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Local Boundary Conditions for Dissipative Symmetric Linear Differential Operators*

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Introduction

This paper is concerned with the linear differential operator

$$L = \sum_{j=1}^{m} A^j \frac{\partial}{\partial x^j} + B,$$

where the A^j and B are $n \times n$ matrix-valued functions defined on a smooth domain G contained in R_m. The operator acts on vector-valued functions likewise defined on G and subject to boundary conditions which can be described as follows: At each point x of the boundary S of G, let $N(x)$ denote a linear subspace of constant dimension which is smoothly varying on S. Then we require $u(x)$ to lie in $N(x)$ at each boundary point x of S.

If we now set $n(x) = (n_1, \cdots, n_m)$ equal to the outer normal to G at $x \in S$, then the boundary matrix $A_n(x)$ is defined by

$$A_n = \sum_{j=1}^{m} n_j A^j.$$

Assuming A_n to be non-singular at all points of S, it is shown in Section 1 that any weak solution of $Lu = f$ satisfying the boundary conditions in a weak sense is actually a strong solution and satisfies the boundary conditions in the strong sense. The main tool used here is the notion of a mollifier which smoothes only the tangential direction and hence does not interfere significantly with the boundary values of the function. In Section 2 the same method is applied to the case where A_n is merely of constant rank near the boundary (but not necessarily non-singular). In this case it is shown that a weak solution of $Lu = f$ satisfying the boundary conditions in a weak sense is actually what we call a semi-strong solution satisfying the boundary conditions in a semi-strong sense.

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Next we restrict the A^j to be symmetric and the operator to be formally dissipative, i.e., we assume

$$D = B + B^* - \sum_{j=1}^{m} A^j_{x^j} \leqq 0, \qquad x \in G.$$

We also take $N(x)$ to be maximal non-positive at each x in S, i.e., we assume that

$$uA_n u \leqq 0, \qquad u \in N(x), \quad x \in S,$$

and that $N(x)$ is not properly contained in any other subspace having this property. Finally we suppose A_n to be of constant rank near S. Under these assumptions it is shown in Section 3 that

$$u - Lu = f$$

has a strong solution satisfying the boundary conditions in the strong sense for each square integrable function f. A solution to this problem has previously been obtained by K. O. Friedrichs [2] as a biproduct of his investigation into the smoothness properties of u as determined by those of $f = u - Lu$.

Following Friedrichs we are also able to extend the above result to the case where the boundary has corners provided that A_n is either positive or negative on one of the sides of the corner. In addition we have developed a technique for extending the result in the presence of certain unessential boundary points. In particular we are able to establish the uniqueness of a strong solution to the Frankl problem for the Tricomi equation and so extend the work of C. S. Morawetz [3] who has obtained the existence, but not uniqueness, of a weak solution to this problem.

1. Weak Implies Strong in the Regular Case

Let L be a first order matrix partial differential operator

$$L = \sum_{j=1}^{m} A^j \frac{\partial}{\partial x^j} + B, \tag{1.1}$$

where the coefficients A^j and B are $n \times n$ matrices not required to be symmetric, the A^j being smooth[1] functions of x and the B being piecewise continuous in x. The domain G of the independent variable x is assumed to have a boundary S which is of class C^2. The regular case is characterized by the fact that S is *nowhere characteristic* with respect to the operator L, i.e., it is assumed that the boundary matrix A_n is nowhere singular; here

[1]A function will be called smooth if it is continuous and piecewise continuously differentiable.

$$A_n = \sum_{j=1}^{m} n_j A^j, \tag{1.2}$$

$(n_1, \cdots, n_m)$ being the normal to the surface S.

We denote the *formal* adjoint of L by L^*:

$$L^* = -\sum_{j=1}^{m} \frac{\partial}{\partial x^j} A^j + B^*. \tag{1.1'}$$

Further, if we denote by parentheses the L_2 scalar product over G of vector-valued functions, then according to Green's formula

$$(v, Lu) - (u, L^* v) = \int_S v A_n u \, dS \tag{1.3}$$

is valid for all vector-valued functions u and v which are, say, absolutely continuous in each variable separately, have square integrable first derivatives, and are continuous up to the boundary in G.

In this paper we are interested in functions subject to certain linear homogeneous boundary conditions. To define these boundary conditions we associate with each boundary point x a linear subspace $N(x)$ of n dimensional space whose dimension is the same at all points of S and which varies smoothly with x, i.e., $N(x)$ is spanned by vectors which vary smoothly with x. Then the *boundary condition* is

$$u(x) \quad \text{in } N(x) \tag{1.4}$$

for all boundary points x.

The *adjoint boundary condition* is

$$v(x) \text{ in } P(x) \tag{1.4'}$$

for all boundary points x, where $P(x)$ is the orthogonal complement of $A_n(x)N(x)$.

If u satisfies the boundary conditions (1.4) and v the adjoint boundary conditions (1.4'), then the boundary terms in Green's formula vanish:

$$(v, Lu) - (L^* v, u) = 0. \tag{1.3'}$$

DEFINITION. Let f be a square integrable function; the square integrable function u is said to be a weak solution of

$$Lu = f$$

and to satisfy in the weak sense the boundary condition (1.4) if it belongs to the domain of the adjoint of L^* defined on all smooth functions satisfying the adjoint boundary conditions (1.4'), i.e., if the relation

$$(v, f)-(L^*v, u) = 0 \tag{1.5}$$

holds for all such functions v.

DEFINITION. Let u and f be square integrable functions; u is said to be a strong solution of

$$Lu = f,$$

satisfying the boundary conditions (1.4) in the strong sense, if the pair $\{u, f\}$ belongs to the closure of the graph of L, more precisely, if u is the limit in the L_2 norm of a sequence of functions $\{u_k\}$ which are continuous up to the boundary, satisfy the boundary conditions (1.4), have square integrable first derivatives, and such that

$$f_k \equiv Lu_k$$

tends to f in the L_2 norm.

Applying Green's formula (1.3′) to u_k and any smooth v satisfying the adjoint boundary conditions, and letting k tend to ∞, we obtain the result that if u satisfies in the strong sense the above differential equation and boundary conditions, then it also satisfies them in the weak sense. In the case where no boundary conditions are imposed Friedrichs has shown in [1] that, conversely, if u satisfies the differential equation $Lu = f$ in the weak sense, then it also satisfies it in the strong sense. In this section we extend this result to include boundary conditions.

THEOREM 1. *Let u be a square integrable function which satisfies equation* (1.1) *and boundary conditions* (1.4) *in a domain G whose boundary is not characteristic; then the equation and boundary conditions are satisfied in the strong sense.*

The first step is to localize the problem; this is easily accomplished. Let $\{\phi_i(x)\}$ be a smooth partition of unity for G, i.e., $\sum \phi_i = 1$ in G. Define u_i as $\phi_i u$; it is easy to show that if u is a weak solution of $Lu = f$, then u_i is a weak solution of $Lu_i = f_i$, where

$$f_i = \phi_i f + u \sum A^j \frac{\partial \phi_i}{\partial x^j}.$$

Furthermore u_i satisfies in the weak sense the same boundary conditions as u does.

From now on we suppose that u vanishes outside of a set with a small diameter. If this set does not intersect the boundary, then the above mentioned Friedrichs result suffices to establish Theorem 1.1 for u. On the other hand, if this set contains a portion of the boundary S, then we introduce new independent variables in terms of which the relevant portion of the boundary becomes a hyperplane. Since u vanishes outside of the small set in question, we may as well assume that the domain G is a slab contained between two hyperplanes $y = 0$ and $y = 1$, and that u vanishes for say $y < \frac{1}{2}$.

Since the boundary space $N(x)$ was assumed to vary smoothly, we can introduce new dependent variables in terms of which the boundary condition becomes

$$u^1(x) = \cdots = u^p(x) = 0, \tag{1.6}$$

where p is the co-dimension of $N(x)$.

We have assumed at the outset that the boundary of G is not characteristic; this means that the coefficient of u_y is non-singular on (and hence near) the boundary. By choosing the set, outside of which u vanishes, sufficiently small, we see that this coefficient will be non-singular on the support of u; however, outside of the support of u we may alter the coefficients of L at will, so we may as well assume that the coefficient of u_y is non-singular throughout the slab G. Multiplying the operator by the inverse of this coefficient we can bring it into the form

$$L = \frac{\partial}{\partial y} + M, \tag{1.7}$$

where M is a first order differential operator in the tangential variables x which varies of course with y.

The adjoint of the boundary condition (1.6) for the operator (1.7) is clearly

$$v^{p+1} = \cdots = v^n = 0. \tag{1.6'}$$

We shall now deduce some properties of u merely from the fact that u satisfies in the weak sense the equation $Lu = f$, i.e. that Green's formula (1.5) holds for all smooth functions v which vanish outside of compact subsets of G.

The first conclusion is: *u satisfies the equation*

$$Lu = f$$

in the sense of the theory of distributions.

For further analysis it is convenient to regard the functions defined in the slab G as functions of the single variable y whose values, however, lie in various spaces of functions of x. For this purpose we introduce the following scalar products, norms and spaces of functions of the x variables:

a) H_0 is the space of square integrable vector-valued functions of x; the scalar product in H_0 is denoted by $[r, s]$, the norm $[r, r]^{\frac{1}{2}}$ by $|r|$.

b) H_1 is the space of vector-valued functions of x with square integrable first derivatives. The norm in H_1 is denoted by $|s|_1$, and is defined, as usual, by

$$|s|_1^2 = |s|^2 + \sum |Ds|^2,$$

the summation being over all first order partial derivatives Ds of s with respect to the x-variables.

c) H_{-1} is the dual space to H_1 with respect to the scalar product of H_0; the norm in H_{-1} is defined by

$$|s|_{-1} = \sup_{r \text{ in } H_1} \frac{[r, s]}{|r|_1}.$$

H_{-1} is a space of distributions. We shall have occasion to use the Schwartz inequality

$$[r, s] \leqq |r|_1 |s|_{-1}$$

valid for any r in H_1, s in H_{-1}.

We shall denote by $L_2(H_0)$, $L_2(H_1)$, $L_2(H_{-1})$ the space of square integrable functions of y, $0 < y < 1$, with values lying in the spaces H_0, H_1 and H_{-1}, respectively.

The space of square integrable functions u over the slab G will be denoted by L_2, and the L_2 norm over G by $||u||$.

We think of all these spaces as being obtained by *completion* in their respective norms of the space of smooth functions defined in the slab G having bounded support.

The following propositions about these spaces are self-evident for continuous functions and follow for the completed spaces by continuity with respect to the appropriate norm:

$L_2(H_0)$ is isometrically isomorphic to L_2.

L_2 is a subspace of $L_2(H_{-1})$.

$L_2(H_{-1})$ is a subspace of the space of distributions over the slab G.

The space of functions over the slab G with square integrable first derivatives is a subspace of $L_2(H_1)$.

Let M denote a first order partial differential operator with respect to the x-variables whose coefficients are bounded, then

M maps H_1 boundedly into H_0.

From this we conclude

M maps $L_2(H_1)$ boundedly into $L_2(H_0)$.

If the principal coefficients of M have piecewise continuous partial derivatives of first order with respect to x, then M^* has bounded coefficients and it follows from the previous result that M^* maps H_1 into H_0. From this it follows by duality that *M maps H_0 boundedly into H_{-1}.*

We sketch the proof:

Integrating by parts and using the Schwarz inequality we have

$$[r, Ms] = [M^*r, s] \leqq |M^*r||s| \leqq \text{const } |r|_1|s|.$$

Since by definition

$$|Ms|_{-1} = \sup \frac{[r, Ms]}{|r|_1},$$

we have

$$|Ms|_{-1} \leqq \text{const } |s|.$$

From this result we conclude

M maps $L_2(H_0)$ boundedly into $L_2(H_{-1})$.

The following result is an immediate consequence of the Schwartz inequality:

If w belongs to $L_2(H_{-1})$, v to $L_2(H_1)$, then $[w(y), v(y)]$ is a square integrable scalar function of y.

Combining this with previous results we have:

If u belongs to $L_2(H_0)$, v to $L_2(H_1)$, then both $[Mu, v]$ and $[u, M^*v]$ are square integrable scalar functions of y and they are equal.

LEMMA 1.1. *Suppose that both u and f belong to L_2 and $Lu = f$ weakly; then $u(y)$ considered as a function of y with values in H_{-1} is continuous in $0 \leqq y \leqq 1$ and u_y lies in $L_2(H_{-1})$. Furthermore, Greens' formula*

$$[u(1), v(1)] - [u(0), v(0)] = (f, v) - (u, L^*v) \tag{1.8}$$

holds for all smooth functions $v(x, y)$ with bounded support.

Proof: Let $v(x, y)$ be any smooth function with bounded support; form the expression

$$\int_0^{y_1} \{[f, v] - [u, L^*v]\} dy, \tag{1.9}$$

y_1 any value between 0 and 1. If integration by parts were permissible, we could easily show that the above expression is $[u(y_1), v(y_1)]$. Instead we proceed as follows:

If $v(y_1) = 0$, then we can continue v as identically zero for $y > y_1$ and rewrite (1.9) as

$$(f, v) - (u, L^*v)$$

which is zero since $Lu = f$ weakly. It follows then that if v_1 and v_2 are two test functions which agree at $y = y_1$, then the value of (1.9) for v_1 and v_2 is the same. Take now $v = r$ to be independent of y. Then $L^*v = M^*r$; applying the Schwarz inequality we see that (1.9) is a bounded linear functional of r in the H_1 norm; by the duality of H_1 and H_{-1} this linear functional can be represented as

$$[\tilde{u}, r],$$

where $\tilde{u} = \tilde{u}(y_1)$ is some element of H_{-1}.

We claim that $\tilde{u}(y)$ depends continuously on y; for, by the Schwarz inequality

$$[\tilde{u}(y_1)-\tilde{u}(y_2)r] = \int_{y_1}^{y_2} \leqq |y_1-y_2|^{1/2} \text{ const. } |r|_1,$$

so by the definition of the H_{-1} norm

$$|\tilde{u}(y_1)-\tilde{u}(y_2)|_{-1} \leqq \text{const. } |y_1-y_2|^{1/2}.$$

Next we verify that $\tilde{u}(y)$, regarded as a distribution in the slab is equal to $u(y)$; for this it is sufficient to show that

$$(\tilde{u}, v) = (u, v)$$

for all smooth v. By definition of $\tilde{u}$ we have

$$(\tilde{u}, v) = \int_0^1 [\tilde{u}(y_1), v(y_1)]dy_1 = \int_0^1 \int_0^{y_1} [f, v]-(u, L^*v]dy\,dy_1,$$

which can be transformed into

$$\int_0^1 [f(y), (1-y)v(y)]-[u(y), (1-y)L^*v]dy.$$

Define

$$(1-y)v = w.$$

Noting that

$$L^*w = (1-y)L^*v+v$$

we can rewrite the above integral as

$$(f, w)-(u, L^*w)+(u, v).$$

Since the function w does vanish at $y = 1$ and since u satisfies weakly $Lu = f$, the first two terms in the above expression add up to zero. This shows that $(\tilde{u}, v) = (u, v)$ for all smooth v, and so $u \equiv \tilde{u}$.

Relation (1.8) can be obtained from (1.9) by putting $y_1 = 1$. Since u is a weak solution of $Lu = f$ and satisfies in the weak sense the boundary conditions (1.6), the relation (1.5) holds for all smooth v satisfying the adjoint boundary conditions (1.6′). Comparing (1.5) with (1.8) we see that

$$[u(1), v(1)] = 0$$

for all smooth vector valued functions $v(1)$ of x whose last $n-p$ components are zero. From this we conclude the

COROLLARY TO LEMMA 1.1. *If u is a weak solution and satisfies in the weak sense the boundary conditions* (1.6), *then the first p components of $u(1)$ are zero as distributions in the x-variables.*

We are now ready to proceed with the proof of the main theorem.

Let $j(x)$ be any infinitely differentiable non-negative function with compact support and total mass one:

$$\int j(x)dx = 1.$$

The functions j_ε are defined, for positive ε, as

$$j_\varepsilon(x) = \varepsilon^{-k} j\left(\frac{x}{\varepsilon}\right),$$

where k is the dimension of the x-space, in this case $m-1$.

The operator J_ε (called mollifier) is defined as convolution with j_ε:

$$J_\varepsilon s = j_\varepsilon * s = \int j_\varepsilon(z)s(x-z)dz.$$

The following lemmas are classical:

LEMMA 1.2. *J_ε maps H_0 into H_0, and its norm is equal to* 1.

COROLLARY. *J_ε maps L_2 into L_2 and its norm is equal to* 1.

LEMMA 1.2′. *Denote partial differentiation with respect to any of the x-variables by D; DJ_ε maps H_0 into H_0 and L_2 into L_2.*

LEMMA 1.3. *For any s in H_0, $|J_\varepsilon s-s|$ tends to zero as ε tends to zero.*

COROLLARY. *For any u in L_2, $\|J_\varepsilon u-u\|$ tends to zero as ε tends to zero.*

Lemma 1.3 is easily verified if s and u are continuous functions; to prove it for any square integrable s and u, it is sufficient to approximate them by continuous functions and use Lemma 1.2.

The key results in Friedrichs' paper [1] are the following two lemmas:[2]

[2]Friedrichs' argument can be extended in two ways: (i) the A^j need only be smooth in $\bar{G}$ and (ii) the boundary $\dot{G}$ need only be Lipschitz continuous. (i) is readily verified simply by checking through Friedrichs' proof of the main theorem in [1] which actually holds under the hypothesis of (i). As for (ii), we note that the theorem deals with a local property. Hence, suppose we have a boundary patch and after a suitable rotation of coordinates the patch can be represented by

$$f(x') = f(x^1, x^2, \cdots, x^{m-1}) < x^m < 1, |x'| = \left[\sum_{i=1}^{m-1} (x^i)^2\right]^{1/2} < 1,$$

where f satisfies the condition

$$|f(x')-f(\bar{x}')| \leqq M|x'-\bar{x}'|$$

for arbitrary x', $\bar{x}'$ of distance less than 1 from the origin and fixed $M \geqq 1$. Let z be a function in the domain of L^* which vanishes near the inner portions of the boundary of the patch. Then in terms of the Friedrichs' mollifier (see Section 2)

LEMMA 1.4. *J_ε "almost commutes" with any first order operator M with piecewise continuous B and smooth A^i's; i.e., the commutator*

$$MJ_\varepsilon - J_\varepsilon M$$

maps H_0 into H_0 and is bounded by a constant independent of ε.

COROLLARY. *The operators $\{MJ_\varepsilon - J_\varepsilon M\}$ map L_2 into L_2 and are uniformly bounded in norm.*

LEMMA 1.5. *For every s in H_0,*

$$|\{MJ_\varepsilon - J_\varepsilon M\}s|$$

tends to zero with ε.

COROLLARY. *For every u in L_2,*

$$||\{MJ_\varepsilon - J_\varepsilon M\}u||$$

tends to zero with ε.

Lemma 1.5 is easily verified if s and u are smooth and have compact support; it follows for any square integrable s and u by approximating them by smooth functions, and using Lemma 1.4.

Suppose now that u and $Lu = f$ are square integrable, and the first p components of u vanish as distributions at $y = 1$. We define the sequence of functions $\{u_\varepsilon\}$ as

$$u_\varepsilon = J_\varepsilon u.$$

We claim that

i) u_ε tends to u and $f_\varepsilon \equiv Lu_\varepsilon$ to f in the L_2 norm,

ii) u_ε has square integrable first derivatives,

iii) u_ε is continuous in the closure of G and satisfies the boundary conditions (1.7) at every point of the boundary.

These properties of the approximating sequence establish u as a strong solution of $Lu = f$, satisfying in the strong sense the boundary conditions.

i) The corollary of Lemma 1.3 implies that u_ε tends to u in the L_2 norm. Next we compute $f_\varepsilon = Lu_\varepsilon$; using the fact that the operators J_ε and $\partial/\partial y$ commute:

$$j_\varepsilon(x) = \varepsilon^{-m} \prod_{i=1}^{m} j(x^k/\varepsilon)$$

we set

$$z_\varepsilon(x) = \int_G j_\varepsilon(x - \bar{x} + \delta_\varepsilon) z(\bar{x}) d\bar{x},$$

where $\delta_\varepsilon = (0, 0, \cdots, 0, 2M\varepsilon)$. Then for any x in the patch we see that $j_\varepsilon(x - \bar{x} + \delta_\varepsilon)$ is different from zero only for $\bar{x}^m > f(\bar{x}')$. With this choice of mollifier the Friedrichs' argument goes through essentially as before to give the desired result.

$$
\begin{aligned}
f_\varepsilon = Lu_\varepsilon = LJ_\varepsilon u &= \left(\frac{\partial}{\partial y} + M\right) J_\varepsilon u \\
&= J_\varepsilon Lu + \{MJ_\varepsilon - J_\varepsilon M\}u \\
&= J_\varepsilon f + \{MJ_\varepsilon - J_\varepsilon M\}u.
\end{aligned}
$$

It follows now from the corollaries of Lemmas 1.3 and 1.5 that f_ε tends to f in the L_2 norm.

ii) According to Lemma 1.2′ all x partial derivatives $Du_\varepsilon = DJ_\varepsilon u_\varepsilon$ of u_ε are square integrable. Since

$$\frac{\partial u_\varepsilon}{\partial y} = f_\varepsilon - Mu_\varepsilon$$

and, as we have already shown, both f_ε and Mu_ε are square integrable, it follows that $\partial u_\varepsilon / \partial y$ is also square integrable.

iii) We shall show that u_ε is continuous separately in y and in x, uniformly for fixed ε.

For fixed x, $j_\varepsilon(x-z)$ as function of z belongs to H_1; denoting this element of H_1 by $j_\varepsilon(x)$ we can write

$$u_\varepsilon(x, y) = [j_\varepsilon(x), u(y)],$$

so we have

$$u_\varepsilon(x, y) - u_\varepsilon(x, y') = [j_\varepsilon(x), u(y) - u(y')].$$

By the Schwartz inequality this is less than

$$|j_\varepsilon|_1 |u(y) - u(y')|_{-1}.$$

Since according to Lemma 1.1, $u(y)$ as function of y with values in H_{-1} is continuous, it follows that u_ε is continuous in y, uniformly in x for fixed ε.

Likewise

$$u_\varepsilon(x, y) - u_\varepsilon(x', y) = [j_\varepsilon(x) - j_\varepsilon(x'), u(y)]$$

which by the Schwartz inequality is less than

$$|j_\varepsilon(x) - j_\varepsilon(x')|_1 |u(y)|_{-1}.$$

Since $u(y)$—being continuous—is bounded in the H_{-1} norm, and since $j(x)$ is continuously differentiable in the H_1 norm, it follows that u_ε is a continuous function of x, uniformly in y for fixed ε.

According to the corollary of Lemma 1.1 the first p components of $u(1)$ are zero as distributions. Since the value of any component of u_ε at a point on the boundary is an integral involving the corresponding component of $u(1)$, it follows that the first p components of u_ε vanish for $y = 1$. This completes the proof of Theorem 1.1.

Lemma 1.1 is a special and more specific instance of the following proposition:

Let L be a differential operator with C^∞ coefficients for which the hyperplanes $y =$ const. are not characteristic. Let u be a distribution solution of

$$Lu = f,$$

f in C^∞. Then u, regarded as a function of y, whose values are distributions in x, belongs to C^∞.

2. Weak Implies Semi-Strong in the Singular Case

In this section we shall extend the previous development to the case in which the boundary matrix A_n is possibly singular but of *constant rank in a neighborhood of the boundary* S. For this case the same approach yields a somewhat weakened version of the theorem of the previous section.

We now assume that the linear subspace $N(x)$ defining the linear homogeneous boundary condition not only has the same dimension at all points x of S and varies smoothly with x on S, but that it contains the null space of $A_n(x)$ at each point x of S (This condition is automatically satisfied in the applications considered in Section 3.) We note that $N(x)$ can then be described as the orthogonal complement of $A_n'(x)P(x)$, where $P(x)$ defines the adjoint boundary condition as in (1.4′).

DEFINITION. Let u and f be square integrable functions; u is said to be a semi-strong solution of

$$Lu = f$$

satisfying the boundary conditions (1.4) in the semi-strong sense if u is the limit in the L_2 norm of a sequence of functions $\{u_k\}$ such that (i) u_k is a weak solution of $Lu_k = f_k$, where f_k tends to f in the L_2 norm, (ii) there exist patches $G_1, \cdots, G_q$ covering G and functions $u_{k,i}$ with support in G_i such that $u_k = \sum_{i=1}^q u_{k,i}$, (iii) the $u_{k,i}$ are continuously differentiable for interior patches G_i, and (iv) for boundary patches G_i and suitable local coordinates in G_i, A_n is of the form (2.1), $A_n u_{k,i}$ is continuous up to the boundary and orthogonal to $P(x)$ on the boundary, $u_{k,i}$ is continuous and has square integrable first derivatives in the tangential directions, and in the normal direction $A_n u_{k,i}$ is continuous and has a square integrable first derivative.

THEOREM 2.1. *If u satisfies the equation $Lu = f$ and boundary conditions* (1.4) *in the weak sense, then it also satisfies them in the semi-strong sense.*

Proof: As before the discussion can be localized and because of the previously mentioned Friedrichs result [1] for interior patches, it will suffice to consider only the situation where G is a slab contained between two hyperplanes $y = 0$ and $y = 1$, and where u vanishes for $y < \frac{1}{2}$ and has a bounded support in the slab. Since by hypothesis A_n is of constant rank near S, we may, without loss of generality, also assume that A_n is of constant rank r

throughout the slab. A smooth orthogonal transformation of the dependent variable will therefore bring A_n into the form

$$A_n = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix}, \tag{2.1}$$

where A_{11} is an $r \times r$ matrix-valued function on G.

Finally we transform the first r components of the dependent variable so that the boundary conditions become

$$u^1(x) = \cdots = u^p(x) = 0 \qquad \text{on } y = 1, \tag{2.2}$$

where again p is the co-dimension of $N(x)$. Multiplying the operator by a suitable smooth non-singular matrix-valued function[3] will bring Lu into the form

$$Lu = \frac{\partial u_1}{\partial y} + Mu, \tag{2.3}$$

where the first r components of u_1 are the same as those of u but the remaining components vanish.

The adjoint of the boundary condition (2.2) for the operator (2.3) is now

$$v^{p+1}(x) = \cdots = v^r(x) = 0 \qquad \text{for } y = 1. \tag{2.3'}$$

LEMMA 2.1. *Suppose that both u and f belong to L_2 and $Lu = f$ weakly, then $u_1(y)$ considered as a function of y with values in H_{-1} is continuous in $0 \leqq y \leqq 1$ with u_{1_y} in $L_2(H_{-1})$, Furthermore, Green's formula*

$$[u_1(1), v(1)] - [u_1(0), v(0)] = (f, v) - (u, L^*v) \tag{2.4}$$

holds for all smooth functions v with bounded support.

The proof proceeds more or less as before if we replace u by u_1 in the appropriate places. For instance in the present case

$$L^*w = \begin{cases} v_1 + (\bar{y} - y)L^*v & \text{for } y \leqq \bar{y}, \\ 0 & \text{for } y > \bar{y}, \end{cases}$$

so that (1.9) becomes

$$[u_1, v] = [u, v_1] = \int_0^{\bar{y}} \{[f, v] - [u, L^*v]\} dy.$$

[3]Multiplying A_n of (2.1) by a suitable orthogonal matrix-valued function on the left will bring it into the form

$$A'_n = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where A_1 is an $r \times r$ non-singular matrix-valued function on G. Finally multiplying on the left by

$$\begin{pmatrix} A_1^{-1} & 0 \\ 0 & I \end{pmatrix}$$

will bring the operator L into the form (2.2).

Again we note that

$$u_{1y} = f - Mu$$

belongs to $L_2(H_{-1})$.

We may also conclude as before that the corollary to Lemma 1.1 holds in this more general case.

The proof of the main theorem now proceeds much as before. We show that

i) u_ε tends to u and $f_\varepsilon \equiv Lu_\varepsilon$ to f in the L_2 norm,

ii) u_ε is absolutely continuous and has square integrable derivatives in the x-directions, whereas $(u_\varepsilon)_1$ is absolutely continuous and has a square integrable derivative in the y-direction,

iii) $(u_\varepsilon)_1$ is continuous in the closure of G and satisfies the boundary conditions (2.2) at every point of the boundary,

iv) u_ε satisfies the boundary conditions (2.2) at every point of the boundary.

We shall remark only on points of difference between the proofs of the above propositions and the proofs of their analogues in Section 1. In the case of i) we note that

$$\frac{\partial}{\partial y}(J_\varepsilon u)_1 = \frac{\partial}{\partial y}(J_\varepsilon u_1) = J_\varepsilon u_{1y},$$

where the last equality essentially asserts the equivalence between $u_1(y)$ being differentiable in the weak and strong H_{-1} topologies and follows from Lemma 2.1. In proposition ii), the square integrability of $(u_\varepsilon)_{1y}$ follows from

$$\frac{\partial}{\partial y}(u_\varepsilon)_1 = f_\varepsilon - Mu_\varepsilon .$$

Proposition iii) follows as before if we simply replace u_ε by $(u_\varepsilon)_1$ in the previous argument. Finally u_ε satisfies the boundary conditions whenever $(u_\varepsilon)_1$ does, since these conditions are independent of the last $n-r$ components.

3. Symmetric Dissipative Operators

A first order operator L is called symmetric if its coefficients A^j are (real) symmetric matrices. Symmetric operators with smooth A^j and piecewise continuous B, satisfy the following identity:

$$uLu = \tfrac{1}{2}\{uA^j u\}_{x^j} + uKu,$$

where $K = B - \frac{1}{2}\sum A^j_{x^j}$. Integrating the above relation over G we obtain the so-called energy identity

$$(u, Lu) = (u, Ku) + \int uA_n u\, dS, \tag{3.1}$$

where the parentheses denote the L_2 scalar product for vector-valued functions defined on G, and A_n denotes as before the boundary matrix (1.2).

This formula is valid for all functions u which have continuous first derivatives in G and which are continuous up to the boundary of G; it is then also valid for all u which have square integrable first derivatives in G and are continuous up to the boundary, since such functions can be approximated, through mollification, by continuously differentiable ones.

To be able to deal with the case in which A_n is singular on the boundary we need the following result.

Let u be a square integrable function which is *semi-strong* in the sense of Section 2, i.e., has square integrable first derivatives in the interior, $A_n u$ is continuous up to the boundary and has square integrable first derivatives while u itself has square integrable first derivatives in the tangential directions.

LEMMA. *The energy identity* (3.1) *holds for semi-strong functions.*

Proof: First we localize the problem. Let $\{\phi_i\}$ be a smooth partition of unity in G in this sense:

$$\sum_i (\phi_i)^2 \equiv 1 \qquad \text{on } G.$$

Define

$$u_i = \phi_i u.$$

We claim that

$$\sum (u_i, Lu_i) = (u, Lu);$$

for,

$$\begin{aligned}(u_i, Lu_i) &= (\phi_i u, L\phi_i u) = (\phi_i u, \phi_i Lu + \phi_{i_{x^j}} A^j u)\\ &= (\phi_i^2 u, Lu) + \tfrac{1}{2}((\phi_i^2)_{x^j} u, A^j u).\end{aligned}$$

Sum with respect to i and use the relation $\sum \phi_i^2 \equiv 1$; the sum of the first terms is (u, Lu) the sum of the second terms is zero. Similarly we find that

$$\sum (u_i, Ku_i) + \sum \int u_i A_n u_i = (u, Ku) + uA_n u,$$

so we conclude: if the energy relation (3.1) holds for the functions u_i, then it holds for u.

Since u is assumed to be smooth on interior patches, there is no difficulty in deriving the energy relation for u_i whose support lies in the interior of G. Consider now $u_i = u$ whose support lies in a boundary patch; change coordinates and map the patch into a slab with tangential coordinates x and normal coordinate y. Denote by u_1 the projection of u into the range of A_n. Clearly,

$$uA_n u = u_1 A_n u_1.$$

By assumption, $A_n u$—and therefore also u_1—is continuous up to the boundary and has square integrable first derivatives.

Consider now the divergence expression

$$uLu = \tfrac{1}{2}\{uA_n u\}_y + \{uA^j u\}_{x^j} + uKu.$$

Since $uA_n u = u_1 A_n u_1$ is continuous up to the boundary and is differentiable with respect to y, the first term can be integrated with respect to y. Since u was assumed differentiable with respect to the tangential variable x^j, the integration of the rest of the terms can be performed too, yielding (3.1).

If we impose the additional condition

$$D = K + K^* \leqq 0, \tag{3.2}$$

the operator L becomes *formally dissipative*. Friedrichs [2], dealing with $I-L$, calls the latter operator positive if (3.2) is satisfied.

LEMMA 3.1. *If L is formally dissipative, then so is its adjoint L^*.*

Proof: It follows from the formula (1.1) for L^* that the K corresponding to L^* is just the adjoint of the K for L and hence L and L^* have the same D.

DEFINITION. The boundary condition

$$u(x) \text{ in } N(x) \qquad \text{for } x \text{ on } S$$

is called non-positive if the matrix $A_n(x)$ is non-positive over $N(x)$, i.e. if

$$uA_n u \leqq 0 \tag{3.3}$$

for all $u(x)$ in $N(x)$, $x \in S$. A formally dissipative operator with domain limited by non-positive boundary conditions will be called *dissipative*.

THEOREM 3.1. *Let L be a formally dissipative operator, u a function with square integrable first derivatives which is continuous up to the boundary and satisfies non-positive boundary conditions. For such u the inverse of $I-L$ is bounded, in fact the following inequality holds*:

$$||u|| \leqq ||(I-L)u||. \tag{3.4}$$

Proof: It follows from (3.1), (3.2) and (3.3) that

$$(u, Lu) + (Lu, u) \leqq 0.$$

Consequently

$$2||u||^2 \leqq (u, (I-L)u) + ((I-L)u, u) \leqq 2||u||\,||(I-L)u||,$$

which yields (3.4).

Recalling the definition of strong solutions we can derive the following corollaries by passing to the limit:

COROLLARY 1. *The inequality* (3.4) *holds for all strong* (*semi-strong*) *solutions of* $u-Lu = f$ *which satisfy in the strong sense non-positive boundary conditions.*

COROLLARY 2. *If* u *is a strong* (*semi-strong*) *solution of* $u-Lu = 0$ *and satisfies non-positive boundary conditions, then* $u = 0$.

COROLLARY 3. *The equation*

$$u-Lu = f$$

has at most one strong (*semi-strong*) *solution satisfying given non-positive boundary conditions.*

We turn now to the question of existence of strong solutions. Clearly, we cannot expect the equation $u-Lu = f$ to have a solution for every f if we impose too many boundary conditions. This is certain to be the case if we can enlarge the domain of the operator and still maintain uniqueness, thus in particular if the spaces $N(x)$ can be enlarged without violating non-positivity. Hence we assume:

The boundary spaces $N(x)$ *are maximal non-positive.* We note that when $N(x)$ is maximal, then it necessarily contains the null space of $A_n(x)$ at each x on S.

The following lemma will be needed (cf. R. Phillips [4], Lemma 3.2 and K. O. Friedrichs [2], Section 5):

LEMMA 3.2. *If the boundary conditions* (1.4) *are maximal non-positive for* L, *then the adjoint boundary conditions* (1.4') *are non-positive for* L^*.

Proof: According to formula (1.1') the boundary matrix for L^* is equal to $-A_n$. Hence it suffices to show that vA_nv is non-negative for each v satisfying the adjoint boundary condition. Suppose the contrary were true, i.e., for such a v suppose that

$$vA_nv < 0.$$

Consider the linear space $N \oplus v$ spanned by N and v; it consists of elements of the form

$$u+av, \qquad u \text{ in } N \text{ and real } a.$$

Now

$$(u+av)A_n(u+av) = uA_nu+2auA_nv+a^2vA_nv.$$

Since v satisfies the adjoint boundary condition, the term linear in a is zero and since vA_nv was assumed negative, we may conclude that $N \oplus v$ is non-positive. On the other hand, N being maximal non-positive, it follows that v belongs to N. Finally since v also satisfies the adjoint boundary condition, we imply $vA_nv = 0$, which is contrary to our assumption; this proves the lemma.

We now come to our main theorem.

THEOREM 3.2. *Let L be a formally dissipative symmetric operator, G a domain whose boundary is of class C^2, and $N(x)$ smoothly varying boundary spaces which are maximal non-positive. Then for any given square integrable function f the equation*

$$u - Lu = f$$

has a unique strong solution satisfying in the strong sense the given boundary conditions.

As mentioned in the introduction, essentially the same existence theorem has been derived by Friedrichs [2] via his differentiability theorem.

Proof: We claim that the range of $u - Lu$ for smooth functions u which satisfy the prescribed boundary conditions at every point of S is not dense in L_2. For suppose the contrary; then there is a non-trivial function v orthogonal to this range, i.e.,

$$(Lu, v) = (u, v)$$

for all such u. Recalling the definition from Section 1, we see that v is a weak solution of

$$L^*v = v$$

satisfying in the weak sense the adjoint boundary conditions. According to Theorem 1.1 of the first section, v is then a strong (semi-strong) solution and satisfies in the strong sense the adjoint boundary conditions. According to Lemmas 3.1 and 3.2 the operator L^* is formally dissipative and the adjoint boundary conditions are non-positive. Thus by Corollary 2 of Theorem 3.1, v is necessarily zero. This contradiction allows us to conclude that the range of $(I - L)$ is dense in the L_2 norm.

Let f be any square integrable function; since functions of the form $u - Lu$ are dense, there exists a sequence $\{u_n\}$ of smooth functions satisfying the prescribed boundary conditions such that $u_n - Lu_n$ tends to f. According to the inequality (3.4) of Theorem 3.1 the functions u_n also converge in the L_2 norm and it follows that the limit u is the desired strong solution.

4. Exceptional Corners

Following Friedrichs one can extend the previous results to the case where the boundary contains corners provided the coefficients A^j are suitably restricted. More precisely we permit boundary patches which map into half-slabs of the form

$$0 < y < 1, \qquad x^1 < 0,$$

with the associated portion of S mapping into $y = 1$ and $x^1 = 0$. Again we assume that the y-coefficient A_n is of constant rank throughout the half-slab and now, in addition, we assume that A^1 is either positive or negative on $x^1 = 0$.

If $A^1 \geqq 0$ we extend L onto the entire slab by setting

$$\left.\begin{aligned} A^j(x^1, x^2, \cdots, y) &= A^j(0, x^2, \cdots, y) \\ B(x^1, x^2, \cdots, y) &= B(0, x^2, \cdots, y) \end{aligned}\right\} \quad \text{for } x^1 > 0.$$

We then proceed precisely as before except that we redefine j_ε as

$$j'_\varepsilon(x) = \frac{1}{\varepsilon^k} j\left(\frac{x+2\varepsilon^1}{\varepsilon}\right),$$

where ε^1 has its first component equal to ε and its other components zero. Suppose now that u is a weak solution satisfying the maximal negative boundary condition weakly. Then the adjoint boundary condition along $x^1 = 0$ is unrestricted. After localizing u to the half-slab so that it vanishes for $y < \frac{1}{2}$ and has bounded support, we extend it to the entire slab by defining it to be 0 throughout the upper half-slab. Extending f in the same way, it is readily seen that u is now a weak solution of $Lu = f$ on the entire slab; here we make use of the unrestricted nature of the adjoint boundary condition. The u_ε which we now obtain by mollifying will vanish for $x^1 > -\varepsilon$ and hence satisfies the negative boundary condition trivially. Theorems 2.1 and 3.2 go through essentially as before.

In case $A^1 \leqq 0$ on $x^1 = 0$ we proceed differently. Starting in Section 1 we redefine the spaces H_0 and H_1 so that they consist of vector-valued functions of x defined for the half-space $x^1 < 0$. In order to perform the necessary integrations by parts, it is necessary that H_1 be the completion of smooth functions which vanish outside of compact subsets of this half-space. The space H_{-1} is defined as before to be dual to H_1 with respect to the H_0 inner product. The smooth functions utilized in Lemma 2.1 are also restricted to have compact support in the half-space. This condition again appears when we construct the mollifiers and will be satisfied if we now replace j_ε by

$$j''_\varepsilon(x) = \frac{1}{\varepsilon^k} j\left(\frac{x-2\varepsilon^1}{\varepsilon}\right).$$

The development of Section 2 goes through just as before and the resulting function u_ε is absolutely continuous in the x^1-direction for $x^1 \leqq 0$. These functions obviously satisfy the condition $u_\varepsilon A^1 u_\varepsilon \leqq 0$ along $x^1 = 0$ if $A^1 \leqq 0$ on this portion of the boundary. We then proceed exactly as in Theorem 3.2.

5. Unessential Boundary Points

In this and the next section we will consider certain other exceptional situations for which Theorem 3.2 continues to hold. Thus we again suppose that L is formally dissipative and that the boundary spaces $N(x)$ are maximal non-positive. Suppose further that there exists a sequence of real-valued smooth scalar multipliers $\{\phi_n\}$ such that for each v, satisfying $L^*v = f$ weakly and the adjoint boundary conditions in the weak sense, we have

(i) $(L^*\phi_n v, \phi_n v)+(\phi_n v, L^*\phi_n v)-(D\phi_n v, \phi_n v) \leqq 0, \qquad n > 0,$

(ii) $(L^*v, v)+(v, L^*v)-(Dv, v)$
$$= \lim_{n\to\infty} [(L^*\phi_n v, \phi_n v)+(\phi_n v, L^*\phi_n v)-(D\phi_n v, \phi_n v)].$$

It then follows that
$$(L^*v, v)+(v, L^*v) \leqq 0,$$

and we may conclude as in the proof of Theorem 3.2 that for any given square integrable function f the equation $u-Lu = f$ has a unique strong solution satisfying in the strong sense the given boundary conditions.

Condition (i) will be satisfied if each ϕ_n vanishes at all points of the boundary S in the neighborhood of which S is not of class C^2, or $N(x)$ does not vary smoothly or at corner points of the kind not treated in the previous section. Condition (ii) holds if the points, at which all of the ϕ_n vanish, are what we shall call "unessential points".

We note that

$$(5.1) \quad (L^*\phi_n v, \phi_n v)+(\phi_n v, L^*\phi_n v)-(D\phi_n v, \phi_n v)$$
$$= [((\phi_n)^2L^*v, v)+((\phi_n)^2v, L^*v)-((\phi_n)^2Dv, v)]-2\sum_j (\phi_n\phi_{n_{x^j}}A^j v, v).$$

We shall choose the ϕ_n such that $0 \leqq \phi_n \leqq 1$ and $\lim_n \phi_n(x) = 1$ for each x in G. In this case the bracketed expression in the right member of (5.1) obviously converges to $[(L^*v, v)+(v, L^*v)-(Dv, v)]$ and hence condition (ii) will be satisfied if it can be shown that

$$(5.2) \qquad \underline{\lim}_{n\to\infty} |\sum_j (\phi_n\phi_{n_{x^j}}A^j v, v)| = 0.$$

The expression $\sum_j (\phi_n\phi_{n_{x^j}}A^j v, v)$ can be thought of as a smeared-out boundary integral around the unessential points of S.

We are not prepared to give a general theory of unessential boundary points but we shall only present two possibilities:

1. The quadratic form vAv in the boundary integral in the sum of terms of the form $a\eta\zeta$, where a is a coefficient and η and ζ are a pair of components

of the dependent variables of which one, say ζ, is known to have square integrable first derivatives.

2. The coefficients A_n tend to zero fast enough when the boundary point in question is approached.

Thus there is present in both cases some feature compensating for the singularity of the boundary or boundary conditions. Condition 1 is frequently satisfied when the operator L results from converting a second order equation into a first order system. Condition 2 is satisfied in the Frankl problem discussed in Section 6.

With (i) as our basic assumption, let us further assume that the boundary patch in question maps into a wedge G of the form

$$\sum_{i=1}^{n} (x^i)^2 < 1,$$

$$0 \leqq \alpha < \tan^{-1} \frac{x^1}{x^n} < \beta \leqq 2\pi,$$

where the mapping is of class C^2 on the closure of the patch except perhaps along the edge $x^1 = 0 = x^n$, where it need only be of class C^1. The portion of S in the patch maps into $\tan^{-1} (x^1/x^n) = \alpha$ and β. In addition to allowing S to have an edge of this sort we permit $N(x)$ to have a discontinuity along $x^1 = 0 = x^n$.

We now set

$$\rho = [(x^1)^2 + (x^n)^2]^{1/2}$$

and define

$$\phi_n = \begin{cases} 1 & \text{for } \rho > \dfrac{1}{n-1}, \\ n(n-1)\left(\rho - \dfrac{1}{n}\right) & \text{for } \dfrac{1}{n} \leqq \rho \leqq \dfrac{1}{n-1}, \\ 0 & \text{for } \rho < \dfrac{1}{n}. \end{cases}$$

Without loss of generality we can suppose that η and ζ vanish near what corresponds to the inner boundary of G, that is near the non-planar portions of the boundary of G. Setting $\theta = \tan^{-1} (x^1/x^n)$ and $x' = (x^2, \cdots, x^{n-1})$, we see that

$$\zeta(x', \rho, \theta) = \int_1^{\rho} \frac{\partial \zeta(x', s, \theta)}{\partial s}\, ds$$

so that

$$|\zeta(x', \rho, \theta)|^2 \leqq |\log \rho| \int_0^1 \left| \frac{\partial}{\partial s} \zeta(x', s, \theta) \right|^2 s\, ds,$$

and hence

$$\int_\alpha^\beta \int_{1/n}^{1/n-1} |\zeta|^2 \rho\, d\rho\, d\theta \leqq \log n \left[\left(\frac{1}{n-1} \right)^2 - \left(\frac{1}{n} \right)^2 \right] \int_\alpha^\beta \int_0^1 |\nabla\zeta|^2 s\, ds\, d\theta.$$

Finally

$$\int_G |\phi_n \phi_{n_{x^j}} a\eta\zeta| dx \leqq n(n-1) ||a|| \left[\int_{\Delta_n} |\eta|^2 dx \int_{\Delta_n} |\zeta|^2 dx \right]^{1/2},$$

where

$$||a|| = \sup |a(x)| \quad \text{and} \quad \Delta_n = \left[x;\ |x| < 1,\ \alpha < \theta < \beta,\ \frac{1}{n} < \rho < \frac{1}{n-1} \right].$$

Consequently

$$\int_G |\phi_n \phi_{n_{x^j}} a\eta\zeta| dx \leqq ||a|| \left[n \log n \int_G |\nabla\zeta|^2 dx \int_{\Delta_n} |\eta|^2 dx \right]^{1/2}.$$

Suppose now that

$$\underline{\lim} \int_G |\phi_n \phi_{n_{x^j}} a\eta\zeta| dx \geqq 2\varepsilon.$$

Then for n sufficiently large

$$\int_{\Delta_n} |\eta|^2 dx \geqq \left(\frac{\varepsilon}{||a||} \right)^2 \frac{c}{n \log n},$$

where

$$c = \left[\int_G |\nabla\zeta|^2 dx \right]^{-1} > 0.$$

However this is impossible since

$$\infty = \sum \left(\frac{\varepsilon}{||a||} \right)^2 \frac{c}{n \log n} \leqq \sum \int_{\Delta_n} |\eta|^2 dx \leqq \int_G |\eta|^2 dx < \infty.$$

We remark that a similar technique can often be used to extend the results of this paper to unbounded domains G.

6. The Frankl Problem

As an illustration of the second possibility indicated in Section 5 we treat the Frankl problem for a mixed type partial differential equation. We shall make extensive use of a recent paper by C. S. Morawetz [3] in which the existence of a weak solution is established for this problem. In particular we shall follow Morawetz in her choice of operator and domain and show that our technique can be used to prove that the solution obtained by Morawetz is a strong solution and that it is unique.

The equation to be studied is

$$K \frac{\partial^2 w}{\partial (x^1)^2} + \frac{\partial^2 w}{\partial (x^2)^2} = g, \tag{6.1}$$

where K (unrelated to the K in Section 3) is a function which depends only on x^2 and

$$\frac{\partial K}{\partial x^2} > 0 \quad \text{for } x^2 > x_0^2 \quad \text{for some } x_0^2 < 0,$$

$$x^2 K(x^2) \geqq 0 \quad \text{for all } x^2.$$

The domain G is shown in Figure 1: C_0 is essentially star shaped in the

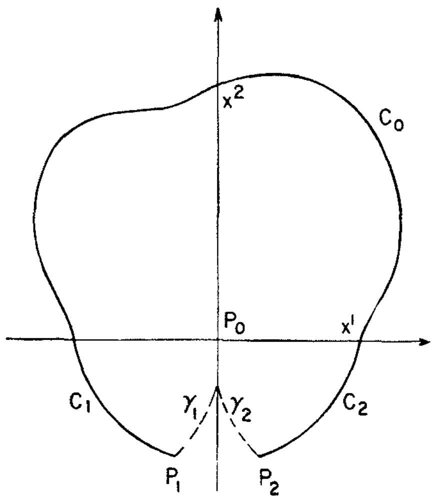

Figure 1

x^1, y-coordinate system, where

$$y = \int_0^{x^2} [K(\sigma)]^{1/2} \, d\sigma;$$

more precisely it is assumed that

$$(x^2)^{-1/2} (x^1 \, dy - y \, dx^2) \geqq k_0 \, ds > 0$$

on C_0. On C_1 and C_2 it is assumed that

$$K \left(\frac{dx^2}{dx^1}\right)^2 + 1 > 0, \quad x^1 dx^2 \geqq k_0 \, ds > 0.$$

Finally γ_1 and γ_2 are taken to be characteristics on which

$$K\left(\frac{dx^2}{dx^1}\right)^2+1=0.$$

Morawetz transforms equation (6.1) into a system by introducing

$$u^1=w_{x^1}, \qquad u^2=w_{x^2},$$

as new unknowns; (6.1) becomes

$$Ku^1_{x^1}+u^2_{x^2}=0,$$

while the compatibility equation is

$$u^2_x-u^1_y=0.$$

Combining these two equations by weight functions a, b and b, $-a$, we obtain a system of two new equations which in the matrix notation (1.1) can be written as $Ju-Lu=f$ with

$$\begin{aligned} &A^1=-\begin{pmatrix}Ka & Kb\\ Kb & -a\end{pmatrix}, \qquad A^2=-\begin{pmatrix}-Kb & a\\ a & b\end{pmatrix},\\ &B=JI, \qquad\qquad (6.2)\\ &J=\tfrac{1}{2}[A^1_{x^1}+A^2_{x^2}]. \end{aligned}$$

The appropriate choice for the as yet undetermined factors a and b, i.e., a choice which permits the application of the foregoing theory, is as follows:

$$\begin{aligned} &a=x^1, \qquad b=c|x^1| \quad \text{for } x^2\leqq 0, \qquad (6.3)\\ &a=x^1, \quad K^{1/2}b=cK^{1/2}|x^1|+\int_0^{x^2}K^{1/2}\,d\sigma \quad \text{for } x^2\geqq 0, \end{aligned}$$

c being a constant to be determined later. The elements of A^1 and A^2 are readily seen to be smooth.

Because of the singular behavior of the system at P_0, Morawetz treats L as an operator on a Hilbert space $H_1(G)$ with elements $u^1, v^1, \cdots$ to the Hilbert space $H_2(G)$ with elements $u^2, v^2, \cdots$; the inner products for these spaces are defined for $u^i=[\eta^1,\eta^2]$ and $v^i=[\zeta^1,\zeta^2]$ as

$$\begin{aligned} &(u^1,v^1)=\int_G(r\eta^1\zeta^1+\eta^2\zeta^2)dx \qquad \text{in } H_1(G), \qquad (6.4)\\ &(u^2,v^2)=\int_G(r^{-1}\eta^1\zeta^1+\eta^2\zeta^2)dx \qquad \text{in } H_2(G); \end{aligned}$$

here $r=[(x^1)^1+(x^2)^2]^{1/2}$. We note that $H_1(G)$ and $H_2(G)$ are dual Hilbert spaces with respect to the inner product

$$(u^1,v^2)=\int_G(\eta^1\zeta^1+\eta^2\zeta^2)dx. \qquad (6.5)$$

It is easy to show that J maps $H_1(G)$ into $H_2(G)$ boundedly; in addition, as Morawetz has shown

$$(6.6) \qquad (Ju^1, u^1) \geqq c_1 \|u^1\|_1^2$$

for some $c_1 > 0$.

The use of dual Hilbert spaces for the domain and range, respectively, of the differential operator L introduces no essential difficulties (see [6]). A dissipative operator L on $H_1(G)$ to $H_2(G)$ is now defined by the property

$$(Lu^1, u^1) + (u^1, Lu^1) \leqq 0$$

for all u^1 in its domain. If L is dissipative and

$$Ju^1 - Lu^2 = f^2,$$

then it is readily proven as in Theorem 3.1 that

$$(6.7) \qquad c_1 \|u^1\| \leqq \|f^2\|.$$

For the present operator L with coefficients (6.2), the matrix D of (3.2) is equal to zero so that the differential operator L is formally dissipative. We proceed to choose $N(x)$ on S to be maximal negative and smoothly varying except at the points P_0, P_1, and P_2. The resulting operator is now dissipative.

Given any f^2 in $H_2(G)$, we wish to show that

$$Ju^1 - Lu^1 = f^2$$

has a unique strong solution in $H_1(G)$ satisfying in the strong sense the given boundary conditions. As before it suffices to show that the range of $Ju^1 - Lu^1$ is dense in $H_2(G)$ as u^1 varies over all smooth functions which satisfy the prescribed boundary conditions. Again, if this were not the case, then there would exist a non-zero v^1 in $H_1(G)$ such that

$$(Ju^1, v^1) = (Lu^1, v^1)$$

for all such u^1. Taking the adjoint L^* of L relative to the mixed inner product, we see that v^1 is a weak solution of $L^* v^1 = Jv^1$, satisfying the adjoint boundary conditions in the weak sense. A contradiction will be reached if it can be shown that

$$(6.8) \qquad (L^* v^1, v^1) + (v^1, L^* v^1) < 0.$$

Proceeding as in Sections 2, 3, and 4 we can show that (6.8) is satisfied by all functions v^1 in the domain of L^* (in the weak sense) which satisfy the adjoint boundary conditions (also in the weak sense) and have support in an interior patch or in a boundary patch not containing P_0. In fact as long as a patch does not contain P_0, the corresponding portions of $H_1(G)$ and $H_2(G)$ are equivalent to the corresponding portions of L_2 and so no additional argument is required. It suffices merely to show that the boundary points for such patches are either regular boundary points or corners of the type considered in Section 4. To this end we note that

$$A_n = -\begin{pmatrix} Ka & Kb \\ Kb & -a \end{pmatrix} n_1 - \begin{pmatrix} -Kb & a \\ a & b \end{pmatrix} n_2 .$$

One readily computes that the product of the eigenvalues of A_n is

$$-[Kb^2+a^2][K(n_1)^2+(n_2)^2].$$

On C_0, where $K \gtreqless 0$, it is clear that this product is always negative and hence that A_n has one positive and one negative eigenvalue. Along C_1 we see that

$$K(n_1)^2+(n_2)^2 > 0$$

and that

$$(Kb^2+a^2) = (Kc+1)(x^1)^2 > 0$$

for c positive but sufficiently small. Hence, with this choice of c, the form A_n also has one positive and one negative eigenvalue on C_1. A similar argument applies to C_2. On the characteristics γ_1 and γ_2 we have

$$K(n_1)^2+(n_2)^2 = 0$$

so that the product of the eigenvalues is zero; since the sum of the eigenvalues is $(K-1)(bn_2-an_1)$ and since this expression is negative for c sufficiently small, we see that A_n is negative and of rank one along both γ_1 and γ_2 for such c. It follows that the points P_1 and P_2 correspond to corners of the type treated in Section 4. On the other hand, the points X_1 and X_2 are critical only in the sense that some of the coefficients are small near these points. However, for our choice of S the eigenvalues are bounded away from zero and since the coefficients are smooth so are the eigenspaces. This is all that is required of a regular boundary point. Thus if we make use of a suitable partition of unity, we may conclude by (3.6) that the relation (6.8) holds for all functions v^1 in the domain of L^* (in the weak sense) which satisfy the adjoint boundary conditions (also in the weak sense) and which vanish in the neighborhood of P_0.

We now show that P_0 is an unessential point in the sense of Section 5. This will establish (6.8) for our original choice of v^1 which satisfies

$L^* v^1 = J v^1$ and we can then conclude that $v^1 = 0$; this shows that our assumption—that the range of $Ju^1 - Lu^1$ was not dense—was false and hence proves the existence of a strong solution to the Frankl problem. The uniqueness follows from (6.7). It should be remarked that showing P_0 to be an unessential point is the only thing actually novel in the present section.

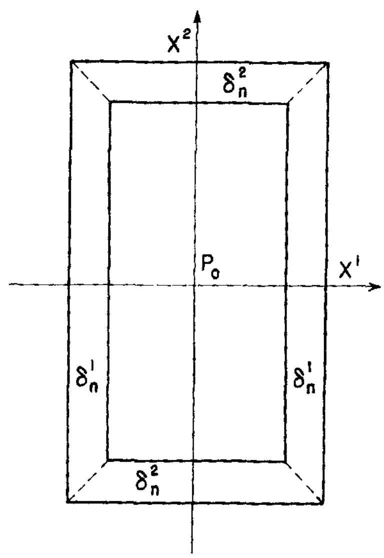

Figure 2

The scalar multipliers ϕ_n will be defined by means of the rectangular frame Δ_n shown in Figure 2. The vertical sides are

$$\delta_n^1: \qquad \frac{1}{n+1} < |x^1| < \frac{1}{n},$$

and the horizontal sides are

$$\delta_n^2: \qquad \frac{1}{(n+1)^{2/3}} < |x^2| < \frac{1}{n^{2/3}}.$$

Strangely enough the exponent 2/3 is rather critical. We now define ϕ_n to be 1 outside of the rectangular frame, zero inside of it, and extended linearly on the frame itself. We note that

$$\begin{aligned} \phi_{n_{x^1}} &= O(n^2) \quad \text{and } \phi_{n_{x^2}} = 0 \qquad \text{on } \delta_n^1, \\ \phi_{n_{x^1}} &= 0 \qquad\quad \text{and } \phi_{n_{x^2}} = O(n^{5/3}) \quad \text{on } \delta_n^2. \end{aligned} \tag{6.9}$$

We shall show that

$$\underline{\lim} \int_G |\phi_n \phi_{n_{x^j}} v^1 A^j v^1| dx = 0$$

for all v^1 in $H_1(G)$ and $j = 1, 2$.

It is readily seen from (6.2) and (6.3) that on δ_n^1 the elements of A^1 are of the order

$$\begin{pmatrix} \frac{r}{n} & \frac{r^{1/2}}{n} \\ \frac{r^{1/2}}{n} & \frac{1}{n} \end{pmatrix},$$

whereas on δ_n^2 the elements of A^2 are of the order

$$\begin{pmatrix} \frac{r}{n^{2/3}} & \frac{r^{1/2}}{n^{2/3}} \\ \frac{r^{1/2}}{n^{2/3}} & \frac{1}{n^{2/3}} \end{pmatrix}.$$

Combining this with the estimates (6.9) we see that

$$\begin{aligned} \sum_{j=1}^{2} \int_G |\phi_n \phi_{n_{x^j}} v^1 A^j v^1| dx &= \int_{\delta_n^1} |\phi_n \phi_{n_{x^1}} v^1 A^1 v^1| dx + \int_{\delta_n^2} |\phi_n \phi_{n_{x^2}} v^1 A^2 v^1| dx \\ &\leqq C \Big\{ n^2 n^{-1} \int_{\delta_n^1} [r|\eta^1|^2 + 2r^{1/2}|\eta^1 \eta^2| + |\eta^2|^2] dx \\ &\quad + n^{5/3} n^{-2/3} \int_{\delta_n^2} [r|\eta^1|^2 + 2r^{1/2}|\eta^1 \eta^2| + |\eta^2|^2] dx \Big\} \\ &\leqq 2Cn \int_{\Delta_n} [r|\eta^1|^2 + |\eta^2|^2] dx, \end{aligned}$$

where C is independent of n. Hence if

$$\underline{\lim} \sum_{j=1}^{2} \int_G |\phi_n \phi_{n_{x^j}} v^1 A^j v^1| dx \geqq 2\varepsilon > 0,$$

the above inequality shows that

$$||v^1||^2 \geqq \sum_{n=1}^{\infty} \int_{\Delta_n} [r|\eta^1|^2 + |\eta^2|^2] dx \geqq \sum \frac{\varepsilon}{2Cn} = \infty,$$

which is impossible if v^1 belongs to $H_1(G)$.

Bibliography

[1] Friedrichs, K. O., *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc., Vol. 55, 1944, pp. 132–151.

[2] Friedrichs, K. O., *Symmetric positive linear differential equations*, Comm. Pure Appl. Math., Vol. 11, 1958, pp. 333–418.

[3] Morawetz, C. S., *A weak solution for a system of equations of elliptic-hyperbolic type*, Comm. Pure Appl. Math., Vol. 11, 1958, pp. 315–331.

[4] Phillips, R. S., *Dissipative hyperbolic systems*, Trans. Amer. Math. Soc., Vol. 86, 1957, pp. 109–173.

[5] Phillips, R. S., *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc., Vol. 90, 1959, pp. 193–254.

[6] Phillips, R. S., *Dissipative operators and parabolic partial differential equations*, Comm. Pure Appl. Math., Vol. 12, 1959, pp. 249–276.

[7] Dezin, A. A., *Existence and uniqueness theorems for solutions in various function spaces of boundary value problems for partial differential equations*, Uspehi Mat. Nauk, Vol. 14, No. 3, 1959, pp. 21–73.

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