# Positive Subspaces

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## 1 Definitions

Suppsoe that $\mathbb{V}$ is a finite dimensional complex scalar product space with scalar product $\langle\cdot,\cdot\rangle$. Suppose that $A : \mathbb{V} \to \mathbb{V}$ is an invertible self adjoint linear map. Denote by $\mathbb{E}_+$ the spectral subspace of $A$ associated to positive eigenvalues and $\mathbb{E}_-$ the negative spectral subspace so $\mathbb{E}_+ \oplus_\perp \mathbb{E}_- = \mathbb{V}$. Deonte by $\Pi_\pm : \mathbb{V} \to \mathbb{V}$ orthogonal projection on $\mathbb{E}_\pm$.

**Definition 1.1** *A linear subpace $\mathcal{N} \subset \mathbb{V}$ is* **positive** *when for all $v \in \mathcal{N}$, $\langle Av, v\rangle \geq 0$. It is* **strictly positive** *when one has strict positivity for all $v \neq 0$. A postive subspace is* **maximal postivie** *when it is not a proper subset of a larger postive subspace.*

**Example 1.1** The subspace $\mathbb{E}_+$ is strictly postive and maximal positive. To prove the second, observe that any larger subpace has nontrivial intersection with $\mathbb{E}_-$ so cannot be postive.

**Exercise 1.1** *Prove that every postive subspace is contained in a maximal positive subspace.* **Discussion.** There is a proof of this in the proof of the next proposition, but it is easy to prove directly from the definition.

**Example 1.2** *If $\Gamma$ is a linear subspace of $\mathbb{E}_+$ and $M : \Gamma \to \mathbb{E}_-$ define*

$$\mathcal{N} := \Big\{ u + Mu \ : \ u \in \Gamma \Big\}.$$

*On $\mathbb{E}_+$ introduce the scalar product $\langle A(\cdot),\cdot\rangle$ and on $\mathbb{E}_-$ the scalar product $\langle -A(\cdot),\cdot\rangle$. With these choices $\mathcal{N}$ is a positive subspace if and only if the norm of $M : \Gamma \to \mathbb{E}_-$ is $\leq 1$. $\mathcal{N}$ is strictly positive if and only if the norm is strictly less than one.* Indeed, since $\mathbb{E}_\pm$ are $A$-invariant and orthogonal

$$\Big\langle A(u+Mu)\,,\, u+Mu\Big\rangle \;=\; \Big\langle Au\,,\, u\Big\rangle + \Big\langle A(Mu)\,,\, Mu\Big\rangle \;=\; \|u\|^2_{\mathbb{E}_+} - \|Mu\|^2_{\mathbb{E}_-}\,.$$

The set of $M$ yielding postive spaces is convex and closed with interior equal to the set yielding strictly postive spaces.

**Proposition 1.1** *Every postive subspace has the form given in Example 1.2.*

**Proof.** If $\mathcal{N}$ is positive then $\mathcal{N} \cap \mathbb{E}_- = \{0\}$. Since $\mathbb{E}_- = \ker \Pi_+$ this shows that $\Pi_+ : \mathcal{N} \to \mathbb{E}_+$ is injective. Define $\Gamma := \Pi_+(\mathcal{N})$ so $\Pi_+$ is invertible from $\mathcal{N}$ to $\Gamma$ is invertible. For $\gamma = \Pi_+ u \in \Gamma$, decompose $u = \Pi_+ u + \Pi_- u$ and define $M\gamma := \Pi_- u$ to yield the desired relation $u = \gamma + M\gamma$. $\square$

## 2 Dimension and maximality

**Proposition 2.1** **i.** *The dimension of a postive subspace is less than or equal to the dimension of* $\mathbb{E}_+$.

**ii.** *A positive subspace is maximal if and only if its dimension is equal to the dimension of* $\mathbb{E}_+$.

**Proof.** **i.** Indeed every postive subspace is as in the example so has dimension equal to the dimension of a subspace $\Gamma \subset \mathbb{E}_+$.

**ii.** The bound in **i** shows that if $\mathcal{N}$ is positive and $\dim \mathcal{N} = \dim \mathbb{E}_+$ then $\mathcal{N}$ is maximal.

It remains to show that if $\dim \mathcal{N} < \dim \mathbb{E}_+$ then $\mathcal{N}$ is not maximal. In this case

$$\mathcal{N} \ = \ \Big\{ u + Mu \ : \ u \in \Gamma \Big\},$$

with $\Gamma$ a proper subspace of $\mathbb{E}_+$ and $M : \mathbb{E}_+ \to \mathbb{E}_+$ with norm less than or equal to one with the spaces normed as in Example 1.2

$\Pi_+(\mathcal{N})$ is a proper subscpace of $\mathbb{E}_+$. Endow $\mathbb{E}_+$ with the scalar product $\langle A(\cdot)\,,\,\cdot\rangle$. Denote by $\mathbb{W} \subset \mathbb{E}_+$ the orthogonal complement of $\Pi_+(\mathcal{N})$ in this scalar product. Extend $M$ to a linear map from $\mathbb{E}_+ \to \mathbb{E}_-$ by defining $M$ to vanish on $\mathbb{W}$. Define the linear subspace

$$\mathcal{N}_2 \ := \ \Big\{ u + Mu \ : \ u \in \mathbb{E}_+ \Big\}.$$

Then $\mathcal{N}$ is a proper subspace of $\mathcal{N}_2$. To complete the proof it suffices to show that $\mathcal{N}_2$ is postive. It suffices to show that $M$ is a contraction. Compute using the orthogonality of $\gamma \in \Gamma$ and $w$,

$$\|M(\gamma + w)\|^2_{\mathbb{E}_-} = \|M\gamma\|^2_{\mathbb{E}_-} \le \|\gamma\|^2_{\mathbb{E}_+} = \|\gamma + w\|^2 - \|w\|^2_{\mathbb{E}_+} \le \|\gamma + w\|^2_{\mathbb{E}_+}\,.$$

$\square$

**Exercise 2.1** *Find all maximal postive subspaces for the map of* $\mathbb{C}^2$ *defined by the matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

## 3 Maximality and the adjoint space

**Definition 3.1** *If* $\mathcal{N}$ *is a maximal postive space define the adjoint space* $\mathcal{N}^\dagger := A(\mathcal{N})^\perp$.

**Proposition 3.1** *If* $\mathcal{N}$ *is a maximal postive subscpace for* $A$*, then* $\mathcal{N}^\dagger$ *is a maximal postive subspace for* $-A$.

**Proof.** The dimension of $\mathcal{N}^\dagger$ is equal to the $\dim \mathbb{E}_-$. This is equal to the dimension of the postive spectral subspace of $-A$. Part **ii** of Proposition 2.1 implies that it suffices to show that $\mathcal{N}^\dagger$ is a positive subspace for $-A$.

Assume not. Then there is a $\zeta \in \mathcal{N}^\dagger$ with

$$\big\langle -A\zeta\,,\,\zeta\big\rangle \;<\; 0\,. \tag{3.1}$$

Define

$$\mathcal{N}_2 \;:=\; \mathcal{N} \;+\; \mathbb{C}\zeta\,.$$

Then $\mathcal{N}_2$ is a positive subspace since for $\eta \in \mathcal{N}$ and $z \in \mathbb{C}$

$$\langle A(\eta + z\zeta)\,,\,\eta + z\zeta\rangle = \langle A\eta, \eta\rangle + |z|^2 \langle A\zeta, \zeta\rangle + 2\mathrm{Re}\langle z\zeta, A\eta\rangle\,.$$

The last term vanishes since $\zeta \perp A(\mathcal{N})$. The fist two terms are positive.

The maximality of $\mathcal{N}$ implies that $\zeta \in \mathcal{N}$. Then since $\zeta \in \mathcal{N}^\dagger = A(\mathcal{N})^\perp$ one has $\langle A\zeta, \zeta\rangle = 0$. This contradicts (3.1). ☐

**Exercise 3.1** *Show that the adjoint space of* $\mathcal{N}^\dagger$ *is the original* $\mathcal{N}$.

## 4 The characteristic case

**Definition 4.1** *If* $A$ *is not invertible the definition of postive and maximal positive remain unchanged. Define* $\mathbb{E}_{\geq 0}$ *to be the spectral subspace associated to nonnegative eigenvalues and* $\mathbb{E}_-$ *to negative eigenvalues, and,* $\mathcal{K} := \ker A$.

The value of the bilinear form $\langle Au, v\rangle$ does not change if each of $u, v$ is modified by adding possibly different elements of $\mathcal{K}$. Thus the form induces a form on $\mathbb{V}/\mathcal{K}$ and the map $A$ induces an invertible self adjoint map from $\mathbb{V}/\mathcal{K}$ to itself. In this way the theory is reduced to the invertible case.

**Proposition 4.1** **i.** *Every maximal postive subspace contains* $\mathcal{K}$.

**ii.** *The dimension of a postive subspace is* $\leq \dim \mathbb{E}_{\geq 0}$. *A positive space is maximal if and only if there is equality.*

**iii.** *If* $\mathcal{N}$ *is maximal positive, then the adjont space* $\mathcal{N}^\dagger := A(\mathcal{N})^\perp$ *is maximal positve for* $-A$.

**Exercise 4.1** *Prove this proposition.*

**Exercise 4.2** *Find all maximal postive subspaces for the map of* $\mathbb{C}^3$ *defined by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$