depend on the type of the equation. Such a method is available: the procedure introduced by P. D. Lax for elliptic and hyperbolic equations, see [5]. This procedure does not depend essentially on the type, and after some adaptations, can be used for our problem.

To explain Lax's method one must introduce the norm $||u||_1$ of order 1 of a function u as

$$||u||_1 = \left[||u||^2 + \sum_{\lambda=1}^{m} \left\| \frac{\partial u}{\partial x_\lambda} \right\|^2 \right]^{1/2},$$

and the norm $||v||_{-1}$ of order -1 of a function v as

$$||v||_{-1} = \text{l.u.b.}_{(u)} \frac{(v, u)}{||u||_1}.$$

For a solution u of $Lu = f$ with $||u||_1 < \infty$ and behaving appropriately at the boundary, an *a priori inequality*

$$||u||_1 \leqq C_1 ||Lu||_1 = C||f||_1$$

holds. It is remarkable that a "dual" inequality

$$||v||_{-1} \leqq C_{-1} ||L^* v||_{-1}$$

also holds. Leray, in the theory of hyperbolic equations, see [3], derives such a dual inequality from the existence of a solution u of $Lu = f$ with $||u||_1 < \infty$. Lax proceeds in the opposite direction; he derives the dual inequality directly and then proves the existence of a solution with $||u||_1 < \infty$.

In carrying this approach over to our problem certain difficulties arise at the boundary of the domain. To overcome these difficulties we modify the definition of the norm $||u||_1$ by using, in place of $\partial/\partial x_\lambda$, differential operators of the first order which at the boundary involve differentiation only in a tangential direction. The major tool in deriving the dual inequality is then the solution of the differential equation

$$(1-\Delta')w = v$$

where Δ' is a differential operator of the second order which is elliptic in the interior of the domain, but does not involve normal derivatives at the boundary. We like to call Δ' a "clipped" Laplacian. The main technical work done in this paper consists in deriving properties of the solution w of this equation; this is done in Part III.

We shall not prove the differentiability theorem for the solutions of all symmetric positive equations. An additional condition will be imposed.

The need for such an additional property can be exemplified in connection with the simple equation

$$Ku \equiv 2\alpha' x \frac{\partial u}{\partial x} + \gamma u = f(x)$$

for a single function u of one variable x. The derivative u' of a solution satisfies the equation

$$K'u' \equiv 2\alpha' x \frac{\partial u'}{\partial x} + \gamma_1 u' = f'(x)$$

with $\gamma_1 = \gamma + 2\alpha'$. In case $\alpha' > 0$ the relation $\kappa = \gamma - \alpha' > 0$ entails the relation $\kappa_1 = \kappa + 2\alpha' > 0$; hence an *a priori* inequality for u' can be set up. In case $\alpha' < 0$, however, $\kappa > 0$ does not necessarily entail $\kappa_1 > 0$, and the condition $\kappa_1 > 0$ must be added as an additional requirement. Of course, if $\kappa_1 < 0$, one can verify from the explicit solution that there are solutions u satisfying an admissible boundary condition but having derivatives which are not quadratically integrable.

It should be mentioned that we can eliminate the additional requirement $\kappa_1 > 0$ by a continuation argument if we are satisfied with establishing the existence of a strong solution without establishing its differentiability; (this is the "detour" referred to above).

If one wishes to establish the existence of strict solutions, i.e. of continuous solutions with continuous derivatives, one may proceed in a standard manner and reduce this question to the question whether or not the solution possesses strong derivatives of a sufficiently high order, depending on the dimension m. We shall not elaborate on this point.

In conclusion a few words may be said about the method of *finite differences*. The aim of this method is to replace the differential equation together with the boundary condition by a finite difference equation whose solution approaches that of the differential equation as the mesh width tends to zero. We shall explain how to set up such a difference equation in the case of the equation

$$Ku \equiv \alpha \frac{\partial u}{\partial x} + \frac{\partial \alpha u}{\partial x} + \kappa u = f$$

for a function $u = \{u_1, \cdots, u_k\} = u(x)$ of one variable x in the interval $x_- \leqq x \leqq x_+$ and the boundary conditions $(\mu \mp \alpha)u = 0$ at $x = x_\pm$. Note that in this case $\beta = \pm\alpha$ so that $\mu \mp \alpha = -2\beta_-$. The coefficient matrix $\alpha = \alpha(x)$ is continuously differentiable, $\kappa(x)$ and $f(x)$ are continuous.

We introduce net points

$$x_\sigma = x_0+\sigma h, \qquad\qquad \sigma_- \leqq \sigma \leqq \sigma_+ ,$$

where h, x_0 and the even integers $\sigma_\pm$ are so chosen that $x_{\sigma_\pm} = x_\pm$. We replace the unknown function $u(x)$ and the given functions $f(x)$ and $\kappa(x)$ by a function $\{u_\sigma\}$ defined for even σ. Also we replace $\kappa(x)$ and $f(x)$ by functions κ_σ and f_σ for even σ such that the linear interpolations of κ_σ and f_σ tend to $\kappa(x)$ and $f(x)$ uniformly and in the mean, respectively; (the κ_σ should be chosen positive definite). The functions $\alpha(x)$ will be replaced by functions α_σ defined for odd values of σ for $\sigma_- -1 \leqq \sigma \leqq \sigma_+ +1$ such that their linear interpolations together with first derivatives tend uniformly to $\alpha(x)$ and $\alpha_x(x)$, respectively.

We then replace the differential equation $Ku = f$ by the equation

$$(K^{(h)}u)_\sigma \equiv (2h)^{-1}[\alpha_{\sigma+1}u_{\sigma+2}-\alpha_{\sigma-1}u_{\sigma-2}]+\kappa_\sigma u_\sigma = f_\sigma$$

for $\sigma_- +2 \leqq \sigma \leqq \sigma_+ -2$, while for $\sigma = \sigma_\pm$ we take

$$(K^{(h)}u)_\sigma = h^{-1}[\mu_\sigma u_\sigma \mp \alpha_{\sigma\mp 1}u_{\sigma\mp 1}]+\kappa_\sigma u_\sigma = f_\sigma ,$$

where $\mu_{\sigma_\pm}$ may be taken as the value of $\mu(x)$ at $x = x_\pm$.

Clearly, when $u(x)$ is in $\mathfrak{C}_1$ and $u_\sigma = u(x_\sigma)$, the function $K^{(h)}u$ tends to Ku uniformly. Also $hK^{(h)}_{\sigma_\pm}$ tends to $(\mu\mp\alpha)u$ at $x = x_\pm$; if $(\mu\mp\alpha)u(x_\pm) = 0$ one readily finds that $(K^{(h)}u)_{\sigma_\pm}$ tends to $Ku(x_\pm)$.

The uniqueness of the solution of the finite difference equation is evident from the identity

$$2h\sum\nolimits_\sigma' u_\sigma(K^{(h)}u)_\sigma = 2h\sum\nolimits_\sigma' u_\sigma\kappa_\sigma u_\sigma + u_{\sigma_+}\mu_{\sigma_+}u_{\sigma_+} + u_{\sigma_-}\mu_{\sigma_-}u_{\sigma_-} ,$$

in which the prime is to indicate that the terms for $\sigma = \sigma_\pm$ are to be supplied with a factor $\frac{1}{2}$. Evidently, the form on the right hand side is positive definite. Therefore the finite difference equations possess a solution $u^{(h)}_\sigma$. Clearly, for the linear interpolation $u^{(h)}(x)$, the integral $\int_{x_-}^{x_+} u^{(h)}(x)\cdot u^{(h)}(x)dx$ remains bounded. A weakly convergent solution can be selected and the linear function $u(x)$ can be shown to satisfy the equation

$$\int_{x_-}^{x_+} K^* v(x)\cdot u(x)dx = \int_{x_-}^{x_+} v(x)\cdot f(x)dx$$

for all functions $v(x)$ in $\mathfrak{C}_1$ satisfying the "adjoint" condition $(\mu'\pm\alpha)v = 0$ at $x = x_\pm$. In other words, the limit function is a "weak solution" (see Section 4).

The set up of finite difference equations as described here can be carried over to the case of functions of several variables without any modification provided the domain is a rectangle[3]. In fact, the attempt at finding such a set up for the Tricomi equation was the starting point for the investigations presented in this paper.

Part I of this paper is concerned with the formulation of the problem, the uniqueness and the weak existence of a solution.

In Part II the problem is formulated for a compact manifold with boundary. Then the differentiability theorem is reduced to theorems on the clipped Laplacian, and finally the existence of the strong solution is derived from the differentiability theorem.

Part III, involving the major technical work, supplies the needed theorems on the clipped Laplacian.

Part IV contains three items: 1) modifications needed to include Cauchy's problem and the mixed problem for hyperbolic and parabolic equations and other problems in which the boundary condition is different on different parts of the boundary, 2) the proof of the differentiability of the solution of a special boundary value problem for the Tricomi equation, 3) an adaptation of a problem for the Cauchy-Riemann equation and of Dirichlet's problem for Laplace's equation.

The work presented in this paper was strongly influenced by discussions with L. Bers, P. D. Lax, and L. Nirenberg. Various suggestions made by P. D. Lax were incorporated in the text, or indicated in footnotes. I am greatly indebted to R. S. Phillips for his painstaking scrutiny of the manuscript.

Part I

UNIQUENESS. WEAK SOLUTIONS

1. Differential Operator

The differential operators to be investigated will act on a set of k functions $u = \{u_1, \cdots, u_k\}$ defined in an m-dimensional differentiable manifold with a smooth or piecewise smooth boundary. To explain the basic notions we shall at first assume the manifold to consist of a region $\mathscr{R}$ in the m-dimensional space of points x with the coordinates $x^1, \cdots, x^m$. Specifically, we shall assume that the region $\mathscr{R}$ can be described with the aid of a function $y(x)$ as consisting of the points x with $y < 0$, so that the boundary $\mathscr{B}$ or $\mathscr{R}$ is given by $y = 0$. The function $y(x)$ should have piecewise con-

[3]Various modifications of this finite difference set-up including finite difference equations for non-rectangular domains will be treated in a forthcoming paper by C. K. Chu.

tinuous derivatives[4] in $\mathscr{R}+\mathscr{B}$ and the gradient ∇y should not vanish at $\mathscr{B}$.

To describe the differential operator we assume that $m+1$ matrices $\alpha^1, \cdots, \alpha^m, \gamma$ are given at each point of $\mathscr{R}+\mathscr{B}$. These matrices are supposed to depend continuously on x and the matrices α^μ are supposed to possess continuous derivatives. With the aid of these matrices we form the differential operator

$$K = 2\alpha^\lambda \frac{\partial}{\partial x^\lambda} + \gamma,$$

which transforms functions $u(x)$ with continuous first derivatives in $\mathscr{R}+\mathscr{B}$ —class $\mathfrak{C}_1$—into functions

$$Ku(x) = 2\alpha^\lambda(x) \frac{\partial u(x)}{\partial x^\lambda} + \gamma(x)u(x),$$

which are continuous in $\mathscr{R}+\mathscr{B}$—class $\mathfrak{C}$.

We impose two "fundamental" requirements on the matrices α and γ—at each point x of $\mathscr{R}+\mathscr{B}$. The first is:

I. The matrices $\alpha^1, \cdots, \alpha^m$ are symmetric.

The second requirement involves the matrix

$$\kappa = \gamma - \frac{\partial}{\partial x^\lambda}\alpha^\lambda;$$

it reads:

II. The symmetric part of the matrix κ is positive definite.

The symmetric part of κ is given as $\frac{1}{2}(\kappa+\kappa')$ in terms of the transpose κ' of κ. We also express condition II simply by the formula

$$\kappa+\kappa' > 0.$$

A differential operator K formed with the aid of the matrices α, γ will be called "symmetric" if the α satisfy condition I; if the α and γ satisfy conditions I and II the operator will be called "symmetric positive".

The role of the matrix κ becomes apparent if the operator K is written in the form

$$K = \alpha^\lambda \frac{\partial}{\partial x^\lambda} + \frac{\partial}{\partial x^\lambda}\alpha^\lambda + \kappa.$$

The formally adjoint operator can then be written as

$$K^* = -\alpha^\lambda \frac{\partial}{\partial x^\lambda} - \frac{\partial}{\partial x^\lambda}\alpha^\lambda + \kappa',$$

[4]By this we mean that $y(x)$ is given as $y = \max\{y_1(x), y_2(x), \cdots\}$ where $y_1(x), y_2(x), \cdots$ stand for a finite number of functions with continuous derivatives in $\mathscr{R}+\mathscr{B}$. Actually, we shall specify the type of domain further; see Section 7.

which makes it evident that the formal adjoint of a symmetric positive operator is also symmetric positive.

Let $f(x) = \{f_1(x), \cdots, f_k(x)\}$ be a function in $\mathfrak{C}$. Then we may ask for a solution $u(x)$ of the differential equation

$$Ku = f;$$

such an equation—or system of equations—will be said to be symmetric positive if the operator K is symmetric positive.

We proceed to show that the standard hyperbolic and elliptic equation of the second order can be written as a symmetric positive system of equations.

A symmetric operator K for which—at a point x_0—a linear combination $\zeta_\lambda \alpha^\lambda$ of the matrices α^λ is positive definite was called hyperbolic symmetric at this point. Such an operator is equivalent to a symmetric positive one in as much as the operator

$$e^{-\Lambda \zeta_\lambda x_\lambda} K e^{\Lambda \zeta_\lambda x_\lambda} = K + 2\Lambda \zeta_\lambda \alpha^\lambda$$

is symmetric positive provided the number Λ is chosen sufficiently large; see [4] and [5]. A hyperbolic equation of the second order can be reduced to a symmetric hyperbolic equation, as is easily verified [4].

An elliptic equation

$$\frac{\partial}{\partial x^\lambda} g^{\lambda\nu} \frac{\partial}{\partial x^\nu} \phi = -\rho,$$

given with the aid of a positive definite matrix $g^{\lambda\nu}$, can be reduced to a symmetric system by setting

$$u = \{u^0, \cdots, u^m\} \quad \text{with } u^0 = \phi, \quad u^\lambda = g^{\lambda\nu} \frac{\partial \phi}{\partial x^\nu}$$

and

$$Ku = \left\{ -\frac{\partial u^\lambda}{\partial x^\lambda}, \quad -\frac{\partial u^0}{\partial x^1} + g_{1\nu} u^\nu, \quad \cdots, \quad -\frac{\partial u^m}{\partial x^m} + g_{m\nu} u^\nu \right\},$$

where $(g_{\lambda\nu})$ stands for the inverse of the matrix $(g^{\lambda\nu})$. The equation is then equivalent to the system

$$Ku = \{\rho, 0, \cdots, 0\}.$$

The matrix κ, given by $\kappa_{\lambda\nu} = g_{\lambda\nu}$ for $\lambda \neq 0$, $\nu \neq 0$, $\kappa_{\lambda\nu} = 0$ for $\lambda = 0$ or $\nu = 0$, is not positive definite. To remedy this defect one may introduce a vector $p^\lambda(x)$ with the property

$$\frac{\partial p^\lambda}{\partial x^\lambda} > 0 \qquad \text{in } \mathscr{R} + \mathscr{B}$$

and then set

$$Ku \equiv \left\{ -\frac{\partial u^\lambda}{\partial x^\lambda} - p^\lambda \frac{\partial u^0}{\partial x^\lambda} + p^\lambda g_{\lambda\nu} u^\nu, \quad -\frac{\partial u^0}{\partial x^1} + g_{1\nu} u^\nu, \quad \cdots \right\}.$$

Now we have

$$\kappa_{00} = \frac{1}{2} \frac{\partial p^\lambda}{\partial x^\lambda} > 0,$$

$$\kappa_{\lambda 0} = 0 \qquad \lambda \neq 0,$$

$$\kappa_{0\nu} = p^\lambda g_{\lambda\nu}, \qquad \nu \neq 0;$$

evidently, $\kappa + \kappa'$ can be made positive definite by taking p sufficiently small.

The Tricomi equation, which we shall write in the form

$$y \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0,$$

is equivalent with the symmetric system

$$\begin{pmatrix} y \dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} = 0$$

for the derivatives $\phi_x = \partial\phi/\partial x$, $\phi_y = \partial\phi/\partial y$. The equation can be made symmetric positive by applying—on the left—a matrix $\left(\begin{smallmatrix} b & yc \\ c & b \end{smallmatrix}\right)$. The equation then assumes the form

$$\begin{pmatrix} by \dfrac{\partial}{\partial x} - cy \dfrac{\partial}{\partial y} & cy \dfrac{\partial}{\partial x} - b \dfrac{\partial}{\partial y} \\ cy \dfrac{\partial}{\partial x} - b \dfrac{\partial}{\partial y} & b \dfrac{\partial}{\partial x} - c \dfrac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} = 0;$$

thus the symmetry is not destroyed. The matrix κ becomes

$$\kappa = \begin{pmatrix} c + c_y y - b_x y & b_y - c_x y \\ by - c_x y & c_y - b_x \end{pmatrix};$$

evidently it can be made positive definite by proper choice of b and c.

2. Semi-Admissible Boundary Condition

To a symmetric positive operator K we shall assign a class of "admissible" boundary conditions and we shall show that the problem of finding a solution u of the equation $Ku = f$ which satisfies these boundary conditions is properly posed. To formulate such boundary conditions we must first introduce an exterior normal vector $n = \{n_1, \cdots, n_m\}$ at each boundary point; we choose the vector with the components

$$n_\lambda = \nabla_\lambda y = \frac{\partial y}{\partial x_\lambda}$$

where $y(x)$ is the function introduced in Section 1; we may do so, having assumed $n \neq 0$. Next we form the matrix

$$\beta = n_\lambda \alpha^\lambda$$

at each boundary point. Finally, we select at each boundary point a matrix μ depending continuously on this point and satisfying the requirement:

III⁰. *$\mu+\mu'$ is non-negative or, simply,*

$$\mu+\mu' \geqq 0.$$

The boundary condition to be imposed is then given as

$$\beta u = \mu u.$$

Introducing the matrix

$$M = \mu - \beta,$$

we may write this condition in the form

$$Mu = 0.$$

The matrix M and the boundary condition $Mu = 0$ will be called "semi-admissible" for the operator K if property III⁰ obtains.

We shall show that the solution u of the equation $Ku = f$ under the condition $Mu = 0$ is unique if M is semi-admissible.

We recall that the formally adjoint operator K^* may be obtained from the operator K by substituting $-\alpha$ and κ' for α and κ. Hence $-\beta$ plays for K^* the role that β plays for K and we may assign the matrix μ' to K^* in place of μ. Accordingly, we set

$$M^* = \mu' + \beta,$$

and impose on the solution v of the adjoint equation $K^* v = g$ the boundary condition

$$M^* v = 0.$$

Evidently, the matrix M^* is semi-admissible for the operator K^* if M is semi-admissible for K.

To be able to prove the existence of the solution of the equation $Ku = f$ under the boundary condition $Mu = 0$ we must impose an additional requirement on this condition. We shall formulate and discuss this additional requirement in Section 5. Before doing so we prove the uniqueness of the solutions.

3. Basic Inequality. Uniqueness

The uniqueness of the solution of $Ku = f$ under the semi-admissible boundary condition formulated in Section 2 is nearly obvious.

We introduce the inner products

$$(v, u) = \int \cdots \int_{\mathscr{R}} v \cdot u \, dx^1 \cdots dx^m$$

with $v \cdot u = v_1 u_1 + \cdots + v_k u_k$ and, in obvious notation,

$$(v, u)_{\mathscr{B}} = \int \cdots \int_{\mathscr{B}} v \cdot u \frac{dx^1 \cdots dx^m}{dy},$$

where $y(x)$ is the function discussed in Section 1. Also we introduce the norm

$$||u|| = (u, u)^{1/2}.$$

The relationship between the operators K, K^* and the associated boundary matrices M, M^* can then be expressed through the "first identity"

$$(v, Ku) + (v, Mu)_{\mathscr{B}} = (K^* v, u) + (M^* v, u)_{\mathscr{B}}$$

which follows immediately from the identity

$$\int \cdots \int_{\mathscr{R}} \frac{\partial}{\partial x^\lambda} (v \alpha^\lambda u) dx^1 \cdots dx^m = \int \cdots \int_{\mathscr{B}} \frac{\partial y}{\partial x^\lambda} (v \alpha^\lambda u) \frac{dx^1 \cdots dx^m}{dy}.$$

We now make use of the relations

$$K + K^* = \kappa + \kappa', \qquad M + M^* = \mu + \mu'$$

which are obvious from the definitions given in Sections 1 and 2. Setting $v = u$ we obtain the "second identity"

$$(u, Ku) + (u, Mu)_{\mathscr{B}} = (u, \kappa u) + (u, \mu u)_{\mathscr{B}}.$$

The right hand side here is non-negative by II and III. In fact, there is a constant $c > 0$ such that $(u, \kappa u) \geqq c(u, u)$. Hence we may conclude

$$||u|| \leqq c||Ku||$$

provided $Mu = 0$. This is the "basic inequality". It implies the

UNIQUENESS THEOREM 3.1. *A solution $u \in \mathfrak{C}_1$ of the differential equation $Ku = f$ under the—semi-admissible—boundary condition $Mu = 0$ is unique.*

Of course, the proof of this uniqueness theorem, as given, is nothing but the natural generalization of the classical uniqueness proofs for elliptic and hyperbolic equations with the aid of energy integrals.

4. Hilbert Space. Weak Solution

With reference to the inner product (v, u) and the norm $||u|| = (u, u)^{1/2}$ we may introduce the Hilbert space $\mathfrak{H}$ of all square integrable functions

$u(x)$. Suppose a function $f \in \mathfrak{H}$ is given. Then we call $u \in \mathfrak{H}$ a "weak" solution of the equation $Ku = f$ with the boundary condition $Mu = 0$ if the relation

$$(v, f) = (K^* v, u)$$

holds for all $v \in \mathfrak{C}_1$ satisfying $M^* v = 0$ at $\mathscr{B}$. As is clear from the second identity, a strict solution—i.e. one in $\mathfrak{C}_1$—is also a weak one.

Consider the subspace w of all functions $w \in \mathfrak{C}$ of the form $w = K^* v$ with $v \in \mathfrak{C}_1$ satisfying $M^* v = 0$. To every such w the function v is uniquely assigned —by the Uniqueness Theorem applied to K^* and M^*; therefore the inner product (v, f) is uniquely assigned to w and linear in w. Moreover, this linear form is bounded,

$$|v, f| \leqq ||f|| \, ||v|| \leqq c||f|| \, ||w||,$$

by virtue of the basic inequality. Hence, as is well known, there exists an element $u \in \mathfrak{H}$ such that $(v, f) = (w, u)$. Thus we have proved

THEOREM 4.1. *To any $f \in \mathfrak{H}$ there exists a weak solution u of $Ku = f$ under the—semi-admissible—boundary condition $Mu = 0$.*

For Tricomi's boundary value problem the existence of a weak solution was first proved in this manner by C. Morawetz [14].

Naturally, one wonders whether or not a weak solution is a strict solution. In this connection one may at first find out which conclusions can be drawn if the weak solution is known to be in $\mathfrak{C}_1$, i.e. if it is known to possess continuous first derivatives.

To do this, let u be such a solution. From the formula $(v, f) = (K^*v, u)$ for all v in $\mathfrak{C}_1$ with $M^* v = 0$ we then may derive the relation

$$(v, Ku-f)+(v, Mu)_{\mathscr{B}} = 0.$$

From the arbitrariness of v we first deduce that u satisfies the differential equation $Ku = f$ and hence the relation $(v, Mu)_{\mathscr{B}} = 0$ for all v satisfying $M^* v = 0$. From this we conclude that at every point of the boundary

$$2v \cdot \beta u = 2\beta v \cdot u = (\beta - \mu')v \cdot u = -v \cdot (\mu - \beta)u = 0.$$

In order to draw further conclusions we shall invoke the additional "admissibility" requirement on the boundary condition $Mu = 0$ to be formulated in the next section. From this requirement we shall deduce that the boundary condition $Mu = 0$ has the property of being the "strict adjoint" of the condition $M^* v = 0$. By this we mean that $Mu = 0$ if u is such that $v \cdot \beta u = 0$ holds for all v with $M^* v = 0$. Using this property we may conclude: *a weak solution $u \in \mathfrak{C}_1$ satisfies the differential equation $Ku = f$ and the boundary condition $Mu = 0$.*

The hypothesis $u \in \mathfrak{C}_1$ made in this last argument is rather strong and can be proved to hold only under additional conditions on the coefficients and the right member of the equation.

Without making severe additional assumptions one certainly can establish stronger properties which a weak solution possesses automatically.

One of the stronger properties one should like to prove is the validity of the basic inequality $||u|| \leqq c||f||$; if it were valid for weak solutions such solutions would be unique.

Moreover, one should like to prove that the weak solution is a "strong" one.

A function $u \in \mathfrak{H}$ will be called a *strong* solution of the equation $Ku = f$ with $f \in \mathfrak{H}$, if a sequence of functions $u^\nu \in \mathfrak{C}_1$ exists such that

$$||u^\nu - u|| \to 0, \qquad ||Ku^\nu - f|| \to 0,$$

as $\nu \to \infty$. The function u will also be said to admit the operator K strongly. If the function u^ν can be so chosen that in addition

$$Mu^\nu = 0,$$

we say that u satisfies the boundary condition $Mu = 0$. Evidently, for functions u, v admitting K and K^* strongly and satisfying $Mu = 0$ and $M^* v = 0$, the two basic identities hold as well as the basic inequality.

In order to prove that a weak solution is strong it is necessary to construct an appropriate sequence of functions u^ν.

For elliptic equations and for Cauchy's problem of hyperbolic equations it is possible to construct such approximating functions by employing mollifiers as smoothing operators. For the general problem considered here this tool does not seem to be strong enough. It is still possible to construct functions u^ν such that $||u^\nu - u|| \to 0$ and $||Ku^\nu - f||' \to 0$, as $\nu \to \infty$, where the prime is to indicate that the integral which enters the definition of the norm is to extended over a proper subregion $\mathscr{R}'$ of $\mathscr{R}$. The difficulty in proving $||Ku^\nu - f|| \to 0$ with reference to $\mathscr{R}$, instead of $\mathscr{R}'$, arises from the fact that different components of the function $u = \{u_1, \cdots, u_k\}$ would seem to require different smoothing operators according to whether they should, or should not, vanish at the boundary.

Nevertheless, under the assumption of a smooth boundary (having a continuous normal n) and of an admissible boundary condition we shall prove the "identity of the strong and the weak solution"; but in doing so, we are forced to employ a detour: we shall first prove a differentiability theorem.*

**Added in proof:* Recently, P. D. Lax has succeeded in proving directly that a weak solution is strong, by employing a modified version of mollifiers, provided that the boundary has no corners and that β does not change type. Subsequently, R. S. Phillips has removed these two restrictions. These results will be published in subsequent issues of this journal.

5. Admissible Boundary Condition

In Section 3 we have described boundary conditions which we called "semi-admissible" and which were sufficient to insure uniqueness. To insure existence we must impose additional requirements which we shall describe in the present section.

We adopt a variant of a proposal made by P. D. Lax in connection with the mixed problem of a hyperbolic system. This proposal was to require that, at each point of the boundary $\mathscr{B}$, the condition $Mu = 0$ be "maximal" with respect to the relation $u \cdot \beta u \geqq 0$. Let $\mathfrak{U}$ be the k-dimensional space of columns $u = \{u_1, \cdots, u_k\}$—at some point of $\mathscr{B}$—and let $\mathfrak{N}(M)$ be the nullspace of M, consisting of all u satisfying $Mu = 0$. Then the condition $Mu = 0$ is called "maximal" with respect to $u \cdot \beta u \geqq 0$ if there exists no subspace of $\mathfrak{U}$ containing $\mathfrak{N}(M)$ properly in which $u \cdot \beta u \geqq 0$.

Instead of adopting this proposal we prefer to use an equivalent but more concrete definition of the "admissibility" of a boundary condition. We assume that the matrix β can be split into two components $\beta_\pm$,

$$\beta = \beta_+ + \beta_- ,$$

in such a way that the following three requirements are met:
1) the matrix

$$\mu = \beta_+ - \beta_-$$

satisfies the condition

III$_0$ $$\mu + \mu' \geqq 0,$$

2) the nullspaces of $\beta_\pm$ span the whole space $\mathfrak{U}$,

III$_1$ $$\mathfrak{N}(\beta_+) \oplus \mathfrak{N}(\beta_-) = \mathfrak{U},$$

3) the ranges of $\beta_\pm$ have only the column $u = 0$ in common,

III$_2$ $$\mathfrak{R}(\beta_+) \cap \mathfrak{R}(\beta_-) = \mathfrak{O}.$$

Such a pair of matrices $\beta_\pm$ will be called "admissible", and so will be the boundary condition $\beta_- u = 0$ formed with such a pair. Evidently, we have

$$M = \mu - \beta = -2\beta_- ,$$

so that condition $Mu = 0$ is equivalent with $\beta_- u = 0$.

Requirements III$_0$, III$_1$, III$_2$ will be combined as requirement III.

While conditions III—in general—will be easily verified in concrete problems, a different form of these conditions will be more handy to work with later on.

A pair of matrices $P_\pm$ will be called a pair of projectors if they satisfy conditions

$$P_+ + P_- = 1, \qquad P_+ P_- = P_- P_+ = 0.$$

It is clear that condition III_1 is equivalent with the condition that *a pair of projectors* $P_\pm$ *exists such that*

$$\beta_\pm = \beta P_\pm .$$

Similarly condition III_2 is equivalent with the condition that *a pair of projectors* $Q_\pm$ *exists such that*

$$\beta_\pm = Q_\pm \beta.$$

The matrices $P_\pm$, $Q_\pm$ are not unique in case the nullspace of β is not empty.

The requirements III are designed to make it evident that *the adjoint boundary condition* $M^* v = (\mu'+\beta)v = 2\beta'_+ v = 0$ *is also admissible.* In fact, the matrices $-\beta$, μ', $Q'_\mp$, $P'_\mp$ play the roles of the matrices β, μ, $P_\pm$, $Q_\pm$ and requirements III'_1, III'_2 are equivalent with III_2, III_1.

It is also evident—as a consequence of III_1—that the *condition* $\beta'_+ v = 0$ *is the strict adjoint of the condition* $\beta_- u = 0$ in the sense explained in Section 3; for, if $v \cdot \beta u = 0$ for all u with $\beta P_- u = \beta_- u = 0$, we have $v \cdot \beta P_+ u$ for all u and hence $\beta'_+ v = P'_+ \beta v = 0$. Similarly, the *condition* $\beta_- u = 0$ *is the strict adjoint of* $\beta'_+ v = 0$, as a consequence of condition III_2.

We shall derive a few properties of admissible pairs of matrices $\beta_\pm$ although we shall not use them explicitly.

First we note that property III_0 leads to

LEMMA 5.1. $\mu u = 0$ *implies* $\mu' u = 0$, *and vice versa.*

Clearly, by III_0 we have, with appropriate $c > 0$,

$$(\mu+\mu')u \cdot (\mu+\mu')u \leqq c(u \cdot (\mu+\mu')u) = 2c(u \cdot \mu u) = 0,$$

hence $\mu' u = (\mu+\mu')u = 0$. From property III_1 we derive

LEMMA 5.2. $\beta u = 0$ *implies* $\beta'_+ u = \beta'_- u = 0$ *and hence* $\mu' u = 0$.

We need only observe that III_1 allows us to write $\beta'_\pm$ in the form $P'_\pm \beta$. Next we shall prove:

Properties III_0 *and* III_1 *imply property* III_2. Let w be in the intersection $\mathscr{R}(\beta_+) \cap \mathscr{R}(\beta_-)$, i.e. $w = \beta_+ u_+ = \beta_- u_-$. By virtue of III_1 we may choose $u_\pm$ such that $\beta_- u_+ = \beta_+ u_- = 0$. Hence $\mu(u_+ + u_-) = \beta_+ u_+ - \beta_- u_- = 0$ whence $\mu'(u_+ + u_-) = 0$ by Lemma 5.1. Furthermore, $\beta(u_+ - u_-) = \beta_+ u_+ - \beta_- u_- = 0$ whence $\mu'(u_+ - u_-) = 0$ by Lemma 5.2. Therefore $\mu' u_\pm = 0$. By Lemma 1 again, $\mu u_\pm = 0$ and thus $w = 0$.

Moreover, we state that *an admissible condition* $Mu = 0$ *is maximal* with respect to $u \cdot \beta u \geqq 0$. For, let u_0 be in an extension of $\mathfrak{N}(\beta_-)$, so that $u_0 \cdot \beta u_0 \geqq 0$. Since $P_- u_0 = u_0 - P_+ u_0$ is also in the extension, we may, by III_1, choose u_0 in $\mathfrak{N}(\beta_+)$ so that $u_0 = P_- u_0$ and hence $u_0 \cdot \beta u_0 \leqq 0$. Consequently, $u_0 \cdot \beta u_0 = 0$. Since $u_0 + \alpha u_+$ with $u_+ = P_+ u$ is in the extension we have

$$0 \leqq (u_0+\alpha u_+) \cdot \beta(u_0+\alpha u_+) = 2\alpha u_0 \cdot \beta u_+ + \alpha^2 u_+ \cdot \beta u_+,$$

and therefore $u_0 \cdot \beta u_+ = 0$; hence $\beta'_+ u_0 \cdot u = u_0 \cdot \beta P_+ u = 0$ for all u, whence $\beta'_+ u_0 = 0$, or u_0 in $\mathfrak{N}(\beta'_+)$. Since in this space the form $-v \cdot \beta v$ is non-negative, we know that $u_0 \cdot \beta u_0 = 0$ implies $\beta u_0 = 0$, hence $\beta_- u_0 = 0$ since u_0 is in $\mathfrak{N}(\beta_+)$ and hence $\beta_+ u_0 = 0$; thus u_0 lies in $\mathfrak{N}(\beta_-)$ and not in a proper extension of this space.

The converse can also be shown: *a maximal boundary condition is admissible*; i.e. to a maximal condition one may assign admissible matrices $\beta_\pm$ such that the condition is equivalent with $\beta_- u = 0$. It should be noted that the matrices $\beta_\pm$ are not unique, in general. To prove the statement we consider the spaces $\mathfrak{N}$ and $\mathfrak{N}^*$ of those u that satisfy the boundary condition and its strict adjoint. Then we let P and Q be any projectors projecting into these spaces, but so chosen that $PQ = QP$ projects into their intersection. We note that i) $Q'\beta P = 0$, ii) $Q'\beta u = 0$ implies $u = Pu$, iii) $P'\beta v = 0$ implies $v = Qv$. Then we set

$$\mu = (1-Q')\beta(P+Q-PQ)-(P'+Q'-P'Q')\beta(1-P).$$

Using i), one easily verifies $\mu+\mu' = 2P'\beta P - 2Q'\beta Q$, so that III_0 holds. Furthermore, using i) again, one verifies $Q'(\mu+\beta) = (\mu-\beta)P = 0$. From these relations together with ii) one concludes that condition $u = Pu$, or, what is the same, $\mu u = 0$, is equivalent with condition $(\mu-\beta)u = 0$.

In order to show that the nullspaces of $\mu+\beta$ and $\mu-\beta$ span the whole space $\mathfrak{U}$, see III_1, we need only construct a u_1 to a given u_0 such that $(\mu+\beta)(u_0-Pu_1) = 0$. From iii) one infers that the range of βP consists just of $\mathfrak{N}^{*\perp}$, i.e. of all w with $Q'w = 0$. In view of $Q'(\mu+\beta) = 0$ we see that $(\mu+\beta)u_0$ is in that range, so that u_1 exists as desired.

Property III_1, thus established, together with III_0, implies III_2, as shown above. Also, it follows that condition $M^*v = 0$ is admissible and hence maximal.

We have elaborated on these matters, primarily because considerations concerning maximality play a major role in the theory of R. Phillips, the difference being that we assume local maximality while he considers global maximality. That local maximality implies global maximality could be derived from our result about the existence of a strong solution.*

The question arises whether or not an admissible boundary condition can be assigned to a given symmetric-positive differential operator.

We should like to emphasize that the answer to this question is affirmative: *with each symmetric positive differential operator admissible boundary conditions can be associated.*

It is inherent in the concept of maximality that at each point of $\mathscr{B}$ a maximal condition exists. But the statement can also be verified in a more

**Added in proof:* In fact, without our "detour", see footnote on page 22.

concrete manner. One need only choose the projectors $P_\pm$ which project into the eigenspaces with positive and non-negative eigenvalues of the symmetric matrix β. Then $P'_\pm = P_\pm$ and the matrices $\beta_\pm = P_\pm \beta P_\pm$ obviously satisfy $\beta_+ + \beta_- = \beta$ and conditions III_0, III_1.

While it is thus clear that with each symmetric positive operator admissible boundary conditions can be associated, it is not at all obvious that any given boundary condition to be associated with a differential equation can be identified with one which is admissible in the sense here introduced.

For such a condition $Mu = 0$, one must first verify whether or not $u \cdot \beta u \geqq 0$ in the null space of M. Next one may verify whether or not it is maximal. Instead of doing this it is, in general, easier to construct the matrices $\beta_\pm$ or the projectors $P_\pm$ and to verify conditions $\mathrm{III}_{0,1}$.

As an example, let $u = \{u_1, u_2\}$ and $\beta u = \{u_2, u_1\}$. Then the condition $u_2 = 0$ is admissible. We have $P_+ u = \{u_1, 0\}$, $P_- u = \{0, u_2\}$ and the associated matrix μ is given by $\mu u = \{-u_2, u_1\}$ and thus unsymmetric. Furthermore $Q'_\pm = Q_\pm = P'_\mp = P_\mp$, so that the adjoint condition is again $u_2 = 0$. We have mentioned this case mainly to point out that it can happen that $M^* = M$ and also that μ may be unsymmetric.

In the following section we shall show that the standard conditions imposed on the solutions of elliptic and hyperbolic equations are admissible. Under certain restrictions on the domain the same will be shown for the condition customarily imposed on the solutions of the Tricomi equation.

6. Elliptic, Hyperbolic, Mixed Equations

Let us first consider a symmetric hyperbolic system, or what is equivalent, an equation of the form $Ku = f$ in which the matrices α^λ are such that a linear combination $\xi_\lambda \alpha^\lambda$ of them is positive-definite. As was mentioned in Section 1, such an equation can be so modified that it is symmetric positive. We ask for the solution u in a lens-shaped region $\mathscr{R}$ described by $y^-(x) \leqq y \leqq y^+(x)$, where the two functions $y^\pm(x)$ are space-like, i.e., such that for the normal vector $\nabla y^\pm = \partial y^\pm / \partial x^\lambda$ the matrix $\alpha^\lambda \nabla_\lambda y^\pm$ is positive definite. The values of u on the initial surface $\mathscr{B}^-$: $y^-(x) = 0$, $y^+(x) \geqq 0$ are prescribed; no values are prescribed on the end surface $\mathscr{B}^+$: $y^-(x) \geqq 0$, $y^+(x) = 0$. With $n = \nabla y^+$ on $\mathscr{B}^+$ and $n = -\nabla y^-$ on $\mathscr{B}^-$ we find $\beta = \pm(\nabla_\lambda y^\pm)\alpha^\lambda$ to be positive definite on $\mathscr{B}^+$, negative definite on $\mathscr{B}^-$. We therefore may take $\mu = \beta$ on $\mathscr{B}^+$, $\mu = -\beta$ on $\mathscr{B}^-$. The boundary condition becomes $2\mu u = (\mu - \beta)u = 0$, i.e. $u = 0$ on $\mathscr{B}^-$, while no condition is imposed on $\mathscr{B}^+$ since there $\mu - \beta = 0$. Evidently, property III obtains with $P_\pm = 1$ on $\mathscr{B}^\pm$. Thus we are led to Cauchy's problem.

Next consider an elliptic equation

$$\frac{\partial}{\partial x^\lambda} g^{\lambda\nu} \frac{\partial}{\partial x^\nu} \phi = -\rho$$

and write it, as in Section 1, in the form $Ku = f$. On the boundary $\mathscr{B}$ of the region $\mathscr{R}$ we then have

$$\beta_{00} = -p, \qquad \beta_{0\nu} = \beta_{\nu 0} = -n_\nu, \qquad \beta_{\nu\lambda} = 0, \qquad \nu \neq 0, \quad \lambda \neq 0,$$

where, as an abbreviation, we have set $p = p^\rho n_\rho$. As in the introduction we set $p = q^- - q^+$ with $q^\pm > 0$. The projectors $P_\pm$, $Q_\pm$ are then given by

$$P_- u = \left\{u_0, \ -q^+ \frac{n_1 u_0}{n_\lambda n_\lambda}, \cdots, -q^+ \frac{n_m u_0}{n_\lambda n_\lambda}\right\}, \quad Q_- u = \left\{q^- \frac{n_\lambda u_\lambda}{n_\lambda n_\lambda}, u_1, \cdots, u_m\right\},$$

$$P_+ u = \left\{0, u_1, -q^+ \frac{n_1 u_0}{n_\lambda n_\lambda}, \cdots, u_m - q^+ \frac{n_m u_0}{n_\lambda n_\lambda}\right\}, \quad Q_+ u = \left\{u_0 - q^- \frac{n_\lambda u_\lambda}{n_\lambda n_\lambda}, 0, \cdots, 0\right\}.$$

The matrix μ given by $\mu_{00} = p$, $\mu_{0\nu} = -\mu_{\nu 0} = -n_\nu$, $\mu_{\nu\lambda} = 0$, $\nu \neq 0$, $\lambda \neq 0$, satisfies condition III_0 provided $p \geqq 0$. This last requirement is consistent with the requirement $\partial p^\lambda / \partial x^\lambda > 0$ imposed in Section 1. Equation $(\mu - \beta)u = 0$ reduces to $u_0 = 0$, i.e. to the "first" boundary condition $\phi = 0$. The equation $(\mu + \beta)u = 0$ reduces to $n_\nu u_\nu = 0$, $\nu \neq 0$. Hence requirement III_1 is satisfied. Also note that the matrix μ is not symmetric.

Suppose we want to impose the boundary condition $\phi_n + q\phi = 0$, where $\phi_n = n_\lambda g^{\lambda\nu} \partial\phi/\partial x^\nu$ and q is a given positive function on $\mathscr{B}$. To recognize this condition as being admissible we write

$$\begin{aligned} u \cdot \beta u &= -p\phi^2 - 2\phi\phi_n \\ &= [2q - p]^{-1} \{(\phi_n + (p - q)\phi)^2 - (\phi_n + q\phi)^2\}, \end{aligned}$$

assuming $p < 2q$, and define the matrix μ by

$$u \cdot \mu u = [2q - p]^{-1} \{(\phi_n + (p - q)\phi)^2 + (\phi_n + q\phi)^2\}.$$

Evidently requirements III are fulfilled.

Next consider the Tricomi equation, written in the form given at the end of Section 1. The matrix β then becomes

$$\begin{pmatrix} byn_x - cyn_y & cyn_x - bn_y \\ cyn_x - bn_y & bn_x - cn_y \end{pmatrix}.$$

Suppose the boundary condition $\phi = 0$ or, in terms of the derivatives ϕ_x, ϕ_y, the condition $\phi_x n_y - \phi_y n_x = 0$, is to be imposed on a section $\mathscr{B}_0$ of the boundary $\mathscr{B}$. In order to find out whether or not this condition is admissible, we write the quadratic form $u \cdot \beta u$ as the difference

$$\begin{aligned} u \cdot \beta u &= (byn_x - cyn_y)\phi_x^2 + 2(cyn_x - bn_y)\phi_x\phi_y + (bn_x - cn_y)\phi_y^2 \\ &= (bn_x + cn_y)^{-1} [(b^2 - c^2 y)(n_y \phi_x - n_x \phi_y)^2 - (n_y^2 - y n_x^2)(b\phi_x + c\phi_y)^2], \end{aligned}$$

assuming $bn_x+cn_y \neq 0$ on $\mathscr{B}_0$. Specifically we assume $bn_x+cn_y < 0$ on $\mathscr{B}_0$. We now see that the condition $n_y\phi_x-n_x\phi_y = 0$ is admissible on $\mathscr{B}_0$ if

$$b^2-c^2y > 0, \qquad n_y^2-yn_x^2 > 0 \qquad \text{on } \mathscr{B}_0.$$

We may then take as the matrix μ the matrix defined by

$$u\cdot\mu u = -(bn_x+cn_y)^{-1}[(b^2-c^2y)(n_y\phi_x-n_x\phi_y)^2+(n_y^2-yu_x^2)(b\phi_x+c\phi_y)^2].$$

Requirements III are met, as is immediately verified.

Consider specifically a region in the x, y-plane as indicated in the figure on page 334. We assume $\mathscr{B}_0$ and $\mathscr{B}_+$ to be such that $n_y^2-yn_x^2 > 0$ on $\mathscr{B}_0$, $n_y^2-yn_x^2 < 0$ on $\mathscr{B}_+$; furthermore, we require b and c to be such that

$$bn_x+cn_y < 0 \qquad \text{on } \mathscr{B}_0, \quad > 0 \text{ on } \mathscr{B}_+.$$

Throughout we assume $b^2-c^2y > 0$. The boundary condition $n_y\phi_x-n_x\phi_y = 0$ is then admissible on $\mathscr{B}_0$. Since β is positive definite on $\mathscr{B}_+$, we may take $\mu = \beta$ there, so that no condition is to be imposed on $\mathscr{B}_+$.

For domains of the indicated type, under certain restrictions, the functions $b = b_0$, $c = c_0(1+\varepsilon y)$ with constant $b_0 > 0$, $c_0 < 0$, $\varepsilon > 0$, satisfy all conditions imposed. The peculiar character of the Tricomi equation, described in the introduction, is thus exhibited for such domains.

Similar arguments can be given for the domain originally considered by Tricomi and variants of it, as follows from the work of Protter and Morawetz.

An interesting question arises with regard to the uniqueness proofs using the functions b and c. Consider any region for which the uniqueness of the solution of Tricomi's equation can be proved under the condition $\phi = 0$ on a part $\mathscr{B}_0$ of its boundary. The question is whether or not this uniqueness can always be proved with the aid of two functions b and c. The answer is that this is not so, as was mentioned in the introduction [14].

In some of the cases where this is not possible, one may reduce the Tricomi equation with $\phi = 0$ on $\mathscr{B}_0$ in a different manner to a symmetric-positive system with an admissible boundary condition, namely by properly selecting three linear combinations of the four equations

$$\frac{y\partial\phi_x}{\partial x} - \frac{\partial\phi_y}{\partial y} = 0, \qquad \frac{\partial\phi_x}{\partial y} - \frac{\partial\phi_y}{\partial x} = 0$$

$$\frac{\partial\phi}{\partial x} - \phi_x = 0, \qquad \frac{\partial\phi}{\partial y} - \phi_y = 0.$$

Then six functions will enter in place of the two functions b, c. The method employed by Ou and Ding and by Protter may be so interpreted.

We have mentioned these facts, since we wanted to emphasize that the theory here presented is essentially only a framework. *This theory reduces* *the analytic question of unique existence of a solution of a boundary value problem to an essentially algebraic question,* namely to the question whether or not the problem can be transformed into that of a symmetric positive equation with admissible boundary condition.

Part II

EXISTENCE. STRONG SOLUTIONS

7. Differential Operator on a Manifold

Before proving the existence of the solution of our boundary value problem, and its differentiability, we shall extend the notions involved in this problem to manifolds with a boundary. There is no loss of simplicity in this generality since in our existence proof, even for a domain of the type considered so far, the domain will be regarded as built up of patches just as a manifold.

Specifically, we assume that the manifold $\mathscr{R}$ is covered by a finite number of patches $\mathscr{P}$ of two kinds: "full", or "interior", patches—one-to-one images of the full sphere $x_1^2 + \cdots + x_m^2 < R^2$—and "half", or "boundary", patches—one-to-one images of the hemisphere $x_1^2 + \cdots + x_m^2 = R^2$, $x_m \leqq 0$. The points in a half-patch $\mathscr{P}$ for which $x_1^2 + \cdots + x_m^2 = R^2$, $x_m < 0$, form the "inner" boundary of $\mathscr{P}$, those for which $x_m = 0$ form its outer boundary. All outer boundary points form the boundary $\mathscr{B}$ of $\mathscr{R}$.

The closure of a patch $\mathscr{P}$ will be called a "closed patch" $\bar{\mathscr{P}}$; the closure $\bar{\mathscr{R}} = \mathscr{R}+\mathscr{B}$ of the manifold M will be called the "closed manifold".

In the intersection of two (full or half) closed patches an identification transformation is assumed to be defined which possesses continuous second derivatives and a Jacobian bounded away from zero. Outer boundary points of one half-patch in such an intersection are to correspond to outer boundary points in the other half-patch.

The characterization of the manifold here given implies that the normal vector on its boundary is continuous, while only piecewise continuity of the normal was required of the regions introduced in Section 1. The continuity of the normal seems to be essential for our method; our method does not extend itself automatically to manifolds with corners or edges. Still, by modifying our approach we can handle a certain class of problems involving edges such as Cauchy's problem and the mixed problem of (symmetric) hyperbolic systems and certain problems for the Tricomi equation. This will be explained in Part IV.

A remark concerning notation should be made here. In principle, of course, we should distinguish between the points x of the manifold and its representers $\{x^1, \cdots, x^m\}$ with reference to a patch. We also should distinguish between the function $u(x)$ and its representer $\{u_1(x^1, \cdots, x^m), \cdots, u_k(x^1, \cdots, x^m)\}$; but we shall not do so. With reference to a patch $\mathscr{P}$ under consideration we shall set $x = \{x^1, \cdots, x^m\}$ and $u = \{u_1, \cdots, u_k\}$. The representers associated with a neighboring patch $\tilde{\mathscr{P}}$ will be denoted by $\tilde{x}$ and $\tilde{u}$.

The identification transformation in the intersection of two closed patches is to be given by functions $\tilde{x}^\lambda = \tilde{x}^\lambda\{x^1, \cdots, x^m\}$ in $\mathfrak{C}_2$.

The connection between u and $\tilde{u}$ is to be given in the form

$$u = V\tilde{u}$$

with the aid of a matrix $V = V(x) \in \mathfrak{C}_1$ defined in the overlap of the two closed patches and possessing there an inverse in $\mathfrak{C}_1$. We introduce the transformation

$$\tilde{V} = \left|\frac{dx}{d\tilde{x}}\right| V'$$

where V' is the transpose of V and

$$\left|\frac{dx}{d\tilde{x}}\right| = \left|\frac{\partial(x^1, \cdots, x^m)}{\partial(\tilde{x}^1, \cdots, \tilde{x}^m)}\right|$$

is the absolute value of the Jacobian. We then require that $\tilde{V}$ be the inverse of V,

$$\tilde{V}V = 1,$$

so that V is orthogonal except for the factor $dx/d\tilde{x}$. As a consequence of this requirement the expression $u \cdot u dx^1 \cdots dx^m$ is invariant:

$$\tilde{u} \cdot \tilde{u}\, d\tilde{x}^1 \cdots d\tilde{x}^m = u \cdot u\, dx^1 \cdots dx^m.$$

We now assume that, in each patch, matrices α^μ and γ are given which satisfy requirements I and II of Section 1 and possess continuous derivatives. In each half-patch we set $y = x^m$ and, accordingly,

$$\beta = \alpha^m;$$

we then assume that at the outer boundary of the half-patch a matrix $\mu = \mu(x^1, \cdots, x^{m-1})$ with continuous derivatives is given which satisfies condition III of Section 5, so that the boundary condition $Mu = (\mu - \beta)u = 0$ is admissible. The projectors $P_\pm$ and $Q_\pm$ defined for $x^m = 0$, are required to possess continuous derivatives with respect to $x^1, \cdots, x^{m-1}$.

The transformation of the matrices α and γ will be so chosen that $Ku = f$

goes over into $\tilde{K}\tilde{u} = \tilde{f}$. For this reason, the set of matrices $\{\alpha^1, \cdots, \alpha^m\}$ will be transformed essentially as a contravariant vector density:

$$\tilde{\alpha}^\lambda = \tilde{V} \frac{\partial \tilde{x}^\lambda}{\partial x^\nu} \alpha^\nu V,$$

and the transform of γ will be given by

$$\tilde{\gamma} = \tilde{V} \gamma V + 2\tilde{V} \alpha^\lambda \left(\frac{\partial}{\partial x^\lambda} V\right).$$

One verifies that then

$$\tilde{K}\tilde{u} = \tilde{V}Ku$$

and that the transform of κ is given by

$$\tilde{\kappa} = \tilde{V}\kappa V + \left[V' \alpha^\rho \left(\frac{\partial}{\partial x^\rho} V\right) - \left(\frac{\partial}{\partial x^\rho} V'\right) \alpha^\rho V\right] \frac{dx}{d\tilde{x}}.$$

Note that symmetry of the matrices α (I) and positive definiteness of $\kappa+\kappa'$ (II) is preserved; in fact[5], we have

$$(\kappa+\kappa') = \tilde{V}(\kappa+\kappa')V.$$

The requirement that an outer boundary point $y = x^m = 0$ of a half-patch corresponds to an outer boundary point in an overlapping patch implies the relation $\partial \tilde{x}^m / \partial x^\mu = 0$ for $\mu \neq m$ at $\mathscr{B}$. Therefore we have at $\mathscr{B}$,

$$\tilde{\beta} = \tilde{V} \frac{\partial \tilde{y}}{\partial y} \beta V.$$

Correspondingly, we stipulate that the matrix μ at $\mathscr{B}$ is transformed by

$$\tilde{\mu} = \tilde{V} \frac{\partial \tilde{y}}{\partial y} \mu V,$$

and that $P_\pm$, $Q_\pm$ are transformed by

$$\tilde{P}_\pm = \tilde{V} P_\pm V, \qquad \tilde{Q}_\pm = \tilde{V} Q_\pm V.$$

Evidently, we have

$$\tilde{\mu}+\tilde{\mu}' = \tilde{V} \frac{\partial \tilde{y}}{\partial y} (\mu+\mu')V.$$

Condition III is preserved.

[5]As was observed by C. K. Chu, the same remains true even if the matrix $\tilde{V} = V' \, dx/d\tilde{x}$ does not satisfy the relation $\tilde{V}V = 1$.

If one wants to work with an invariant form (u, u) even in such a case, one may set

$$(u, u) = \Sigma_\rho \int_{\mathscr{P}^\rho} \eta_\rho (u \cdot r^\rho u) dx^1 \cdots dx^m$$

where the positive definite matrices $r \in \mathfrak{C}$ are to be transformed according to

$$\tilde{r} = \tilde{V} r V.$$

The boundary value problem now consists in finding in each patch a function $u = u(x)$ satisfying the equation $Ku = f$ there, and the boundary condition $Mu = 0$ on $x^m = 0$ if the patch is a half-patch. On the intersection of two patches the functions u, f, $\tilde{u}$, $\tilde{f}$ should be related as described.

We assume the existence of a partition of the identity, i.e. a set of functions $\eta_\rho(x) \in \mathfrak{C}_2$ with support in the patch $\mathscr{P}_\rho$ such that

$$\sum_\rho \eta_\rho(x) = 1.$$

We require a little more of the functions $\eta_\rho(x)$, namely that they are supported by a sub-patch $\mathscr{Q}_\rho$ of the patch $\mathscr{P}_\rho$ characterized by the restriction $x_1^2 + \cdots + x_m^2 < R_1^2$ with $R_1 < R$.

With the aid of the functions $\eta_\rho(x)$ we define the inner products

$$(v, u) = \sum_\rho \int \cdots \int \eta_\rho v \cdot u \, dx^1 \cdots dx^m,$$

$$(v, u)_{\mathscr{B}} = \sum_\rho \int \cdots \int \eta_\rho v \cdot u \, \frac{dx^1 \cdots dx^{m-1}}{dy}.$$

Evidently the first of these inner products does not depend on the coordinate system.[6]

From the transformation formulas, one verifies the relations

$$\tilde{v} \cdot \tilde{K}\tilde{u} \, d\tilde{x}^1 \cdots d\tilde{x}^m = v \cdot Ku \, dx^1 \cdots dx^m,$$

$$\tilde{v} \cdot \tilde{M}\tilde{u} \, d\tilde{x}^1 \cdots d\tilde{x}^{m-1} = v \cdot Mu \, dx^1 \cdots dx^{m-1},$$

where v transforms like u; furthermore,

$$\tilde{u} \cdot \tilde{\kappa}\tilde{u} \, d\tilde{x}^1 \cdots d\tilde{x}^m = u \cdot \kappa u \, dx^1 \cdots dx^m,$$

$$\tilde{u} \cdot \tilde{\mu}\tilde{u} \, d\tilde{x}^1 \cdots d\tilde{x}^{m-1} = u \cdot \mu u \, dx^1 \cdots dx^{m-1}.$$

One then sees that the two basic identities hold just as they were formulated in Section 3. The same therefore is true of the basic inequality. Consequently, *the uniqueness theorem holds for manifolds.*

With the aid of the inner product (v, u) we define a norm $\|u\| = (u, u)^{1/2}$ and denote the associated Hilbert space by $\mathfrak{H}$.

The notion of weak and strong applicability of the operator K with the boundary condition $Mu = 0$ is then the same as in Section 4.

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[6]According to a remark made by M. Gaffney, we might have made the expression $(v, u)_{\mathscr{B}}$ independent of the coordinates by defining y independently, e.g. such that $y = \Sigma'_\rho \eta_\rho x^m_\rho$ near $\mathscr{B}$; here the summation runs over all half-patches $\mathscr{P}_\rho$ and x^m_ρ refers to a particular coordinate system in $\mathscr{P}_\rho$.

8. Differentiability. A Priori Inequalities

Suppose the solution of the equation $Ku = f$ is in $\mathfrak{C}_2$, with f in $\mathfrak{C}_1$. Then the derivatives $\nabla_\lambda u = \partial u/\partial x^\lambda$ of u satisfy the differential equations

$$2\alpha^\nu \frac{\partial}{\partial x^\nu} \nabla_\lambda u + 2\left(\frac{\partial \alpha^\nu}{\partial x^\lambda}\right) \nabla_\nu u + \gamma \nabla_\lambda u + \frac{\partial \gamma}{\partial x^\lambda} u = \nabla_\lambda f$$

which, together with $Ku = f$, form again a symmetric system of equations for the $(m+1)k$ functions $\{u, \nabla_1 u, \cdots, \nabla_m u\}$. Accordingly, one might try to derive an analogue of the basic inequality for the system $\{u, \nabla_1 u, \cdots, \nabla_m u\}$. There are two difficulties however. First, the matrix κ associated with the new system does not necessarily have a positive definite symmetric part and we are, therefore, forced to impose an additional condition. Secondly, the system of derivatives $\nabla_\lambda u$ does not necessarily satisfy a homogeneous boundary condition of the form $Mu = 0$.

The tangential derivatives, however, do satisfy a condition of this form. Using this fact, we can overcome the second difficulty by not employing all derivatives $\nabla_1 u, \cdots, \nabla_m u$, but by restricting ourselves to a set of differential operators which at the boundary $\mathscr{B}$ involve only differentiation in tangential directions. To this end we shall introduce a set of differential operators of the first order D_σ, $\sigma = 0, \cdots, m$, which in each patch $\mathscr{P}$ are of the form

$$D_\sigma = d_\sigma^\tau \frac{\partial}{\partial x^\tau} + d_\sigma;$$

here the numbers d_σ^τ and the matrices d_σ are functions in $\mathfrak{C}_1$ of the variables $x^1, \cdots, x^m$ defined in the patch $\mathscr{P}$.

In the overlap of two patches the relation $u = V\tilde{u}$ should of course imply $D_\sigma u = V\tilde{D}_\sigma \tilde{u}$, in obvious notation. For convenience we include the identity among the operators D_σ; specifically, we set $D_0 = 1$.

The main property of the operators D_σ, i.e. the property that they involve differentiation only in tangential direction at the boundary, is expressed by the relation

(1) $$d_\sigma^m = 0 \quad \text{at } \mathscr{B}.$$

An operator D_σ having this property will simply be said to be "tangential".

Next we require that

(2) the operators D_σ form a *complete* system of "tangential" operators at the boundary; i.e., that every operator of the form $d^\tau \partial/\partial x^\tau + d$ with d^τ and d in $\mathfrak{C}$ and for which $d^m = 0$ at $\mathscr{B}$ with reference to a boundary patch, is a linear combination of operators D_σ with coefficients in $\mathfrak{C}$.

Furthermore, we require that

(3) the commutator of any one of the operators D_σ with the operator K,

defined as $[D_\sigma, K] = D_\sigma K - K D_\sigma$, should be a linear combination of the operators $\{D_\tau\}$ and of the operator K; i.e., there should[7] be matrices p_σ^τ in $\mathfrak{C}$ and t_σ in $\mathfrak{C}_1$ such that

$$[\mathrm{K}] \qquad [D_\sigma, K] = p_\sigma^\tau D_\tau + t_\sigma K.$$

Finally, we require that

(4) the commutator of any one of the operators D_σ with the operator $M = \mu - \beta$ at the boundary $\mathscr{B}$ should be a linear combination of the operators D_σ and M in such a way that there exists a matrix q_σ^τ so that

$$[\mathrm{M}] \qquad [D_\sigma, M] = q_\sigma^\rho D_\rho + t_\sigma^{\mathscr{B}} M$$

where the matrix $t_\sigma^{\mathscr{B}}$ is connected with the matrix t_τ which enters requirement (3) by the formula

$$t_\sigma^{\mathscr{B}} = t_\sigma + \mathring{d}_\sigma$$

where

$$\mathring{d}_\sigma = \frac{\partial}{\partial x^m} d_\sigma^m .$$

The reason for introducing this matrix will be made clear in Section 9.

Note that for $u \in \mathfrak{C}_1$ the operator D_σ can be applied on the function Mu, defined only on the boundary $\mathscr{B}$, since there D_σ involves differentiation only in a tangential direction.

The identities postulated here represent actually only relations between the coefficients of the operators involved and their derivatives; but they may be regarded as operator identities when applied to functions u in $\mathfrak{C}_2$.

The commutator relations [K] and [M] can be written in a different way. To this end we introduce "compound" systems $\underset{1}{u} = \{u_\sigma\}$ of functions $u_0, u_1, \cdots$ in the same number as there are operators D_σ. Then we introduce an operator $\underset{1}{K}$ acting on such systems by

$$\underset{1}{K}\,\underset{1}{u} = \{K u_\sigma + p_\sigma^\tau u_\tau\}.$$

To this operator we assign the boundary operator $\underset{1}{M}$ by

$$\underset{1}{M}\,\underset{1}{u} = \{M u_\sigma + q_\sigma^\tau u_\tau\}.$$

Furthermore we introduce the operators D and $D - t$ by

$$Du = \{D_\sigma u\}, \qquad (D-t)u = \{D_\sigma u - t_\sigma u\};$$

they transform simple systems u into compound systems. The commutator identities [K] and [M] can then be written in the form

[7]Since the final inequalities do not depend on the derivatives of t, it is perhaps possible to eliminate the requirement that t in $\mathfrak{C}$ be in $\mathfrak{C}_1$. We shall not attempt to do so.

$$\text{(K)} \qquad (D-t)K = \underset{1}{K}D,$$

$$\text{(M)} \qquad (D-t^{\mathscr{B}})M = \underset{1}{M}D.$$

The operator $\underset{1}{K}$ corresponds evidently to a symmetric system with $\underset{1}{\alpha}^{\rho}$ and $\underset{1}{\beta}$ given by

$$\underset{1}{\alpha}^{\rho}\underset{1}{u} = \{\alpha^{\rho}u_{\sigma}\}, \qquad \underset{1}{\beta}\underset{1}{u} = \{\beta u_{\sigma}\}.$$

Clearly, $\underset{1}{M}$ is an associated boundary matrix. The corresponding matrices $\underset{1}{\kappa}$ and $\underset{1}{\mu}$ are given by

$$\underset{1}{\kappa}\underset{1}{u} = \{\kappa u_{\sigma} + p_{\sigma}^{\tau}u_{\tau}\},$$

$$\underset{1}{\mu}\underset{1}{u} = \{\mu u_{\sigma} + q_{\sigma}^{\tau}u_{\tau}\}.$$

We now impose the *additional conditions* that

(5) $\qquad \underset{1}{\kappa}+\underset{1}{\kappa}'$ should be *positive definite,*

and that

(6) $\qquad \underset{1}{\mu}+\underset{1}{\mu}'$ should be *non-negative*

so that the conditions II and III_0 are satisfied for $\underset{1}{K}$ and $\underset{1}{M}$. We shall refer to these conditions as $\underset{1}{\text{II}}$ and $\underset{1}{\text{III}}_0$.

A *set of differential operators* D_{σ} associated with a differential operator K and its boundary operator M *will be called "satisfactory" if it satisfies the conditions* (1) *to* (6). We do not require admissibility of the boundary matrix $\underset{1}{M}$.

At the end of this section we shall show under which condition a satisfactory set of operators D_{σ} exists. At present we shall derive a basic inequality for the derivatives $D_{\sigma}u$ of a function u which admits K and M.

To this end we apply the "second identity" of Section 3 with $\underset{1}{K}, \underset{1}{M}, \underset{1}{\kappa}, \underset{1}{\mu}$ to the compound system $\underset{1}{u} = Du = \{D_{\sigma}u\}$ derived from a function $u(x)$ in $\mathfrak{C}_2$ which satisfies the boundary condition $Mu = 0$. We find

$$\begin{aligned}(Du, \underset{1}{\kappa}\, Du) + (Du, \underset{1}{\mu}\, Du)_{\mathscr{B}} &= (Du, \underset{1}{K}\, Du) + (Du, \underset{1}{M}\, Du)_{\mathscr{B}} \\ &= (Du, (D-t)Ku) + (Du, (D-t^{\mathscr{B}})Mu)_{\mathscr{B}} \\ &= (Du, (D-t)Ku) \leqq ||Du||\; ||(D-t)Ku||.\end{aligned}$$

With a constant $\underset{1}{c}$ so chosen that $||\underset{1}{u}||^2 = (\underset{1}{u}, \underset{1}{u}) \leqq \underset{1}{c}(\underset{1}{u}, \underset{1}{\kappa}\underset{1}{u})$ we then have

$$||Du|| \leqq \underset{1}{c}||(D-t)Ku||.$$

Since $D_0 = 1$ is contained among the $\{D_\sigma\}$ we can find a constant c_1, viz. $c_1 = [1+\max|t|]\underset{1}{c}$, such that

$$||Du|| \leqq c_1||DKu||.$$

This is the basic inequality for the system $\{D_\sigma u\}$, to which we shall also refer as the "direct *a priori* inequality".

Note that we have not used requirement (2) in deriving it; this requirement will be used only in Section 9.

We proceed to investigate *the circumstances under which a set of "tangential" operators D_σ exists, which satisfy requirements* (2) *to* (6).

First we shall show that one can always find a system of operators D_σ which possesses the *completeness property* (2). In each interior patch one may take the operators

$$(\mathscr{I}) \qquad \eta\frac{\partial}{\partial x^1}, \cdots, \eta\frac{\partial}{\partial x^m},$$

where $\eta(x)$ is the function assigned to the patch in a partition of unity. In each half-patch one may take the operators

$$(\mathscr{B}) \qquad \eta\frac{\partial}{\partial x^1}, \cdots, \eta\frac{\partial}{\partial x^{m-1}}, \ \eta x^m\frac{\partial}{\partial x^m}.$$

The operators thus assigned to all patches forming the manifold $\mathscr{R}$ may then be taken as the operators D_σ for $\sigma \neq 0$, in addition to $D_0 = 1$.

In the overlap of any two (full or half) patches the operators $\partial/\partial x^1, \cdots, \partial/\partial x^{m-1}$ and $\partial/\partial x^m$ or $x^m\,\partial/\partial x^m$, respectively, associated with the two patches, can be expressed linearly in terms of each other. Using the identity $1 = \sum_\rho \eta_\rho$ we, therefore, can express any of these operators as linear combinations of the D_σ. From this fact one immediately infers that requirement (2) is met.

The existence of a complete set of tangential operators D_σ which has the *commutator property* (3) will be shown only under the condition

(3_A) In every boundary patch, matrices $\sigma_1, \cdots, \sigma_{m-1}$ $\quad \tau_1, \cdots, \tau_{m-1}$ in $\mathfrak{C}$ exist such that the relations

$$(\mathrm{A}) \qquad \frac{\partial}{\partial x^\lambda}\alpha^m = \tau_\lambda\alpha^m + \alpha^m\sigma_\lambda$$

hold for $\lambda = 1, \cdots, m-1$.

As before we take the operators defined in interior patches by $(\mathscr{I})$ into the set of operators D_σ; but we modify the definition ($\mathscr{B}$) of the operators assigned to the half-patches and take them as

$$(\mathscr{B}_{\text{A}}) \qquad \eta\left(\frac{\partial}{\partial x^1}+\sigma_1\right), \cdots, \eta\left(\frac{\partial}{\partial x^{m-1}}+\sigma_{m-1}\right), \ \eta x^m \frac{\partial}{\partial x^m}.$$

Setting $t_\lambda = \eta(\tau_\lambda+\sigma_\lambda)$, we observe that

$$\left[\eta\left(\frac{\partial}{\partial x^\lambda}+\sigma_\lambda\right), K\right] - t_\lambda K = \left[\eta \frac{\partial}{\partial x^\lambda}, K\right] - K\eta\sigma_\lambda - \eta\tau_\lambda K$$

$$= 2\eta\left\{\left(\frac{\partial}{\partial x^\lambda}\alpha^m\right) - \alpha^m\sigma_\lambda - \tau_\lambda\alpha^m\right\}\frac{\partial}{\partial x^m} + \sum_{\rho\neq m}\xi^\rho_\lambda \frac{\partial}{\partial x^\rho},$$

whence relation $[D_\sigma, K] = p^\tau_\sigma D + t_\sigma K$ for $D_\sigma = \eta(\partial/\partial x^\lambda+\sigma_\lambda)$ follows by (A) with t_λ in place of t_σ. For the operator $\eta x^m \partial/\partial x^m$ the same relation follows from

$$\left[\eta x^m \frac{\partial}{\partial x^m}, K\right] + \eta K = \sum_{\rho\neq m}\xi^\rho \frac{\partial}{\partial x^\rho} + \left\{\frac{\partial\alpha^m}{\partial x^m}\eta - K\eta\right\} x^m \frac{\partial}{\partial x^m}$$

with $-\eta$ in place of t_σ.

To verify the commutator property (4) we need only make use of the fact that the operators D_σ involve differentiation in a tangential direction at the boundary. Therefore, D_σ is applicable on $Mu = (\mu-\beta)u$, although μ is defined only at $\mathscr{B}$. Since the operator $[D_\sigma, M] - t^{\mathscr{B}}_\sigma M$ does not involve any differentiation, it is a matrix and can be written in the form $q^\rho_\sigma D_\rho = q^0_\sigma$ with $q^\rho_\sigma = 0$ for $\rho \neq 0$.

Thus we have shown that operators having properties (1) to (4) exist if matrices σ_λ, τ_λ exist for which (A) holds.

Such matrices σ_λ, τ_λ do always exist in a half-patch if the matrix α^m has an inverse in the closed half-patch; in fact in this case we may take $\sigma_\lambda = 0$. However, the existence of this inverse is not necessary. Matrices σ_λ, τ_λ always exist if the matrices α^m, at different points of the half-patch, are equivalent, i.e. if there exists a matrix $W = W(x)$ in $\mathfrak{C}_1$ with an inverse $W^{-1}(x)$, in the closed patch, such that

$$\text{(W)} \qquad \alpha^m(x) = W(x)\alpha^m W'(x)$$

where α^m is independent of x. Here W' is the transpose of W. Then we may[8] simply take

$$\tau_\lambda = \frac{\partial W}{\partial x^\lambda} W^{-1} \quad \text{and} \quad \sigma_\lambda = \tau'_\lambda.$$

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[8]In this case we may even assume α^m constant to begin with, and set $\tau_\lambda = \sigma_\lambda = 0$. This corresponds to an approach suggested by P. D. Lax.

Condition (W) implies that the matrix α^m—and hence the operator β on $\mathscr{B}$ represented by α^m in $\mathscr{P}$—does not change its "inertial" type along the boundary. This is a severe restriction, which—or at least condition (A)—seems to be required by our method.

In the construction of the operators D_σ with the properties (1) to (4) we have essentially made use of the assumption that the boundary of the manifold has a continuous normal, implied by the assumption that the neighborhood of the boundary can be covered by half-patches. It is only for this construction that this severe assumption was made. If operators D_σ satisfying (1) to (4) could be found for a manifold whose boundary has rims or corners, our theory would apply.

On occasion we shall modify the set of operators D_σ by multiplying all of them except D_0 by a constant factor chosen small enough so as to attain a particular aim. We then shall say that we make the operators D_σ, $\sigma \neq 0$, "sufficiently small". In this way, for example, we can achieve that the coefficients p^0_σ, q^0_σ and t_σ entering the two commutator relations in (3) and (4) are as small as we wish.

Our next question concerns the conditions (5) and (6) requiring that the new matrices $\underset{1}{\kappa}$ and $\underset{1}{\mu}$ be *positive definite* and *non-negative.*

In the introduction we have described a case in which one can verify explicitly that the differentiability statement does not hold unless a condition similar to $\underset{1}{\kappa}+\underset{1}{\kappa}' > 0$ is added to the condition $\kappa+\kappa' > 0$. Accordingly, we take the attitude that *the condition* (5), $\underset{1}{\kappa}+\underset{1}{\kappa}' > 0$, *is imposed as an additional requirement.* It should be mentioned, however, that it is sufficient that the quadratic form $\underset{1}{u}'\underset{1}{\kappa}\underset{1}{u}'$ with $\underset{1}{u}' = \{0, u_1, \cdots\}$ be positive definite. For then, one can make sure that condition (5) holds by making the operators D_σ "sufficiently small", i.e. by replacing D_σ by εD_σ for $\sigma \neq 0$. Evidently, the factor ε can be chosen so small that $\underset{1}{u}\underset{1}{\kappa}\underset{1}{u}$ becomes positive definite.

As to the boundary matrix $\underset{1}{\mu}$, let us first assume that the matrix μ is positive definite. Then again one may replace D_σ by εD_σ for $\sigma \neq 0$ and choose the factor ε sufficiently small.

If the matrix μ is not positive definite, the argument just given applies provided an inequality $\underset{1}{u}_\tau \underset{1}{q}^\tau_\sigma u_\sigma \leq c(\underset{1}{u}\mu\underset{1}{u})$ obtains. *Otherwise, the condition* (6) *that* $\underset{1}{\mu}$ *be non-negative must be imposed as an additional requirement.*

9. Dual Inequality

Having established the *a priori* inequality for the differential operators $D_\sigma u$, one may be inclined to believe that a solution u exists possessing derivatives, for which this inequality holds—at least if the notion of solution and derivative is understood in a sufficiently general sense. Such a belief is supported by the experience in analysis—in particular since Schauder's work.

For elliptic equations one has constructed such solutions in different ways making specific use of the elliptic character of the equation. For Cauchy's problem for hyperbolic equations three methods have been used. In the first method one approximates the coefficients of the equation and the initial data by analytic functions, and proves that the solution of the so modified problem, obtained from Cauchy-Kowalewski's theorem leads in the limit to a solution of the original problem. In this proof one employs the *a priori* inequality. It seems impossible to extend this approach to the problem of general symmetric-positive equations. In the second approach one approximates the differential equation by finite difference equations in such a way that the basic inequality and the *a priori* inequality carry over, and proves that the solutions of the finite difference equation problem converges to a solution of the differential equation problem. Such an approach can probably be developed for our problem. The third approach due to P. D. Lax, involves nearly exclusively tools of Hilbert space theory; it was used by Lax for symmetric hyperbolic systems and elliptic equations and, in a modified way, by Gårding for general hyperbolic equations. This third approach can be adapted to the present problem.

In his method Lax employs a new *a priori* inequality, called "dual" inequality by Gårding, which involves the concept of norm of negative order, suggested by notions introduced by L. Schwartz. We shall discuss the modifications needed to adapt these tools to our problem.

To explain the concept of negative norm let us introduce as norm of order 1 of the function u the norm of its derivatives

$$||u||_1 = ||Du||,$$

defined for all u in $\mathfrak{C}_1$. In a standard manner we extend the space $\mathfrak{C}_1$ to a space $\mathfrak{H}_1$ in which the strong extensions of all operators D_σ are applicable and which is closed with respect to the norm $||\ ||_1$.

The norm of order -1 of a function $v \epsilon \mathfrak{H}$ can be defined in two ways. The "abstract" definition is

$$||v||_{-1} = \underset{u \neq 0}{\text{l.u.b.}} \frac{(v, u)}{||u||_1},$$

where the functions u are restricted to $\mathfrak{H}_1$. The inequality

$$(v, u) \leqq ||v||_{-1}||u||_1,$$

implied by this definition will be referred to as the "generalized" Schwarz inequality.

The notions $\| \ \|_{\pm 1}$ introduced here differ from those employed by Lax inasmuch as we restrict the operators D_σ to tangential operators, while he uses all differentiations $\partial/\partial x^1, \cdots, \partial/\partial x^m$.

The closure of the space $\mathfrak{H}$ with respect to the norm $\| \ \|_{-1}$ was first introduced by L. Schwartz, [2]. The norms $\| \ \|_{\pm s}$—in obvious notation—were also employed by[9] Leray [3].

The "concrete" definition of the norm $\| \ \|_{-1}$ is more suitable for our purposes. To explain it we must introduce the operator

$$D^* \cdot D = \textstyle\sum_\sigma D_\sigma^* D_\sigma$$

which involves the adjoints

$$D_\sigma^* = -\frac{\partial}{\partial x^\lambda} d_\sigma^\lambda + d'_\sigma = -d_\sigma^\lambda \frac{\partial}{\partial x^\lambda} + d'_\sigma - \frac{\partial}{\partial x^\lambda} d_\sigma^\lambda$$

of the operator D_σ.

If one writes the operator $D^* \cdot D$ in the form

$$D^* \cdot D = 1 + \sum_{\sigma \neq 0} D_\sigma^* D_\sigma$$

one recognizes that it is a natural generalization of the operator $1 - c\Delta$ where Δ is the Laplacian. By virtue of the completeness property (2) of the set of operators D_σ, the operator $D^* \cdot D$ is elliptic in the interior of $\mathscr{R}$ but not at the boundary $\mathscr{B}$, since no normal derivatives at the boundary enter its definition. This defect is advantageous since it makes it unnecessary to associate a boundary condition with the operator $D^* \cdot D$.

We shall make essential use of the fact that the operator $D^* \cdot D$ possesses an inverse. Specifically, we state

THEOREM 9.1. *To every function v in $\mathfrak{H}$ there exists a function $\tilde{v}$ in $\mathfrak{H}_1$ such that the relation*

$$D^* \cdot D\tilde{v} = v$$

holds in the weak sense. I.e., for every u in $\mathfrak{H}_1$ the relation

$$(D\tilde{v}, Du) = (v, u)$$

holds.

[9]The analogue of the "dual inequality" for Cauchy's problem of a hyperbolic equation is contained among a number of inequalities which Leray formulates, see Lemmas 88.1 and 88.2. However, Leray deduces these inequalities from an existence theorem, derived with the aid of the Cauchy-Kovalevsky Theorem. Lax proceeds in the opposite direction and derives an existence theorem from a dual inequality.

This theorem, called Theorem 12.1 in Section 12, and various other statements about the solution $\tilde{v}$, will be proved there.

Using the inverse of $D^* \cdot D$ we may introduce the norm $|| \ ||_{-1}$ simply by the formula

$$||v||_{-1} = ||\tilde{v}||_1 = ||(D^* \cdot D)^{-1} v||_1 .$$

This is the "concrete" definition of the norm of order -1. Using this definition we find for u in $\mathfrak{H}_1$ the inequality

$$(v, u) \leqq ||D\tilde{v}|| \, ||Du||,$$

which is nothing but the generalized Schwarz inequality

$$(v, u) \leqq ||v||_{-1} ||u||_1 ;$$

this shows that $||v||_{-1}$ is an upper bound for $(v, u)/||u||_1$. Since the bound is assumed for $u = \tilde{v}$, it is seen that the abstract definition of $|| \ ||_{-1}$ agrees with the concrete one.

The notions introduced here again differ somewhat from those employed by P. D. Lax. Where he takes the Laplacian operator, we take the operator $-\sum_{\sigma \neq 0} D_\sigma^* D_\sigma$, which may be regarded as a "clipped" Laplacian in as much as the normal derivatives along the edge have been eliminated.

In terms of the notion of norm of order 1 one may evidently write the direct *a priori* inequality of Section 8 in the form

$$||u||_1 \leqq c_1 ||Ku||_1 \qquad \text{if } Mu = 0.$$

The dual inequality of P. D. Lax, adapted to our problem, may now simply be written as

$$||v||_{-1} \leqq c_{-1} ||K^* v||_{-1} \qquad \text{if } M^* v = 0,$$

in terms of the norm $|| \ ||_{-1}$ of order -1. Specifically, we state that a number c_{-1} exists such that this inequality holds for all functions v in $\mathfrak{C}_1$. We have written the inequality in terms of K^* and M^* since we need it in this form; of course, it holds just as well for K and M.

The proof of this *dual inequality* is the major piece of the present work. In this proof we employ the operator

$$(D-t)^* \cdot D = \sum_\sigma (D_\sigma - t_\sigma)^* D_\sigma = 1 + \sum_{\sigma \neq 0} (D_\sigma - t_\sigma)^* D_\sigma ,$$

which is similar to the clipped operator $D^* \cdot D$, but involves the operators $D_\sigma - t_\sigma$ introduced in Section 8. The operator $(D-t)^* \cdot D$ is not self-adjoint as is $D^* \cdot D$, but its leading part is. As we shall show in Section 12 an analogue of Theorem 9.1 (12.1) holds (Theorem 12.2) which we shall formulate here as

THEOREM 9.2. *To every function v in $\mathfrak{H}$ there exists a function $\hat{v}$ in $\mathfrak{H}_1$ such that the relation*

$$(D-t)^* \cdot D\hat{v} = v$$

holds in the weak sense; i.e., for every u in $\mathfrak{H}_1$ the relation

$$(D\hat{v},\ (D-t)u) = (v,\ u)$$

holds.

Here it is assumed that the operators D_σ, $\sigma \neq 0$ are chosen "sufficiently small", in the sense explained in Section 8.

We need stronger statements however, which we formulate in Theorems 9.3 to 9.6. These theorems refer to an arbitrary function v in $\mathfrak{C}_1$ and the solution $\hat{v}$ of the equation $(D-t)^* \cdot D\hat{v} = v$, which exists according to Theorem 9.2.

THEOREM 9.3. *The functions $\hat{v}$ and $D\hat{v}$ admit the operators K and $\underset{1}{K}$ in the strong sense.*

THEOREM 9.4. *The function $K\hat{v}$ admits the operator D (strongly) and the identity $(D-t)K\hat{v} = \underset{1}{K}D\hat{v}$ holds.*

The next statement will refer to the boundary values of the functions $\hat{v}$ and $D\hat{v}$. We first introduce the space $\mathfrak{H}_{\mathscr{B}}$ of functions $u_{\mathscr{B}}$ defined on $\mathscr{B}$ which is complete with respect to the norm given by

$$||u_{\mathscr{B}}||^2_{\mathscr{B}} = (u_{\mathscr{B}},\, u_{\mathscr{B}})_{\mathscr{B}} = \int_{\mathscr{B}} u^2_{\mathscr{B}} \frac{dx}{dy}.$$

We then say that a function u admitting the operator K strongly has boundary values $u_{\mathscr{B}}$ (in $\mathfrak{H}_{\mathscr{B}}$) if a sequence u^ν in $\mathfrak{C}_1$ exists such that

$$||u^\nu - u|| \to 0, \qquad ||Ku^\nu - Ku|| \to 0, \qquad ||u^\nu - u_{\mathscr{B}}||_{\mathscr{B}} \to 0.$$

In the following we shall suppress the subscript $\mathscr{B}$. We then formulate

THEOREM 9.5. *The functions $\hat{v}$ and $D\hat{v}$ have boundary values.*

As an evident consequence of this fact, the second basic identity of Section 3 holds for $\hat{v}$ and for $D\hat{v}$. Thus in particular

$$(D\hat{v},\ \underset{1}{\kappa}D\hat{v}) + (D\hat{v},\ \underset{1}{\mu}D\hat{v})_{\mathscr{B}} = (D\hat{v},\ \underset{1}{K}D\hat{v}) + (D\hat{v},\ \underset{1}{M}D\hat{v})_{\mathscr{B}}\,.$$

According to identity (K), which holds by Theorem 9.4, we have

$$(D\hat{v},\ \underset{1}{K}D\hat{v}) = (D\hat{v},\ (D-t)K\hat{v}).$$

In view of identity (M) it is tempting to assume that similarly the relation

$$(D\hat{v},\ \underset{1}{M}D\hat{v})_{\mathscr{B}} = (D\hat{v},\ (D-t^{\mathscr{B}})M\hat{v})_{\mathscr{B}}$$

holds.

Furthermore, since $(D-t)^* \cdot D\hat{v} = v$, we have

$$(D\hat{v}, (D-t)K\hat{v}) = (v, K\hat{v}).$$

It is also tempting to assume that a similar operation can be performed on $(D\hat{v}, (D-t^{\mathscr{B}})M\hat{v})_{\mathscr{B}}$ since the operator D is purely tangential on $\mathscr{B}$ and therefore might be regarded as a differential operator acting on functions in $\mathfrak{H}_{\mathscr{B}}$. The adjoint of the operator $(D-t^{\mathscr{B}})$, when so regarded, is however not the operator $(D-t^{\mathscr{B}})^*$ since the definition of the operator D^* involves contributions from the terms $d^m \partial/\partial x^m$, which vanish on $\mathscr{B}$. A little consideration will show that the adjoint of $(D-t^{\mathscr{B}})$ with respect to $\mathscr{B}$ is just the operator $(D-t)^*$. As a matter of fact, the definition $t^{\mathscr{B}} = t+\hat{d}$ of $t^{\mathscr{B}}$ given in Section 8 was chosen so as to achieve this result. Accordingly, one is led to assume the identity

$$(D\hat{v}, (D-t^{\mathscr{B}})M\hat{v})_{\mathscr{B}} = (v, M\hat{v})_{\mathscr{B}}.$$

The results of the two operations performed on $\mathscr{B}$ are formulated as

THEOREM 9.6. *The identity*

$$(D\hat{v}, \underset{1}{M}D\hat{v})_{\mathscr{B}} = (v, M\hat{v})_{\mathscr{B}}$$

holds.

We shall give the proof of Theorems 9.3 to 9.6 somewhat later in this section. Assuming their validity we can state the identity

$$(D\hat{v}, \underset{1}{\kappa}D\hat{v}) + (D\hat{v}, \underset{1}{\mu}D\hat{v})_{\mathscr{B}} = (v, K\hat{v}) + (v, M\hat{v})_{\mathscr{B}} = (K^* v, \hat{v})$$

if $M^* v = 0$. In the last step we have made use of the first basic identity of Section 3, which certainly holds for v in $\mathfrak{C}_1$ since $\hat{v}$ has boundary values in the sense described above.

It is now easy to derive the dual inequality from the last identity.

From the positive definiteness requirement imposed on $\underset{1}{\kappa}$ and $\underset{1}{\mu}$ in Section 8 we may infer that there is a constant $\underset{1}{c}$ such that

$$||D\hat{v}||^2 \leqq \underset{1}{c}(K^* v, \hat{v}).$$

From the generalized Schwarz inequality we have

$$(K^* v, \hat{v}) \leqq ||K^* v||_{-1}||\hat{v}||_1 .$$

Thus, we obtain the inequality

$$||D\hat{v}|| \leqq \underset{1}{c}||K^* v||_{-1} .$$

This is not yet the desired dual inequality which involves $||v||_{-1} = ||D\tilde{v}||$ in place of $||D\hat{v}||$. We estimate $||D\tilde{v}||$ in terms of $||D\hat{v}||$ as follows. First we have

$$\begin{aligned} ||\tilde{v}||_1^2 = (D\tilde{v}, D\tilde{v}) &= (v, \tilde{v}) = (D\hat{v}, (D-t)\tilde{v}) \\ &\leqq ||D\hat{v}||[||D\tilde{v}||+\tau||\tilde{v}||] \\ &\leqq (1+\tau)||\hat{v}||_1||\tilde{v}||_1, \end{aligned}$$

where τ is a bound of $|t|$. Hence

$$||v||_{-1} = ||\tilde{v}||_1 \leqq (1+\tau)||\hat{v}||_1$$

and thus we obtain, with $c_{-1} = c_1(1+\tau)$, the desired dual inequality:[10]

THEOREM 9.7. *For any function v in $\mathfrak{C}_1$ which satisfies the boundary condition $M^* v = 0$, the inequality*

$$||v||_{-1} \leqq c_{-1}||K^* v||_{-1}$$

holds.

We proceed to supply the proofs of Theorems 9.3 to 9.6 by reducing them to various theorems which will be proved in Part III. Before we can formulate these theorems we must introduce a class of differential operators $E = \{E_\tau\}$ given by

$$E_\tau = e_\tau^\lambda \frac{\partial}{\partial x^\lambda} + e_\tau$$

with reference to a patch in terms of numbers $e_\tau^\lambda(x)$ and matrices $e_\tau(x)$ in $\mathfrak{C}_1$. The system $\{E_\tau\}$ is required to be complete in the sense that every operator of first order with coefficients in $\mathfrak{C}$ or $\mathfrak{C}_1$ is a linear combination of the E_τ with coefficients in $\mathfrak{C}$ or $\mathfrak{C}_1$, respectively; also it is assumed that $E_0 = 1$. We then formulate the following

SUMMARY OF THEOREMS 12.4, 13.2, 16.1, AND 16.2.

Suppose the function v admits the operator E (strongly) and let $\hat{v}$ be the solution of the equation $(D-t)^* \cdot D\hat{v} = v$. Then:

1) *The functions $\hat{v}$ and $D\hat{v}$ admit the operator E and $E\hat{v}$ admits D.*

2) *Let the matrices p be such that for all σ, τ the identity*

$$(D_\sigma E_\tau - E_\tau D_\sigma)w = p_{\sigma\tau}^\rho E_\rho w$$

holds for all w in $\mathfrak{C}_2$, then it holds for $w = \hat{v}$.

3) *The functions $\hat{v}$ and $D\hat{v}$ possess boundary values.*

4) *There exists a function σ^ν in $\mathfrak{C}_1$ for every $\nu > 0$ such that*

$$(w, \sigma^\nu w) \to (w, w)_{\mathscr{B}},$$

$$(w, (\mathring{D}\sigma^\nu)w)_{\mathscr{B}} \to -(w, \hat{d}w)_{\mathscr{B}},$$

as $\nu \to 0$ for all w in $\mathfrak{H}(E)$. Here $\hat{d} = \{\hat{d}_\sigma\}$ stands for the set of numbers $\hat{d}_\sigma = \partial d_\sigma^m / \partial x^m$ as described in Section 8 and $D_\sigma^0 = e_\sigma^\lambda \partial/\partial x^\lambda$ in a patch.

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[10] A somewhat shorter reasoning, involving constant matrices α^m in boundary patches, was suggested by P. D. Lax, cf. footnote 8, page 369.

It is easy to derive Theorems 9.3 to 9.6 from these statements.

Since the operators E_τ are complete, the operators $\underset{1}{K}$ and K are linear combinations of them. Since the functions $D\hat{v}$ and $\hat{v}$ admit E strongly, it follows that they admit $\underset{1}{K}$ and K strongly. Thus Theorem 9.3 follows.

Since $E\hat{v}$ admits D strongly the same is true of $K\hat{v}$. Evidently, the expression $A_\sigma \equiv (D_\sigma - t_\sigma)K - \underset{1}{K}D_\sigma$ is a linear combination of the expressions

$$B_{\sigma\tau} = D_\sigma E_\tau - E_\tau D_\sigma - p^\rho_{\sigma\tau} E_\rho .$$

Hence $B_{\sigma\tau}\hat{v} = 0$ implies $A_\sigma \hat{v} = 0$. Thus Theorem 9.4 follows.

Theorem 9.5 is the third statement of the Summary.

To prove Theorem 9.6 we first extend the matrix M, defined on $\mathscr{B}$ into the interior. We can do this by first extending M for each patch as a function u^ρ in its interior which vanishes outside of a neighborhood of $\mathscr{B}$ and then setting $M = \sum_\rho \eta_\rho u^\rho$. Next we define the matrix $\underset{1}{M}$ in $\mathscr{R}$ by

$$\underset{1}{M}\underset{1}{w} = M\underset{1}{w} + [(DM) - Md - t^{\mathscr{B}} M]w_0 ,$$

where $d = \{d^\sigma\}$ is the matrix entering the definition

$$D_\sigma = d^\mu_\sigma \frac{\partial}{\partial x^\mu} + d_\sigma .$$

This matrix agrees with the matrix $\underset{1}{M}$ introduced through (M) in Section 8. Evidently, we have $\underset{1}{M}D = (D - t^{\mathscr{B}})M$ and hence the identity

$$(D\hat{v}, \sigma^\nu \underset{1}{M} D\hat{v}) = (D\hat{v}, \sigma^\nu (D - t^{\mathscr{B}}) M\hat{v}) = (v, \sigma^\nu M\hat{v}) - (D\hat{v}, [\hat{d} + (\mathring{D}\sigma^\nu)]M\hat{v}).$$

From the two statements of item 4 we then deduce the relation

$$(D\hat{v}, \underset{1}{M} D\hat{v})_{\mathscr{B}} = (v, M\hat{v})_{\mathscr{B}}$$

which is the statement of Theorem 9.6.

10. Existence of Differentiable Solutions

Having established the dual *a priori* inequality we can prove that a weak solution exists which admits the operators D_σ. We need only follow the reasoning used by P. D. Lax for symmetric hyperbolic systems.

To every function $v \in \mathfrak{C}_1$ satisfying the boundary condition $M^* v = 0$ we assign the function $w = (D^* \cdot D)^{-1} K^* v$ which exists by Theorem 1 of Section 9 and belongs to the space $\mathfrak{H}_1$; i.e., w admits the operators D_σ strongly. It is to be observed that $w = 0$ implies $K^* v = 0$ and hence $v = 0$, in view of $M^* v = 0$ and the uniqueness theorem which holds for v in $\mathfrak{C}_1$ when employed for K^* and M^*. Consequently, the function v is assigned uniquely and linearly to the function w. The inner product (v, f) of v with a function $f \epsilon \mathfrak{H}$ may therefore be regarded as a linear form in w,

$$(v, f) = l_f(w).$$

We now assume $f \epsilon \mathfrak{H}_1$. Then we may apply the generalized Schwarz inequality and the dual inequality (see Section 9 for both) obtaining

$$|l_f(w)| \leqq ||v||_{-1}||f||_1 \leqq c_f||K^* v||_{-1},$$

with $c_f = c_{-1}||f||_1$. By definition of w and of the norm $||\ \ ||_{-1}$ we have

$$||K^* v||_{-1} = ||(D^* \cdot D)^{-1}K^* v||_1 = ||w||_1.$$

Hence we have

$$||l_f(w)|| \leqq c_f||w||_1.$$

In other words, the form l_f is bounded with respect to the norm $||\ \ ||_1$; it therefore could be extended to a bounded linear form defined in all of $\mathfrak{H}_1$. In any case it follows from this boundedness that a function $u \epsilon \mathfrak{H}_1$ exists such that the relation

$$(v, f) = l_f(w) = (Dw, Du)$$

holds for all functions w. Since $D^* \cdot Dw = K^* v$, this relation becomes

$$(v, f) = (K^* v, u),$$

valid for all v in $\mathfrak{C}_1$ with $M^* v = 0$. In the terminology introduced in Section 4, this means that the function u is a solution of the equation $Ku = f$ with $Mu = 0$ in the weak sense. Thus we have proved

LEMMA 10.1. *To every function f in $\mathfrak{H}_1$ the equation $Ku = f$ with the boundary condition $Mu = 0$ possesses a weak solution u which also belongs to $\mathfrak{H}_1$.*

Since the relation

$$(Dw, Du) = l_f(w) \leqq c_f||w||_1 = c_{-1}||f||_1||w||_1$$

holds for all w in $\mathfrak{H}_1$, we may set $w = u$ obtaining the

COROLLARY TO LEMMA 10.1. *The solution u obeys the inequality*

$$||u||_1 \leqq c_{-1}||f||_1.$$

Our next aim is to show that the *weak solution* of Lemma 10.1 is also a *strong solution*. That means we want to construct to this solution u a sequence of functions $u^\nu(x)$ in $\mathfrak{H}_1$ such that the relations

$$\text{(S)} \qquad \begin{gathered} ||u^\nu - u|| \to 0, \\ ||Ku^\nu - f|| \to 0, \qquad Mu^\nu = 0 \end{gathered}$$

hold as $\nu \to \infty$.

We maintain that it is sufficient to construct for each patch $\mathscr{P}_\rho$ a sequence of functions $u_\rho^\nu(x)$ in $\mathfrak{C}_1$ such that the relations

$$(\mathrm{S}_\rho) \qquad \begin{aligned} &\|\zeta_\rho(u_\rho^\nu - u)\| \to 0, \\ &\|\zeta_\rho(Ku_\rho^\nu - f)\| \to 0, \qquad Mu_\rho^\nu = 0, \end{aligned}$$

hold as $\nu \to \infty$, where ζ_ρ stands for any function in $\mathfrak{C}$ with support in the subpatch $\mathscr{Q}_\rho$ of $\mathscr{P}_\rho$. We recall that the functions $\eta_\rho(x)$, which form a partition of unity as described in Section 7, were assumed to have their supports in this subpatch $\mathscr{Q}_\rho$.

To show that it is sufficient to construct such functions $u_\rho^\nu(x)$ we set

$$u^\nu = \sum_\rho \eta_\rho u_\rho^\nu$$

and then prove that relations (S) hold. From the relation $\sum_\sigma \eta_\sigma = 1$ we have

$$\sum_\rho \mathring{K}\eta_\rho = 0$$

where $\mathring{K}$ is the operator which, with respect to a patch, is represented by $\mathring{K} = \sum_\lambda \alpha^\lambda \,\partial/\partial x^\lambda$. Consequently, we have

$$\begin{aligned} Ku^\nu - f &= \sum_\rho (K\eta_\rho u_\rho^\nu - \eta_\rho f) = \sum_\rho \eta_\rho (Ku_\rho^\nu - f) + \sum_\rho (\mathring{K}\eta_\rho)(u_\rho^\nu - u), \\ Mu^\nu &= \sum_\rho M\eta_\rho u_\rho^\nu = \sum_\rho \eta_\rho M u_\rho^\nu . \end{aligned}$$

Applying relations (S_ρ) with $\zeta_\rho = \eta_\rho$ and $\zeta_\rho = \mathring{K}\eta_\rho$ we may conclude that relations (S) hold.

For an interior patch $\mathscr{P}_\rho$ the existence of an appropriate sequence u_ρ^ν is implied by the theorem that any weak operator of first order is strong [1].

It is therefore sufficient to prove a corresponding statement for each boundary patch. To this end we make use of the completeness assumption (2) made in Section 8. Since in every boundary patch $\mathscr{P}$ each differentiation $\partial/\partial x^\lambda$, $\lambda = 1, \cdots, m-1$, is tangential, it is a linear combination of the tangential operators D_σ by virtue of this assumption. In particular, therefore, the operator

$$\tilde{K} = K - \beta \frac{\partial}{\partial y},$$

represented in the patch by

$$\tilde{K} = \sum_{\lambda=1}^{m-1} \alpha^\lambda \frac{\partial}{\partial x^\lambda} + \gamma,$$

is such a linear combination:

$$\tilde{K} = \tilde{b}_i D_i .$$

Since the weak solution u of Theorem 10.1 is in $\mathfrak{H}_1$, it admits the operator D_σ strongly and therefore admits the operator $\tilde{K}$ strongly. Consequently, the relation

$$(K^* v, u) = -\left(\frac{\partial}{\partial y}\beta v, u\right) + (v, \tilde{K}u)$$

holds for every function $v \in \mathfrak{C}_1$ which satisfies the boundary condition $M^* v = 0$ and vanishes at the inner boundary of the half-patch $\mathscr{P}$. From the fact that u is a weak solution we therefore may deduce the relations

$$\left(\frac{\partial}{\partial y}\beta v, u\right) + (v, \tilde{f}) = 0 \tag{B_0}$$

with $\tilde{f} = f - \tilde{K}u$, for all functions v in $\mathfrak{C}_1$ with $M^* v = 0$. Thus, the statement to be proved has been reduced to the identity of the weak and the strong—essentially one-dimensional—differential operator $\beta\,\partial/\partial y$ with the boundary condition $(\mu-\beta)u = 0$.

We may prove this identity in different ways. We find it convenient to employ reflection at the boundary, $y = 0$. We introduce the reflected half-patch $\mathscr{P}^*$ consisting of all points $(x^1, \cdots, x^{m-1}, y)$ for which the point $(x^1, \cdots, x^{m-1}, -y)$ is in $\mathscr{P}$. We then continue the functions u and f into the half-patch $\mathscr{P}^*$ by continuing $P_- u$ and $Q_+ \tilde{f}$ as odd functions, $P_+ u$ and $Q_- \tilde{f}$ as even functions of y. Here $P_\pm$ and $Q_\pm$ are the projectors introduced and discussed in Section 5, extended into the half-patches $\mathscr{P}$ and $\mathscr{P}^*$ independently of y.

Let w be an arbitrary function in $\mathfrak{C}_1$ defined in the full patch, $\mathscr{P}+\mathscr{P}^*$, assumed to vanish at the interior boundary of the half-patch and its image. We maintain that then the relation

$$\left(\frac{\partial}{\partial y}\beta w, u\right)_{\mathscr{P}+\mathscr{P}^*} + (w, \tilde{f})_{\mathscr{P}+\mathscr{P}^*} = 0 \tag{B}$$

holds. Note that w need not satisfy a boundary condition on $y = 0$.

To prove this relation we transfer the functions defined for $y > 0$ to the original patch y by setting

$$w^*(y) = w(-y) \qquad \text{for } y \leqq 0$$

and, similarly,

$$u^*(y) = u(-y) = (P_+ - P_-)u(y),$$
$$\tilde{f}^*(y) = \tilde{f}(-y) = (Q_- - Q_+)\tilde{f}(y).$$

The contribution from $\mathscr{P}^*$ to the left member of relations (B) then becomes

$$-\left(\frac{\partial}{\partial y}\beta w^*, (P_+ - P_-)u\right) + (w^*, (Q_- - Q_+)\tilde{f}),$$

where the integration is to be performed over the original half-patch $\mathscr{P}$. We now make use of the fact that the operators $P_\pm$ are independent of y.

Furthermore, we use the relation $P'_{\pm}\beta = \beta Q'_{\pm}$, mentioned in Section 5. The contribution from $\mathscr{P}^*$ then reduces to

$$-\left(\frac{\partial}{\partial y}\,\beta(Q'_+-Q'_-)w^*,\,u\right)+\left((Q'_--Q'_+)w^*,\tilde{f}\right).$$

In terms of the function

$$v = w+(Q'_--Q'_+)w^*,$$

we may write the left member of (B) in the form

$$\left(\frac{\partial}{\partial y}\,\beta v,\,u\right)+(v,\tilde{f}),$$

and thus identify it with the left member of relation (B_0). Since w is continuous in $\mathscr{P}+\mathscr{P}^*$, we have $w^* = w$ on the boundary $y = 0$ and hence $v = 2Q'_-w$ there. Because of $Q_-Q_+ = 0$ (see Section 5) v therefore satisfies the boundary condition

$$M^*v = 2\beta Q'_+v = 4\beta Q'_+Q'_-w = 0.$$

Consequently, relation (B) follows from relation (B_0).

Now we are prepared to construct functions u^ν_ρ for which relations (S_ρ) hold.

Relation (B) implies that the function u in $\mathscr{P}+\mathscr{P}^*$ admits the operator $\beta\partial/\partial y$ in the weak sense with the result $\beta\,\partial u/\partial y = \tilde{f}$. Since the weak operator is strong, one knows that there exists a sequence u^ν in $\mathfrak{C}_1$ such that

$$\|\zeta(u^\nu-u)\|_{\mathscr{P}+\mathscr{P}^*}\to 0,\qquad \left\|\zeta\left(\beta\frac{\partial}{\partial y}\,u^\nu-\tilde{f}\right)\right\|_{\mathscr{P}+\mathscr{P}^*}\to 0\qquad \text{as } \nu\to\infty.$$

In order to attain $Mu^\nu = 0$ we shall choose the functions u^ν in a special manner.

We introduce mollifiers J^ν with kernels $j^\nu(x-x')$. We may choose the kernel $j^\nu(x-x')$ such that as a function of x' it has its support in $\mathscr{P}$ provided x lies in $\mathscr{Q}$. Also $j^\nu(x-x')$ should be even in $y-y'$. Then we set

$$u^\nu(x) = J^\nu P_-u(x)+P_+J^\nu u(x)$$

for x in $\mathscr{Q}$. Since P_-u is odd in y and $j^\nu(x-x')$ is even in $y-y'$, the function $J^\nu P_-u$ is also odd in y and hence vanishes on $y = 0$. Therefore, since $P_-P_+ = 0$, we have $P_-u^\nu = 0$ and hence $Mu^\nu = 0$ on $\mathscr{B}$. From the properties of J^ν we conclude $\|\zeta(J^\nu P_-u-P_-u)\|\to 0$ and $\|\zeta(P_+J^\nu u-P_+u)\|\to 0$, whence $\|\zeta(u^\nu-u)\|\to 0$.

Next we observe that $\beta\,\partial u/\partial y = \tilde{f}$ in the weak sense implies $Q_+\beta\,\partial u/\partial y = Q_+\tilde{f}$ in the weak sense since Q_+ was assumed in $\mathfrak{C}_1$. Hence, from the known properties of J^ν,

$$\left\|\zeta\left(Q_+\beta\frac{\partial}{\partial y}J^\nu u-Q_+\tilde{f}\right)\right\|\to 0$$

or, since $Q_+\beta=\beta P_+$ and P_+ is independent of y,

$$\left\|\zeta\left(\beta\frac{\partial}{\partial y}P_+J^\nu u-Q_+\tilde{f}\right)\right\|\to 0.$$

Furthermore we observe that P_-u admits the operator $\beta\,\partial/\partial y$ in the weak sense with the result $\beta\,\partial P_-u/\partial y=Q_-\tilde{f}$, since we have

$$(v, Q_-\tilde{f})=(Q'_-v,\tilde{f})=\left(\frac{\partial}{\partial y}\beta Q'_-v, u\right)=\left(\frac{\partial}{\partial y}P'_-\beta v, u\right)=\left(P'_-\frac{\partial}{\partial y}\beta v, u\right)$$
$$=\left(\frac{\partial}{\partial y}\beta v, P_-u\right)$$

for v in $\mathfrak{C}_1$ with support in $\mathscr{P}$. As a consequence,

$$\left\|\zeta\left(\beta\frac{\partial}{\partial y}J^\nu P_-u-Q_-\tilde{f}\right)\right\|\to 0.$$

Thus we obtain the desired relation

$$\left\|\zeta\left(\beta\frac{\partial}{\partial y}u^\nu-\tilde{f}\right)\right\|\to 0.$$

Since the function u admits the operators D_σ strongly, the relations

$$||\zeta(D_\sigma u^\nu-D_\sigma u)||\to 0$$

hold and, consequently, also

$$||\zeta(\tilde{K}u^\nu-\tilde{K}u)||\to 0.$$

Because of $K=\beta\,\partial/\partial y+\tilde{K}$ and $f=\tilde{f}+\tilde{K}u$ we finally have

$$||\zeta(Ku^\nu-f)||\to 0.$$

At the same time we have

$$||\zeta(u^\nu-u)||\to 0 \qquad \text{and } Mu^\nu=0.$$

Thus, with $\mathscr{P}=\mathscr{P}_\rho$ and $u^\nu=u^\nu_\rho$, we have established the relations (S_ρ) as desired.

Using the terminology introduced in Section 8, we summarize the result in

THEOREM 10.1. *Suppose the operator pair (K, M) possesses a satisfactory set of operators D_σ. Then for every f in the space $\mathfrak{H}_1$ there exists, in $\mathfrak{H}_1$, a strong solution u of the equation $Ku=f$ with the boundary condition $Mu=0$. The solution u obeys the inequality*

$$||u||_1\leqq c_{-1}||f||_1\,.$$

This theorem combines an existence statement with a differentiability statement.

11. Existence of Strong Solutions

In establishing the existence of a strong solution u of the boundary value problem it was also shown that this solution possesses strong first derivatives $D_\sigma u$. This result was paid for primarily by requiring that the matrix $\underset{1}{\kappa}$ associated with the operator $\underset{1}{K}$ be positive definite.

In this section we shall show that the existence of a strong solution is independent of this assumption.

We require of our operator K that the matrices α^ρ and μ involved have those properties that are implied by the conditions (1) to (6) imposed in Section 8, except condition (5). A simple way of formulating this requirement is this:

The symmetric positive operator K and the associated boundary operator M should be such that with the operator $K+b$ together with M a satisfactory set of operators D_σ can be associated provided the positive constant b is chosen large enough. Then we shall prove

THEOREM 11.1. *Under the assumption just stated, the equation $Ku = f$ with the boundary condition $Mu = 0$ possesses a strong solution for each $f \in \mathfrak{H}$.*

We prove this theorem by the *continuation method.* We connect the operator $K = K_0$ with the operator $K+b = K_1$ by the set of operators

$$K_\lambda = K+\lambda b, \qquad 0 \leqq \lambda \leqq 1,$$

for which the matrix

$$\kappa_\lambda = \kappa+\lambda b$$

evidently satisfies requirement II.

We shall use the following two lemmas.

LEMMA 11.1. *Let K and $K+\zeta$ be two symmetric positive operators differing only in the matrix $\zeta \in \mathfrak{C}$. Then a strong solution u of $(K+\zeta)u = f$ with $Mu = 0$ for $f \in \mathfrak{H}$ is a strong solution of $Ku = f-\zeta u$ with $Mu = 0$.*

This follows from the fact that the sequence in $\mathfrak{C}_1$ which approximates u as required for a strong solution of the one problem, see Section 4, evidently may serve as approximating sequence for the other problem.

LEMMA 11.2. *Let u^ν be a sequence of strong solutions of $Ku = f^\nu$, $Mu = 0$, where K is symmetric positive and $f^\nu \in \mathfrak{H}$ converges to a function $f \in \mathfrak{H}$, i.e. $||f^\nu-f|| \to 0$ as $\nu \to \infty$. Then there exists a strong solution of the problem $Ku = f$, $Mu = 0$.*

This follows from the fact that out of the approximating sequence in $\mathfrak{C}_1$ associated with each u^ν a sequence can be extracted, converging to a limit function u, which evidently is a strong solution of $Ku = f$, $Mu = 0$.

We now consider the set $\mathscr{S}$ of values λ in $0 \leqq \lambda \leqq 1$ for which the problem $K_\lambda u = f$, $Mu = 0$, has a strong solution for every $f \in \mathfrak{H}$. We then prove 1) $\lambda = 1$ is in this set, 2) the set is open, 3) the set is closed.

1) Since the operator K_1 satisfies the requirements of Section 8 by assumption, a strong solution u of $K_1 u = f$, $Mu = 0$, exists provided $f \in \mathfrak{H}_1$. The last requirement can be eliminated. To this end one may approximate any $f \in \mathfrak{H}$ by a sequence $f^\nu \in \mathfrak{H}_1$ such that $||f^\nu - f|| \to 0$ as $\nu \to \infty$ and let u^ν be the corresponding solution. Then Lemma 11.2 yields the desired result, namely that $\lambda = 1$ is in $\mathscr{S}$.

2) Let λ_0 be in $\mathscr{S}$ and set $\lambda = \lambda_0 + \delta\lambda$, $\kappa_\lambda = \kappa_0 + \delta\kappa$. The equation $K_\lambda u = f$, $Mu = 0$, may be re-written as

$$K_{\lambda_0} u = f - \delta\kappa u, \qquad Mu = 0,$$

and solved by iterations provided $\delta\kappa = b\delta\lambda$ is taken sufficiently small. This follows immediately from the basic inequality for K_{λ_0} and Lemmas 11.2 and 11.1. Hence the set $\mathscr{S}$ is open.

3) Let the sequence λ_ν be in $\mathscr{S}$ and converge to λ_0. Let u^ν be the corresponding solution of $K_\lambda u = f$, $Mu = 0$. Lemma 11.1 then implies that u^ν is a strong solution of $K_{\lambda_0} u = f - \delta\kappa_\nu u^\nu$. Clearly $||u^\nu||$ is bounded independently of ν so that $f - \delta\kappa_\nu u^\nu$ tends to f. Hence Lemma 11.2 yields a strong solution of $K_{\lambda_0} u = f$, $Mu = 0$. I.e., the set $\mathscr{S}$ is closed.

Since, as a consequence, $\lambda = 0$ is in the set $\mathscr{S}$, Theorem 11.1 is proved.

PART III

THE CLIPPED LAPLACIAN

12. Basic Theorems

We like to refer to the operator $-\sum_{\sigma \neq 0} D_\sigma^* D_\sigma$ as the "clipped" (generalized) Laplacian in as much as it differs from a generalized Laplacian $\sum_{\lambda,\nu} \partial/\partial x^\lambda g^{\lambda\nu} \partial/\partial x^\nu$ by the absence of normal derivatives along the edge. The operator $-D^* \cdot D = -1 - \sum_{\sigma \neq 0} D_\sigma^* D_\sigma$ should then be called "augmented" clipped Laplacian.

Our aim in this section is to establish various properties of this clipped Laplacian used in the preceding sections. The first of these was formulated in Section 9 as Theorem 9.1:

THEOREM 12.1. *To every function v in $\mathfrak{H}$ there exists a function $\tilde{v}$ in $\mathfrak{H}_1$ such that the relation*

$$D^* D\tilde{v} = v$$

holds in the weak sense. I.e., for every u in $\mathfrak{H}_1$ the relation

$$(D\tilde{v}, Du) = (v, u)$$

holds.

To prove this theorem[11] it is sufficient to note that the form (v, u), generated by any v in $\mathfrak{H}$, is bounded with respect to the norm $||u||$ and hence with respect to the norm $||u||_1 \geqq ||u||$ for u in $\mathfrak{H}_1$. Therefore, since $\mathfrak{H}_1$ is closed with respect to $|| \ ||_1$, a function $\tilde{v} \in \mathfrak{H}_1$ exists such that $(v, u) = (D\tilde{v}, Du)$, so that $\tilde{v}$ is a weak solution of $D^* \cdot D\tilde{v} = v$. This is the statement of Theorem 12.1.

Theorem 12.2 (Theorem 9.2) refers to the operator

$$(D-t)^* \cdot D = \textstyle\sum_\sigma (D_\sigma - t_\sigma)^* D_\sigma$$

which is not self-adjoint, but whose leading part involves the clipped Laplacian.

THEOREM 12.2. *To every function v in $\mathfrak{H}$ there exists a function $\hat{v}$ in $\mathfrak{H}_1$ such that the relation*

$$(D-t)^* D\hat{v} = v$$

holds in the weak sense. I.e., for every u in $\mathfrak{H}_1$ the relation

$$(D\hat{v}, (D-t)u) = (v, u)$$

holds.

Here it is assumed that the operators D_σ, $\sigma \neq 0$, are chosen "sufficiently small", in the sense explained in Section 8.

This statement can be reduced to that of Theorem 12.1 by a continuation argument.[12] One connects the operator $(D-t)^* \cdot D$ with the operator $D^* \cdot D$ through the operators $(D-\lambda t)^* \cdot D$ for $0 \leqq \lambda \leqq 1$ and proves that the set of values λ for which the statement of Theorem 12.2 holds is open and closed.

We recall that in the formulation of Theorem 12.2, we reserved the right to take the operators D_σ, $\sigma \neq 0$, sufficiently small, in the sense explained in Section 8. We now do this in such a way that a number $\theta < 1$ exists so that $(v \cdot tu)^2 \leqq \theta(v \cdot v)(u \cdot u)$ for all u, v at each point of the manifold $\mathscr{R}$. For the solution v_λ of the equation $(D-\lambda t)^* \cdot Dv_\lambda = v$ we then have

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[11]Equation $D^* \cdot Dv = v$ could be regarded as a symmetric positive system

$$\begin{pmatrix} 1 & D_1^* & \cdots \\ -D_1 & 1 & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} v \\ D_1 v \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

without boundary condition. It seems, however, that doing so would not lead to a simplification of the arguments given in Part III.

[12]It could also be deduced from the Lax-Milgram theorem.

$$||v_\lambda||_1^2 \leqq \lambda\theta ||v_\lambda||\, ||v_\lambda||_1 + ||v_\lambda||\, ||v|| \leqq ||v_\lambda||_1 \{\theta ||v_\lambda||_1 + ||v||\}$$

and hence

$$||v_\lambda||_1 \leqq (1-\theta)^{-1} ||v||.$$

Clearly, since this *a priori* inequality is valid, the continuation process can be carried out. Thus we obtain the statement of Theorem 12.2.

The main effort to be made here (in Part III) consists in establishing Theorems 9.3—9.6. The first two major steps in doing this consist in proving Theorems 12.3 and 12.4.

THEOREM 12.3. *Let $\hat{v}$ be the (weak) solution of the equation*

$$(D-t)^* \cdot D\hat{v} = v$$

for v in $\mathfrak{H}$ according to Theorem 12.2; then each function $D_\sigma \hat{v}$ is in $\mathfrak{H}_1$. Furthermore, with an appropriate constant c, the inequality

$$||\hat{v}||_2 = ||D\hat{v}||_1 = ||DD\hat{v}|| \leqq c||v||$$

—in obvious notation—holds for any such function v.

The second theorem involves a complete set of operators of the first order, denoted by E_τ and given by

$$E_\tau = e_\tau^\rho \frac{\partial}{\partial x^\rho} + e_\tau, \quad E_0 = 1$$

with respect to each patch, where the numbers e_τ^ρ and the matrices e_τ are in $\mathfrak{C}_1$ when considered as functions on the manifold $\mathscr{R}$. Completeness means that every operator of the first order with coefficients in $\mathfrak{C}$ or $\mathfrak{C}_1$ is a linear combination of the operators E_τ with coefficients in $\mathfrak{C}$ or $\mathfrak{C}_1$, respectively. If the manifold consists of a region in the m-dimensional space we may simply take $E_\tau = \partial/\partial x_\tau$, $E_0 = 1$. We set

$$||Ev||^2 = \textstyle\sum_\tau ||E_\tau v||^2,$$

and denote by $\mathfrak{H}_1(E)$ the space of those functions $v \in \mathfrak{H}$ for which the operator $E = \{E_\tau\}$ is applicable in the strong sense. We then may formulate the second statement:

THEOREM 12.4. *Let v be in $\mathfrak{H}_1(E)$ and let $\hat{v}$ be the (weak) solution of the equation $(D-t)^* \cdot D\hat{v} = v$. Then each function $D_\sigma \hat{v}$ is in $\mathfrak{H}_1(E)$ and, with an appropriate constant c_1, the inequality*

$$||ED\hat{v}|| \leqq c_1 ||Ev||$$

holds—in obvious notation.

Furthermore, the function $\hat{v}$ is in $\mathfrak{H}_1(E)$, each function $E\hat{v}$ is in $\mathfrak{H}_1(D)$, and the inequality

$$||DE\hat{v}|| \leqq c_2 ||Ev||$$

holds with an appropriate constant c_2.

In the following sections (13 and 14) we shall develop the tools needed for the proofs of Theorems 12.3 and 12.4 which will be given in Sections 14 and 15.

13. Commutator Identities and Formal Inequalities

In addition to the particular tangential operators D_σ, introduced in Section 8, we must consider other systems of differential operators $\{G_\tau\}$ of the first order given, with respect to each patch, as

$$G_\tau = g_\tau^\rho(x) \frac{\partial}{\partial x^\rho} + g_\tau(x), \quad G_0 = 1$$

with numbers $g_\tau^\rho(x)$ and matrices $g_\tau(x)$ in $\mathfrak{C}_1$. Since the coefficients $d_\sigma^\rho(x)$ of the operators D_σ are numbers—and not (non-diagonal) matrices—the formal commutators of the operators D_σ and G_τ are also operators of the first order. We use the notation $[D_\sigma, G_\tau]$ for these formal commutators; i.e., we define

$$[D_\sigma, G_\tau] \equiv [d_\sigma^\nu g_{\tau/\nu}^\rho + d_\sigma g_\tau^\rho - g_\tau^\nu d_{\sigma/\nu}^\lambda - g_\tau d_\sigma^\lambda]\partial/\partial x^\lambda + d_\sigma^\lambda g_{\tau/\lambda} - g_\tau^\nu d_{\sigma/\nu} + d_\sigma g_\tau - g_\tau d_\sigma.$$

Similarly, we define the formal commutator $[D_\sigma^*, G_\tau]$.

We also say that the "formal identities"

$$D_\sigma G_\tau - G_\tau D_\sigma = [D_\sigma, G_\tau],$$
$$D_\sigma^* G_\tau - G_\tau D_\sigma^* = [D_\sigma^*, G_\tau]$$

hold. By this we mean that the identities

$$D_\sigma G_\tau w - G_\tau D_\sigma w = [D_\sigma, G_\tau]w,$$
$$D_\sigma^* G_\tau w - G_\tau D_\sigma^* w = [D_\sigma^*, G_\tau]w$$

hold for functions w in $\mathfrak{C}_2$.

We now require of the operators G_τ that the commutators $[D_\sigma, G_\tau]$ and $[D_\sigma^*, G_\tau]$ are linear combinations of the operators G. Specifically, there should exist matrices $[dg]_{\sigma\tau}^\rho$ and $[d^*g]_{\sigma\tau}^\rho$ in $\mathfrak{C}$ such that the relations

[DG] $$[D_\sigma, G_\tau] = [dg]_{\sigma\tau}^\rho G_\rho,$$

[D*G] $$[D_\sigma^*, G_\tau] = [d^*g]_{\sigma\tau}^\rho G_\rho$$

hold.

We shall take three particular systems of operators as operators G.

First, we maintain that we may take $G = D$. To ascertain that we may do this we first observe that the commutator of two operators D_σ is again tangential, as is immediately seen by expressing them in terms of $\partial/\partial x^\lambda$, $\lambda = 1, \cdots, m-1$, and $x^m \partial/\partial x^m$ with reference to boundary patches. Then we invoke the completeness of the set of operators D_σ, our second requirement imposed in Section 8; in fact, now is the first time that this requirement is invoked. We conclude from this completeness that identities of the form [DD] and [D* D] hold with appropriate coefficients $[dd]$ and $[d^* d]$.

As our next choice of the operators G_τ we take the operators E_τ described at the end of Section 13. The existence of the coefficients $[de]$ and $[d^* e]$ is guaranteed by the assumed completeness of the operators E_τ.

The commutator relations for $G = D$ and $G = E$ do not only hold formally, i.e. for functions w in $\mathfrak{C}_2$; they also hold for the solution $\hat{v}$ of the equation $(D-t)^* \cdot D\hat{v} = v$ with v in $\mathfrak{C}_1$. Specifically, we formulate as corollaries to the Theorems 12.3 and 12.4

THEOREM 13.1. *The identities*

$$D_\sigma D_\tau \hat{v} - D_\tau D_\sigma \hat{v} = [D_\sigma, D_\tau]\hat{v},$$
$$D^*_\sigma D_\tau \hat{v} - D_\tau D^*_\sigma \hat{v} = [D^*_\sigma, D_\tau]\hat{v}$$

hold.

THEOREM 13.2. *The identity*

$$D_\sigma E_\tau \hat{v} - E_\tau D_\sigma \hat{v} = [D_\sigma, E_\tau]\hat{v}$$

holds.

These theorems will be proved in Sections 14 and 15 together with Theorems 12.3 and 12.4.

From our formal commutator identities we can derive two inequalities, which will also be called "formal" in as much as they hold for functions in $\mathfrak{C}_2$. The *first inequality* is

$$(\mathrm{I}_1) \qquad ||GDw|| + ||DGw|| \leqq c_1 ||G(D-t)^* \cdot Dw||,$$

valid for w in $\mathfrak{C}_3$ provided the functions t_σ are in $\mathfrak{C}_1$ and provided the coefficients of the operators D_σ, $\sigma \neq 0$, are made sufficiently small.

The *second inequality* refers to the case that the operators G_τ themselves are tangential; we then call them F_τ. This inequality is

$$(\mathrm{I}_2) \qquad ||FDw||^2 \leqq c_1 ||F^* \cdot Fw|| \; ||(D-t)^* \cdot Dw||,$$

valid for w in $\mathfrak{C}_2$, with sufficiently small D_σ.

The proof of these inequalities is immediate: From [DG] and [D* G] we have

$$\begin{aligned}(G_\tau D_\sigma w, G_\tau D_\sigma w) &+ ([dg]^\rho_{\sigma\tau} G_\rho w, G_\tau D_\sigma w) \\ &= (D_\sigma G_\tau w, G_\tau D_\sigma w) = (G_\tau w, D^*_\sigma G_\tau D_\sigma w) \\ &= (G_\tau w, G_\tau D^*_\sigma D_\sigma w) + (G_\tau w, [d^* g]^\rho_{\sigma\tau} G_\rho D_\sigma w) \\ &= (G_\tau w, G_\tau (D_\sigma - t_\sigma)^* D_\sigma w) + (G_\tau w, \{[d^* g]^\rho_{\sigma\tau} + t^*_\sigma \delta^\rho_\tau\} G_\rho D_\sigma w) \\ &+ (G_\tau w, \{(\overset{\circ}{G}_\tau t^*_\sigma) + g_\tau t^*_\sigma - t^*_\sigma g_\tau\} D_\sigma w).\end{aligned}$$

Here $\mathring{G}_\tau = g^\lambda_\tau \partial/\partial x^\lambda = G_\tau - g_\tau$—with reference to a patch. Hence, in view of $||Gw|| \leqq ||GDw||$, $||Dw|| \leqq ||GDw||$, we have

$$(1-\delta)||GDw|| \leqq |(Gw, G(D-t)^* \cdot Dw)|$$

where the constant δ depends on bounds of $[dg]^\rho_{\sigma\tau}$, $[d^*g]^\rho_{\sigma\tau}$, t and $\mathring{G}t$. By making the coefficients of D_σ, $\sigma \neq 0$, sufficiently small, we can obviously make δ arbitrarily small. Using

$$||DGw|| \leqq ||GDw|| + \delta||Gw|| \leqq (1+\delta)||GDw||$$

we are led to the first inequality (I_1).

If the operators G themselves are tangential (at $\mathscr{B}$), $G = F$, we may carry out integration by parts in $(Fw, F(D-t)^* \cdot Dw)$; thus the second inequality (I_2) results for w in $\mathfrak{C}_3$. Since it involves only second derivatives it carries over to $\mathfrak{C}_2$.

In (I_2) we may take $F = D$. Since $||tDw|| \leqq \theta||Dw|| \leqq \theta||DDw||$, we obtain the inequality

$$||DDw|| \leqq c_2||(D-t)^* \cdot Dw||,$$

valid for w in $\mathfrak{C}_2$. This is the inequality, which was stated in the formulation of Theorem 12.3 to hold for the solution $w = \hat{v}$ of the equation

$$(D-t)^* \cdot D\hat{v} = v.$$

The two inequalities stated in the formulation of Theorem 12.4 would result from (I_1) and $G = E$ if we were permitted to set $w = \hat{v}$.

In order to establish the statements of Section 12, we shall approximate the function $D\hat{v}$ involved in these statements by functions in $\mathfrak{C}_1$ and repeat the steps that led to inequalities (I_1) and (I_2) in this section. For this approximation we shall employ mollifiers.

14. Mollifiers. Proof of Theorem 12.3

We have already referred to mollifiers in Section 10, without describing them in detail. Since for the present purpose we must adjust them in a particular way, we must describe them now.

We introduce the mollifier J^ρ associated with the patch $\mathscr{P}_\rho$ as an integral operator having a kernel

$$j_\varepsilon(x-x') = \varepsilon^{-m} j\left(\frac{x-x'}{\varepsilon}\right)$$

defined for x' in the patch $\mathscr{P}_\rho$ and x in the sub-patch $\mathscr{Q}_\rho$, see Section 8. The function $j(\xi)$ is to be infinitely differentiable, should vanish for $|\xi| \geqq 3$, and satisfy $\int j(\xi)d\xi = 1$. The special requirement which we want to add is that $j(\xi) = 0$ for $\xi^m \leqq 1$, so that $j_\varepsilon(x-x') = 0$ if x is in a neighborhood of the (inner and outer) boundary of $\mathscr{P}_\rho$, provided ε is small enough.

We finally set

$$J_\varepsilon = \sum_\rho \eta_\rho J_\varepsilon^\rho ,$$

where the functions η_ρ form a partition of unity as described in Section 8.

With this operator J_ε we then have the relation

$$(\mathrm{J}_1) \qquad ||J_\varepsilon w - w|| \to 0 \quad \text{as } \varepsilon \to 0 \quad \text{for } w \text{ in } \mathfrak{H},$$

as follows from the general properties of mollifiers.

In the following we shall on occasion use the abbreviation

$$R_\varepsilon \to R(\mathfrak{H}_l)$$

for $||R_\varepsilon w - Rw|| \to 0$ as $\varepsilon \to 0$ for w in $\mathfrak{H}_1$. Thus we can express J_1 simply as

$$(\mathrm{J}_1) \qquad J_\varepsilon \to 1 \ (\mathfrak{H}).$$

Suppose the function $w(x)$ in $\mathfrak{H}$ admits the operators D_σ in the weak sense; i.e., suppose that there are functions in $\mathfrak{H}$, denoted by $D_\sigma w(x)$, such that

$$\int \phi(x) D_\sigma w(x) dx = \int (D_\sigma^* \phi(x)) w(x) dx$$

for all functions $\phi(x)$ which vanish at the boundary of the patch $\mathscr{P}_\rho$ (and outside of it). By virtue of the special requirement we have imposed on the kernels j_ε, we may set $\phi(\hat{x}) = j_\varepsilon(x - \hat{x})$ provided x is in the subpatch $\mathscr{Q}_\rho$. Consequently, we have

$$J_\varepsilon^\rho D_\sigma w(x) = \int (D^* j_\varepsilon(x - \hat{x})) w(\hat{x}) d\hat{x}$$

for x in $\mathscr{Q}_\rho$. We denote the integral operator on the right by $(J_\varepsilon^\rho D_\sigma)$ and, using this notation, we introduce the integral operator

$$(J_\varepsilon D) = \left\{ \sum_\rho \eta_\rho (J_\varepsilon^\rho D_\sigma) \right\}.$$

We then evidently have

$$(\mathrm{J}_2) \qquad J_\varepsilon D w = (J_\varepsilon D) w \qquad \text{if } w \text{ in } \mathfrak{H} \text{ admits } D \text{ weakly}.$$

For the difference of the two integral operators DJ_ε and $(J_\varepsilon D)$ we shall use, somewhat improperly, the notation

$$DJ_\varepsilon - (J_\varepsilon D) = [D, J_\varepsilon].$$

From the main lemma on which the identity of strong and weak extensions of first order operators was based [1], we may infer the relation

$$(\mathrm{J}_3) \qquad [D, J_\varepsilon] \to 0 \quad (\mathfrak{H}).$$

In Section 13 we introduced the notation $[D_\sigma, D_\tau]$ for the first order operator $[dd]_{\sigma\tau}^\rho D_\rho$ which equals $D_\sigma D_\tau - D_\tau D_\sigma$ for w in $\mathfrak{C}_2$. Clearly, $[D_\sigma, D_\tau]$ is applicable in $\mathfrak{H}_1$ and hence we may formulate the statement

$$(\mathrm{J}_4) \qquad D_\sigma(J_\varepsilon D_\tau) - D_\tau(J_\varepsilon D_\sigma) \to [D_\sigma, D_\tau] \quad (\mathfrak{H}_1).$$

In fact, the left hand side here may be written as

$$([D_\sigma, J_\epsilon]D_\tau)-([D_\tau, J_\epsilon]D_\sigma)+J_\epsilon[D_\sigma, D_\tau]$$

in obvious notation. (J_4) then follows from (J_3) and (J_1).

Relations (J_1) to (J_4) are sufficient for the proof of Theorem 12.3. Let $\hat{v}$ be the (weak) solution of $(D-t)^* \cdot D\hat{v} = v$, which exists according to Theorem 12.2, proved in Section 12. Then $D\hat{v}$ is in $\mathfrak{H}$ and admits the operator $(D-t)^*$ in the weak sense. From relation (J_2) we therefore have the identity

$$Jv = (J(D-t)^*) \cdot D\hat{v} = [J, D^*] \cdot D\hat{v} - Jt^* \cdot D\hat{v} + D^* \cdot JD\hat{v}.$$

Here, and in the following we set J for $J_{\epsilon_1}-J_{\epsilon_2}$.

We now follow the steps that led to the proof of the second *a priori* inequality, setting

$$H_{\tau\sigma} = D_\tau(JD_\sigma)-D_\sigma(JD_\tau).$$

In obvious notation, we then have

$$\begin{aligned}(D_\tau JD_\sigma v, D_\tau JD_\sigma v) &= (H_{\tau\sigma}\hat{v}, D_\tau JD_\sigma\hat{v}) + ((JD_\tau)\hat{v}, [d^*d]^\rho_{\sigma\tau}D_\rho JD_\sigma\hat{v}) \\ &\quad + (([D^*_\tau, J]+Jt^*_\tau)D_\tau\hat{v}+Jv, ([D^*_\sigma, J]+Jt^*_\sigma)D_\sigma v+Jv).\end{aligned}$$

With an appropriate constant δ, which can be made arbitrarily small by making the D_σ, $\sigma \neq 0$, sufficiently small, we obtain the inequality

$$(1-\delta)||DJD\hat{v}|| \leqq ||H\hat{v}||+||[D^*, J] \cdot D\hat{v}||+||Jt \cdot D\hat{v}||+||Jv||.$$

Hence, by virtue of (J_1), (J_3), (J_4), we have

$$||D(J_{\epsilon_1}-J_{\epsilon_2})D\hat{v}|| \to 0 \qquad \text{as } \epsilon_1, \epsilon_2 \to 0.$$

Since $J_\epsilon D\hat{v}$ is in $\mathfrak{C}_1$ it follows that $D\hat{v}$ admits the operator D strongly; the main statement of Theorem 12.3 is thus proved. At the same time the inequality $||D_2 v|| \leqq c||v||$ follows, since we now know that

$$D_\sigma(J_\epsilon D_\tau)\hat{v} = D_\sigma J_\epsilon D_\tau \hat{v} \to D_\sigma D_\tau \hat{v} \text{ and } D_\tau(J_\epsilon D_\sigma)\hat{v} \to D_\tau D_\sigma \hat{v}.$$

Furthermore the statement of Theorem 13.1 results from (J_4).

15. Proof of Theorem 12.4

Theorem 12.4 involved the set of operators E not all of which are tangential. Nevertheless, we shall prove this theorem with the aid of Theorem 12.3 which does refer to tangential operators. To this end we replace the operators E_τ for $\tau \neq 0$ by operators $\omega^\nu E_\tau$ formed with the aid of functions $\omega^\nu(x)$ which vanish at the boundary $\mathscr{B}$ and are equal to 1 except in a neighborhood of the boundary. We set $E^\nu_\tau = \omega^\nu E_\tau$ for $\tau \neq 0$ and $E^\nu_0 = 1$. Clearly, the operators E^ν_τ are linear combinations of the operators D_σ.

The functions $\omega^\nu(x)$ which we shall introduce will be in $\mathfrak{C}_1$ and, as ν tends to zero, they will tend to 1, nondecreasingly, in the interior of $\mathscr{R}$. Furthermore, these functions will have the following—very essential—property: There are functions $r_\sigma^\nu(x)$ in $\mathfrak{C}$, bounded as $\nu \to 0$,

$$(\mathrm{R}^0) \qquad |r_\sigma^\nu(x)| \leqq r^0,$$

such that the relation

$$(\mathrm{R}) \qquad \mathring{D}_\sigma \omega^\nu = r_\sigma^\nu \omega^\nu$$

holds. Here $\mathring{D}_\sigma$ is the main part of the operator D_σ:

$$\mathring{D}_\sigma = d_\sigma^\lambda \frac{\partial}{\partial x^\lambda} \qquad \text{with reference to a patch.}$$

To exhibit such functions we introduce, for each patch $\mathscr{P}_\rho$, a function ω_ρ^ν given by $\omega_\rho^\nu = 1$ if $\mathscr{P}_\rho$ is an interior patch and by

$$\omega_\rho^\nu = \begin{cases} 1 & \text{for } y \leqq -\nu, \\ -2\frac{y}{\nu} - \frac{y^2}{\nu^2} & \text{for } -\nu \leqq y \leqq 0, \end{cases}$$

if $\mathscr{P}_\rho$ is a boundary patch. Then we set

$$\omega^\nu = \sum_\rho \eta_\rho \omega_\rho^\nu$$

where the functions η_ρ constitute a partition of unity.

We first write $\mathring{D}_\sigma \omega^\nu$ in the form

$$\mathring{D}_\sigma \omega^\nu = \sum_\rho \eta_\rho (\mathring{D}_\sigma \omega_\rho^\nu) + \sum_\rho (\mathring{D}_\sigma \eta_\rho) \omega_\rho^\nu$$

and then write each term separately in the form (R).

The function $\mathring{D}_\sigma \omega_\rho^\nu$ in $\mathscr{P}_\rho$ can be written in the form

$$\mathring{D}_\sigma \omega_\rho^\nu = r_{\rho\sigma}^\nu \omega_\rho^\nu$$

with

$$r_{\rho\sigma}^\nu = \begin{cases} 0 & \text{where } \omega_\rho^\nu = 1 \\ d_\sigma^m \dfrac{1}{y} \dfrac{2\nu+2y}{2\nu+y} & \text{for } -\nu \leqq y \leqq 0 \end{cases}$$

if $\mathscr{P}_\rho$ is a boundary patch. Since $(2-2z)(2-z)^{-1} \leqq 1$ for $0 \leqq z \leqq 1$ and d_σ^m/x^m is bounded—because $d_\sigma^m = 0$ on $\mathscr{B}$—we find that $|r_{\rho\sigma}^\nu|$ is bounded, independently of ν; hence there is a constant r_1 such that

$$|r_{\rho\sigma}^\nu| \leqq r_1 \qquad \text{in } \mathscr{R}.$$

To handle the expression $\sum_\rho (\mathring{D}_\sigma \eta_\rho) \omega_\rho^\nu$ we consider a point x^0 where not all $\mathring{D}_\sigma \eta_\rho$ are zero and let $\mathscr{P}$ be a patch such that x^0 is in the subpatch $\mathscr{Q}^\lambda$, where $\eta_\lambda = 1$. Only a finite number of patches have points in common with

$\mathscr{Q}^\lambda$ and, since the transformation from one to another patch has a non-vanishing Jacobian, there is a constant $c_{\rho\lambda}$ such that

$$\omega_\rho^\nu \leqq c_{\rho\lambda}\omega_\lambda^\nu ,$$

independently of ν. Consequently, we have

$$|\sum_\rho (\mathring{D}_\sigma \eta_\rho)\omega_\rho^\nu| \leqq \sum_\lambda (\sum_\rho |\mathring{D}_\rho \eta_\rho| c_{\rho\lambda})\omega_\lambda^\nu \leqq r_2 \omega^\nu,$$

with

$$r_2 = \max \sum_\rho |\mathring{D}_\rho \eta_\rho| c_{\rho\lambda} .$$

This maximum is finite since the manifold is covered by a finite number of patches. Thus we obtain

$$|\mathring{D}_\sigma \omega^\nu| \leqq (r_1 + r_2)\omega^\nu,$$

and hence (R).

With the aid of the functions ω^ν we modify the operators $\{E_\tau\}$ introduced in Section 13 by forming the operators $\{\omega^\nu E_\tau\}$, which are evidently tangential and hence are linear combinations of the operators D_σ. Therefore the operator $\omega^\nu E$ is applicable to ω in $\mathfrak{H}_1$. We now formulate

LEMMA 15.1. *Let w in $\mathfrak{H}_1$ be such that $||E^\nu w||$ is bounded independently of ν; then w is in $\mathfrak{H}_1(E)$.*

To prove it we first observe that we may introduce a function in $\mathfrak{H}$, denoted by Ew, such that $Ew = E^\nu w$ whenever $\omega^\nu = 1$. Clearly, $||E^\nu w|| \leqq ||Ew||$. Next we take the sequence of functions $\omega_\varepsilon = J_\varepsilon w$ and establish the relation

$$||Ew_\varepsilon - Ew|| \to 0 \qquad \text{as } \varepsilon \to 0.$$

To this end we again make use of the assumed special property of the kernel $j_\varepsilon(x-\hat{x})$, viz. that it vanishes if $\hat{x}$ is in a neighborhood of the boundary $\mathscr{B}$ while x is in $\mathscr{Q}^\rho$. As a consequence of this property we have

$$J_\varepsilon E^\nu w = J_\varepsilon Ew$$

if ν is sufficiently small for a given $\varepsilon > 0$, and as another consequence we observe that there exists an integral operator—we denote it by $(J_\varepsilon E)$—such that

$$(J_\varepsilon E)w = J_\varepsilon Ew$$

for our function w.

Next we use the relation

$$EJ_\varepsilon w - (J_\varepsilon E)w \to 0 \qquad \text{as } \varepsilon \to 0,$$

which holds because of $||w|| < \infty$; it corresponds to relation (J_3) and is proved in the same way. From it and (J_1) we find

$$Ew_\varepsilon = EJ_\varepsilon w \to Ew \qquad \text{as } \varepsilon \to 0,$$

and thus the statement of Lemma 15.1.

We now are ready to prove Theorem 12.4. We first consider the operators E_τ, in place of G_τ, introduced in Section 12. As was explained in Section 13, coefficients $[de]$ and $[d^* e]$ in $\mathfrak{C}$ exist such that the commutator identities [DE] and [D*E] hold. Next we recognize the operators E^ν_τ to be operators of the type F_τ.

In view of the relations $\mathring{D}_\sigma \omega^\nu = r^\nu_\sigma \omega^\nu$ and $\mathring{D}^*_\sigma \omega^\nu = -\mathring{D}_\sigma \omega^\nu = -r^\nu_\sigma \omega^\nu$ we realize that the operators $F_\tau = E^\nu_\tau$ satisfy relations (DE$^\nu$) and (D*E$^\nu$) with

$$[de^\nu]^\rho_{\sigma\tau} = [de]^\rho_{\sigma\tau} + r^\nu_\sigma \delta^\rho_\tau\,,$$
$$[d^* e^\nu]^\rho_{\sigma\tau} = [d^* e]^\rho_{\sigma\tau} - r^\nu_\sigma \delta^\rho_\tau\,.$$

By making D_σ, $\sigma \neq 0$, sufficiently small we can make the coefficients r^ν_σ, $[de]^\rho_{\sigma\tau}$, $[d^*e]^\rho_{\sigma\tau}$, and hence $[de^\nu]^\rho_{\sigma\tau}$, $[d^*e^\nu]^\rho_{\sigma\tau}$ so small that the first formal inequality (I_1) holds with $F = E^\nu$. This inequality was established for functions w in $\mathfrak{C}_2$. We now shall establish it for $w = \hat{v}$.

We first notice that the operators E^ν_τ, being tangential, are linear combinations of the operators D_σ with coefficients in $\mathfrak{C}_1$—by virtue of the completeness of the D_σ. It then follows from Theorem 12.3 that the functions $D_\sigma \hat{v}$ admit the operators E^ν_τ strongly and, at the same time, that the functions $E^\nu_\tau \hat{v}$ admit the operators D_σ strongly. Furthermore, it follows from Theorem 13.1 that the relations

$$D_\sigma E^\nu_\tau \hat{v} - E^\nu_\tau D_\sigma \hat{v} = [D_\sigma\,, E^\nu_\tau]\hat{v},$$
$$D^*_\sigma E^\nu_\tau \hat{v} - E^\nu_\tau D^*_\sigma \hat{v} = [D^*_\sigma\,, E^\nu_\tau]\hat{v}$$

hold.

Next we observe that the special identity

$$(D_\sigma E^\mu_\tau \hat{v},\, E^\nu_\tau D_\sigma \hat{v}) = ({}^\nu E^*_\tau E^\mu_\tau \hat{v},\, D^*_\sigma D_\sigma \hat{v}) + (E^\mu_\tau \hat{v},\, [d^* e^\nu]^\rho_{\sigma\tau} E^\nu_\rho D_\sigma \hat{v})$$

holds, where ${}^\nu E^* = (E^\nu)^*$. In fact, for any two functions w_1, w_2 in $\mathfrak{H}_1$ and any tangential operator F, the identity

$$(D_\sigma w_1 F_\tau w_2) = (F^*_\tau w_1\,, D^*_\sigma w_2) + (w_1\,, [d^* f]^\rho_{\sigma\tau} F_\rho w_2)$$

holds. To verify this one need only mollify w_1 and w_2, carry out integration by parts, and use relation [D*F]. Then the last identity follows from (J_3). The special identity results if we set $w_1 = E^\mu_\tau \hat{v}$, $w_2 = D_\sigma \hat{v}$, $F = E^\nu$. Carrying out integration by parts in the special identity, we obtain the relation

$$\begin{aligned}(E^\mu_\tau D_\sigma \hat{v},\, E^\nu_\tau D_\sigma \hat{v}) &= ({}^\nu E^*_\tau E^\mu_\tau \hat{v},\, v) + ({}^\nu E^*_\tau E^\mu_\tau \hat{v},\, t^*_\sigma D_\sigma \hat{v}) \\ &\quad + (E^\mu_\tau \hat{v},\, [d^* e^\nu]^\rho_{\sigma\tau} E^\nu_\rho D_\sigma \hat{v}) - ([de^\nu]^\rho_{\sigma\tau} E^\mu_\rho \hat{v},\, E^\nu_\tau D_\sigma \hat{v}).\end{aligned}$$

Since v is assumed to admit the operator E and hence the operator E^ν we may transfer ${}^\nu E^*_\tau$ to the other side. From the resulting identity

$$(E^{\mu}_{\tau}D_{\sigma}\hat{v}, E^{\nu}_{\tau}D_{\sigma}\hat{v}) = (E^{\mu}_{\tau}\hat{v}, E^{\nu}_{\tau}v)+(E^{\mu}_{\tau}\hat{v}, E^{\nu}_{\tau}t^{*}_{\sigma}D_{\sigma}\hat{v})$$
$$+ (E^{\mu}_{\tau}\hat{v}, [d^{*}e^{\nu}]^{\rho}_{\sigma\tau}E^{\nu}_{\rho}D_{\sigma}\hat{v})-([de^{\mu}]^{\rho}_{\sigma\tau}E^{\mu}_{\rho}\hat{v}, E^{\nu}_{\tau}D_{\sigma}\hat{v})$$

we may derive the first inequality (I_1) as before setting $\mu = \nu$. This inequality may now be written in the form

$$||DE^{\nu}\hat{v}||+||E^{\nu}D\hat{v}|| \leqq 4c_1||E^{\nu}v||.$$

In view of $||E^{\nu}v|| \leqq ||Ev||$ we conclude that $||E^{\nu}D\hat{v}||$ is bounded independently of ν. According to Lemma 15.1, therefore, the function $D\hat{v}$ is in $\mathfrak{H}_1(E)$ and thus admits the operator E.

We now set

$$E^{\mu\nu} = E^{\mu}-E^{\nu}, \quad r^{\mu\nu} = r^{\mu}-r^{\nu},$$

use

$$[de^{\mu\nu}]^{\rho}_{\sigma\tau} = -[d^{*}e^{\mu\nu}]^{\rho}_{\sigma\tau} = r^{\mu\nu}_{\sigma}\partial^{\rho}_{\tau},$$

and derive in an obvious manner the identity

$$(D_{\sigma}E^{\mu\nu}_{\tau}\hat{v}, D_{\sigma}E^{\mu\nu}_{\tau}\hat{v}) = (E^{\mu\nu}_{\tau}\hat{v}, E^{\mu\nu}_{\tau}v)+(E^{\mu\nu}_{\tau}\hat{v}, E^{\mu\nu}_{\tau}t^{*}_{\sigma}D_{\sigma}\hat{v})$$
$$+(E^{\mu\nu}_{\tau}\hat{v}, [d^{*}e^{\mu}]^{\rho}_{\sigma\tau}E^{\mu\nu}_{\rho}D_{\sigma}\hat{v})-(E^{\mu\nu}_{\tau}\hat{v}, r^{\mu\nu}_{\sigma}E^{\nu}_{\tau}D_{\sigma}\hat{v})$$
$$+(D_{\sigma}E^{\mu\nu}_{\tau}\hat{v}, [de^{\mu}]^{\rho}_{\sigma\tau}E^{\mu\nu}_{\rho}\hat{v})+(D_{\sigma}E^{\mu\nu}_{\tau}\hat{v}, r^{\mu\nu}_{\sigma}E^{\nu}_{\tau}\hat{v}).$$

From it we then obtain the estimate

$$||DE^{\mu\nu}\hat{v}|| \leqq 4c_1||E^{\mu\nu}v||+c_2|| \, |r^{\mu\nu}|E^{\nu}D\hat{v}||.$$

Now, $|E^{\nu}D\hat{v}| \leqq |ED\hat{v}|$ at every point of $\mathscr{R}$, since $\omega^{\nu} \leqq 1$; hence

$$||DE^{\mu\nu}\hat{v}|| \leqq 4c_1||E^{\mu\nu}v||+c_2|| \, |r^{\mu\nu}|ED\hat{v}||.$$

We now use the fact just proved that $||Ed\hat{v}||$ is finite. Since $|r^{\mu\nu}|$ is bounded, $|r^{\mu\nu}| \leqq 2r^0$, and vanishes outside of a boundary strip which shrinks to nothing as $\mu, \nu \to \infty$, it is clear that $|| \, |r^{\mu\nu}|ED\hat{v}|| \to 0$ as $\mu, \nu \to \infty$. Also, $||E^{\mu\nu}v|| \to 0$ since $||Ev|| < \infty$. Hence

$$||DE^{\mu\nu}\hat{v}|| \to 0 \text{ as } \mu, \nu \to \infty.$$

Since the functions E^{ν} admit D strongly, they can be approximated by functions w in $\mathfrak{C}_1$ such that Dw approximates $DE^{\nu}\hat{v}$. From this we see that $E\hat{v}$ admits the operator D strongly. Obviously, the inequality

$$||DE\hat{v}||+||ED\hat{v}|| \leqq 4c_1||Ev||$$

holds. Thus Theorem 12.4 is proved.

To prove Theorem 13.2 we need only show that

$$[D_{\sigma}, E^{\nu}_{\tau}]\hat{v} \to [D_{\sigma}, E_{\tau}]\hat{v},$$
$$[D^{*}_{\sigma}, E^{\nu}_{\tau}]\hat{v} \to [D^{*}_{\sigma}, E_{\tau}]\hat{v},$$

or, what is equivalent, that

$$r^{\nu}_{\sigma}E_{\tau}\hat{v} \to 0 \qquad \text{as } \nu \to 0.$$

Now $|r^\nu_\sigma|$ is bounded independently of ν and vanishes outside of an arbitrarily small neighborhood of the boundary if ν is made sufficiently small. Since $||E_\tau \hat{v}|| < \infty$, the statement follows.

16. Boundary Values

It is an important property of the functions in $\mathfrak{H}_1(E)$ that they possess boundary values.

To explain what we mean by this we first recall the definition of the space $\mathfrak{H}_{\mathscr{B}}$ given at the end of Section 7. We say that "the function w in $\mathfrak{H}_1(E)$ possesses boundary values in $\mathfrak{H}_{\mathscr{B}}$" if a sequence of functions w^ν in $\mathfrak{C}_1$ and a function $w_{\mathscr{B}}$ in $\mathfrak{H}_{\mathscr{B}}$ exist such that, with $\nu \to \infty$,

$$||w^\nu - w|| \to 0, \qquad ||Ew^\nu - Ew|| \to 0, \qquad ||w^\nu - w_{\mathscr{B}}||_{\mathscr{B}} \to 0.$$

The function $w_{\mathscr{B}}$ is then called the boundary value of w; in the following we shall omit the subscript $\mathscr{B}$. We now formulate

THEOREM 16.1. *A function w in $\mathfrak{H}_1(E)$ possesses boundary values.*

This theorem will be derived from the inequality

$$||w||_{\mathscr{B}} \leqq c_{\mathscr{B}} ||Ew||$$

valid for every function w in $\mathfrak{C}_1$ with an appropriate constant $c_{\mathscr{B}}$.

To prove this inequality it is sufficient to restrict oneself to a boundary patch $\mathscr{P}$. Clearly, for x in the intersection of $\mathscr{B}$ and the subpatch $\mathscr{Q}$ we have

$$w(x^1, \cdots, x^{m-1}, 0) = \int^0 \frac{\partial}{\partial x^m} \xi(x^m) w(x) dx^m$$

with any function $\xi(x^m)$ which vanishes outside of $\mathscr{P}$ and equals 1 on $\mathscr{B} \cap \mathscr{Q}$. Immediately we then obtain

$$\int_{\mathscr{B} \cap \mathscr{Q}} w^2 \frac{dx}{dx^m} \leqq c_0 \int_{\mathscr{P}} \left\{ \left| \frac{\partial}{\partial x^m} w \right|^2 + |w|^2 \right\} dx.$$

Since $\{E_\tau\}$ is complete there is a constant c_E such that $|(\partial/\partial x^m) w|^2 + |w|^2 \leqq c'|Ew|^2$. Thus the inequality results.

Now let w be in $\mathfrak{H}_1(E)$ so that there is a sequence w^ν in $\mathfrak{C}_1$ such that $||E(w^\nu - w)|| \to 0$. Hence $||E(w^{\nu_1} - w^{\nu_2})|| \to 0$. From the inequality we have $||w^{\nu_1} - w^{\nu_2}|| \to 0$. Theorem 16.1 then follows from the completeness of the space $\mathfrak{H}_{\mathscr{B}}$.

The integral $(w, w)_{\mathscr{B}}$ involving the boundary values of a function in $\mathfrak{H}_1(E)$ can also be described as the limit of certain integrals extended over a neighborhood of the boundary. Specifically, we formulate

THEOREM 16.2. *With an appropriate function $\sigma^\nu(x)$ in $\mathfrak{C}_1$ depending on a parameter ν, the relation*

$$(w, \sigma^\nu w) \to (w, w)_{\mathscr{B}}$$

holds for each function w in $\mathfrak{H}_1(E)$. Furthermore

$$(w, (\mathring{D}\sigma^\nu)w) \to -(w, \mathring{d}w)_{\mathscr{B}}$$

where $\mathring{D} = \{\mathring{D}_\sigma\}$ is given by $\mathring{D}_\sigma = d^\mu_\sigma \partial/\partial x^\mu$ and $\mathring{d} = \{\mathring{d}_\sigma\}$ by $\mathring{d}_\sigma = d^m_\sigma \partial/\partial x^m$ with reference to a patch.

To prove this theorem we set

$$\sigma^\nu = \frac{3}{\nu}(1-\omega^\nu)$$

where $\omega^\nu = \sum_\rho \eta_\rho \omega^\nu_\rho$ is the function introduced in Section 15. Clearly, the function $\sigma^\nu_\rho = (3/\nu)(1-\omega^\nu_\rho)$ vanishes for $y \leqq -\nu$ with reference to $\mathscr{P}_\rho$. Furthermore

$$\int_{-\infty}^0 \sigma^\nu_\rho(y)dy = 3\int_{-1}^0 (z+1)^2 dz = 1$$

and, since $d^m = 0$ on $\mathscr{B}$, we have

$$\int_{-\infty}^0 \mathring{D}\sigma^\nu_\rho(y)dy = -3\int_{-1}^0 \mathring{d}\Big|^{y=\nu z}(1+z)^2 dz = -\mathring{d}^\nu \to -\mathring{d}\Big|^{y=0}.$$

Now let w be a function in $\mathfrak{C}_1$. By an argument, similar to that employed in the proof of Theorem 16.1, we find

$$|w(y)-w(0)|^2 \leqq c_0 v \int_\nu^0 \left|\frac{\partial}{\partial x^m} w\right|^2 dx^m$$

for $y \geqq -\nu$, where we have set

$$w = w(y) \qquad \text{for } x^m = y.$$

Consequently, we have

$$\int_{\mathscr{Q}} \sigma^\nu_\rho(y)|w(y)-w(0)|^2 dx \leqq c_0 \nu \int_{\mathscr{P}} |Ew|^2 dx,$$

and therefore

$$\left|\left[\int_{\mathscr{Q}} \sigma^\nu_\rho w^2 dx\right]^{\frac{1}{2}} - \left[\int_{\mathscr{Q}\cap\mathscr{B}} w^2 \frac{dx}{dy}\right]^{\frac{1}{2}}\right|^2 \leqq c_0 \nu \int_{\mathscr{P}} |Ew|^2 dx.$$

Similarly we have

$$\int_{\mathscr{Q}} |\mathring{D}\sigma^\nu_\rho||w(y)-w(0)|^2 dx \leqq c_1 v \int_{\mathscr{P}} |Ew|^2 dx$$

whence, in view of

$$||w||_{\mathscr{B}\cap\mathscr{Q}} \leqq c_{\mathscr{B}} ||Ew||_{\mathscr{P}},$$

we may deduce

$$\left|\int_{\mathscr{Q}} \mathring{D}\sigma_\rho^\nu w^2\,dx - \int_{\mathscr{Q}\cap\mathscr{B}} \partial w^2 \frac{dx}{dx^m}\right| \leqq \varepsilon_\nu^\rho \int_{\mathscr{P}} |Ew|^2\,dx$$

with $\varepsilon_\nu^\rho \to 0$ as $\nu \to 0$.

Going over to the whole manifold and using $||w|| \leqq c_{\mathscr{B}}||Ew||$, we may derive the inequalities

$$|(w, \sigma^\nu w) - (w, w)_{\mathscr{B}}| \leqq c\sqrt{\nu}||Ew||^2,$$
$$|(w, (\mathring{D}\sigma^\nu)w) - (w, \partial w)_{\mathscr{B}}| \leqq \varepsilon_\nu||Ew||^2$$

with $\varepsilon_\nu \to 0$ as $\nu \to 0$.

These inequalities were derived for functions w in $\mathfrak{C}_1$. By virtue of Theorem 16.1—and the definition of $\mathfrak{H}_1(E)$—they carry over to w in $\mathfrak{H}_1(E)$. Theorem 16.2 is then an immediate consequence.

From Theorem 16.1 and Theorem 12.2 we have

THEOREM 16.3. *Let $\hat{v}$ be the solution of $(D-t)^* \cdot D\hat{v} = v$ for $v \in \mathfrak{C}_1$. Then the function $D\hat{v}$ possesses boundary values and the inequality*

$$||D\hat{v}||_{\mathscr{B}} \leqq c_{\mathscr{B}}||ED\hat{v}||$$

holds with appropriate $c_{\mathscr{B}}$.

Part IV

MODIFICATIONS

17. Boundaries With Edges

In the theory developed in Parts II and III the boundary of the manifold was assumed to possess a continuous normal. In this part we shall show that under certain circumstances problems can also be handled where the boundary possesses edges along which the normal is discontinuous. We assume that the boundary $\mathscr{B}$ of the manifold consists of three parts, $\mathscr{B}_0$, $\mathscr{B}_+$, $\mathscr{B}_-$; the intersections of $\mathscr{B}_0$ with $\mathscr{B}_+$ and $\mathscr{B}_-$ are then the edges. Specifically we assume that some of the boundary patches, called quarter-patches, are one-to-one images of a quarter-sphere:

$$x_1^2 + \cdots + x_m^2 \leqq R^2, \qquad x_1 \leqq 0, \qquad x_m \leqq 0.$$

The image of $x_m = 0$ in this patch then belongs to $\mathscr{B}_0$ that of $x_1 = 0$ to $\mathscr{B}_+$ or $\mathscr{B}_-$. Everything that was said in Section 7 about the manifold and the entities defined in it is to be carried over in an obvious way.

We add the requirement on the identification transformation $\hat{x}_\lambda \cdot \hat{x}_\lambda(x)$ for two overlapping half- or quarter-patches adjacent to $\mathscr{B}_\pm$, viz. that

$\partial \hat{x}_\lambda / \partial x_1 = 0$ on $\mathscr{B}_\pm$ for $\lambda \neq 1$. This requirement will enable us to carry out a reflection procedure in a very simple manner; probably it could be eliminated.

Of the parts $\mathscr{B}_+$ and $\mathscr{B}_-$ of $\mathscr{B}$ we now assume that the matrix β is non-negative on $\mathscr{B}_+$ and non-positive on $\mathscr{B}_-$. Accordingly, we may choose $\mu = \pm\beta$ on $\mathscr{B}_\pm$ and the boundary condition becomes

$$\beta u = 0 \qquad \text{on } \mathscr{B}_-,$$

while no condition should be imposed on $\mathscr{B}_+$.

A situation as described here arises for example in the case of the mixed problem of a hyperbolic system when $\mathscr{B}_-$ is the initial surface at which $u = 0$ is given while nothing is prescribed at the end surface $\mathscr{B}_+$. The case of Cauchy's problem for a lens-shaped region would result if the mantle $\mathscr{B}_0$ of the boundary were absent. Although this case will not be explicitly covered in the following, it could be handled by an obvious modification of our treatment; it could also be reduced to the case treated in this section by modifying the problem in the neighborhood of the intersection of $\mathscr{B}_+$ and $\mathscr{B}_-$.

The statements about uniqueness and weak existence made in Part I carry over immediately. The only essential modification that must be made concerns the tangential operators D introduced in Section 8. It is no longer possible to define these operators in such a way that the commutators $[D, K]$ are linear combinations of the D and K. For example, the operators $D = x^m \partial/\partial x^m$, $x^1 \partial/\partial x^1$, and $\partial/\partial x^\lambda$, $\lambda = 2, \cdots, m-1$, are tangential in a corner patch; but each commutator $[D, K]$ involves a combination of the operators $\partial/\partial x^m$, $\partial/\partial x^1$, which, in general, cannot be expressed in terms of K and $\partial/\partial x^\lambda$, $\lambda = 2, \cdots, m-1$.

For this reason we shall give up the requirement that the operators D be tangential at the surfaces $\mathscr{B}_\pm$ but retain this requirement for the surface $\mathscr{B}_0$. Moreover, near $\mathscr{B}_0$ we retain all requirements on the matrices α^ρ which in Section 8 were imposed along the boundary $\mathscr{B}$. It then follows as in Section 8 that the commutators $[D, K]$ are indeed linear combinations of D and K. Accordingly we may introduce matrices t and an operator $\underset{1}{K}$ acting on compound systems such that

$$(D-t)K = \underset{1}{K}D.$$

Similarly, we may introduce matrices $t_{\mathscr{B}}$ and $\underset{1}{M}$ along the boundary $\mathscr{B}_0$ such that

$$(D-t_{\mathscr{B}})M = \underset{1}{M}D \qquad \text{on } \mathscr{B}_0.$$

We first establish an *a priori* inequality for the derivatives Du of a solution of the equation $Ku = f$, under the boundary condition $Mu = 0$ on $\mathscr{B}_0$ and $u = 0$ on $\mathscr{B}_-$. We denote by $\mathfrak{C}_{1,2}^-$ the subspaces of $\mathfrak{C}_{1,2}$ consisting of those functions u in $\mathfrak{C}_{1,2}$ which vanish on $\mathscr{B}_-$. Then we set

$$||u||_1 = ||Du|| \qquad \text{for } u \text{ in } \mathfrak{C}_1^-$$

and state

THEOREM 17.1. *Suppose the function u is in $\mathfrak{C}_2^-$ and such that Ku is in $\mathfrak{C}_1^-$ and satisfies $Mu = 0$ on $\mathscr{B}$. Then with an appropriate constant c the inequality*

$$||u||_1 \leqq c||Ku||_1$$

holds.

The requirement that Ku should vanish on $\mathscr{B}_-$ is not serious since one can always subtract from u an appropriate function so that this requirement is satisfied by the difference.

The requirement that u should vanish on $\mathscr{B}_-$ appears to be stronger than the boundary condition $\beta u = 0$ on $\mathscr{B}_-$. Actually, the relation $u = 0$ follows from $\beta u = 0$ and $Ku = 0$ on $\mathscr{B}_-$ and $Mu = 0$ on $\mathscr{B}_0$, provided $\tilde{K} = K + \partial\beta/\partial y > 0$ on $\mathscr{B}_-$; for,

$$0 = (u, Ku)_{\mathscr{B}_-} = \left(u, 2\beta \frac{\partial u}{\partial y}\right)_{\mathscr{B}_-} + (u, \tilde{K}u)_{\mathscr{B}_-}$$

$$= 2\left(\beta u, \frac{\partial u}{\partial y}\right)_{\mathscr{B}_-} + (u, \tilde{\kappa}u)_{\mathscr{B}_-} = (u, \tilde{\kappa}u)_{\mathscr{B}_-}.$$

The proof of Theorem 17.1 is given through the following steps: First we set $\underset{1}{\mu} = \pm\beta$ on $\mathscr{B}_\pm$; hence $\underset{1}{M} = 0$ on $\mathscr{B}_+$, $\underset{1}{M} = -2\beta + \rho$ on $\mathscr{B}_-$, where the matrix ρ is such that $u = 0$ implies $\rho\{u, u_1, \ldots\} = 0$. Furthermore, we note that the assumptions $u = 0$ and $Ku = 0$ on $\mathscr{B}_-$ imply $\beta\,\partial u/\partial y = 0$ and $Du = d^\nu\,\partial u/\partial y$ where $d^\nu = d^1$ if $y = x^1$. Hence we have

$$\underset{1}{M}\,Du = -2\beta d^\nu \frac{\partial}{\partial y} u = -2d^\nu \beta \frac{\partial}{\partial y} u = 0 \qquad \text{on } \mathscr{B}_-.$$

Consequently, we may derive

$$\begin{aligned}
(Du, \underset{1}{\kappa} Du) &\leqq (Du, \underset{1}{\kappa} Du) + (Du, \underset{1}{\mu} Du) = (Du, \underset{1}{K} Du) + (Du, \underset{1}{M} Du) \\
&= (Du, \underset{1}{K} Du) + (Du, \underset{1}{M} Du)_{\mathscr{B}_0} \\
&= (Du, (D-t)Ku) + (Du, (D-t^{\mathscr{B}})Mu)_{\mathscr{B}_0} \\
&= (Du, (D-t)Ku) \leqq ||Du||\;||(D-t)Ku|| \\
&\leqq (1+\theta)||Du||\;||DKu||.
\end{aligned}$$

Using a constant c_1 such that $||Du||^2 \leqq c_1(Du, \underset{1}{\kappa} Du)$ we find

$$||Du|| \leqq (1+\theta)c_1||DKu||$$

as desired.

In a similar fashion we may derive a dual inequality. To this end we must introduce a partially clipped Laplacian. Specifically, we introduce the operator

$$(D-t)^* \cdot D$$

applicable to functions w in $\mathfrak{C}_2^-$ which satisfy the condition

$$d^\nu \cdot Dw = 0 \qquad \text{on } \mathscr{B}_+ .$$

For any function w satisfying these conditions and any function $\hat{w}$ in $\mathfrak{C}_1^-$ we evidently have

$$(\hat{w}, (D-t)^* \cdot Dw) = \big((D-t)\hat{w}, Dw\big).$$

We denote by $\mathfrak{H}_1^-$ the extension of the space $\mathfrak{C}_1^-$ with respect to the norm $\|\ \|_1$ and by $\mathfrak{H}_1^+$ the space in which the adjoint D^* of D in $\mathfrak{H}^-$ is defined. Then we formulate

THEOREM 17.2. *To every v in $\mathfrak{H}$ there exists a $\hat{v}$ in $\mathfrak{H}_1^-$ with $D\hat{v}$ in $\mathfrak{H}_1^+$ such that*

$$(D-t)^* \cdot D\hat{v} = v.$$

This theorem is proved in the standard manner. In fact, it can be reduced to Theorem 12.1 by a reflection device, which we shall also use in order to establish the needed differentiability properties of the solution $\hat{v}$. Before discussing these matters we shall derive the dual inequality formally.

Let v be a function in $\mathfrak{C}_1$ satisfying the conditions $M^* v = 0$ on $\mathscr{B}$. Without restriction we may further assume $v = 0$ on $\mathscr{B}_+$. We then define $\hat{v}$ according to Theorem 17.2 and assume that $\hat{v}$ has the differentiability properties needed for the following argument. The first basic identity then yields

$$\begin{aligned}(K^* v, \hat{v}) &= (v, K\hat{v}) + (v, M\hat{v})_{\mathscr{B}} = (v, K\hat{v}) + (v, M\hat{v})_{\mathscr{B}_0} \\ &= (D\hat{v}, (D-t)K\hat{v}) + (D\hat{v}, (D-t^{\mathscr{B}})M\hat{v})_{\mathscr{B}_0} - (D\hat{v}, d^\nu K\hat{v})_{\mathscr{B}_- + \mathscr{B}_+}.\end{aligned}$$

Here we have omitted terms involving $D\hat{v} \cdot dM\hat{v}$ on the intersection of $\mathscr{B}_0$ with $\mathscr{B}_+$ and $\mathscr{B}_-$ resulting from integration by parts; for, these terms vanish because of $\hat{v} = 0$ on $\mathscr{B}_-$ and $D\hat{v} = 0$ on $\mathscr{B}_+$. Now on $\mathscr{B}_-$ we have

$$d^\nu K\hat{v} = 2d^\nu \beta \frac{\partial}{\partial y}\hat{v} = 2\beta d^\nu \frac{\partial}{\partial y}\hat{v} = 2\beta D\hat{v} = -\underset{1}{M} D\hat{v},$$

while $d^\nu \cdot D\hat{v} = 0$ on $\mathscr{B}_+$. Hence, in view of $\underset{1}{M} = 0$ on $\mathscr{B}_+$, we have

$$\begin{aligned}(K^* v, \hat{v}) &= (D\hat{v}, \underset{1}{K} D\hat{v}) + (D\hat{v}, \underset{1}{M} D\hat{v})_{\mathscr{B}} \\ &= (D\hat{v}, \underset{1}{\kappa} D\hat{v}) + (D\hat{v}, \underset{1}{\mu} D\hat{v})_{\mathscr{B}} \\ &\geqq c_1^{-1}\|D\hat{v}\|^2,\end{aligned}$$

whence

$$||\hat{v}||_1 \leqq c_1||K^* v||_{-1} .$$

As before we may conclude

THEOREM 17.3. *A function v in $\mathfrak{C}_1$ with $M^* v = 0$ on $\mathscr{B}$, $v = 0$ on $\mathscr{B}_+$ satisfies the dual inequality*

$$||v||_{-1} \leqq c_{-1}||K^* v||_{-1} .$$

Before justifying the arguments that led to this theorem we draw conclusions from it. The first is

LEMMA 17.1. *To every function f in $\mathfrak{H}_1^-$ on $\mathscr{B}_-$ there exists a weak solution u of the equation $Ku = f$ with the boundary condition $Mu = 0$, which also belongs to $\mathfrak{H}_1^-$; it obeys the inequality*

$$||u||_1 \leqq c_{-1}||f||_1 .$$

Its proof is literally the same as that of Lemma 10.1. In deriving from this lemma the fact that u is a strong solution a slight modification is necessary.

Such a modification must be made in the quarter-patches adjacent to the edges where $\mathscr{B}_0$ intersects $\mathscr{B}_+$ or $\mathscr{B}_-$. Since this modification affords a simplification, it might just as well be made in all half-patches adjacent to $\mathscr{B}_\pm$, which is possible because $\mu = \pm\beta$ there.

In Section 10 we introduced the operator $\tilde{K}$, with reference to any patch, and stated that it is a linear combination of tangential operators. We define again the operator $\tilde{K}$ as in Section 10; it then remains true that $\tilde{K}$ is a linear combination of the operators D_σ. This operator $\tilde{K}$ plays the same role as in Section 10 with respect to $\mathscr{B}_0$ but not with respect to $\mathscr{B}_\pm$. The procedure of reflection in the boundary used in Section 10 should be used now with respect to $\mathscr{B}_0$ only.

It is necessary to adjust the definition of the mollifiers J^ν. The kernels $j^\nu(x-x')$ should be so chosen that

$$j^\nu(x-x') = 0 \qquad \begin{array}{c} \text{if } x' \text{ on } \mathscr{B}_+,\ x \text{ in } \mathscr{R}, \\ \text{and} \\ \text{if } x' \text{ in } \mathscr{R},\ x \text{ on } \mathscr{B}_- . \end{array}$$

This can be achieved, if necessary, by a small shift of the variable x_1. As a consequence of this adjustment we can again conclude that $u^\nu = 0$ on $\mathscr{B}_-$ and furthermore that the relation

$$||\zeta(D_\sigma u^\nu - D_\sigma u)|| \to 0$$

holds, although not all operators D_σ are tangential at $\mathscr{B}_+$.

The same arguments that led to Theorem 10.1 then lead to its analogue

THEOREM 17.4. *Suppose the operator pair (K, M) possesses a satisfactory* *set of operators D_σ. Then for every f in the space $\mathfrak{H}_1^-$ there exists in the space $\mathfrak{H}_1^-$ a strong solution u of the equation $Ku = f$ with the boundary condition $Mu = 0$. This solution satisfies the inequality*

$$||u||_1 \leqq c_{-1}||f||_1.$$

We finally supply the argument needed to justify the derivation of the dual inequality, formulated in Theorem 17.3. The properties of the solution $\hat{v}$ of the equation $(D^*-t^*) \cdot D\hat{v} = 0$ used for this purpose could be established essentially by the methods of Part III. Additional considerations, however, are needed in view of the boundary conditions imposed on $\hat{v}$ along $\mathscr{B}_+$ and $\mathscr{B}_-$. These considerations may be supplied by the method of reflection.

We first reflect the manifold in the boundary segment $\mathscr{B}_-$, and then reflect the resulting double manifold in the two boundary segments $\mathscr{B}_+$. In doing so, each half- or quarter-patch adjacent to $\mathscr{B}_\pm$ goes over into a full or half-patch. We want to make sure that, after this reflection, the identification transformation of two overlapping patches has continuous derivatives.

To this end we denote by x the variables different from y, writing (x, y) in place of x. The identification transformation then takes the form

$$\hat{x} = f(x, y), \qquad \hat{y} = g(x, y).$$

In the reflected patch we evidently have

$$f(x, y) = f(x, -y), \qquad g(x, y) = -g(x, -y).$$

Now, the derivatives f_x and g_y are evidently continuous at $y = 0$; furthermore, g_x is continuous there because $g(x, 0) = 0$. Continuity of f_y is insured by virtue of the relation $f_y(x, 0) = 0$ which expresses the additional requirement made at the beginning of this section.

The operators D may now be extended in an obvious way into the quadruple manifold: if in an image patch $\mathscr{P}'$ of $\mathscr{P}$ the same coordinates are assigned to corresponding points, the operator D in the image patch has the same coefficients as the operator D in the original patch when expressed in terms of differentiation with respect to these coordinates. We also assign the same function t to corresponding points.

The operators D_σ prescribed in this way to images $\mathscr{P}'$ of boundary half- or quarter-patches $\mathscr{P}$ along $\mathscr{B}_\pm$ are not necessarily continuous continuations of the operators in $\mathscr{P}$. We want to achieve that either D_σ or $-D_\sigma$ is such a continuous continuation. To this end, we take in each such half- and quarter-patch $\mathscr{P}$ operators $\eta\,\partial/\partial x$ and $\eta\,\partial/\partial y$ with $\eta x^m\,\partial/\partial x^m$ in place of $\eta\,\partial/\partial x^m$ in a quarter-patch. If the image $\mathscr{P}'$ is described by $y \geqq 0$, the continued operators are $\eta\,\partial/\partial\tilde{x}$ and $\eta\,\partial/\partial y$ while the assigned operators are $\eta\,\partial/\partial\tilde{x}$ and $-\eta\,\partial/\partial y$.

We also want to make sure that the assigned operator $D_\sigma - t_\sigma$, or its negative, gives a continuous continuation. If $D_\sigma = \eta\,\partial/\partial\tilde{x}$, the first is the case, but if $D_\sigma = \eta\, \partial/\partial y$, the assigned operator, $D_\sigma - t_\sigma$ is $-\eta\, \partial/\partial y - t_\sigma$ and hence the continued operator is $\eta\, \partial/\partial y + t_\sigma$. In this case continuity is achieved if $t_\sigma = 0$ on $\mathscr{B}_\pm$.

In half-patches adjacent to $\mathscr{B}_\pm$ we can set $t_\sigma = 0$ since enough derivatives are available among the D_σ to express $[D, K]$ in terms of them. Special considerations are needed at the quarter-patches at an edge at which $\mathscr{B}_+$ or $\mathscr{B}_-$ meets $\mathscr{B}_0$. Here we are not free to choose $D_\sigma = \eta\, \partial/\partial x_1$, nor are we free to choose $t_\sigma = 0$. The only contribution to $[\partial/\partial x_1, K]$ to be considered in this connection is $2(\partial \alpha^m/\partial x_1)(\partial/\partial x^m)$. From the assumptions made in Section 8 we conclude that this term may be written in the form

$$2(R'\alpha^m + \alpha^m R)\, \frac{\partial}{\partial x^m} \cdot$$

We now make the *additional assumption* that the matrix R vanishes along the edge $x_1 = x_m = 0$. This is not a serious restriction; for, with the aid of an appropriate transformation V of the columns u, the matrix α^m could be made constant in a quarter-patch, as follows from the assumptions made in Section 8.

Since now the matrix R vanishes along the edge we can write it in the form

$$R = x_1 R_1 + x_m R_m \,.$$

The operator $x_m \partial/\partial x_m$ is one of the operators D_σ; hence the crucial contribution to $[\partial/\partial y, K]$ can be reduced to

$$2(x_1 R_1' \alpha^m + \alpha^m x_1 R_1)\, \frac{\partial}{\partial x^m}$$

and hence to

$$x_1 R_1' K + K x_1 R_1 \,.$$

In other words, the expression

$$\left(\frac{\partial}{\partial x_1} - x_1 R_1'\right) K - K \left(\frac{\partial}{\partial x_1} + x_1 R_1\right)$$

can be expressed in terms of the D_σ. Accordingly we now take $\eta(\partial/\partial x_1 + x_1 R_1)$ as one of the D and set $t_\sigma = x_1(R_1 + R_1')$. Evidently this operator D_σ, as well as the operator $D_\sigma - t_\sigma$, has the property that the negative of the operator, assigned to it in the image (with respect to the surface $x_1 = 0$) of the patch, affords a continuous continuation of it.

Thus the derived property is achieved.

Next we observe that the operators $(D_\sigma - t_\sigma)^*$ have the same behavior on reflection as the operators $D_\sigma - t_\sigma$. Consequently, in forming the operator $(D-t)^* \cdot D = \sum_\sigma (D_\sigma - t_\sigma)^* \cdot D_\sigma$ it makes no difference whether the operator

D_σ in an image $\mathscr{P}'$ of $\mathscr{P}$ is obtained by "continuation" or by "assignment". If the first interpretation is adopted, it is clear that this operator is of the form denoted by $(D-t)^* \cdot D$ in the theory of Part III, with respect to the quadruple manifold. The statements made there about the solution $\hat{v}$ of the equation $(D-t)^* \cdot D\hat{v} = 0$ can therefore be taken over.

To explain the consequences of the second interpretation, let w be a function defined in the quadruple manifold. We say it has the "proper oddity" if it has the same value at an image point with respect to $\mathscr{B}_+$ and the opposite value at an image point with respect to $\mathscr{B}_-$.

It now follows from the "second" interpretation of the operator $(D-t)^* \cdot D$ that if a function w, admitting this operator, has the proper oddity, the function $(D-t)^* \cdot Dw$ has the same property.

Vice versa, if v has the proper oddity, the same is true of the solution $\hat{v}$ of the equation $(D-t)^* \cdot D\hat{v} = v$. For, if $\hat{v}$ is such a solution, the functions obtained from v by even and odd reflections in $\mathscr{B}_+$ and $\mathscr{B}_-$ are also solutions; the statement then follows from the uniqueness of the solution $\hat{v}$.

By arguments similar to those used in deriving Theorem 16.1, one shows that the functions $\hat{v}$ and $D\hat{v}$ have boundary values on $\mathscr{B}_+$ and $\mathscr{B}_-$.

From the fact that $\hat{v}$ has the proper oddity it then follows that this solution satisfies the boundary condition $\hat{v} = 0$ on $\mathscr{B}_-$, and consequently, the relation $d^\nu K\hat{v} = 2\beta D\hat{v} = -\underset{1}{M}D\hat{v}$ holds here. It is also clear that on $\mathscr{B}_+$ the condition $d^\nu \cdot D\hat{v} = 0$ is satisfied. This is easily verified by a simple argument in which one uses the specific nature of the operators D_σ assigned to the half- or quarter-patches adjacent to $\mathscr{B}_+$. In order to justify the derivation of the dual inequality one need only justify the contribution due to the section $\mathscr{B}_0$ of the boundary. This is done in the same way as in Part III.

By these considerations the validity of the dual inequality (Theorem 17.3) is established.

From the arguments of Section 10 we then infer the existence of a solution u in $\mathfrak{H}_1^-$ of the equation $Ku = f$ with the boundary condition $Mu = 0$, provided f is in $\mathfrak{H}_1^-$ and if we assume that the matrix $\underset{1}{\kappa}+\underset{1}{\kappa}'$ is positive definite and $\underset{1}{\mu}+\underset{1}{\mu}'$ is non-negative. Without making the first of these two assumptions we infer from the arguments of Section 11 that a strong solution of the problem exists.

18. A Boundary Problem for the Tricomi Equation

The Tricomi equation—slightly generalized and modified—is

$$G\varphi_{xx}-\varphi_{yy} = 0,$$

where the function $G = G(y)$ is such that

$$G \gtreqless 0 \qquad \text{for} \quad y \gtreqless 0$$

and $G' \neq 0$, so that the equation is elliptic for $y < 0$ and hyperbolic for $y > 0$. For the solution of this equation we shall impose conditions on the boundary $\mathscr{B}$ of a region $\mathscr{R}$, which may be described by the inequalities

$$q(x, y) \leqq 0, \qquad x \leqq x_+ .$$

See the figure on page 334. The boundary consists of two parts

$$\mathscr{B}_0 : q(x, y) = 0, \qquad x \leqq x_+ ,$$
$$\mathscr{B}_+ : q(x, y) \leqq 0, \qquad x = x_+ .$$

Of the function $q(x, y)$—in $\mathfrak{C}_2$—we require that it satisfies the conditions $q(x_+, y) > 0$ for $y \leqq 0$, further $q_x^2+q_y^2 > 0$, $q_y^2-q_x^2 G > 0$ throughout. Moreover, we require for convenience that

$$q_x+q_y \leqq 0 \qquad \text{on } \mathscr{B}_0 .$$

We prescribe the values of the function ψ along $\mathscr{B}_0$ and nothing on $\mathscr{B}_+$. In order to be able to reduce this problem to one in which the boundary condition is homogeneous, we assume that a function ψ_0 in $\mathscr{R}$ exists which has these boundary values on $\mathscr{B}_0$. We then introduce the function $\phi = \psi-\psi_0$, in place of ψ. Furthermore, we introduce the pair $u = \{u_1, u_2\}$ of derivatives $u_1 = \phi_x$, $u_2 = \phi_y$. This pair satisfies the differential equation

$$Lu = \begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial u}{\partial x} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial u}{\partial y} = f$$

where $f = \{f_1, f_2\}$ with $f_1 = -L\psi_0$ and $f_2 = 0$. The boundary condition for the function u is then $u_1 dx+u_2 dy = 0$ or

$$n_y u_1-n_x u_2 = 0 \qquad \text{on } \mathscr{B}_0 .$$

We maintain that *a strong solution of this problem exists.* Furthermore we should like to show that *this solution has a strong derivative if the function f has a strong derivative.* We shall do so *only for a special region* $\mathscr{R}$ of the type described.

First we must transform the equation into a positive symmetric one. To this end we employ a matrix

$$Z = 2 \begin{pmatrix} b & cG \\ c & b \end{pmatrix}$$

and replace the operator L by the operator

$$K = ZL = 2 \begin{pmatrix} bG & cG \\ cG & b \end{pmatrix} \frac{\partial}{\partial x} - 2 \begin{pmatrix} cG & b \\ b & c \end{pmatrix} \frac{\partial}{\partial y} .$$

On the functions b, c we impose the conditions $b > 0$, $c > 0$, and

$$b^2-c^2G > 0 \qquad \text{in } \mathscr{R};$$

the latter relation insures that Z has an inverse.

It is a consequence of these relations that the matrix $\left(\begin{smallmatrix} bG & cG \\ cG & b \end{smallmatrix}\right)$ is positive definite on $\mathscr{B}_+$, so that indeed no boundary conditions should be imposed there.

The quadratic form $u \cdot \beta u$ on $\mathscr{B}_0$ becomes

$$u \cdot \beta u = (bGn_x - cGn_y)u_1^2 + 2(cGn_x - b_{n_y})u_1 u_2 + (bn_x - c_{n_y})^2 u_2^2,$$

where $n_x = q_x$, $n_y = q_y$ except for a positive factor. As was explained in Section 4 we may write this expression in the form

$$u \cdot \beta u = [bn_x + cn_y]^{-1}\{(b^2-c^2G)(n_y u_1 - n_x u_2)^2 - (n_y^2 - n_x^2 G)(bu_1 + cu_2)^2\}.$$

In view of $b^2-c^2G > 0$ and $n_y^2 - n_x^2 G > 0$ on $\mathscr{B}_0$ it is seen that the boundary condition $n_y u_1 - n_x u_2 = 0$ in $\mathscr{B}_0$ is admissible if b and c are such that

$$bn_x + cn_y < 0 \qquad \text{on } \mathscr{B}_0 .$$

The boundary matrix $M = \mu - \beta$ is seen to be

$$M = -2[bn_x + cn_y]^{-1}(b^2-c^2G)\begin{pmatrix} n_y^2 & -n_x n_y \\ -n_x n_y & n_x^2 \end{pmatrix}.$$

To satisfy the condition $bn_x + cn_y > 0$ on $\mathscr{B}_0$, we choose

$$b = 1-2\sigma x, \qquad c = 1-\sigma(x+y)$$

with a positive constant σ at our disposal. By restricting the region $\mathscr{R}$ to lie in a sufficiently small neighborhood $\mathscr{N}$ of the origin we can achieve that

$$b > 0 \text{ and } c > 0 \qquad \text{in } \mathscr{R}.$$

We assume that the region $\mathscr{R}$ is such that $n_y \gtrless 0$ for $y \gtrless x$. By virtue of condition $n_x + n_y \leqq 0$, this requirement can always be met by a shift of the origin along the x-axis. Condition $bn_x + cn_y < 0$ on $\mathscr{B}_0$ reduces to $b(n_x + n_y) - \sigma n_y(y-x) < 0$ on $\mathscr{B}_0$. By a shift of the origin along the x-axis we may further achieve that

$$n_y(y-x) \leqq 0 \qquad \text{on } \mathscr{B}_0 ,$$

as follows from $n_x + n_y \leqq 0$. Then condition $bn_x + cn_y < 0$ on $\mathscr{B}_0$ is satisfied.

Next we show that the constants x_+, $|x_0|$, R can be so chosen that the matrix $\kappa + \kappa'$ is positive-definite. Evidently we have

$$\kappa = \kappa' = \begin{pmatrix} cG' + \sigma G & -\sigma G \\ -\sigma G & \sigma \end{pmatrix}.$$

Again requiring the region $\mathscr{R}$ to lie in a sufficiently small neighborhood of 0 we can insure that G is so small in $\mathscr{R}$ that κ is positive definite there.

Thus the positive symmetric character of the operator K with the boundary condition $M = O$ has been established.[13]

We now apply the results of Section 11 in the modified form described in Section 17. To this end we must first show that one can introduce operators D which have the properties required in Section 9. This can be done since on $\mathscr{B}_+$ the matrix $\beta = n_x \alpha_1 + n_y \alpha_2 = \alpha_1$ is positive definite. The only one among the conditions formulated in Sections 8 and 17 which perhaps is not satisfied, is the condition that the matrix $\underset{1}{\kappa} + \underset{1}{\kappa'}$ be positive definite; but this is exactly the one condition which was not required in Section 11. Therefore Theorem 11.1 is applicable and *the existence of a strong solution of the problem is established.*

Next we proceed to show the strong differentiability of the solution in case f has strong derivatives.

We shall derive this statement only for a somewhat special region $\mathscr{R}$, namely one for which

$$q(x, y) = (x-y)^2 - R^2 \qquad \text{for } x_0 \leqq x \leqq x_+ ,$$

with $x_0 < -R$, so that $y \leqq 0$ on the segment $x = x_0$, $q(x, y) \leqq 0$.

As tangential operators we may introduce operators which for $x_0 \leqq x$, are given by

$$D_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$D_2 = [R^2 - (x-y)^2] \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right],$$

and which are appropriately continued for $x \leqq x_0$. We then could set up identities expressing the commutators $[D_1, K]$ and $[D_2, K]$ in terms of D_1, D_2 and K. One might try to do this in such a way that the associated matrix $\underset{1}{\kappa} + \underset{1}{\kappa'}$ is positive definite.

It is doubtful whether this is possible. In any case we are not able to proceed in this manner. In order to establish our differentiability statement we shall make use of the fact that the differential equation is elliptic for $y < 0$.

Since the pair $\{u_1, u_2\}$ satisfies the equation $\partial u_1/\partial y - \partial u_2/\partial x = 0$ in the weak—in fact even in the strong—sense and satisfies the boundary condition $n_y u_1 - n_x u_2 = 0$ on $\mathscr{B}_0$, there exists a function[14] ϕ which has strong derivatives $\partial\phi/\partial x = u_1$, $\partial\phi/\partial y = u_2$, and vanishes on $\mathscr{B}_0$. This function

[13] The adaptation here employed is a variant of that employed by Protter and Morawetz.

[14] Describing $\mathscr{R}$ by $y_-(x) \leqq y \leqq y_+(x)$, we may set

$$\varphi(x, y) = \int_{y_-(x)}^{y} u_2(x, y')dy'$$

and then prove that $\varphi_x = u_1$, $\varphi_y = u_2$ in the weak and hence in the strong sense.

satisfies the elliptic equation $G\,\partial^2\phi/\partial x^2-\partial^2\phi/\partial y^2=f_1$ for $y<0$ in the strong sense. From the theory of elliptic equations it then follows that in the interior of the region $\mathscr{R}$ for $y<0$, the function admits strong derivatives of any order, assuming G and f_1 sufficiently differentiable.

As part of this theory we obtain inequalities for these derivatives. Specifically, let $\xi(x)\geqq 0$ be a function in $\mathfrak{C}_\infty$ having the properties:

$$\xi(x)=\begin{cases}0 & \text{for } x\leqq x_0\,,\\ 1 & \text{for } x_0+\delta\leqq x\leqq x_0+4\delta,\\ 0 & \text{for } x_0+5\delta\leqq x,\end{cases}$$

where $\delta<-\frac{1}{5}(R+x_0)$ so that $y<0$ on the segment $x=x_0+5\delta$, $|y-x|\leqq R$.

By virtue of the fact that $(\partial/\partial x+\partial/\partial y)\phi=0$ on $\mathscr{B}_0$ for $x\geqq x_0$ we then obtain in the standard manner the inequality

$$\left\|\xi\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\phi\right\|^2+\left\|\xi\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\phi\right\|^2\leqq c\|\phi\|_1^2+c\|f_1\|_1^2\,.$$

From the fact that ϕ satisfies the differential equation we deduce a similar inequality for all second derivatives of ϕ. Assuming f_1 to possess strong higher derivatives the same can be concluded of ϕ, at least in the segment $x_0+\delta\leqq x\leqq x_0+4\delta$.

We now modify the differential equation $Lu=f$. Let $\zeta(x)\geqq 0$ be a function in $\mathfrak{C}_\infty$ such that

$$\zeta(x)\equiv\begin{cases}0 & \text{for } x_0+\delta\leqq x\leqq x_0+2\delta,\\ 1 & \text{for } x_0+3\delta\leqq x.\end{cases}$$

Then we replace the operator L by

$$\tilde{L}=\zeta\begin{pmatrix}G & 0\\ 0 & 1\end{pmatrix}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)-\begin{pmatrix}G & 1\\ 1 & 1\end{pmatrix}\frac{\partial}{\partial y}+\tilde{\gamma}$$

where the matrix $\tilde{\gamma}$ in $\mathfrak{C}_\infty$ is such that $\gamma=0$ for $x_0+3\delta\leqq x$, and kept at our disposal for $x\leqq x_0+3\delta$.

This operator will be applied to the function

$$\tilde{u}=\zeta u.$$

We maintain that the differentiability properties of u for $x\leqq x_0+4\delta$, just established, imply that the function

$$\tilde{f}=\tilde{L}\tilde{u}$$

possesses strong derivatives. For $x\geqq x_0+3\delta$ this follows from the fact that $\tilde{f}=f$ there; for $x_0+\delta\leqq x\leqq x_0+4\delta$ this follows from the differentiability property of u; for $x\leqq x_0+2\delta$ we have $\tilde{f}=0$.

The operator $\tilde{L}$ is so designed that on the sides $|y-x| = R$ the matrix β associated with it is the same as the matrix β associated with L, so that again boundary condition $u_1+u_2 = 0$ is admissible for it. On the segment $x_0 = 0$, on the other hand, the matrix $\tilde{\beta}$ vanishes; there no boundary condition should be imposed.

The equation $\tilde{L}\tilde{u} = f$ in the region $\tilde{\mathscr{R}}$: $x_0 \leqq x \leqq x_+$, $|x-y| \leqq R$, is thus one of the equations treated in Section 17. We can choose the matrix $\tilde{\gamma}$ such that the matrix $\tilde{\kappa}+\tilde{\kappa}'$ becomes positive definite. Therefore the solution $\tilde{u} = \zeta u$ of the equation $\tilde{L}\tilde{u} = f$ is unique.

As operators $\tilde{D}$ to be associated with the operator $\tilde{L}$ according to the theory of Section 17, the operators

$$\tilde{D}_0 = 1, \qquad \tilde{D}_1 = \varepsilon_1 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \qquad \tilde{D}_2 = \varepsilon_2(R^2-(x-y)^2)\frac{\partial}{\partial y}$$

may be chosen since they are tangential on the sides $|x-y| = R$. Evidently, we may express the commutators $[\tilde{D}_1, \tilde{K}]$ and $[\tilde{D}_2, \tilde{K}]$ in terms of D_0, $\tilde{D}_1$ and $\tilde{K}$. Our aim is to show that the matrix $\underset{1}{\tilde{\kappa}}+\underset{1}{\kappa'}$ can be made positive definite by the choice of $\tilde{\gamma}$, ε_1 and ε_2 and of the domain $\mathscr{R}$.

Note that the matrix K acts on compound systems $\underset{1}{u} = \{u_0, u_1, u_2\}$. We maintain that it is sufficient to make the contribution from u_1 alone positive, $\underset{1}{\kappa_{11}} > 0$, in addition to making κ positive. Suppose this is done; then one may take $|\varepsilon_1|$ so small that the matrix

$$\begin{pmatrix} \underset{1}{\kappa_{00}} & \underset{1}{\kappa_{01}} \\ \underset{1}{\kappa_{10}} & \underset{1}{\kappa_{11}} \end{pmatrix}$$

has a positive definite symmetric part, since $\underset{1}{\kappa_{10}}$ is proportional to ε_1 and $\underset{1}{\kappa_{01}} = 0$. Finally, we note that $\underset{1}{\kappa_{22}} = \kappa$ since the expression of the commutator $[\tilde{D}_1, \tilde{K}]$ does not involve $\tilde{D}_2$. Hence by making $|\varepsilon_2|$ sufficiently small we can make the whole matrix $\kappa+\kappa'$ positive definite, provided $\underset{1}{\kappa_{11}}+\underset{1}{\kappa'_{11}}$ is positive definite.

Clearly, we can make $\underset{1}{\kappa_{11}}$ positive definite for $x < x_0+3\delta$ by proper choice of $\tilde{\gamma}$ once σ is chosen. Note that the derivatives of $\tilde{\gamma}$ do not enter the expression of $\underset{1}{\kappa_{11}}$ (they enter $\underset{1}{\kappa_{10}}$). It is therefore sufficient to make $\underset{1}{\kappa_{11}}$ positive definite for $x \geqq x_0+3\delta$.

To this end we first evaluate the commutator $[D_1, K]$. In the following we use the congruence symbol "$\equiv$" to express the fact that an operator is a multiple of L or K:

$$QL \equiv 0,$$

where Q is any matrix. We find

$$[D_1, K] \equiv Z[D_1, L] = ZG' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial y} \equiv \begin{pmatrix} 0 & 1 \\ G & 0 \end{pmatrix} \frac{\partial}{\partial x},$$

$$D_1 \equiv \begin{pmatrix} 1 & 1 \\ G & 1 \end{pmatrix} \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial x} \equiv (1-G)^{-1} \begin{pmatrix} 1 & -1 \\ -G & 1 \end{pmatrix} D_1 .$$

Certainly, we can make $1-G > 0$ by contracting the region $\mathscr{R}$ further, if necessary. Finally, we obtain

$$[D_1, K] \equiv \lambda D_1$$

where

$$\lambda = (1-G)^{-1} G' Z \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = 2(1-G)^{-1} G' \begin{pmatrix} b & -b \\ c & -c \end{pmatrix}.$$

Consequently,

$$\underset{1}{\kappa}_{11} = \kappa_{11} + \lambda.$$

At the origin we have

$$\underset{1}{\kappa}_{11}(0) = \begin{pmatrix} 3G' & -2G' \\ 2G' & \sigma-2G' \end{pmatrix}$$

and hence

$$\underset{1}{\kappa}_{11}(0) + \underset{1}{\kappa}'_{11}(0) = \begin{pmatrix} 3G'(0) & 0 \\ 0 & \sigma-2G'(0) \end{pmatrix}.$$

By choosing $\sigma > 2G'(0)$ we can make this matrix positive definite and by again restricting the domain $\mathscr{R}$—if necessary—we can make $\underset{1}{\kappa}_{11} + \underset{1}{\kappa}'_{11}$ positive definite everywhere in $\mathscr{R}$ for $x \geqq x_0+3\delta$. As was explained above we then can make the whole matrix $\underset{1}{\kappa} + \underset{1}{\kappa}'$ positive definite.

Now the theory of Section 17 is applicable. It follows that the function $\tilde{u}$ possesses strong first derivatives in the region $\tilde{\mathscr{R}}$. Since in the region $x \leqq x_0+3\delta$ the function u, being the solution of an elliptic equation there, has strong derivatives in the part $x \leqq x_0+3\delta$ of $\mathscr{R}$, it follows that *u has strong derivatives in the whole region $\mathscr{R}$.*

This is the desired main result.

One may strengthen this result. One observes that the function $D_1\tilde{u}$ satisfies in $\tilde{\mathscr{R}}$ the equation

$$(\tilde{K}+\lambda)D_1\tilde{u} = \tilde{f}^{(1)}$$

where the function $\tilde{f}^{(1)}$ is given in terms of f and u. The boundary conditions for D_1u are the same as for u along the straight sides $|x-y| = R$. The argument to establish differentiability just given could then be carried out just as well, the only difference being that the matrix $\kappa+2\lambda$ takes the place of $\kappa+\lambda$. After restricting the region further, if necessary, one may conclude again that D_1u has strong derivatives. The same then holds for the derivatives $\partial u/\partial x$ and $\partial u/\partial y$ being linear combinations of D_1u, Ku, and u. Continuing one finds that the solution u has strong derivatives of any order provided the data have sufficient differentiability properties. As was explained in the introduction, it is a consequence of this fact that *the solution of our problem is continuous and possesses continuous derivatives of any desired order if the data possess sufficient differentiability properties.*

19. Cauchy-Riemann and Laplace Equations

In this section we shall show that problems of the Cauchy-Riemann equation and the Laplace equation can be fitted into our framework.

In treating the Cauchy-Riemann equation we confine ourselves to the non-homogeneous equation

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) w = \frac{\partial}{\partial \bar{z}}\, w = f = f_1 + if_2$$

for a function $w = w_1+iw_2$ of two variables x_1, x_2 in the rectangle $x_1^- \leqq x_1 \leqq x_1^+$, $x_2^- \leqq x_2 \leqq x_2^+$, at whose boundary we impose the conditions

$$w_1 = 0 \qquad \text{for } x_1 = x_1^+ ,$$
$$w_2 = 0 \qquad \text{for } x_1 = x_1^- \text{ and } x_2 = x_2^\pm .$$

Of the function f we require that $f_2 = 0$ on $x = x_1^+$ and $f_1 = 0$ on $x = x_1^-$.

To be sure, the problem of the homogeneous equation with values w_1 and w_2 prescribed at $x_1 = x_1^+$ and $x_1 = x_1^-$, $x_2 = x_2^\pm$ respectively, can be reduced to this problem.

We double the rectangle by reflection in the sides $x_2 = x_2^\pm$, taking w_1 as even, w_2 as odd in $x_2 - x_2^\pm$. Then the boundary consists of two disconnected parts, the two double segments $x_1 = x_1^\pm$, $x_2^- \leqq x_2 \leqq x_2^+$.

We introduce the operator C which transforms w into $Cw = \bar{w}$ and take the inner product $w^{(1)} \cdot w = \mathscr{R}e\, \bar{w}^{(1)}w = \mathscr{R}e\,(Cw^{(1)})w = \mathscr{R}e\, w^{(1)}Cw$. We then could write the differential equation in the form

$$C\,\frac{\partial}{\partial \bar{z}}\, w = Cf$$

noting that the differential operator $C\,\partial/\partial\bar{z}$ is "symmetric". This operator is not positive symmetric however.

To reduce the equation to a positive symmetric one we introduce new functions

$$\chi = \frac{\partial}{\partial z} w$$

and a diagonal matrix

$$p = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

in which the terms $p_{1,2}$ are to have continuous derivatives. For the system $\{w, \chi\}$ we then set up the system of equations

$$2\begin{pmatrix} -pC\dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial \bar{z}} + pC \\ -\dfrac{\partial}{\partial z} & 1 \end{pmatrix}\begin{pmatrix} w \\ \chi \end{pmatrix} = 2\begin{pmatrix} -pCf - \dfrac{\partial}{\partial z} f \\ 0 \end{pmatrix}$$

which is satisfied for a solution of $\partial w/\partial\bar{z} = f$ with $\partial w/\partial z = \chi$ by virtue of $\partial/\partial x_1 = (\partial/\partial z)+(\partial/\partial\bar{z})$ and $(\partial/\partial z)(\partial/\partial\bar{z}) = (\partial/\partial\bar{z})(\partial/\partial z)$. The quantities $p_{1,2}$ and χ_1 are to be reflected as even functions in $x_2 - x_2^{\pm}$, while χ_2 is to be an odd function.

This system is symmetric positive since $\partial/\partial z$ is adjoint to $\partial/\partial\bar{z}$ provided the functions $p_{1,2}$ are so chosen that the matrix $\kappa = C\,\partial p/\partial x_1$ is positive definite.

On the boundary $x_1 = x_1^{\pm}$ the matrix β becomes

$$\pm\frac{1}{2}\begin{pmatrix} -2pC & -1 \\ -1 & 0 \end{pmatrix};$$

setting

$$\mu = \frac{1}{2}\begin{pmatrix} 2p & -C \\ C & 0 \end{pmatrix}$$

we verify that the boundary condition $(\mu-\beta)(w, \chi) = 0$ is admissible provided p is positive on $x_1 = x_1^{\pm}$. Evidently, this boundary condition reduces to $w_1 = \chi_2 = 0$ on $x_1 = x_1^{+}$ and to $w_2 = \chi_1 = 0$ on $x_1 = x_1^{-}$, as desired.

It is then clear that the theory of Sections 9, 10 and 11 is applicable. Therefore a strong—and even strongly differentiable—solution exists, provided f has square integrable first and second derivatives.

Thus the function $\partial w/\partial z = \chi$ admits $\partial/\partial\bar{z}$ strongly. Hence $\partial w/\partial\bar{z}$ admits $\partial/\partial z$ strongly and $\partial^2 w/\partial z\,\partial\bar{z} = \partial^2 w/\partial\bar{z}\,\partial z$. Consequently, the function $g = \partial w/\partial\bar{z} - f$ satisfies the equation

$$\left(p + \frac{\partial}{\partial z} C\right) C \left(\frac{\partial w}{\partial \bar{z}} - f\right) = 0.$$

This is a symmetric positive equation for $C(\partial w/\partial \bar{z} - f)$ provided $p_1, p_2 > 0$ which we can achieve without restriction. The associated matrix β is $\pm C$ on $x_1 = x_1^{\pm}$. We set $\mu = 1$. Then the boundary conditions $(\partial w/\partial \bar{z} - f)_2 = 0$ on $x_1 = x_1^+$, and $(\partial w/\partial \bar{z} - f)_1 = 0$ on $x_1 = x_1^-$ are admissible. Now, in view of $f_2 = 0$, $\chi_2 = 0$, and $w_1 = 0$ on $x_1 = x_1^+$ we have

$$\left(\frac{\partial}{\partial \bar{z}} w - f\right)_2 = \left(\left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\right) w\right)_2 = \frac{\partial w_1}{\partial x_2} = 0 \qquad \text{on } x_1 = x_1^+ .$$

Similarly, we derive $(\partial w/\partial \bar{z} - f)_1 = 0$ from $f_1 = 0$, $\chi_1 = 0$, and $w_2 = 0$. It then follows that w satisfies the relation $\partial w/\partial \bar{z} = f$ on $x_1 = x_1^-$.

20. The Dirichlet Problem

In conclusion we want to show that the Dirichlet problem for an elliptic equation of the second order can be handled by our method. We consider the equation $Lu = f$ in a manifold $\mathscr{R}$ with boundary $\mathscr{B}$ and ask for a solution which vanishes at $\mathscr{B}$. With reference to each patch we write the equation in the form

$$\frac{\partial}{\partial x^\nu}\left(g^{\lambda\nu} \frac{\partial \phi}{\partial x^\lambda}\right) = (g^{\lambda\nu}\phi_{/\lambda})_{/\nu} = -f(x),$$

where $g^{\lambda\nu}$ is a positive definite contravariant tensor density; then we rewrite the equation as a symmetric positive system in the manner explained in the introduction and in Section 6. We introduce the functions $u^0 = \phi$, $u^\lambda = g^{\lambda\nu}\phi_{/\nu}$, and define the matrix $g_{\lambda\nu}$ by $g^{\lambda\iota} g_{\iota\nu} = \delta^\lambda_\nu (\det\{g_{\lambda\nu}\})^{1/2}$. We also introduce the functions $p^\lambda(x)$ with $p^\lambda_{/\lambda} > 0$ in $\mathscr{R}$, $p^\lambda n_\lambda > 0$ on $\mathscr{B}$ and set $p_\nu = g_{\nu\iota} p^\iota$. We can now write the equation in the form

$$K \begin{pmatrix} u_0 \\ \\ u^\nu \end{pmatrix} = \begin{pmatrix} -p^\iota \dfrac{\partial}{\partial x^\iota} & p_\nu - \dfrac{\partial}{\partial x^\nu} \\ -\dfrac{\partial}{\partial x^\lambda} & g_{\lambda\nu} \end{pmatrix} \begin{pmatrix} u^0 \\ \\ u^\nu \end{pmatrix} = \begin{pmatrix} f \\ \\ 0 \end{pmatrix}.$$

Here and in the following, the subscripts λ and ν run from 1 to m. Near the boundary $\mathscr{B}$, in a half-patch with $x^m = 0$ on $\mathscr{B}$, we have

$$2\beta = -\begin{pmatrix} p^m & \delta^m_\nu \\ \delta^m_\lambda & 0 \end{pmatrix}, \qquad 2\mu = \begin{pmatrix} p^m & -\delta^m_\nu \\ \delta^m_\lambda & 0 \end{pmatrix}$$

and hence

$$M = \begin{pmatrix} p^m & 0 \\ \delta^m_\lambda & 0 \end{pmatrix},$$

so that the boundary condition $Mu = 0$ reduces to $u^0 = 0$, as desired.

We proceed to define tangential operators $D = \{D_\sigma\}$ such that a relation $(D-t)K = \underset{1}{K}D$ holds with an appropriate matrix t and an appropriate operator $\underset{1}{K}$. We shall adopt a notation which somewhat differs from the one introduced in Section 8 and let the operator $\underset{1}{K}$ act only on the system $\{D_\sigma\}$ with $\sigma \neq 0$, from which $D_0 = 1$ is omitted; the relation in question will then have the form

$$(D-t)K = \underset{1}{K}D + r$$

where r is an appropriate matrix.

It is a remarkable fact that the matrices D_σ can be introduced in such a way that $\underset{1}{K}$ reduces to K, making it possible to write the last relation in the form

$$(D_\sigma - t_\sigma)K = KD_\sigma + r_\sigma \,.$$

In other words, the component $(\underset{1}{K}D)_\sigma$ of $\underset{1}{K}D$ depends only on D_σ itself and is given by KD_σ. As a consequence, the matrix $\underset{1}{\kappa}$ reduces to the matrix κ, already assumed to have a positive definite symmetric part. The condition that the matrix, denoted by $\underset{1}{\kappa}$ in Section 8 and now to be written as

$$\begin{pmatrix} \kappa & r \\ 0 & \kappa \end{pmatrix},$$

be positive definite can be achieved by multiplying all D_σ, and thus r, by a sufficiently small factor.

Furthermore we shall show that the relation $(D - t^{\mathscr{B}})M = \underset{1}{M}D$ at the boundary reduces to

$$(D_\sigma - t_\sigma^{\mathscr{B}})M = MD_\sigma \,,$$

so that $\underset{1}{M} = M$ and $\underset{1}{\mu} = \mu$ and hence $\underset{1}{\mu} + \underset{1}{\mu}' = \mu + \mu' \geqq 0$.

Thus, the fact that the solutions of elliptic equations of the form $(g^{\lambda\mu}\phi_{/\lambda})_{/\mu} = -f$ admit differentiation of every order is—in our framework—reduced to the existence of operators D with the properties described.

Consider first an interior patch and take the operator

$$\eta \frac{\partial}{\partial x^s} = \eta \begin{pmatrix} 1 & 0 \\ 0 & \delta_{\lambda\nu} \end{pmatrix} \frac{\partial}{\partial x^s} \,,$$

where η is the contribution of the partition of unity associated with the patch. We readily compute the commutator of this operator with K:

$$\left[\eta \frac{\partial}{\partial x^s}, K\right] = \begin{pmatrix} p^\iota \eta_{/\iota} \dfrac{\partial}{\partial x^s} - \eta p^\iota_{/s} \dfrac{\partial}{\partial x^\iota} & \eta p_{\nu/s} + \eta_{/\nu} \dfrac{\partial}{\partial x^s} \\ \eta_{/\lambda} \dfrac{\partial}{\partial x^s} & \eta g_{\lambda\nu/s} \end{pmatrix}.$$

By virtue of the relations

$$\begin{pmatrix} 0 & p_s - \dfrac{\partial}{\partial x^s} \\ 0 & g_{\lambda s} \end{pmatrix} = K \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\lambda s} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ -\dfrac{\partial}{\partial x^s} & g_{s\nu} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{s\nu} \end{pmatrix} K,$$

$$\begin{pmatrix} -\dfrac{\partial}{\partial x^s} & g_{s\nu} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{s\nu} \\ 0 & 0 \end{pmatrix} K$$

we may write the commutator relation in the form

$$\left[\eta \frac{\partial}{\partial x_s}, K\right] = -K \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\lambda s}\eta_{/\nu} \end{pmatrix} + \begin{pmatrix} 0 & \eta p^\nu_{/s} - \delta_{s\nu}\eta_{/\iota} p^\iota \\ 0 & -\delta_{s\nu}\eta_{/\lambda} \end{pmatrix} K - r_s$$

where

$$r_s = \begin{pmatrix} 0 & r_{0\nu} \\ 0 & r_{\lambda\nu} \end{pmatrix}$$

with

$$r_{0\nu} = \eta p^\iota g_{\iota\nu/s} + \eta_{/\nu} p_s + \eta_{/\iota} p^\iota g_{s\nu},$$

$$r_{\lambda\nu} = -\eta g_{\lambda\nu/s} + \eta_{/\nu} g_{\lambda s} + \eta_{/\lambda} g_{s\nu}.$$

Introducing the operators D and $D-t$ by

$$D_s = \eta \frac{\partial}{\partial x^s} + \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\lambda s}\eta_{/\nu} \end{pmatrix},$$

$$D_s - t_s = \eta \frac{\partial}{\partial x^s} - \begin{pmatrix} 0 & -\eta p^\nu_{/s} + \delta_{s\nu}\eta_{/\iota} p^\iota \\ 0 & \delta_{s\nu}\eta_{/\lambda} \end{pmatrix},$$

we arrive at the relations

$$(D_s - t_s)K = KD_s - r_s$$

as desired.

For a half-patch with $x^m = 0$ on the boundary $\mathscr{B}$ we modify the definition of the operators D_σ simply by substituting $x^m\eta$ for η in the definitions of D_m, $D_m - t_m$ and r_m. The tangential character of D_m is then assured. According to the definition of the matrix $t^{\mathscr{B}}$ given in Section 8 we have

$$t_m^{\mathscr{B}} - t_m = \eta \text{ and } t_s^{\mathscr{B}} - t_s = 0 \qquad \text{for } s \neq m.$$

A simple calculation shows that the relation

$$(D-t)_s M = MD_s$$

holds on $\mathscr{B}$.

The operators D_s are assigned to the patch $\mathscr{P}_\rho$ and should, therefore, be denoted by D_s^ρ. Assigning a number σ to the pair ρ, s and setting $D_\sigma = D_s^\rho$ we obtain the operators D_σ as desired.

The definition of the operators D is given in connection with a definite representation of the column u with reference to a patch, so chosen that u^λ, for $\lambda \neq 0$, is the density of the contravariant gradient of $u^0 = \phi$. The identification transformation for the column associated with different patches is thus the transformation of a contravariant density. This transformation is not of the character of those introduced in Section 7, which were orthogonal except for a factor. Here we make use of the fact that this restriction, made to simplify the description, was actually not necessary, see footnote 5 on page 363.

The final result of these considerations is that *the equation in question possesses a unique solution ϕ in $\mathfrak{H}$ with strong first and second derivatives in $\mathfrak{H}$.* The function ϕ and its first derivatives possess boundary values; those of ϕ and its tangential derivatives vanish. Here it is assumed that the identification transformation has continuous second derivatives; the coefficients $g_{\lambda\nu}$ are assumed to have continuous first derivatives. The right member f is to have strong first derivatives.

Note that we have established immediately the existence of second derivatives of the solution, and not, as in other procedures, the existence of just first derivatives as a first step. Our method does not seem suitable to answer this more modest question. The fact that our method immediately yields stronger results is, of course, one of the reasons for its complicated character. Indeed, among the many different approaches to the Dirichlet problem the one here presented seems to be the most intricate one. Still it is gratifying that it is possible to press this problem into the general framework of our theory.

Bibliography

[1] Friedrichs, K. O., *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc., Vol. 55, 1944, pp. 132–151.

[2] Schwartz, L., *Théorie des Distributions*, 1st Edit. 1950-1951, Hermann et Cie, Paris. 2nd Edit. 1957.

[3] Leray, J., (A) *Lectures on hyperbolic equations with variable coefficients*, Institute for Advanced Study, Princeton, Fall, 1952. (B) *On linear hyperbolic differential equations with variable coefficients on a vector space*, Contributions to the Theory of

Partial Differential Equations, Ann. Math. Studies, No. 33, Princeton University Press, 1954, pp. 201–210.

[4] Friedrichs, K. O., *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math., Vol. 7, 1954, pp. 345–392.

[5] Lax, P. D., *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math., Vol. 8, 1955, pp. 615–633.

[6] Phillips, R. S., (A) *Dissipative hyperbolic systems*, Trans. Amer. Math. Soc., Vol. 86, 1957, pp. 109–173. (B) *Dissipative hyperbolic systems of partial differential equations*. To appear.

[7] Lewy, H., *An example of a smooth linear partial differential equation without solution*, Annals of Mathematics, Vol. 66, No. 1, 1957, pp. 155–158.

[8] Lelong-Ferrand, J., *Application of Hilbert space method to the groups acting on a differentiable manifold*, Proc. Nat. Acad. Sci. U.S.A., Vol. 43, 1957, pp. 249–252.

With reference to the Tricomi equation, see

[9] Tricomi, F., *Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto*, Rend. Atti dell'Accad. Nazionale Lincei, Ser. 5, Vol. 14, 1923, pp. 134–247.

[10] Protter, M. H., *Uniqueness theorems for the Tricomi problem*, J. Ratl. Mech. Analysis, Vol. 2, 1953, pp. 107–114. Part II, Vol. 4, 1955, pp. 721–732.

[11] Morawetz, C. S., *A uniqueness theorem for Frankl's problem*, Comm. Pure Appl. Math., Vol. 7, 1954, pp. 697–703.

[12] Ou, S., and Ding, S., *Sur l'unicité du problème de Tricomi de l'équation d Chaplygin* (Chinese, French Summary), Acta Math. Sin., Vol. 5, 1955, pp. 697–703.

[13] Morawetz, C. S., *On the non-existence of continuous transonic flows past profiles*, Comm. Pure Appl. Math., Part I, Vol. 9, 1956, pp. 45–68. Part II, Vol. 10, 1957, pp. 107–131. Part III, Vol. 11, 1958, pp. 129–144.

[14] Morawetz, C. S., *A weak solution for a system of equations of elliptic-hyperbolic type*, Comm. Pure Appl. Math., Vol. 11, 1958, this issue.

[15] Bers, L., *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Surveys in Applied Mathematics, Vol. 4, John Wiley, New York, 1958.
This book contains an extensive bibliography. In addition to those quoted above see in particular the following papers quoted in its bibliography: Agmon [1], Bers [35], Bitsadze [48] and [365], Frankl [105], Gellerstedt [136], Morawetz [257] and [260], Protter [286], Filipov [370].

For clipped elleptic equations, see also

[16] Višik, M. I., *Boundary problems for elliptic equations degenerating on the boundary of the domain*, Mat. Sbornik N.S., Vol. 35, 1954, pp. 513–573, and references given there.

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