Chapter 3. Dispersive Behavior

3.1. Orientation.

In this chapter we return to Fourier analysis techniques as in Chapter 1. The Fourier transform of the solution is written exactly and then analysed.

The results show how the geometry of the characteristic variety of $L = L_1(\partial_y)$ is reflected in qualitative properties of the solutions of $Lu = 0$. The main idea is that when the characteristic variety is curved, the corresponding solutions tend to spread out in space. This dispersive effect is reflected in solutions becoming smaller in $L^\infty(\mathbb{R}^d)$ in contast to $L^2(\mathbb{R}^d)$ conservation.

Three simple examples illustrate the theme. The scalar advection operator

$$L \ := \ \partial_t \ + \ \mathbf{v}.\partial_x \,, \tag{3.1.1}$$

in dimension d and the system

$$\frac{\partial v}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial v}{\partial x} = 0 \tag{3.1.2}$$

in dimension $d = 1$ have only purely translating modes. The characteristic variety of (3.1.1) is the hyperplane $\tau + \mathbf{v}.\xi = 0$ and for (3.1.2) it is the pair of lines $\tau \pm \xi = 0$. In both cases the variety is not curved at all.

The system analogue of $\Box_{1+2}$,

$$L \ := \ \partial_t \ + \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 \ + \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 \tag{3.1.3}$$

behaves differently. Each component satisfies $\Box_{1+2} u = 0$. For smooth compactly supported data, they decay (in sup norm) as $t^{-1/2}$. The characteristic variety is $\tau^2 - |\xi|^2 = 0$. Since all charateristic varieties are conic their Gauss curvatures vanish. The present variety intersects $\tau = 1$ in a strictly convex set. So the variety is as curved as a conic set can be.

Exercise. Prove the decay rate for compactly supported solutions of $\Box_{1+2} u = 0$ by expressing solutions as convolutions with fundamental solution(s).

For all three examples the $L^2(\mathbb{R}^d)$ norm is perserved during the time evolution.

For the solutions of the transport equation associated to (3.1.1), the size of the support of solutions does not change in time. For (3.1.3), solutions spread out over a set whose two dimensional area grows with time. The spread together with L^2 conservation, explains the decay.

In optics, the word dispersion is used to mean that the speed of light depends on its wavelength. In that sense, none of the above models is dispersive. The dispersion relations of the first and third models are

$$\tau \ = \ -\mathbf{v}.\xi \,, \qquad \text{and} \qquad \tau \ = \ \pm |\xi| \,.$$

Both are positive homogeneous of degree one in ξ. Therefore the group velocities $-\nabla_\xi \tau$ satisfy

$$\big|\nabla_\xi \tau\big| \;=\; |\mathbf{v}|\,, \qquad \text{and} \qquad \big|\nabla_\xi \tau\big| \;=\; 1\,,$$

respectively. In neither case does the speed depend on the wavelength. However for (3.1.2), the velocity depends strongly on ξ, though not on $|\xi|$. The fact that the group velocities point in different directions has the effect of spreading the solution, and for large time the solutions decay.

The variation of the group velocity with ξ is given by the matrix of second derivatives $\nabla_\xi^2 \tau$. For our homogeneous operators, $\nabla_\xi \tau$ is homogeneous of degree zero, so ξ belongs to the kernel of matrix. The rank can be at most $d-1$. The D'Alembertian $\Box_{1+d}$ achieves this maximal rank so is as dispersive as a homogeneous operator can be.

At the extreme opposite is $\nabla_\xi^2 \tau \equiv 0$, in which case the dispersion relation is linear in ξ. The associated graph is a hyperplane which belongs to the characteristic variety. The characteristic variety for (3.1.1) and (3.1.2) consist of hyperplanes while for (3.1.3) the variety is curved. Where the variety is flat, $\tau = -\mathbf{v}.\xi$, the group velocity is identically equal to $\mathbf{v}$ so does not depend on ξ. This is the completely nondispersive situation. Solutions translate without spread.

If the variety contains no hyperplanes, the variation of the group velocity spreads wavepackets. We will show that as $t \to \infty$, solutions decay in L^∞. These results, presented in §3.2-§3.3, are taken from [JMR, Indiana Math. J. 1998].

An even stronger notion of uniform dispersion is when the rank of $\nabla_\xi^2 \tau$ is everywhere equal to $d-1$. In this case, the sheets of the characteristic variety are uniformly convex cones and smooth compactly supported solutions decay at the rate $t^{-(d-1)/2}$ as $t \to \infty$. This is investigated in §3.4. In §3.4.1 $L^1 \to L^\infty$ decay estimates are proved. These are applied in §6.7 to prove global solvability of for nonlinear problems with small initial data and high dimension. In §3.4.3 they are used to derive Strichartz estimates. In §6.8, these estimates are applied to study the nonlinear Klein-Gordon equation in the natural energy space.

§3.2. Spectral decomposition of solutions.

Since $(\tau, 0)$ is noncharacteristic for L, any hyperplane $\{a\tau + b.\xi = 0\}$ contained in the characteristic variety must have $a \neq 0$. Therefore, it is necessarily a graph $\{\tau = -\mathbf{v}.\xi\}$.

Over each $\xi \in \mathbb{R}^d$ there are at most N points in the characteristic variety. Therefore, the number of distinct hyperplanes in the variety can be no larger than N. Denote by $0 \le M \le N$ the number of such hyperplanes, $H_1, \ldots, H_M$,

$$H_j = \{\, (\tau, \xi) \,:\, \tau = -\mathbf{v}_j.\xi \,\}\,, \qquad j = 1, \ldots, M \le N\,. \tag{3.2.1}$$

Examples. 1. When $d = 1$ the characteristic variety is a union of lines so consists only of hyperplanes. There are no curved sheets.

2. The operator from (3.1.3) has characteristic variety is given by $\tau^2 = |\xi|^2$ so the variety is the classical light cone, and there are no hyperplanes.

3. The characteristic varieties of Mawell's Equations and the the linearization at $u = 0$ of the compressible Euler equations are the union of a convex light cone and a single horizontal hyperplane $\tau = 0$. ∎

Definition. *A wave number $\omega \in \mathbb{R}^d \setminus \{0\}$ is* **good** *when there is a neighborhood Ω of ω and a finite number of real valued real analytic functions $\lambda_1(\xi) < \lambda_2(\xi) < \cdots < \lambda_m(\xi)$ so that the spectrum of $\sum_{j=1}^d A_j \xi_j$ is $\{\lambda_1(\xi), \ldots, \lambda_m(\xi)\}$ for $\xi \in \Omega$. The complementary set consists of* **bad** *wave numbers. The set of bad wave numbers is denoted $\mathcal{B}(L)$.*

Over a good ξ, the characteristic variety of L cotains exactly m nonintersecting sheets $\tau = -\lambda_j(\xi)$. At bad points, eigenvalues cross and multiplicities change. The examples above have no bad points.

Examples. Consider the characteristic equation $(\tau^2 - |\xi|^2)(\tau - c\xi_1) = 0$ with $c \in \mathbb{R}$. If $|c| < 1$ then the variety is a cone and a hyperplane intersecting only at the origin and all points are good. If $|c| > 1$ the plane and cone intersect in a cone whose projection on ξ space is the set of bad points

$$\mathcal{B} = \left\{ \xi \, : \, (c^2 - 1)\xi_1^2 = \xi_2^2 + \ldots + \xi_d^2 \right\}.$$

When $|c| = 1$, $\mathcal{B}(L)$ degenerates to a line of tangency. ∎

Proposition 3.2.1. i. *$\mathcal{B}(L)$ is a closed conic set of measure zero in $\mathbb{R}^d \setminus \{0\}$.*

ii. *The complementary set, $\mathbb{R}^d \setminus (\mathcal{B} \cup \{0\})$, is the disjoint union of a finite family of conic connected open sets $\Omega_g \subset \mathbb{R}^d \setminus \{0\}$, $g \in \mathcal{G}$.*

iii. *The mulitplicity of $\tau = -\mathbf{v}_j.\xi$ as a root of $\det L(\tau, \xi) = 0$ is independent of $\xi \in \mathbb{R}^d \setminus (\mathcal{B} \cup \{0\})$.*

iv. *If $\lambda(\xi) \in C^\omega(\Omega_g)$ is an eigenvalue of $\sum A_j \xi_j$ depending real analytically on ξ, then either there is $j \in \{1, \ldots, M\}$ such that $\lambda(\xi) = -\mathbf{v}_j \cdot \xi$ or $\nabla^2 \lambda \neq 0$ almost everywhere on Ω_g.*

Proof. i. Use the basic stratification theorem of real algebraic geometry (see [BR], [CR]). The characteristic variety is a conic real algebraic variety in $\mathbb{R}^{1+d} \setminus \{0\}$.

Over each ξ it contains at least 1 and at most N points. Therefore its projection on $\mathbb{R}^d_\xi$ is the whole space so the variety has dimension at least d. On the other hand it has measure zero by Fubini's Theorem so the dimension is at most d, since $d+1$ dimensional algebraic sets contain open sets.

The singular points are therefore a stratum of dimension at most $d-1$. The bad frequencies are exactly the projection of this singular locus and so is a real algebraic subvariety of $\mathbb{R}^d_\xi$ of dimension at most $d - 1$ and **i** follows.

ii. That there are at most a finite number of components in the complementary set is a classical theorem of Whitney (see [BR], [CR]).

iii. Denote by m the mulitplicity on Ω_g and m' the mulitplicity on $\Omega_{g'}$. by definition of mulitplicity,

$$\xi \in \Omega_g \quad \text{and} \quad k < m \quad \Longrightarrow \quad \frac{\partial^k \det L(\tau,\xi)}{\partial \tau^k}\Bigg|_{\tau=-\mathbf{v}_j.\xi} = 0\,. \tag{3.2.2}$$

Then $\partial_\tau^k\, L(-\mathbf{v}_j.\xi,\xi)$ is a polynomial in ξ which vanishes on the nonempty open set Ω_g, so must vanish identically. Thus it vanishes on $\Omega_{g'}$ and it follows that $m' \ge m$. by symmetry one has $m \ge m'$.

iv. If λ is a linear function $\lambda = -\mathbf{v}.\xi$ on Ω_g, then $\det L(-\mathbf{v}.\xi,\xi) = 0$ for $\xi \in \Omega_g$ so by analytic continuation, must vanish for all ξ. It follows that the hyperplane $\tau = -\mathbf{v}.\xi$ lies in the characteristic variety and therefore that $\lambda = -\mathbf{v}_j.\xi$ for some j.

If λ is not a linear function, then the matrix $\nabla_\xi^2\lambda$ is a real analytic function on Ω_g which is not identically zero and therefore vanishes at most on a set of measure zero in Ω_g. ∎

Definitions. *Enumerate the roots of* $\det\, L(\tau,\xi) = 0$ *as follows. Let* $\mathcal{A}_f := \{1,\ldots,M\}$ *denote the indices of the flat parts, and for* $\alpha \in \mathcal{A}_f$, $\tau_\alpha(\xi) := -\mathbf{v}_\alpha.\xi$. *For* $g \in \mathcal{G}$ *and* $\xi \in \Omega_g$, *number the roots other than the* $\{\tau_\alpha : \alpha \in \mathcal{A}_f\}$ *in order* $\tau_{g,1}(\xi) < \tau_{g,2}(\xi) < \cdots < \tau_{g,M(g)}$. *Multiple roots are not repeated in this list. Let* $\mathcal{A}_c$ *denote the indices of the curved sheets*

$$\mathcal{A}_c := \big\{\,(g,j)\ :\ g \in \mathcal{G} \ \text{ and } \ 1 \le j \le M(g)\,\big\}\,. \tag{3.2.3}$$

Let $\mathcal{A} := \mathcal{A}_f \cup \mathcal{A}_c$. *For* $\alpha \in \mathcal{A}_f$ *and* $\xi \in \mathbb{R}^d$ *define* $E_\alpha(\xi) := \pi(-\mathbf{v}_j.\xi,\xi)$. *For* $\alpha \in \mathcal{A}_c$ *define*

$$E_\alpha(\xi) := \begin{cases} \pi\big(\tau_\alpha(\xi),\xi\big) & \text{for} \quad \xi \in \Omega_g \\ \\ \quad 0 & \text{for} \quad \xi \notin \Omega_g\,. \end{cases} \tag{3.2.4}$$

The next proposition decomposes an arbitrary solution of $Lu = 0$ as a finite sum of simpler waves.

Proposition 3.2.2. 1. *For each* $\alpha \in \mathcal{A}$, $E_\alpha(\xi) \in C^\omega(\mathbb{R}^d \setminus (\mathcal{B}\cup\{0\}))$ *is an orthogonal projection valued function positive homogeneous of degree zero.*

2. *For each* $\xi \in \mathbb{R}^d \setminus (\mathcal{B}\cup\{0\})$, $\mathbb{C}^N$ *is equal to the orthogonal direct sum*

$$\mathbb{C}^N = \oplus_{\alpha\in\mathcal{A}}\ \text{Image}\, E_\alpha(\xi)\,. \tag{3.2.5}$$

3. *The operators* $E_\alpha(D_x) := \mathcal{F}^* E(\xi)\mathcal{F}$ *are orthogonal projectors on* $H^s(\mathbb{R}^d)$, *and for each* $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ *is equal to the orthogonal direct sum,*

$$H^s(\mathbb{R}^d) = \oplus_{\alpha\in\mathcal{A}}\ \text{Image}\, E_\alpha(D_x)\,. \tag{3.2.6}$$

4. *If* $f \in \mathcal{S}'(\mathbb{R}^d)$ *has Fourier transform equal to a locally integrable function, then the solution of the initial value problem*

$$L(\partial_y)\, u = 0\,, \qquad u|_{t=0} = f \tag{3.2.7}$$

is given by the formula

$$\hat{u}(t,\xi) = \sum_{\alpha\in\mathcal{A}} \hat{u}_\alpha(t,\xi) := \sum_{\alpha\in\mathcal{A}} e^{it\tau_\alpha(\xi)}\, E_\alpha(\xi)\, \hat{f}(\xi)\,. \tag{3.2.8}$$

Remarks. 1. The last decomposition is also written

$$u := \sum_{\alpha\in\mathcal{A}} u_\alpha := \sum_{\alpha\in\mathcal{A}} e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x) f\,.$$

2. Since τ_α is real valued on the support of $E_\alpha(\xi)$ the operator $e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x)$ is a contraction on $H^s(\mathbb{R}^d)$ for all s.

3. If $\alpha\in\mathcal{A}_f$ then $-i\tau_\alpha(D_x) = \mathbf{v}_\alpha.\partial_x$. For $\alpha=(g,j)\in\mathcal{A}_c$, $|\tau_\alpha(\xi)|\le C|\xi|$, so the operator $\tau_\alpha(D_x)f$ is continuous from H^s to H^{s-1}. The mode $u_\alpha = e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x) f$ satisfies $\partial_t u_\alpha = i\tau_\alpha(D_x)u_\alpha$. For $\alpha\in\mathcal{A}_f$ this is $\big(\partial_t + \mathbf{v}_\alpha.\partial_x\big)u_\alpha = 0$, so

$$u_\alpha \;=\; \Big(E_\alpha(D)f\Big)\big(x-\mathbf{v}_\alpha t\big)\,.$$

4. Over $\mathcal{B}(L)$ only the E_α corresponding to the hyperplanes are defined. One does not have a decomposition of $\mathbb{C}^N$. It is important that $\mathcal{B}$ is a negligible set for $\hat f$. The $\hat f\in L^1_{\rm loc}$ assumption in **4** is essential.

§**3.3. Large time asymptotics.**

Definition. *Define $\mathbb{A}$ as the set of tempered distributions whose Fourier transforms belong to $L^1(\mathbb{R}^d)$. Then $\mathbb{A}$ is a Banach space with norm*

$$\|f\|_{\mathbb{A}} := (2\pi)^{-d/2}\int_{\mathbb{R}^d} |\hat f(\xi)|\; d\xi\,. \tag{3.3.2}$$

The Fourier Inversion Formula implies that $\mathbb{A}\subset L^\infty(\mathbb{R}^d)$ and

$$\|f\|_{L^\infty(\mathbb{R}^d)} \;\le\; \|f\|_{\mathbb{A}}\,. \tag{3.3.3}$$

The elements of $\mathbb{A}$ are continuous and tend to zero as $x\to\infty$. Moreover, the Fourier transform of f^2 is a multiple of $\hat f * \hat f$ and therefore in L^1, so $\mathbb{A}$ is an algebra. It is called the *Wiener algebra*. It was a centerpiece of the Tauberian Theorems of N. Wiener.

Theorem 3.3.1 (L^∞ asymptotics for symmetric systems). *Suppose that $f\in\mathbb{A}$ and u is the solution of the initial value problem $L(\partial_x)u=0$, $u|_{t=0}=f$. Then with the notation introduced in the preceding section,*

$$\lim_{t\to\infty}\ \Big\| u(t) - \sum_{\alpha\in\mathcal{A}_f} \left(E_\alpha(D_x)f\right)(x-\mathbf{v}_\alpha t)\Big\|_{L^\infty(\mathbb{R}^d)} = 0\,. \tag{3.3.1}$$

Remarks. 1. This result shows that a general solution of the Cauchy problem is the sum of M rigidly translating waves, one for each hyperplane in the characteristic variety, plus a term which tends to zero in sup norm. The last part decays because of the dispersion of waves.

2. The Theorem does not extend to f whose Fourier Transform is a bounded measure. For example, $u := \big(e^{i(x_1-t)}, 0\big)$ is a solution of (3.1.3) with $\hat f$ equal to a point mass. The characteristic variety contains no hyperplanes so (3.3.1) asserts that solutions with $\hat f \in L^1$ tend to zero in $L^\infty(\mathbb{R}^d)$ while $u(t)$ has sup norm equal to 1 for all t.

Proof of Theorem. Step 1. Approximation-decomposition. Symmetric hyperbolicity implies that for each t, ξ, $\exp\big(it\sum A_j\xi_j\big)$ is unitary on $\mathbb{C}^N$. Therefore $S(t) := \exp\big(-t\sum_j A_j\partial_j\big)$ is isometric on $\mathbb{A}$. Since the family of linear maps

$$f \quad\longmapsto\quad S(t)f - \sum_{\alpha\in\mathcal{A}_f} \big(E_\alpha(D_x)f\big)\,(x - \mathbf{v}_\alpha t)$$

is uniformly bounded from $\mathbb{A}$ to $L^\infty(\mathbb{R}^d)$, it suffices to prove (3.3.1) for a set of f dense in $\mathbb{A}$.

For $\alpha \in \mathcal{A}_c$, Propostion 3.2.1.iv shows that the matrix of second derivatives, $\nabla_\xi^2\tau_\alpha$ can vanish at most on a closed set of measure zero. The set of f we choose is those with

$$\hat f \;\in\; C_0^\infty\Big(\mathbb{R}^d\setminus\big\{\,\mathcal{B}\;\cup\;\{0\}\;\cup\;\bigcup_{\alpha\in\mathcal{A}_c}\{\xi\in\Omega_g\;:\;\nabla_\xi^2\tau_\alpha(\xi)=0\,\}\;\;\big\}\Big)\,.$$

Since the removed set is a closed null set, these f are dense.

To prove (3.3.1) for such f decompose

$$f = \sum_{\alpha\in\mathcal{A}} f_\alpha := \sum_{\alpha\in\mathcal{A}} E_\alpha(D_x)\,f\,,\qquad u(t) = S(t)f = \sum u_\alpha(t) := \sum S(t)\,f_\alpha\,. \tag{3.3.4}$$

For $\alpha \in \mathcal{A}_f$, $u_\alpha(t) = \big(E_\alpha(D_x)f\big)\,(x - \mathbf{v}_\alpha t)$ which recovers the summands in (3.15). To prove (3.15) it suffices to show that for $\alpha \in \mathcal{A}_c$

$$\lim_{t\to\infty}\;\|u_\alpha(t)\|_{L^\infty(\mathbb{R}^d)} = 0\,. \tag{3.3.5}$$

Step 2. Stationary and nonstationary phase. Part 4 of Proposition 3.2. shows that for $\alpha \in \mathcal{A}_c$,

$$u_\alpha(t,x) = \int_{\Omega_g} e^{i(\tau_\alpha(\xi)t + x.\xi)}\;\hat f_\alpha(\xi)\;d\xi\,,\qquad \hat f_\alpha \in C_0^\infty(\Omega_g)\,. \tag{3.3.6}$$

For each ξ in the support of $\hat f_\alpha$, there is a vector $\mathbf{r} \in \mathbb{R}^d$ so that $\langle\nabla_\xi^2\tau(\xi)\,\mathbf{r}, \mathbf{r}\,\rangle \neq 0$ on a neighborhood of ξ. Using a partition of unity we can write $\hat f_\alpha$ as a finite sum of functions

$h_\mu \in C_0^\infty(\Omega_g)$ so that for each μ there is a $\mathbf{r}_\mu \in \mathbb{C}^N$ so that on an open ball containing the support of h_μ, $\langle \nabla^2_\xi \tau(\xi)\, \mathbf{r}_\mu, \mathbf{r}_\mu \rangle \neq 0$. It suffices to show that for each μ

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \int e^{i(\tau_\alpha(\xi)t + x.\xi)}\, h_\mu(\xi)\, d\xi \;=\; 0\,. \tag{3.3.7}$$

For ease of reading we suppress the subscripts. Write $x = tz$. For each $t > 0$, the sup in x is equal to the sup in z so it suffices to show that

$$\lim_{t\to\infty} \sup_{z\in\mathbb{R}^d} \Big| \int e^{it(\tau(\xi) + z.\xi)}\, h(\xi)\, d\xi \Big| \;=\; 0\,.$$

Choose

$$\sigma \;>\; \sup_{\xi\in\operatorname{supp} h} \big|\nabla_\xi \tau(\xi)\big|\,.$$

There is a $\delta > 0$ so that for all $|z| \geq \sigma$,

$$\big|\nabla_\xi (\tau(\xi) + z.\xi)\big| \;\geq\; \delta\,.$$

The method of nonstationary phase implies that

$$\forall N > 0,\ \exists C_N,\ \forall |z| \geq \sigma,\ t > 1, \qquad \Big| \int e^{it(\tau(\xi)+z.\xi)}\, h(\xi)\, d\xi \Big| \;\leq\; C_N\, t^{-N}\,.$$

It remains to show that

$$\lim_{t\to\infty} \sup_{|z|\leq\sigma} \Big| \int e^{it(\tau(\xi)+z.\xi)}\, h(\xi)\, d\xi \Big| \;=\; 0\,. \tag{3.3.8}$$

Make a linear change of variables in ξ so that $\mathbf{r} = (1, 0, \ldots, 0)$ and therefore

$$\frac{\partial^2 \tau}{\partial^2 \xi_1} \neq 0\,, \qquad \text{on} \qquad \operatorname{supp}\, h\,.$$

Choose $R > 0$ so that for $\xi \in \operatorname{supp} h$, $|\xi| \leq R$. Set

$$\Gamma \;:=\; \{|z_1| \leq \sigma\} \times \{|\xi_2, \ldots, \xi_d| \leq R\} \;\subset\; \mathbb{R}^1 \times \mathbb{R}^{d-1}\,.$$

Define

$$\begin{aligned} K(t) \;&:=\; \sup_{|z|\leq\sigma,\ |\xi_2,\ldots,\xi_d|\leq R} \Big| \int e^{it(\tau(\xi)+z_1.\xi_1)}\, h(\xi)\, d\xi_1 \Big| \\ &=\; \sup_{\Gamma} \Big| \int e^{it(\tau(\xi)+z_1.\xi_1)}\, h(\xi)\, d\xi_1 \Big|\,. \end{aligned}$$

Then

$$\sup_{|z|\le\sigma} \ \Big| \int \ e^{it(\tau(\xi)+z.\xi)} \ h(\xi) \ d\xi \ \Big|$$

$$\le \int_{|\xi_2,\ldots,\xi_d|\le R} e^{i(z_2\xi_2+\ldots+z_d.\xi_d)} \ \Big(\int e^{it(\tau+z_1\xi_1)} \ h(\xi) \ d\xi_1\Big) d\xi_2\ldots d\xi_d$$

$$\le \Big|\Big\{|\xi_2,\ldots,\xi_d| \le R\Big\}\Big| \ K(t)\,.$$

It therefore suffices to show that

$$\lim_{t\to\infty} \ K(t) \ = \ 0\,. \tag{3.3.9}$$

The points of Γ are split according to whether the phase $\tau(\xi)+z_1\xi_1$ has a stationary point with respect to ξ_1 or not. If $\underline{\gamma}\in\Gamma$ is such that

$$\left|\frac{\partial\tau}{\partial\xi_1} \ + \ z_1\right| \ > \ \delta \ > \ 0 \quad \text{for all} \quad |z_1|\le\sigma, \ |\xi|\le R,$$

the same is true on a neighborhood of $\underline{\gamma}$. The principal of nonstationary phase shows that

$$\int \ e^{it(\tau_\alpha(\xi)+z.\xi)} \ \hat h_\mu(\xi) \ d\xi_1 \ = \ O(t^{-N})$$

uniformly on such a neighborhood.

On the other hand if for $\underline{\gamma}$ there is a stationary point, then the strict convexity of τ in ξ_1 shows that it is unique and nondegenerate. Therefore for nearby γ there is a nearby unique and nondegenerate stationary point. The inequality of stationary phase (see Appendix) implies that

$$\int \ e^{it(\tau_\alpha(\xi)+z.\xi)} \ \hat h_\mu(\xi) \ d\xi_1 \ = \ O(t^{-1/2})$$

uniformly on a neighborhood of $\underline{\gamma}$.

Covering the compact set Γ by a finite family of neighborhoods proves (3.39) and therefore the Theorem. ∎

Definition. *The operator L* **purely dispersive** *when its characteristic variety contains no hyperplanes. It is call* **nondispersive** *when its characteristic variety is equal to a union of hyperplanes.*

The nondispersive operators have a discrete set of group velocities. The characteristic variety of purely dispersive operators have only curved sheets. The latter name is justified by the next Corollary.

Corollary 3.3.2. *If $L=L_1(\partial_x)$ is a constant coefficient homogeneous symmetric hyperbolic operator, then the following are equivalent.*

1. *The characteristic variety of L contains no hyperplanes (i.e. L is purely dispersive).*

2. *Every solution of $Lu = 0$ with $u\big|_{t=0} \in C_0^\infty(\mathbb{R}^d)$ satisfies,*

$$\lim_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\to\; 0\,. \tag{3.3.10}$$

3. *Every solution of $Lu = 0$ with $u\big|_{t=0} \in \mathbb{A}$ satisfies (3.3.10).*

4. *If $\tau(\xi)$ is a C^∞ solution of $\det L(\tau,\xi) = 0$ defined on a open set of $\xi \in \mathbb{R}^d$ then for every $\mathbf{v} \in \mathbb{R}^d$, $\{\xi \in \mathbb{R}^d : \nabla_\xi \tau = -\mathbf{v}\}$ has measure zero.*

Proof. Theorem 3.3 shows that **1** $\Leftrightarrow$ **3**. To complete the proof we show that **3** $\Leftrightarrow$ **2** and **1** $\Leftrightarrow$ **4**.

The assertions **2** and **3** are equivalent because the family of mappings $u(0) \mapsto u(t)$ is uniformly bounded from $\mathbb{A} \to L^\infty$, and C_0^∞ is dense in $\mathbb{A}$.

That $\sim$ **1** $\Longrightarrow \sim$ **4** is immediate.

If **4** is violated there is a smooth solution τ so that $\nabla_\xi \tau = -\mathbf{v}$ on a set of positive measure. It follows from the Fundamental Stratification Theorem (see [BR],[CR]) that $\nabla_\xi \tau = -\mathbf{v}$ on a conic open real algebraic set of dimension d in $\mathbb{R}^d \setminus 0$. Then $\tau = -\mathbf{v}.\xi$ on this set and we conclude that the polynomial $\det L(-\mathbf{v}.\xi, \xi)$ vanishes on this set and therefore everywhere. Thus the hyperplane $\{\tau = -\mathbf{v}.\xi\}$ is contained in the characteristic variety and **1** is violated.

Thus **1** and **4** are equivalent. ∎

Remark. Part four of this Corollary shows that for any velocity $\mathbf{v}$ the group velocity $-\nabla_\xi \tau$ associated to a curved sheet of the characteristic variety takes the value $\mathbf{v}$ for at most a set of frequencies ξ of measure zero. ∎

The nondispersive evolutions are described in the next results.

Corollary 3.3.3. *If $L = L_1(\partial_y)$ is a constant coefficient homogeneous symmetric hyperbolic operator with $A_0 = I$, then the following are equivalent.*

1. *The characteristic variety of L is a finite union of hyperplanes.*

2. *(Motzkin and Tausky) The matrices A_j commute.*

3. *If u satisfies $Lu = 0$ with $u(0) \in \mathbb{A}$ and $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$, then u is identically equal to zero.*

Proof. **2** $\Rightarrow$ **3**. A unitary change of variable $u = Vv$ replaces the equation $Lu = 0$ with the equivalent equation $\tilde{L}v = 0$ with $\tilde{A}_j := V^* A_j V$. When the A_j commute, V can be chosen so that the $\tilde{A}_j$ are all real diagonal matrices. Property **3** is clear for the tilde equation as each component of the solution rigidly translates as time goes on. The only way its supremum can tend to zero at $t \to \infty$ is for it to vanish.

3 $\Rightarrow$ **1**. This is an immediate consequence of Theorem 3.3.

1 $\Rightarrow$ **2**. This is a result of Motzkin and Tausky.

Theorem 3.3.4. (Motzkin and Tausky) *Suppose that A and B are hermitian $N \times N$ matrices. The eigenvalues of $\xi A + \eta B$ are linear functions of ξ, η if and only if A and B commute.*

Proof. We must show that linear eigenvalue implies commutation. The proof is by induction on N. The case $N = 1$ is trivial. We suppose that $N > 1$ and the result is known for dimensions $\leq N - 1$.

Consider the characteristic variety $\det(\tau + \xi A + \eta B) = 0$. Choose a good point $(\underline{\xi}, \underline{\eta})$ so that above this point the variety has $k \leq N$ real analytic sheets. If $\eta = 0$, leave the spatial coordinates as they are. If $\eta \neq 0$, change orthogonal coordinates in $\mathbb{R}^2$ so that $(\underline{\xi}, \underline{\eta})$ is a mutiple of dy_1. In this way we can without loss of generality assume that above $\eta = 0$ the variety consists of k real analytic sheets.

For s small the eigenvalues of $A + sB$ are real analytic function $\lambda_j(s)$ with $\lambda_j(0) < \lambda_{j+1}(0)$ for $1 \leq j < k - 1$. Denote by μ_j the multiplicity of $\lambda_j(0)$ and therefore of $\lambda_j(s)$ for s small. By hypothesis the $\lambda_j(s)$ are affine functions of s so $\lambda'' = 0$. We use this only at $s = 0$.

By a unitary change of variable in $\mathbb{C}^N$ one can arrange that A is block diagonal with diagonal entries $\lambda_j(0) I_{\mu_j \times \mu_j}$.

Corresponding to this block structure and the eigenvalue λ_1, one has one has

$$\begin{aligned} \pi &= \operatorname{diag}\Big(I_{\mu_1 \times \mu_1}, 0_{\mu_2 \times \mu_2}, \dots, 0_{\mu_k \times \mu_k}\Big), \\ Q &= \operatorname{diag}\Big(0_{\mu_1 \times \mu_1}, \frac{1}{\lambda_2 - \lambda_1} I_{\mu_2 \times \mu_2}, \dots, \frac{1}{\lambda_k - \lambda_1} I_{\mu_k \times \mu_k}\Big). \end{aligned} \tag{3.3.11}$$

The matrix B has block structure

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \dots\dots & B_{1,k} \\ B_{2,1} & B_{2,2} & \dots\dots & B_{2,k} \\ & & & \\ B_{k,1} & B_{k,2} & \dots\dots & B_{k,k} \end{pmatrix},$$

with B_{ij} a $\mu_i \times \mu_j$ matrix and $B_{ij} = B^*_{ji}$.

The fundamental formula of second order perturbation theory (see Appendix) yields $\lambda'' \pi = 2\pi B Q B \pi$. By hypothesis this is equal to zero.

Straightforward calculation shows that

$$\pi B = \begin{pmatrix} B_{1,1} & B_{1,2} & \dots\dots & B_{1,k} \\ 0 & 0 & \dots\dots & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad QB\pi = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \frac{1}{\lambda_2 - \lambda_1} B_{2,1} & 0 & \dots & 0 \\ & & & \\ \frac{1}{\lambda_k - \lambda_1} B_{k,1} & 0 & \dots & 0 \end{pmatrix}.$$

Therefore, the $\mu_1 \times \mu_1$ upper left hand block block of $\pi Q B Q \pi$ is equal to

$$\sum_{j=2}^{k} \frac{1}{\lambda_j - \lambda_1} B_{1,j} B^*_{1,j}.$$

Conclude that this sum of positive square matrices vanishes. Thus, for $j \geq 2$, $B_{1,j} = 0$ and $B_{j,1} = 0$.
Thus B and A are reduced by the splitting

$$\mathbb{C}^N = \mathbb{C}^{\mu_1} \times \mathbb{C}^{N-\mu_1}\,.$$

The commutation then follows by the inductive hypothesis applied to the diagonal blocks. This proves the case N and completes the induction. ▮

This completes the proof of Corollary 3.3.3. ▮

Example. The implication **1** $\Rightarrow$ **2** is not true without the symmetry hypothesis. For example, the hypberbolic system

$$\partial_t \;+\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 \;+\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \partial_2$$

has flat characteristic variety with equation

$$(\tau + \xi_1 + \xi_2)(\tau - \xi_1 + \xi_2) \;=\; 0\,,$$

and the coefficient matrices do not commute. The conclusion is correct assuming that the hyperbolic system generates a semigroup in $L^2(\mathbb{R}^d)$ (see [GR, Hyperbolic multipliers are translations]).

Theorem (P. Brenner). *If $L = L(\partial_y)$ is a constant coefficient homogeneous symmetric hyperbolic operator then the conditions of Corollary 3.3.3 are equivalent to each of the following.*
i. *For all $t \in \mathbb{R}$ and $p \in [1,\infty]$ the Fourier multiplication operator*

$$S(t) := \mathcal{F}^{-1}\, e^{-it\sum A_j\xi_j}\, \mathcal{F}$$

is a bounded from $L^p(\mathbb{R}^d)$ to itself.
ii. *For some $\underline{t} \in \mathbb{R}\setminus 0$ and $2 \neq \underline{p} \in [1,\infty]$ the operator $S(\underline{t})$ is bounded from $L^{\underline{p}}(\mathbb{R}^d)$ to itself.*

Remark. The Fourier multiplication operators are unitary on L^2. The properties **ii** means that the restriction to $\mathcal{S}(\mathbb{R})$ extend to bounded operators on L^p, equivalently

$$\sup_{f\in\mathcal{S}(\mathbb{R}^d)\setminus 0} \frac{\|S(t)f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} \;<\; \infty\,.$$

Proof. The conditions of Corollary 3.3.3 implies that after an orthogonal change of basis, the A_j are all real diagonal matrices. It is then elementary to verify that **i** is satisfied.

Clearly **i** implies **ii**. It remains to show that **ii** implies the conditions of the Corollary.

If the conditions of the the Corollary are violated, then **ii** is violated. The first remark is that **ii** is stronger than it appears. Since $S(\underline{t})$ is unitary on L^2, if **ii** is satisfied then $S(t)$ is bounded on L^p for all p between 2 and $\underline{p}$. Thus we may assume that $\underline{p}$ in not equal to 1 or ∞.

For $\sigma \in \mathbb{R} \setminus 0$, $Lu = 0$ if and only if $u^\sigma(t,x) := u(\sigma t, \sigma x)$ satisfies $Lu^\sigma = 0$. It follows that if **ii** is satisfied then

$$\|S(t)\|_{\mathrm{Hom}(L^{\underline{p}})} \;=\; \|S(\underline{t})\|_{\mathrm{Hom}(L^{\underline{p}})} \;<\; \infty\,, \qquad \forall\, t \neq 0\,. \tag{3.3.12}$$

If $\underline{q}$ is the conjugate index to $\underline{p}$, that is $\frac{1}{\underline{p}} + \frac{1}{\underline{q}} = 1$, then

$$\begin{aligned} \|S(t)\|_{\mathrm{Hom}(L^{\underline{q}}(\mathbb{R}^d))} \;&=\; \sup_{f,g\in\mathcal{S}\setminus 0} \frac{\big(S(t)f\,,\,g\big)}{\|f\|_{L^{\underline{q}}(\mathbb{R}^d)}\ \|g\|_{L^{\underline{p}}(\mathbb{R}^d)}} \\ &=\; \sup_{f,g\in\mathcal{S}\setminus 0} \frac{\big(f\,,\,S(-t)g\big)}{\|f\|_{L^{\underline{q}}(\mathbb{R}^d)}\ \|g\|_{L^{\underline{p}}(\mathbb{R}^d)}} \;=\; \|S(-t)\|_{\mathrm{Hom}(L^{\underline{p}}(\mathbb{R}^d)}\,. \end{aligned}$$

Thus when **ii** is satisfied for $\underline{p}$ it is satisfied for $\underline{q}$ so we may suppose that $\underline{p} > 2$.

When the conditions of Corollary 3.3.3 are violated, there is a conic set of good points Ω_g and a sheet $\tau = \tau(\xi)$ over Ω_g with $\nabla^2_{\xi\xi}\tau \neq 0$ for almost all $\xi \in \Omega_g$. Denote by $\pi(\xi)$ with associated spectral projection. Choose an $f \in \mathcal{S}(\mathbb{R}^d)$ with $\hat f$ compactly supported in Ω_g. Replacing $\hat f$ by $\pi(\xi)\hat f$ we may assume that $\pi(D)f = f$. Theorem 3.3.1 implies that

$$\lim_{t\to\infty} \ \|S(t)\,f\|_{L^\infty(\mathbb{R}^d)} \;=\; 0\,.$$

Then

$$\|S(t)f\|^{\underline{p}}_{L^{\underline{p}}(\mathbb{R}^d)} \;\le\; \|S(t)f\|^{\underline{p}-2}_{L^\infty(\mathbb{R}^d)}\ \|S(t)f\|^2_{L^2(\mathbb{R}^d)} \;=\; \|S(t)f\|^{\underline{p}-2}_{L^\infty(\mathbb{R}^d)}\ \|f\|^2_{L^2(\mathbb{R}^d)} \;\to\; 0\,,$$

as $t \to \infty$.

Therefore,

$$\big\|S(-t)\big\|_{\mathrm{Hom}(L^{\underline{p}})} \;\ge\; \frac{\big\|S(-t)\big(S(t)f\big)\big\|_{L^{\underline{p}}}}{\big\|S(t)f\big\|_{L^{\underline{p}}}} \;=\; \frac{\big\|f\big\|_{L^{\underline{p}}}}{\big\|S(t)f\big\|_{L^{\underline{p}}}} \;\to\; \infty\,.$$

Thus (3.3.12) is violated and the proof is complete. ∎

Example. It may seem that (3.3.12) together with $\lim_{t\to 0} S(t)f = f$ might imply that $S(t)$ has norm equal to 1. That this is not true can be seen from the one dimensional example

$$\partial_t \;+\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_x\,,$$

and L^p norm chosen so that for $p=2$ one has unitarity,

$$\|(u_1,u_2)\|_p \ := \ \int \|(u_1,u_2)\|^p \ dx\big)^{1/p}\,, \qquad \|(u_1,u_2)\| := \big(|u_1|^2+|u_2|^2\big)^{1/2}\,.$$

Choosing $u_1(0)=u_2(0)=f\in C_0^\infty(\{|x|\le\rho\})$ one has

$$\|u(0)\|_p^p \ = \ (\sqrt{2})^p\|f\|_p^p\,,$$

and for $|t|>\rho$,

$$\|u(t)\|_p^p = 2\,\|f\|_p^p\,.$$

It follows that for all $t\neq 0$ and $p<2$, $\|S(t)\|^p_{\mathrm{Hom}(L^p)}\ge 2^{1-p/2}>1$. Reversing time, treats $p>2$.

§3.4. Maximally dispersive systems.

§3.4.1. The $L^1\to L^\infty$ decay estimate.

If $\tau=\tau(\xi)$ parametrizes a real analytic patch of the characteristic variety of a hyperbolic operator then τ is homogeneous of degree 1 in ξ. The group velocity $\mathbf{v}(\xi)=-\nabla_\xi\tau(\xi)$ is homogeneous of degree 0. Therefore $\xi.\nabla_\xi\mathbf{v}=0$ so ξ belongs to the kernel of the symmetric matrix $\nabla_\xi\mathbf{v}(\xi)=\nabla_\xi^2\tau(\xi)$. Thus the rank of $\nabla_\xi^2\tau$ is at most $d-1$. When the rank is equal to $d-1$ the group velocity depends as strongly on ξ as possible. The dispersion is as strong as possible.

Definition. *The homogeneous constant coefficient symmetric hyperbolic operator is* **maxmally dispersive** *when*

$$\mathrm{Char}\,L \ = \ \cup_{j=1}^m\Big\{(\tau,\xi)\ :\ \tau=\tau_j(\xi)\Big\}$$

where for $\xi\in\mathbb{R}^d\setminus 0$

$$\tau_1(\xi)<\tau_2(\xi)\ldots<\tau_m(\xi)\,,$$

the τ_j *are real analytic, positive homogeneous of degree one in* ξ*, and*

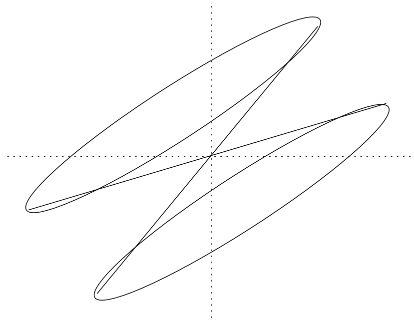
$$\forall j,\ \ \forall\,\xi\in\mathbb{R}^d\setminus 0\,,\qquad \mathrm{rank}\,\nabla_\xi^2\tau(\xi)\ =\ d-1\,. \tag{3.4.1}$$

Examples. i. The simplest example is

$$(\tau^2-|\xi|^2)\,(\tau^2-c^2|\xi|^2)\ =\ 0\,,\qquad 0<c\neq 1\,.$$

The variety in this case consists of two sheets $\tau=|\xi|$ and $\tau=c|\xi|$ which have $d-1$ strictly positive principal curvatures. The other sheets bound $\tau\le-|\xi|$ and $\tau\le -c\,|\xi|$ and have $d-1$ strictly negative curvatures.

ii. The next figure gives an example with two sheets bounding strictly convex regions for which the functions τ_j change sign. In particular the generator $G = -\sum A_j \partial_j$ is not elliptic since the points where the cone crosses $\tau = 0$ are characteristic for G.



The next result is closely related to Hadamard's Ovaloid Theorem which is recalled in Appendix III.

Propostion 3.4.1. *If $\tau(\xi)$ is smooth in $\xi \neq 0$, homogeneous of degree one and the hessian has rank equal to $d-1$ at all points, then the nonzero eigenvalues of $\nabla_\xi^2 \tau$ have the same sign. When they are positive (resp. negative) τ is convex (resp. concave).*

Proof. When $d = 2$, $\nabla_\xi^2 \tau$ has only one nonzero eigenvalue and the result is immediate. For $d \geq 3$, consider the mapping

$$\Gamma(\xi) \ := \ \mathbf{v}(\xi) \ = \ -\nabla_\xi \tau(\xi)\,.$$

The differential of the mapping Γ is equal to $-\nabla_\xi^2 \tau$ so ξ is in its kernel and it is invertible when restricted to the orthogonal to ξ.

Since Γ is homogeneous of degree 0, it is natural to consider Γ as a map from $S^{d-1} = \{|\xi| = 1\}$. As such it is an immersion onto a compact $d-1$ dimensional manifold, $\mathcal{M}$. The image is oriented by the image of the orientation of S^{d-1}.

Since ξ is orthogonal to the image of $-\nabla_\xi^2 \tau(\xi)$ it follows that the ξ is the unit normal to $\mathcal{M}$ at $\Gamma(\xi)$. Thus, at least locally, Γ is the inverse of the Gauss map of $\mathcal{M}$. Since the differential is invertible it follows that the Gauss curvature of $\mathcal{M}$ is nowhere vanishing.

Since $\xi \in \ker(\nabla_\xi^2 \tau(\xi)$, the unit normal to $\mathcal{M}$ at $\mathbf{v}(\xi)$ is equal to ξ. Since the map from $\xi \in S^{d-1}$ to $\mathbf{v}(\xi)$ has invertible jacobian, the Gauss curvature of $\mathcal{M}$ is nowhere vanishing.

Since $d \geq 3$, it follows from Hadamard's Ovaloid Theorem, that $\mathcal{M}$ is the boundary of a strictly convex set and $\Gamma : S^{d-1} \to \mathcal{M}$ is a diffeomorphism.

Thus each value $-\nabla_\xi \tau(\xi) \in \mathcal{M}$ is attained at a unique $\xi \in S^{d-1}$.

The normals to $\tau = \tau(\xi)$ are the nonzero multiples of $(1, v(\xi))$. Thus, the hyperplane $\{\tau + v(\underline{\xi}).\xi = 0\}$ is tangent at $\tau = \tau(\underline{\xi})$ and at no other point $\tau = \tau(\underline{\xi}')$ with $\underline{\xi}' \in S^{d-1}$.

It follows that the cone $\tau = \tau(\xi)$ is strictly convex in the sense that its intersection with its tangent plane conists exactly of the line $(\mathbb{R} \setminus 0)(\tau(\xi), \xi)$.

This implies that the $d-1$ nonzero eigenvalues must have one sign. ∎

Examples. The characteristic variety of a maximally dispersive system consists of m disjoint sheets, each the boundary of a strictly convex cone.

Lemma 3.4.2 (Pointwise decay). *If $d \geq 2$, τ is as above and $k \in C_0^\infty(\mathbb{R}^d \setminus 0)$ then there is a constant C so that*

$$u(t,x) \;:=\; \int e^{it\tau(\xi)}\; e^{ix.\xi}\; k(\xi)\; d\xi\,,$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\leq\; C\,(1+|t|)^{-(d-1)/2}\,. \tag{3.4.2}$$

Remark. This is the decay rate for solutions of $\Box_{1+d} u = 0$ which corresponds to the choice $\tau(\xi) = \pm|\xi|$.

Proof. The easy estimate

$$\|u(t,x)\|_{L^\infty(\mathbb{R}^d)} \;\leq \int |k(\xi)|\; d\xi\,,$$

shows that only the decay for $|t| \geq 1$ needs to be proved.

Let

$$y \;:=\; \frac{x}{t}\,, \qquad x \;=\; ty\,.$$

Then

$$\sup_x \big|u(t,x)\big| \;=\; \sup_y \big|u(t,ty)\big| \;=\; \sup_y \Big|\int e^{it(\tau(\xi)+y.\xi)}\; k(\xi)\; d\xi\Big|\,.$$

The phase $\tau(\xi) + y.\xi$ is stationary when

$$-\nabla_\xi \tau(\xi) \;=\; y\,.$$

The left hand side is the group velocity.

As in Lemma 3.4.1, denote by $\mathcal{M}$ the set of attained group velocities which is an embedded strictly convex compact $d-1$ manifold.

For any open neighborhood $\mathcal{O}$ of $\mathcal{M}$, the method of nonstationary phase shows that for any N,

$$\sup_{y\in\mathbb{R}^d\setminus\mathcal{O}} \;\Big|\int e^{it(\tau(\xi)+y.\xi)}\; k(\xi)\; d\xi\Big| \;\leq\; C_N\,|t|^{-N}\,,$$

as $t \to \infty$.

Choose $0 < r_1 < r_2$ so that

$$\operatorname{supp} k \;\subset\; \{r_1 \;\leq\; |\xi| \;\leq\; r_2\}\,.$$

Write

$$\int e^{it(\tau(\xi)+y.\xi)}\ k(\xi)\ d\xi\ =\ \int_{r_1}^{r_2}\Big(\int_{|\xi|=1} e^{it(\tau(\xi)+y.\xi)}\ k(r\,\xi)\ d\sigma(\xi)\Big)\ r^{d-1}\ dr\,.$$

It suffices to show that for any $\underline{y}\in\mathcal{M}$ and $\underline{r}\in[r_1,r_2]$ one has

$$\int_{|\xi|=1} e^{it(\tau(\xi)+y.\xi)}\ k(r\,\xi)\ d\sigma(\xi)\ \le\ C\,|t|^{-(d-1)/2}\,,$$

uniformly for r, y in a neighborhood of $\underline{r}, \underline{y}$.

For $\underline{r}, \underline{y}$ fixed, there is a unique $\underline{\xi}$ with $|\underline{\xi}| = \underline{r}$ for which the phase is stationary and the stationary point is nondegenerate because of the rank equal to $d-1$ hypothesis. It follows that for r, y in a neighborhood, there is a unique uniformly nondegenerate startionary point. The desired estimate follows from the inequality of stationary phase (see Appendix II). ∎

Proposition 3.4.3. *Suppose that* $0 < R_1 < R_2 < \infty$ *and* $\omega := \{\xi\in\mathbb{R}\ :\ R_1 < |\xi| < R_2\}$. *There is a constant* C *so that for all* $f\in L^1(\mathbb{R}^d_x)$ *with* $\operatorname{supp}\hat f\ \subset\ \overline{\omega}$,

$$u(t,x)\ :=\ (2\pi)^{-d/2}\int e^{i(t\tau_j(\xi)+x.\xi)}\ \hat f(\xi)\ d\xi\ :=\ e^{it\tau_j(D_x)}\,f$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)}\ \le\ C\,(1+|t|)^{-(d-1)/2}\,\|f\|_{L^1(\mathbb{R}^d)}. \tag{3.4.3}$$

The proof is based on a simple idea. The solution u is equal to the convolution of the fundamental solution with f. The Fourier transform of the fundamental solution at $t=0$ is equal to a constant. To have an analogous but more regular representation, it is sufficient that one convolve with a solution whose initial data has Fourier Transform equal to this constant on the spectrum of f.

Proof. Choose a $k\in C_0^\infty(\mathbb{R}^d\setminus 0)$ with k equal to $(2\pi)^{-d/2}$ on a neighborhood of $\overline{\omega}$. Define G so that $\hat G := k$. Then since $(2\pi)^{d/2}\,k\,\hat f = \hat f$ one has $G * f = f$. Since $e^{it\tau(D_x)}$ is a Fourier multiplier, one has

$$u(t) := e^{it\tau(D_x)}\,f = e^{it\tau(D_x)}\big(f*G\big) = f*\big(e^{it\tau(D_x)}\,G\big)\,.$$

Then

$$\|u(t)\|_{L^\infty}\ \le\ \|f\|_{L^1}\,\|e^{i\tau(D_x)}\,G\|_{L^\infty}\,.$$

The preceding Lemma shows that

$$\|e^{i\tau(D_x)}\,G\|_{L^\infty}\ \le\ C\,(1+|t|)^{-(d-1)/2}\,.$$ ∎

The next subsections consist of two different paths for exploiting the estimates just proved. The first is more elementary and will be used in Chapter 6 to derive, in the spirit of John-Klainerman, that in high dimensions there is global solvability for maximally dispersive nonlinear problems with small data. The second is devoted to Strichartz estimates which are important in trying to treat existence problems with low regularity data. That in turn is important in trying to pass from local solvability to global solvability for nonlinear problems for which the natural *a priori* estimates control few derivatives.

§3.4.2. Fixed time dispersive Sobolev estimates.*

We next find decay estimates for $\|u(t)\|_{L^1}$ for sources with Fourier transform supported in $\lambda\overline{\omega}$ for $0<\lambda$. The starting point is

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|f\|_{L^1(\mathbb{R}^d)}\,, \qquad \operatorname{supp}\hat f\subset\omega\,. \tag{3.4.4}$$

Proposition 3.4.4. *There is a constant C so that for all $\lambda>0$ and $f\in L^1$ with $\operatorname{supp}\hat f\subset\lambda\omega$, the solution of*

$$L\,u \;=\; 0\,, \qquad u\Big|_{t=0} \;=\; f\,,$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|\,|D|^{(d+1)/2}f\|_{L^1(\mathbb{R}^d)}\,. \tag{3.4.5}$$

First verify the dimensions of the homogeneous estimate (3.4.5). With t,x having the dimensions of a length ℓ, the factor $|t|^{(d-1)/2}$ has dimension $\ell^{(d-1)/2}$. On the other hand, in

$$\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)} \;=\; \int \big||D|^\gamma f\big|\;dx$$

the integrand has dimension $\ell^{-\gamma}$ and dx has dimension ℓ^d. In total the right hand side of (3.4.5) has dimension $\ell^{d-\gamma-(d-1)/2}$. It is dimensionless as is the left hand side exactly when

$$\gamma \;:=\; \frac{d+1}{2}\,.$$

Proof. Choose $\psi\in C_0^\infty(\mathbb{R}^d_\xi)$ so that $\psi_\pm=|\xi|^{\pm\gamma}$ on $\overline{\omega}$. Then

$$|D|^\gamma f \;=\; C\,\hat\psi_+ * f\,, \qquad \text{and} \qquad f \;=\; C\,\hat\psi_- * (|D|^\gamma f)\,.$$

Young's inequality implies that $\|\,|D|^\gamma f\|_{L^1}$ is a norm equivalent to that on the right in (3.4.4) so

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)}\,, \qquad \operatorname{supp}\hat f\subset\omega\,.$$

* The material in this subsection is not needed for the Strichartz estimates in the next subsection

If $u_\lambda(t,x) := u(\lambda t, \lambda x)$ then, $Lu_\lambda = 0$ if and only if $Lu = 0$ and $\hat u_\lambda(\lambda t, \xi) = \lambda^{-d}\hat u(t, \xi/\lambda)$. The spectrum of u is contained in ω if and only if the spectrum of u_λ is contained in $\lambda\omega$.

Exercise. *Show that if* $f_\lambda(x) := f(\lambda x)$,

$$|D|^\gamma f_\lambda(x) \;=\; \lambda^{-\gamma}\big(|D|^\gamma f\big)(\lambda x)\,.$$

The change of variable $z = \lambda x$ yields

$$\|\,|D|^\gamma f_\lambda\|_{L^1(\mathbb{R}^d)} \;=\; \int \lambda^{-\gamma}\big(|D|^\gamma f\big)(\lambda x)\; dx \;=\; \lambda^{-\gamma-d}\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)}\,.$$

Then (3.4.3) yields

$$\begin{aligned}
\big\|u_\lambda(t)\big\|_{L^\infty} \;&=\; \big\|u(\lambda t)\big\|_{L^\infty} \;\le\; C\,\big|\lambda t\big|^{-(d-1)/2}\,\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)} \\
&=\; C\,\big|\lambda t\big|^{-(d-1)/2}\,\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)} \\
&=\; C\,\lambda^{-(d-1)/2+\gamma+d}|t|^{-(d-1)/2}\|\,|D|^\gamma f_\lambda\|_{L^1(\mathbb{R}^d)}\,.
\end{aligned}$$

The choice $\gamma = (d+1)/2$ is made so that the λ factors cancel. ∎

Since $\hat u$ and $\hat f$ are locally integrable functions, the point $\xi = 0$ is negligible so we have the Littlewood-Paley decompositions

$$u \;=\; \sum_{j=-\infty}^{\infty} \chi(2^{-j}D_x)\,u \;:=\; \sum_{j=-\infty}^{\infty} u_j\,, \qquad f \;=\; \sum_{j=-\infty}^{\infty}\chi(2^{-j}D)\,f \;:=\; \sum_{j=-\infty}^{\infty} f_j\,,$$

where the dyadic decomposition of unity is constructed in the Appendix on the stationary phase inequality. This expresses a solution of $Lu = 0$ as a sum of spectrally localized solutions. The estimates of the next exercise show that $|D|^\sigma$ acts like multiplication by $2^{\sigma j}$ on f_j.

Exercise. *Show that there is an integer k and a constant C depending on σ and χ so that for $p \in [1,\infty]$*

$$\big\|\,|D|^\sigma f_j\big\|_{L^p} \;\le\; C\; 2^{\sigma j}\sum_{|n-j|\le k}\|f_n\|_{L^p}\,, \tag{3.4.6}$$

$$\big\|\,f_j\big\|_{L^p} \;\le\; C\,2^{-\sigma j}\sum_{|n-j|\le k}\|\,|D|^\sigma f_n\|_{L^p}\,. \tag{3.4.7}$$

Theorem 3.4.5. i. *If $Lu = 0$ and $u\big|_{t=0} = f$ then,*

$$\|u\|_{L^\infty} \;\le\; C\,|t|^{-(d-1)}\sum_{j=-\infty}^{\infty}\|\,|D|^\gamma f_j\|_{L^1}\,, \qquad \gamma \;=\; \frac{d+1}{2}\,. \tag{3.4.8}$$

ii. *If* $0 < \delta < \gamma$ *there is a constant* $C(\gamma, \delta)$ *so that*

$$\sum_{j=-\infty}^{\infty} \| \, |D|^{\gamma} f_j \|_{L^1} \ \leq \ C \Big(\big\| \, |D|^{\gamma-\delta} f \big\|_{L^1(\mathbb{R}^d)} \ + \ \big\| \, |D|^{\gamma+\delta} f \big\|_{L^1(\mathbb{R}^d)} \Big) \,. \tag{3.4.9}$$

Remarks. 1. The sum on the right of (3.4.8) is the definition of the norm in the homogeneous Besov space $\dot{B}^{\gamma}_{1,1}$. Estimate (3.4.9) yields a bound which is not as sharp but avoids these spaces.

2. A slightly weaker estimate than (3.4.8-3.4.9) was proved by [Lucente-Ziliotti].

S. Lucente and G. Ziliotti, A decay estimate for a class of hyperbolic pseudo-differential equations, Math. App. **10**(1999), 173-190, Atti, Acc. Naz. Linccei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9).

3. It is impossible to have a decay estimate of the form

$$\|u(t)\|_{L^\infty} \ \leq \ g(t) \|f\|_{H^s} \,, \qquad \lim_{t\to\infty} g(t) = 0 \,,$$

with a conserved norm on the right hand side. If there were such an estimate one can apply it to $v(t) = v(t-T)$ at $t = T \to \infty$ to find

$$\|u(0)\|_{L^\infty} \leq g(T) \, \|f\|_{H^s} \ \to \ 0 \,.$$

The appearance of norms which are not propagated by the equation is necessary.

4. An L^1 condition encodes more rapid decay as $|x| \to \infty$ than an L^2 condition. This is natural since the energy in a ring $R < |x| < R+1$ can focus at time $t \sim R$ into a ball of radius $O(1)$. If the amplitude in the initial ring in $\sim a$ the L^2 norm is $\sim a^2 \, R^{d-1}$. If the focused amplitude is $\sim A$ one obtains $A^2 \sim a^2 \, R^{d-1}$. If this focussing is to take place at $t \sim R$ and also $A^2 \leq t^{-(d-1)}$ that yields $a \leq R^{-(d-1)}$ which is on the L^1 borderline. Thus one cannot have $t^{-(d-1)/2}$ decay estimates as in the Theorem with L^p norms on the right with $p > 1$.

Proof of Theorem. i. Estimate (3.4.5) implies

$$\|u_j(t)\|_{L^\infty} \ \leq \ C \, |t|^{-(d-1)} \, \| \, |D|^{\gamma} f_j \|_{L^1} \,.$$

Summing yields

$$\|u\|_{L^\infty} \ \leq \ \sum \|u_j\|_{L^\infty} \ \leq \ C \, |t|^{-(d-1)} \sum \| \, |D|^{\gamma} f_j \|_{L^1} \,.$$

ii. For $j \geq 0$, estimate (3.4.6) implies

$$\big\| \, |D|^{\gamma} f_j \big\|_{L^1} \ \leq \ C \ 2^{\gamma j} \sum_{|n-j| \leq k} \big\| f_n \big\|_{L^1} \,.$$

Estimate (3.4.7) implies

$$\big\|f_n\big\|_{L^1} \;\le\; C\, 2^{-\sigma n} \sum_{|m-n|\le k} \big\|\,|D|^\sigma f_m\big\|_{L^1}\,.$$

Finally,

$$\big\|\,|D|^\sigma f_m\big\|_{L^1} \;\le\; C\big\|\,|D|^\sigma f\big\|_{L^1}\,.$$

Combining yields

$$\sum_{j\ge 0} \|\,|D|^\gamma f_j\|_{L^1} \;\le\; C\,\big\|\,|D|^\sigma f\big\|_{L^1} \sum_{j\ge 0}\;\sum_{|n-j|\le k} 2^{\gamma j-\sigma n}\;\;.$$

With $\sigma=\gamma+\delta$, the sum is finite, so

$$\sum_{j\ge 0} \|\,|D|^\gamma f_j\|_{L^1} \;\le\; C\,\big\|\,|D|^{\gamma+\delta} f\big\|_{L^1}\,.$$

Exercise. *Prove the complementary estimate*

$$\sum_{j< 0} \|\,|D|^\gamma f_j\|_{L^1} \;\le\; C\,\big\|\,|D|^{\gamma-\delta} f\big\|_{L^1}\,.$$

This completes the proof. ∎

Corollary 3.4.6. *For any $d/2>\delta>0$ there is a constant C so that if $Lu=0$, then*

$$\begin{aligned}\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,\langle t\rangle^{-(d-1)/2}\Big(&\|f\|_{H^{d/2+\delta}(\mathbb{R}^d)}\\ &+\|\,|D|^{(d+1)/2+\delta} f\|_{L^1(\mathbb{R}^d)} \;+\; \|\,|D|^{(d+1)/2-\delta} f\|_{L^1(\mathbb{R}^d)}\Big)\,.\end{aligned} \tag{3.4.10}$$

Remark. The smaller is $\delta>0$ the stronger is the conclusion.

Proof. Sobolev's inequality yields

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,\|u(t)\|_{H^{\delta+d/2}(\mathbb{R}^d)} \;=\; C\,\|f\|_{H^{\delta+d/2}(\mathbb{R}^d)}\,.$$

This yields (3.4.10) for $|t|\le 1$.
For $|t|\ge 1$ use the two estimates of the Theorem. ∎

§3.4.3. Strichartz estimates.

The estimates involve norms

$$\|u\|_{L^q_t L^r_x} \ := \ \left(\int_0^\infty \|u(t)\|^q_{L^r(\mathbb{R}^d_x)} \ dt \right)^{1/q}$$

which integrate over space and time. If such a norm is finite, then the integrand must be small for large times. This requires $r > 2$. The estimates express time decay because of dispersion.

The group velocities lie on the strictly convex manifold $\mathcal{M}$. For a typical Fourier Transform, an open set of these velocites is sampled. The method of nonstationary phase shows that for large time the solution is concentrated on the rays with these speeds, starting from the support of the initial data. Thus, a solution is expected to be concentrated on and spread over a region of measure which grows like t^{d-1}. An example is concentration in an annulus $\rho_1 < |x| - t < \rho_2$. Or even finer, concentration on that part of the annulus subtending a fixed solid angle.

The conservation of $L^2(\mathbb{R}^d)$ and also Lemma 3.4.2 show that the expected amplitude is $O(t^{-(d-1)/2})$. Then

$$\|u(t)\|^r_{L^r} \ \sim \ t^{-r(d-1)/2} \ t^{d-1} \,,$$

so

$$\|u\|^q_{L^q_t L^r_x} \ \sim \ \int_1^\infty \left(t^{-r(d-1)/2} \ t^{d-1} \right)^{q/r} \ dt \,.$$

The limiting indices are those for which the power of t is equal to -1, that is with

$$\sigma \ := \ d - 1 \,,$$

$$\left(\frac{-r\sigma}{2} + \sigma \right) \frac{q}{r} \ = \ -1 \,, \qquad \text{equivalently,} \qquad \frac{-\sigma}{2} \ + \ \frac{\sigma}{r} \ = \ \frac{-1}{q} \,.$$

The admissible indices are those for which the power is less than or equal to -1,

$$\frac{-\sigma}{2} \ + \ \frac{\sigma}{r} \ \le \ \frac{-1}{q} \,.$$

Definitions. *The pair* $2 < q, r < \infty$ *is* σ-**admissible** *if*

$$\frac{1}{q} \ + \ \frac{\sigma}{r} \ \le \ \frac{\sigma}{2} \,.$$

It is **sharp** *σ-admissible when equality holds.*

The estimates involve the homogeneous Sobolev norms

$$\| \, |D|^\gamma f \|_{L^2} \ := \ \left(\int \big| \, |\xi|^\gamma \hat{f}(\xi) \big|^2 \ d\xi \right)^{1/2} .$$

Theorem 3.4.7 (Strichartz inequalities). *Suppose that $L(\partial)$ is maximally dispersive, $\sigma = d-1$, q, r is σ-admissible, and γ is the solution of*

$$\frac{1}{q} \;+\; \frac{d}{r} \;=\; \frac{d}{2} - \gamma\,.$$

There is a constant C so that for $f \in L^2$ with $\| \, |D|^\gamma f\|_{L^2} < \infty$, the solution of $Lu = 0$, $u|_{t=0} = f$ satisfies

$$\big\| u \big\|_{L^q_t L^r_x} \;\le\; C \, \big\| \, |D|^\gamma f \big\|_{L^2(\mathbb{R}^d)}\,. \tag{3.4.11}$$

There are two complicated relations in this assertion. The first is the definition of admissibility. It is the crucial one which encodes the rate of decay of solutions. The second is the definition of γ. Once admissible q, r are chosen, γ is forced so that the two sides of (3.4.11) scale the same for $(t, x) \mapsto (at, ax)$. From this perspective the dispersion is key as it constrains the q, r.

There is a diametrically opposite perspective which starts from the scaling relation which is independent of the dispersion. For example if you are obliged to work with a specific γ then the scaling restricts $1/q, 1/r$ to lie on a line. Then the admissability chooses an interval on that line. Changing the dispersion, for example considering a problem with the same scaling but weaker dispersion leaves the line fixed but constrains the $1/q, 1/r$ to lie on a smaller subinterval.

We follow the proof of [Keel-Tao]. Another standard reference is [Ginibre-Velo]. The limit point case is treated in the first reference. The key step is an estimate for spectrally localized data.

Lemma 3.4.8. *Suppose that $\sigma := d-1$, q, r is σ-admissible, and ω is as in the Corollary. There is a constant C so that for all $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat f \subset \overline{\omega}$,*

$$u(t) \;:=\; e^{it\tau_j(D_x)} f \;:=\; U(t) f\,, \qquad U(t)^* = U(-t)\,,$$

satisfies

$$\big\| u \big\|_{L^q_t L^r_x} \;\le\; C \, \big\| f \big\|_{L^2}\,. \tag{3.4.12}$$

Futhermore, for all $F \in L^{q'}_t L^{r'}_x$ with $\operatorname{supp} \hat F(t, \cdot\,) \subset \overline{\omega}$,

$$\Big\| \int_0^\infty U(s)^* \, F(s) \; ds \Big\|_{L^2(\mathbb{R}^d)} \;\le\; C \, \big\| F \big\|_{L^{q'}_t L^{r'}_x} \tag{3.4.13}$$

Discussion. The estimate is true in the sharp admissible case even though for the heuristics given before the definition, the integral diverged. It is not possible to achieve the concentration suggested in the heuristics with data which has spectrum with support in an annulus. For example, if one considers the wave operator $\Box$ on $\mathbb{R}^{1+3}$ with data supported in $|x|\le 1$ the solutions are supported in $|x|-t\le 1$ and decay along with their derivatives exactly as in the heuristic. Thus one gets divergent integrals. However, compact support and compactly supported Fourier transform are not compatible, and the compact spectrum is enough to overcome the divergence.

Proof. Denote by $(\ ,\ )$ the $L^2(\mathbb{R}^d)$ scalar product. Since.

$$\int_0^\infty (U(t)\, f\,,\, F(t))\ dt\ =\ \int_0^\infty (f\,,\, U(t)^*\, F(t))\ dt\ =\ \Big(f\,,\, \int_0^\infty U(t)^*\, F(t)\ dt\Big)\,,$$

estimates (3.4.12) and (3.4.13) are equivalent thanks to the duality representations of the norms,

$$\Big\| \int_0^\infty U(t)^*\, F(t)\ dt\Big\|_{L^2(\mathbb{R}^d)}\ =\ \sup\Big\{\Big(f\,,\, \int_0^\infty U(t)^*\, F(t)\ dt\Big)\ :\ \hat f\in C_0^\infty(\omega),\ \ \|f\|_{L^2}=1\Big\}\,,$$

$$\Big\| U(t) f\Big\|_{L^qL^r}\ =\ \sup\Big\{\int_0^\infty (U(t)\, f\,,\, F(t))\ dt\ :\ \hat F\in C_0^\infty\big(]0,\infty[\times\omega\big),\ \ \|F\|_{L^{q'}L^{r'}}=1\Big\}\,.$$

Estimate (3.4.13) holds if and only if

$$\Big(\int_0^\infty (U(t)^*F(t))\ dt\ ,\ \int_0^\infty (U(s)^*G(s))\ ds\Big)$$

is a continuous bilinear form on $L^{q'}L^{r'}$, that is

$$\Big|\int_0^\infty\int_0^\infty \big(U(s)^*\, F(s)\,,\, U(t)^*\, G(t)\big)\ ds\, dt\Big|\ \le\ C\,\|F\|_{L^{q'}_tL^{r'}_x}\|G\|_{L^{q'}_tL^{r'}_x}\,. \tag{3.4.14}$$

Unitarity implies that

$$\forall s,t\,,\quad B\ :=\ U(t)U^*(s)\,,\qquad \text{satisfies}\qquad \|Bf\|_{L^2}\ \le\ \|f\|_{L^2}\,.$$

The dispersive estimate (3.4.3) is

$$\forall s,t\,,\quad \|Bf\|_{L^\infty}\ \le\ C\langle t-s\rangle^{-\sigma}\|f\|_{L^1}\,.$$

With $r'\in]1,2[$ the dual index to r, choose $\theta\in]0,1[$ so that

$$\frac{1}{r'}\ =\ \theta\,\frac{1}{1}\ +\ (1-\theta)\,\frac{1}{2}\,,\qquad \text{then,}\qquad \theta=\frac{2-r'}{r'}=\frac{r-2}{r}\,. \tag{3.4.15}$$

The Riesz-Thorin Theorem implies that

$$\|Bf\|_{L^r}\ \le\ C^\theta\,\langle t-s\rangle^{-\sigma\theta}\,\|f\|_{L^{r'}}\,.$$

With Hölder's inequality, this yields the interpolated bilinear estimate,

$$\Big|\big(U(s)^* F(s)\,,\,U(t)^* G(t)\big)\Big| \;\le\; C^\theta\,\langle t-s\rangle^{-\sigma\theta}\,\|F(s)\|_{L^{r'}}\|G(t)\|_{L^{r'}}\,.$$

Admissibility implies that

$$\frac{1}{q}\;\le\;\sigma\Big(\frac{1}{2}-\frac{1}{r}\Big)\;=\;\sigma\Big(\frac{r-2}{2r}\Big)\;=\;\frac{\sigma\,\theta}{2}\,.$$

When strict inequality holds in the definition of admissibility, $\langle t-s\rangle^{-\sigma\theta}\in L^{q/2}(\mathbb{R}_t)$. The hypothesis $q>2$ is used here. For the limiting case, it is nearly so. The Hardy-Littlewood inequality shows that convolution with $|t|^{-2/q}$ has the L^p mapping properties that convolution with an element of $L^{q/2}(\mathbb{R})$ would have.

The Hausdorff-Young inequality shows that

$$L^{p_1}\;*\;L^{p_2}\;\subset\;L^{p_3}\,,\qquad\text{provided}\qquad\frac{1}{p_1}\;+\;\frac{1}{p_2}\;=\;1\;+\;\frac{1}{p_3}\,.\tag{3.4.16}$$

The Hardy Littlewood inequality asserts that when $1<p_1,p_2,p_3<\infty$

$$\frac{1}{\langle t\rangle^{1/p_1}}\;*L^{p_2}(\mathbb{R})\;\subset L^{p_3}(\mathbb{R})\,,\qquad\text{provided}\qquad\frac{1}{p_1}\;+\;\frac{1}{p_2}\;=\;1\;+\;\frac{1}{p_3}\,.\tag{3.4.17}$$

Set

$$p_1\;=\;\frac{q}{2}\,,\qquad p_2\;=\;q',\qquad\text{and,}\qquad p_3\;=\;q\,.\tag{3.4.18}$$

The index conditions in (3.4.16)-(3.4.17) become

$$\frac{2}{q}\;+\;\frac{1}{q'}\;=\;1+\frac{1}{q}\,,$$

which holds by definition of q'. Then (3.4.16) in the admissible case and (3.4.17) in the sharp admissible case imply that

$$\Big\|\int_{-\infty}^{\infty}\langle t-s\rangle^{-\sigma\theta}\;\|F(s)\|_{L^{r'}}\;ds\Big\|_{L^q(\mathbb{R}_t)}\;\le\;C\,\big\|F\big\|_{L^{q'}_t L^{r'}_x}\,.\tag{3.4.19}$$

Hölder's inequality yields

$$\int_0^\infty\Big(\int_0^\infty\langle t-s\rangle^{-\sigma\theta}\;\big\|F(s)\big\|_{L^{r'}}\;ds\Big)\;\big\|G(t)\big\|_{L^{r'}}\;dt\;\le\;C\,\big\|F\big\|_{L^{q'}_t L^{r'}_x}\;\big\|G\big\|_{L^{q'}_t L^{r'}_x}\,.$$

This proves the desired estimate (3.4.14). ∎

A scaling yields estimates for sources with Fourier transform suppoerted in $\lambda\,\overline{\omega}$ for $0<\lambda$.

Lemma 3.4.9. *With q, r, ω, σ as in the previous lemma and γ as in the Theorem, there is a C so that for all $0 < \lambda$ and $f \in L^2$ with $\operatorname{supp} \hat{f} \subset \lambda \overline{\omega}$,*

$$u(t) \;:=\; e^{it\tau_j(D_x)} f \;:=\; U(t) f\,,$$

satisfies

$$\big\|u\big\|_{L^q_t L^r_x} \;\le\; C \big\|\,|D|^\gamma f\big\|_{L^2}\,. \tag{3.4.20}$$

Proof of Lemma. If $u_\lambda(t,x) := u(\lambda t, \lambda x)$ then, $L u_\lambda = 0$ and the spectrum of u_λ is contained in $\overline{\omega}$.

The two sides of (3.4.12) scale differently. Compute

$$\big\|u_\lambda(t)\big\|_{L^r} \;=\; \Big(\int |u_\lambda(t,x)|^r \;dx\Big)^{1/r} \;=\; \Big(\int |u(\lambda t\,,\,\lambda x)|^r \;dx\Big)^{1/r}.$$

The substitution $y = \lambda x$, $dx = \lambda^{-d} dx$ yields

$$=\; \lambda^{-d/r}\Big(\int |u(\lambda t\,,\,y)|^r \;dy\Big)^{1/r} \;=\; \lambda^{-d/r}\big\|u(\lambda t)\big\|_{L^r}\,.$$

A similar change of variable for the time integral shows that

$$\big\|u_\lambda\big\|_{L^q_t L^r_x} \;=\; \lambda^{-1/q\,-\,d/r}\big\|u\big\|_{L^q_t L^r_x}\,.$$

For any γ, $\|\,|D|^\gamma f\|_{L^2}$ is a norm equivalent to the norm on the right hand side for sources with spectrum in $\overline{\omega}$. Compute

$$\begin{aligned}\|\,|D|^\gamma f_\lambda\|_{L^2} \;&=\; \Big(\int |\xi|^{2\gamma}\big|\hat{f}_\lambda(\xi)\big|^2 d\xi\Big)^{1/2} = \Big(\int |\xi|^{2\gamma}\big|\lambda^{-d}\hat{f}(\xi/\lambda)\big|^2 d\xi\Big)^{1/2}\\ &=\; \lambda^{\gamma\,-\,d/2}\Big(\int |\xi|^{2\gamma}\big|\hat{f}(\xi)\big|^2 d\xi\Big)^{1/2} \;=\; \lambda^{\gamma\,-\,d/2}\|\,|D|^\gamma f\|_{L^2}\,.\end{aligned}$$

Given q, r, the γ of the Theorem is the unique value so that the two norms scale the same. Therefore the estimate of the present Lemma follows from the preceding Lemma. ∎

Proof of Theorem. With χ from the dyadic partition of unity for $\mathbb{R}^d_\xi \setminus 0$ constructed in the stationary phase inequality, introduce the Littlewood-Paley decomposition of tempered distributions

$$g \;=\; \sum_{J\in\mathbb{Z}} g_j\,, \qquad g_j \;:=\; \chi(D/2^j)\, g \;:=\; (2\pi)^{-d/2}\int e^{ix\xi}\;\chi(\xi/2^j)\;\hat{g}(\xi)\;d\xi\,.$$

Then for $1 < r < \infty$ the classical square function estimate (see [Stein, Singular Integrals]) asserts that there is a $C > 1$ so that

$$C^{-1}\|g\|_{L^r} \;\le\; \Big\|\Big(\sum_{j\in\mathbb{Z}} |g_j|^2\Big)^{1/2}\Big\|_{L^r} \;\le\; C\,\|g\|_{L^r}\,.$$

Lemma 3.4.10. *If $2 \le q, r < \infty$, there is a constant C so that*

$$\big\|F\big\|^2_{L^q_t L^r_x} \;\le\; C\sum_{j\in\mathbb{Z}} \big\|F_j\big\|^2_{L^q_t L^r_x}\,, \tag{3.4.21}$$

where $F(t) = \sum_j F_j(t)$ is the Littlewood-Paley decomposition in x.

Proof of Lemma. The square function estimate yields

$$\big\|F(t)\big\|^2_{L^r_x} \;\le\; C\int\Big(\sum_j |F_j(t)|^2\Big)^{r/2}\,dx \;=\; C\,\Big\|\sum_j |F_j(t)|^2\Big\|_{L^{r/2}}\,.$$

Minkowski's inequality in $L^{r/2}$ shows that this is

$$\le\; C\sum_j \big\|F_j(t)^2\big\|_{L^{r/2}} \;=\; C\sum_j \big\|F_j(t)\big\|^2_{L^r}\,.$$

Using this yields

$$\big\|F\big\|^2_{L^q_t L^r_x} \;\le\; C\Big(\int_0^\infty \Big(\sum_j \big\|F_j(t)\big\|^2_{L^r}\Big)^{q/2}\,dt\Big)^{2/q} \;=\; C\,\Big\|\sum_j \|F_j(t)\|^2_{L^r(\mathbb{R}^d_x)}\Big\|_{L^{q/2}(\mathbb{R}_t)}\,.$$

Minkowski's inequality in $L^{q/2}(\mathbb{R}_t)$ shows this is

$$\le\; C\sum_j \Big\|\|F_j(t)\|^2_{L^r(\mathbb{R}^d_x)}\Big\|_{L^{q/2}(\mathbb{R}_t)} \;=\; C\sum_j \|F_j(t)\|^2_{L^q_t L^r_x}\,. \qquad \blacksquare$$

Return now to the proof of the Theorem. Associate to the sheet $\tau = \tau_k(\xi)$ the projector $\pi_k(\xi) := \pi(\tau_k(\xi), \xi)$ from §3.2. The π_k are real analytic on $\xi \ne 0$ and homogeneous of degree 0 in ξ. In addition $\sum_k \pi_k = I$. The solution u satisfies

$$u \;=\; \sum_k e^{it\tau_k(D)}\,\pi_k(D)\,f \;:=\; \sum_k u_k\,.$$

Apply (3.4.21) to u_k to find using (3.4.20)

$$\big\|u_k\big\|^2_{L^q_t L^r_x} \;\le\; C\sum_j \|u_{k,j}\|^2_{L^q_t L^r_x} \;\le\; C'\sum_j \|\,|D|^\gamma \pi_k(D) f_j\|^2_{L^2} \;\le\; C'\,\|\,|D|^\gamma f\|^2_{L^2}\,.$$

The finite sum on k completes the proof of the Theorem. ∎

Corollary 3.4.11. *Denote by $S(t)$ the L^2 unitary mapping $u(0) \mapsto u(t)$ for solutions of $Lu = 0$. With the indices of the Theorem one has*

$$\Big\| \int_0^\infty S(s)^* F(s)\, ds \Big\|_{L^2(\mathbb{R}^d)} \ \le\ C\, \big\| |D_x|^\gamma F \big\|_{L_t^{q'} L_x^{r'}} \,. \tag{3.4.22}$$

Proof. Estimate (3.4.22) is equivalent to the Strichartz estimate (3.4.11) by a duality like that used to establish the equivalence of (3.4.12) and (3.4.13). ∎

Exercise. *Prove the following complement to (3.4.21) which comes from the other side of the square function inequality. If $1 < p \le 2$ and $1 \le r \le 2$ then there is a C so that*

$$\sum_{j=-\infty}^{\infty} \big\| F_j \big\|^2_{L_t^r L_x^p} \ \le\ C\, \big\| F \big\|^2_{L_t^r L_x^p} \,. \tag{3.4.23}$$

§6.8. The subcritical nonlinear Klein-Gordon equation in the energy space.

§6.8.1. Introductory remarks.

The mass zero nonlinear Klein-Gordon equation is

$$\Box_{1+d} u \ +\ F(u) \ =\ 0 \,. \tag{6.8.1}$$

where

$$F \in C^1(\mathbb{R})\,, \qquad F(0) = 0\,, \qquad F'(0) = 0\,. \tag{6.8.2}$$

The classic examples from quantum field theory are the equations with $F(u) = u^p$ with $p \ge 3$ an odd integer. For ease of reading we consider only real solutions.

The equation (6.8.1) is Lorentz invariant and if

$$G'(s) \ =\ F(s)\,, \qquad G(0) \ =\ 0\,, \tag{6.8.3}$$

The local energy density is defined as

$$e(u) \ :=\ \frac{u_t^2 + |\nabla_x u|^2}{2} \ +\ G(u)\,. \tag{6.8.4}$$

Solutions $u \in H^2_{\rm loc}(\mathbb{R}^{1+d})$ satisfy the differential energy law,

$$\partial_t e - {\rm div}\big(u_t\, \nabla_x u\big) \ =\ u_t\Big(\Box u \ +\ F(u)\Big) = 0\,. \tag{6.8.5}$$

The corresponding integral conservation law for solutions suitably small at infinity is,

$$\partial_t \int_{\mathbb{R}^d} \frac{u_t^2 + |\nabla_x u|^2}{2} \; + \; G(u) \; dx \; = \; 0\,, \tag{6.8.6}$$

is one of the fundamental estimates in this section. Solutions are stationary for the Lagrangian,

$$\int_0^T \int_{\mathbb{R}^d} \frac{u_t^2 - |\nabla_x u|^2}{2} - G(u) \; dt\, dx\,.$$

When F is smooth, the methods of §6.3-6.4 yield local smooth existence.

Theorem 6.8.1. *If $F \in C^\infty$, $s > d/2$, $f \in H^s(\mathbb{R}^d)$, and $g \in H^{s-1}(\mathbb{R}^d)$, then there is a unique maximal solution*

$$u \;\in\; C\big([0,T_*[\;;\; H^s(\mathbb{R}^d)\big) \;\cap\; C^1\big([0,T_*[\;;\; H^{s-1}(\mathbb{R}^d)\big)\,.$$

satisfying

$$u(0,x) = f\,, \quad u_t(0,x) = g\,.$$

If $T_* < \infty$ then

$$\limsup_{t \to T_*} \; \|u(t)\|_{L^\infty(\mathbb{R}^d)} \;=\; \infty\,.$$

In favorable cases, the energy law (6.8.6) gives good control of the norm of $u, u_t \in H^1 \times L^2$. Controling the norm of the difference of two solutions is, in contrast, a very difficult problem for which many fundamental questions remain unresolved.

An easy first case is nonlinearities F which are uniformly lipschitzean. In this case, there is global existence in the energy space.

Theorem 6.8.2. *If F satisfies $F' \in L^\infty(\mathbb{R})$, then for all Cauchy data $f, g \in H^1 \times L^2$ there is a unique solution*

$$u \in C\big(\mathbb{R}\;;\; H^1(\mathbb{R}^d)\big) \;\cap\; C^1\big(\mathbb{R}\;;\; L^2(\mathbb{R}^d)\big)\,.$$

For any finite T, the map from data to solution is uniformly lipschitzean from $H^1 \times L^2$ to $C([-T,T\,;\,H^1) \cap C^1([-T,T]\,;\,L^2)$. If $f, g \in H^2 \times H^1$ then

$$u \;\in\; L^\infty\big(\mathbb{R};\; H^2(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; L^\infty\big(\mathbb{R};\; H^1(\mathbb{R}^d)\big)\,.$$

If $f, g \in H^s \times H^{s-1}$ with $1 \le s < 2$, then

$$u \;\in\; C\big(\mathbb{R};\; H^s(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; C\big(\mathbb{R};\; H^{s-1}(\mathbb{R}^d)\big)\,.$$

Sketch of Proof. The key estimate is the following. If u and v are solutions then

$$\square(u-v) \;=\; F(v)-F(u), \qquad |F(u)-F(v)| \;\le\; C|u-v|\,.$$

Multiplying by $u_t - v_t$ yields

$$\frac{d}{dt}\int (u_t-v_t)^2+|\nabla_x(u-v)|^2dx \;=\; 2\int (u_t-v_t)\,\big(F(v)-F(u)\big)\,dx \;\le\; C\,\|u_t-v_t\|_{L^2}^2\,\|u-v\|_{L^2}^2\,.$$

It follows that for any T there is an *a priori* estimate

$$\sup_{|t|\le T}\ \Big(\|u(t)-v(t)\|_{H^1} \;+\; \|u_t-v_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0)-v(0)\|_{H^1} \;+\; \|u_t(0)-v_t(0)\|_{L^2}\Big)\,.$$

This estimate exactly corresponds to the asserted Lipschitz continuity of the map from data to solutions.

Applying the estimate to $v = u(x+h)$ and taking the supremum over small vectors h, yields an *a priori* estimate

$$\sup_{|t|\le T}\ \Big(\|u(t)\|_{H^2} \;+\; \|u_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0)\|_{H^2} \;+\; \|u_t(0)\|_{H^1}\Big)\,,$$

which is the estimate correponding to the H^2 regularity. ∎

Higher regularity for dimensions $d \ge 10$ is an outstanding open problem. For example, for $d \ge 10$, smooth compactly supported initial data, and $F \in C_0^\infty$ or $F = \sin u$, it is not known if the above global unique solutions are smooth. For $d \le 9$ the result can be found in [Brenner-vonWahl 1982]. Smoothness would follow if one could prove that $u \in L^\infty_{\rm loc}$. What is needed is to show that the solutions do not get large in the pointwise sense. Compared to the analogous regularity problem for Navier-Stokes this problem has the advantage that solutions are known to be unique and depend continuously on the data.

§6.8.2. The ordinary differential equation and nonlipshitzean F.

Concerning global existence for functions $F(u)$ which may grow more rapidly than linearly as $u \to \infty$, the first considerations concern solutions which are independent of x and therefore satisfy the ordinary differential equation,

$$u_{tt} \;+\; F(u) \;=\; 0\,. \tag{6.8.7}$$

Global solvability of the ordinary differential equation is analysed using the energy conservation law

$$\Big(\frac{u_t^2}{2} \;+\; G(u)\Big)' \;=\; u_t\Big(u_{tt}+F(u)\Big) \;=\; 0\,.$$

Think of the equation as modeling a nonlinear spring. The spring force is attractive, that is pulls the spring toward the origin when

$$F(u)>0 \quad \text{when} \quad u>0 \qquad \text{and,} \qquad F(u)<0 \quad \text{when} \quad u<0\,.$$

In this case one has $G(u) > 0$ for all $u \neq 0$. Conservation of energy then gives a pointwise bound on u_t uniform in time

$$u_t^2(t) \;\le\; u_t^2(0) + 2G(u(0))\,, \qquad |u_t(t)| \;\le\; \big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

This gives a pointwise bound

$$|u(t)| \;\le\; |u(0)| + |t|\big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

In particular the ordinary differential equation has global solutions.

In the extreme opposite case consider the replusive spring force $F(u) = -u^2$ and $G(u) = -u^3/3$. The energy law asserts that $u_t^2/2 - u^3/3 := E$ is independent of time. Consider solutions with

$$u(0) > 0, \quad u_t(0) > 0 \qquad \text{so} \quad E > -\frac{u^3(0)}{3}\,.$$

For all $t > 0$,

$$|u_t| \;=\; \Big|\frac{u^3}{3} + E\Big|^{1/2},$$

At $t = 0$ one has

$$u_t(0) \;=\; \Big(\frac{u^3(0)}{3} + E\Big)^{1/2} > 0\,.$$

Therefore u increases and $u^3/3 + E$ stays positive and one has for $t \ge 0$

$$u_t(t) \;=\; \Big(\frac{u^3(t)}{3} + E\Big)^{1/2} > 0\,.$$

Both u and u_t are strictly increasing.

Since

$$\frac{du}{\big(\frac{u^3}{3} + E\big)^{1/2}} \;=\; dt\,,$$

$u(t)$ approaches ∞ at time

$$T \;:=\; \int_{u(0)}^{\infty} \frac{du}{\big(\frac{u^3}{3} + E\big)^{1/2}}\,.$$

Exercise. *Show that if there is an $M > 0$ so that $G(s) < 0$ for $s \ge M$ and*

$$\int_M^\infty \frac{1}{\sqrt{|G(s)|}}\, ds \;<\; \infty$$

then there are solutions of the ordinary differential equation which blow up in finite time.

Proposition 6.8.3 [J.B. Keller 1957]. *If*

$$a, \delta > 0, \qquad d \le 3, \qquad E := \delta^2/2 - a^3/3, \qquad T \ := \int_a^\infty \Big|\frac{u^3}{3} + E\Big|^{-1/2}\, du\,,$$

and $\phi, \psi \in C^\infty(\mathbb{R}^d)$ *satisfy*

$$\phi \ge a \quad \text{and} \quad \psi \ge \delta \qquad \text{for} \quad |x| \le T\,,$$

the the smooth solution of

$$\Box_{1+d} u \ - \ u^2\,, \qquad u(0) = \phi, \quad u_t(0) = \psi$$

blows up on or before time T*.*

Proof. Denote by $\underline{u}$ the solution of the ordinary differential equation with initial data $\underline{u}(0) = a$, $\underline{u}_t(0) = \delta$.

If $u \in C^\infty\big([0, \underline{t}] \times \mathbb{R}^d\big)$, then finite speed of propagation and positivity of the fundamental solution of $\Box_{1+d}$ imply that

$$u \ \ge \ \underline{u} \qquad \text{on} \quad \big\{|x| \le T - \underline{t}\big\}\,.$$

Since $\underline{u}$ diverges as $t \to T$ it follows that $\underline{t} \le T$ ∎

In the case of attractive forces where $G \ge 0$ one can hope that there is global smooth solvability for smooth initial data. This question has received much attention and is very far from being understood. For example even in the uniformly lipschitzean case where solutions H^2 in x exist globally, higher regularity is unknown in high dimensions.

In the remainder of this section we will study solvability in the energy space defined by $u, u_t \in H^1 \times L^2$. This regularity is suggested by the basic energy law. For uniformly lipschitzean nonllinearities the global solvability is given by Theorem 6.8.2. The interest is in attractive nonlinearities with superlinear growth at infinity.

A crucial role is played by the rate of growth of F at infinity. There is a critical growth rate so that for nonlinearities which are subcritical and critical there is a good theory based on Strichartz estimates. The analysis is valid in all dimensions.

To concentrate on essentials, we present the family of attractive (repulsive) nonlinearities $F = u|u|^{p-1}$ ($F = -u|u|^{p-1}$) with potential energies given by $\pm \int |u|^{p+1}/(p+1)dx$. Start with four natural notions of subcriticality. They are in increasing order of strength. One could expect to call p subcritical when

1. $H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ so the nonlinear term makes sense for elements of H^1.

2. $H^1(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$ so the potential energy makes sense for elements of H^1.

3. $H^1(\mathbb{R}^d)$ is compact in $L^{p+1}_{\rm loc}(\mathbb{R}^d)$ so the potential energy is in a sense small compared to the kinetic energy.

4. $H^1(\mathbb{R}^d) \subset L^{2p}(\mathbb{R}^d)$ so the nonlinear term belongs to $L^2(\mathbb{R}^d)$ for elements of H^1.

The Sobolev embedding is

$$H^1(\mathbb{R}^d) \ \subset \ L^q(\mathbb{R}^d)\,, \quad \text{for,} \quad q \ = \ \frac{2d}{d-2}\,. \tag{6.8.8}$$

The above conditions then read (with the values for $d = 3$ given in parentheses),

1. $p \le 2d/(d-2)$, $\qquad (p \le 6)$,
2. $p + 1 \le 2d/(d-2)$, $\qquad \big(\text{equiv.} \quad p \le (d+2)/(d-2)\big)$, $\qquad (p \le 5)$,
3. $p < (d+2)/(d-2)$, $\qquad (p < 5)$,
4. $p \le d/(d-2)$, $\qquad (p \le 3)$.

The correct answer is **3**. Much that will follow can be extended to the critical case $p = (d+2)/(d-2)$. The case **1** in contrast is supercritical and comparatively little is known. It is known that in the supercritical case, solutions are very sensitive to initial data. The dependence is not lipschitzean, and it is lipschitzean in the subcritical and critical cases. The books of Sogge, and Shatah-Struwe and the orignal 1985 article of Ginibre and Velo are good references. The sensitive dependence is a recent result of Lebeau.

Notation. *Denote by* $L^q_t L^r_x([0,T])$ *the space* $L^q_t L^r_x([0,T] \times \mathbb{R}^d)$, *Denote with an open interval*

$$L^q_t L^r_x([0,T[) \ := \ \cup_{0<\underline{T}<T} \ L^q_t L^r_x([0,\underline{T}])\,.$$

Theorem 6.8.4. i. *If p is subcritical for H^1, that is $p < (d+2)/(d-2)$, then for any $f \in H^1(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$ there is $T_* > 0$ and a unique solution*

$$u \in C([0,T_*[\ \ H^1(\mathbb{R}^d)) \ \cap \ C^1([0,T_*[\,;\, L^2(\mathbb{R}^d)) \ \cap \ L^p_t L^{2p}_x([0,T_*[) \tag{6.8.9}$$

of the repulsive problem

$$\Box u \ - \ u|u|^{p-1} \ = \ 0\,, \qquad u(0) = f, \quad u_t(0) = g\,. \tag{6.8.10}$$

If $T_* < \infty$ then

$$\liminf_{t \nearrow T_*} \ \|\nabla_{t,x} u\|_{L^2(\mathbb{R}^d)} \ = \ \infty\,. \tag{6.8.11}$$

The energy conservation law (6.8.6) is satisfied.
ii. *For the attractive problem*

$$\Box u \ + \ u|u|^{p-1} \ = \ 0\,, \qquad u(0) = f, \quad u_t(0) = g\,. \tag{6.8.12}$$

one has the same result with $T_* = \infty$ and with $u \in L^p_t L^{2p}_x(\mathbb{R})$. For any $T > 0$, the map from Cauchy data to solution is uniformly lipschitzean

$$H^1 \times L^2 \quad \to \quad C([-T,T]\,;\, H^1) \ \cap \ C([-T,T]\,;\, L^2) \ \cap \ L^p_t L^{2p}_x([0,T]))\,.$$

In the proof of this result and all that follows a central role is played by the linear wave equation and its solution for which we recall the basic energy estimate

$$\|\nabla_{t,x} u(t)\|_{L^2(\mathbb{R}^d)} \ \leq\ \|\nabla_{t,x} u(0)\|_{L^2(\mathbb{R}^d)} \ +\ \int_0^t \|\Box u(t)\|_{L^2(\mathbb{R}^d)} \ dt\,.$$

This is completed by the L^2 estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \ \leq\ \int_0^t \|u_t(t)\|_{L^2(\mathbb{R}^d)} \ dt\,.$$

In particular, for $h \in L^1_{\rm loc}\big(\mathbb{R}\,;\, L^2(\mathbb{R}^d)\big)$ there is a unique solution

$$u \ \in\ C\big(\mathbb{R}\,;\, H^1(\mathbb{R}^d)\big)\ \cap\ C^1\big(\mathbb{R}\,;\, L^2(\mathbb{R}^d)\big)\,,$$

to

$$\Box u \ =\ h\,, \qquad u(0)\ =\ 0\,, \quad u_t(0)\ =\ 0\,.$$

This solution is denoted

$$\Box^{-1} h\,.$$

In order to take advantage of this we seek solutions so that

$$F_p(u) \ :=\ \pm\, u|u|^{p-1}\ \in\ L^1_t L^2_x\,.$$

Compute

$$\|F_p(u)\|_{L^1_t L^2_x} \ =\ \int_0^T \Big(\int |u^p|^2 \ dx \Big)^{1/2} \ dt\,,$$

where

$$\Big(\int |u|^{2p} \ dx \Big)^{1/2} \ =\ \Big[\Big(\int |u|^{2p} \Big)^{1/2p} \Big]^p \ =\ \|u\|^p_{L^{2p}(\mathbb{R}^d)}\,,$$

so

$$\|F_p(u)\|_{L^1_t L^2_x} \ =\ \int_0^T \|u\|^p_{L^{2p}_t \mathbb{R}^d_x} \ dt \ =\ \|u\|^p_{L^p_t L^{2p}_x}\,. \tag{6.8.13}$$

The above calculation proves the first part of the next lemma.

Lemma 6.8.5. *The mapping $u \mapsto F_p(u)$ takes $L^p_t L^{2p}_x([0,T]$ to $L^1_t L^2_x([0,T])$. It is uniformly Lipshitzean on bounded subsets.*

Proof. Write

$$F_p(v) \ -\ F_p(w) \ =\ G(v,w)\big(v-w\big)\,, \qquad |G(v,w)| \ \leq\ C\Big(|v|^{p-1} \ +\ |w|^{p-1} \Big)\,.$$

Write

$$\big\|G(v,w)(v-w)\big\|_{L^2_x}^2 \;=\; \int |G|^2\;|v-w|^2\;dx\,.$$

Use Hölder's inequality for $L_x^{p/(p-1)}\times L_x^p$ to estimate by

$$\le\;\Big(\int |G(v,w)|^{2p/(p-1)}dx\Big)^{\frac{p-1}{p}}\Big(\int |v-w|^{2p}dx\Big)^{\frac{1}{p}}.$$

Then

$$\|F_p(v)\;-\;F_p(w)\|_{L^2}\;\le\;C\,\|v,w\|^{p-1}_{L^{2p}_x}\;\|v-w\|_{L^{2p}_x}\,.$$

Finally estimate the integral in time using Hölder's inequality for $L_t^{p/(p-1)}\times L_t^p$. ∎

It is natural to seek solutions $u\in L_t^pL_x^{2p}([0,T])$. With that as a goal we ask when it is true that

$$\square^{-1}\big(L^1_tL^2_x\big)\;\subset\;L^p_tL^{2p}_x\,.$$

This is exactly in the family of questions addressed by the Strichartz inequalities. The next Lemma gives the inequalities adapted to the present situation.

Lemma 6.8.6. *If*

$$q>2\,,\qquad\text{and}\qquad \frac{1}{q}\;+\;\frac{d}{r}\;=\;\frac{d}{2}-1\,,\tag{6.8.14}$$

then there is a constant $C>0$ *so that for all* $T>0$, $h,f,g\in L^1_t(L^2_x)\times H^1\times L^2$ *the solution of*

$$\square u\;=\;h\,,\qquad u(0)\;=\;f\,,\qquad u_t(0)\;=\;g\,,$$

satisfies

$$\big\|u\big\|_{L^q_tL^r_x([0,T])}\;\le\;C\Big(\big\|h\|_{L^1_tL^2_x([0,T])}\;+\;\big\|\nabla_xf\big\|_{L^2(\mathbb{R}^d)}\;+\;\big\|g\big\|_{L^2(\mathbb{R}^d)}\Big)\,.\tag{6.8.15}$$

Proof. 1. Rewrite the wave equation as a symmetric hyperbolic pseudodifferential system motivated by D'Alembert's solution of the $1-d$ wave equation. Factor,

$$\partial_t^2-\Delta\;=\;(\partial_t+i|D|)\,(\partial_t-i|D|)\;=\;(\partial_t+i|D|)\,(\partial_t-i|D|)\,.$$

Introduce

$$v_\pm\;:=\;(\partial_t\mp i|D|)u\,,\qquad V\;:=\;(v_+,v_-)\,,$$

so

$$V_t\;+\;\begin{pmatrix}1&0\\0&-1\end{pmatrix}i|D|\,V\;=\;\begin{pmatrix}h\\h\end{pmatrix}.$$

Lemma 3.4.8 implies that for $\sigma = d-1$, $q>2$, (q,r) σ- admissible, and h, f, g with spectrum in $\{R_1 \le |\xi| \le R_2\}$ one has

$$\|u\|_{L^q_tL^r_x} \;\le\; C\,\|\nabla_{t,x}u\|_{L^q_tL^r_x} \;\le\; C\,\|V\|_{L^q_tL^r_x} \;\le\; C\,\Big(\|h\|_{L^1_tL^2_x} \;+\; \||D|f\|_{L^2} \;+\; \|g\|_{L^2}\Big)\,.$$

2. Denote by ℓ the dimensions of t and x. With dimensionless u , the terms on right of this inequality have dimension $\ell^{d/2\,-1}$.
The dimension of the term on the left is equal to

$$\big(\ell^{dq/r}\,\ell\big)^{1/q} \;=\; \ell^{\frac{d}{r}+\frac{1}{q}}\,.$$

The two sides have the same dimensions if and only if

$$\frac{d}{r} \;+\; \frac{1}{q} \;=\; \frac{d}{2} - 1\,. \tag{6.8.16}$$

Under this hypothesis it follows that the same inequality holds, with the same constant C for data with support in $\lambda R_1 \le |\xi| \le \lambda R_2$.
Comparing (6.8.16) with σ-admissibility which is equivalent to

$$\frac{d}{r} \;+\; \frac{1}{q} \;\le\; \frac{d}{2} \;-\; \frac{1}{2} \;-\; \frac{1}{r}\,,$$

shows that (6.8.16) implies admissibility since $r \ge 2$.
3. Lemma 6.8.6 follows using Littlewood-Paley theory as at the end of §3.4.3. ∎

We now answer the question of when $\Box^{-1}$ maps $L^1_tL^2_x$ to $L^p_tL^{2p}_x$. This is the crucial calculation. In Lemma 6.8.6, take $r = 2p$ to find

$$\frac{1}{q} \;+\; \frac{d}{2p} \;=\; \frac{d-2}{2}\,,$$

so,

$$\frac{1}{q} \;=\; \frac{d-2}{2} - \frac{d}{2p} \;=\; \frac{p(d-2)-d}{2p}\,, \qquad q \;=\; p\bigg(\frac{2}{p(d-2)-d}\bigg)\,.$$

We want $q \ge p$, that is

$$\frac{2}{p(d-2)-d} \;\ge\; 1\,, \quad\Leftrightarrow\quad p(d-2)-d \;\le 2 \quad\Leftrightarrow\quad p \;\le\; \frac{d+2}{d-2}\,.$$

The critical case is that of equality, and the subcritical case that we treat is the one with strict inequality. For $d = 3$ the critical power is $p = 5$ and for $d = 4$ it is $p = 3$. In the subcritical case the operator has small norm for $T << 1$.

The strategy of the proof is to write the solution u as a perturbation of the solution of the linear problem, at least for small times. Define u_0 to be the solution of

$$\Box u_0 \;=\; 0\,, \qquad u_0(0) \;=\; f, \quad \frac{\partial u_0}{\partial t}(0) \;=\; g\,. \tag{6.8.17}$$

Write

$$u \;=\; u_0 \;+\; v \tag{6.8.18}$$

with the hope that v will be small at least for t small.

Lemma 6.8.7. *If $u = u_0 + v$ with $v \in L^p_t L^{2p}_x\big([0,T]\big)$ satisfying*

$$v \;=\; \pm\Box^{-1} F_p(u_0 + v)\,. \tag{6.8.20}$$

then

$$u \;\in\; C([0,T]\,;\, H^1(\mathbb{R}^d)) \;\cap\; C^1([0,T]\,;\, L^2(\mathbb{R}^d)) \;\cap\; L^p_t L^{2p}_x([0,T]) \tag{6.8.21}$$

satisfies

$$\Box u \;\pm\; F_p(u) \;=\; 0\,, \qquad u(0) = f, \quad u_t(0) = g\,, \tag{6.8.22}$$

Conversely, if u satisfies (6.8.21)-(6.8.22) then $v := u - u_0 \in L^p_t L^{2p}_x([0,T])$ and satisfies (6.8.21)

Proof. The Strichartz inequality implies that $u_0 \in L^p_t L^{2p}_x$ and by hypothesis the same is true of v. Therefore $u_0 + v$ belongs to $L^p_t L^{2p}_x$ so $F_p(u_0 + v) \in L^1_t L^2_x$.
Therefore $v = \pm\Box^{-1} F_p$ is $C(H^1) \cap C^1(L^2)$. The differential equation and initial condition for v are immediate.
The converse is similar, not used below, and left to the reader. ∎

Proof of Theorem 6.8.4. For $K > 0$ arbitrary but fixed, we prove unique local solvability with continuous dependence for $0 \le t \le T$ with T uniform for all data f, g with

$$\|f\|_{H^1} \;+\; \|g\|_{L^2} \;\le\; K\,.$$

Choose $R = R(K)$ so that for such data,

$$\|u_0\|_{L^p_t L^{2p}_x([0,1])} \;\le\; \frac{R}{2}\,.$$

Define

$$B \;=\; B(T) \;:=\; \Big\{ v \in L^p_t L^{2p}_x([0,T]) \;:\; \|v\|_{L^p_t L^{2p}_x([0,T])} \;\le R \Big\}\,.$$

We show that for $T = T(K)$ sufficiently small, the map $v \mapsto \Box^{-1} F_p(u)$ is a contraction from B to itself.
This is a consequence of three facts.

1. Lemma 6.8.5 shows that F_p is uniformly lipschitzean from B to $L^1_t L^2_x([0,T])$ uniformly for $0 < T \le 1$.

2. Lemma 6.8.6 together with subcriticality shows that there is an $r > p$ so that $\Box^{-1}$ is uniformly lipshitzean from $L^1_t L^2_x$ to $L^r_t L^{2p}_x$ uniformly for $0 < T < 1$.

3. The injection $L^r_t L^{2p}_x \mapsto L^p_t L^{2p}_x$ has norm which tends to zero as $T \to 0$.

This is enough to carry out the existence parts of Theorem 6.8.4.

If there are two solutions u, v with the same initial data, compute

$$\Box(u - v) \;=\; G(u,v)(u-v)\,.$$

Lemma 6.8.6 together with subcriticality shows that with r slightly larger than p,

$$\|u - v\|_{L^r_t L^{2p}_x} \;\le\; C\,\|G(u,v)(u-v)\|_{L^1_t L^2_x} \;\le\; C\,\|u-v\|_{L^p_t L^{2p}_x}\,.$$

Use this estimate for $0 \le t \le T << 1$ noting that Hölder's inequality shows that for $T \to 0$,

$$\|u - v\|_{L^p_t L^{2p}_x} \;\le\; C\,T^\rho\,\|u-v\|_{L^r_t L^{2p}_x} \;\le\; C\,T^\rho \|u-v\|_{L^p_t L^{2p}_x}\,, \qquad \rho > 0\,,$$

to show that the two solutions agree for small times. Thus the set of times where the solutions agree is open and closed proving uniqueness.

To prove the energy law note that $F_p(u) \in L^1_t L^2_x$ so the linear energy law shows that

$$\int \frac{|u_t|^2 + |\nabla_x u|^2}{2}\; dx\Big|_{t=0}^{t} \;=\; \mp \int_0^t \int u_t\; F_p(u)\; dx\, dt\,. \tag{6.8.23}$$

Now

$$u_t \;\in\; L^\infty_t L^2_x\,, \quad \text{and} \quad F_p(u) \in L^1_t L^2_x\,.$$

Hölder's inequality shows that

$$\int |u_t\; F_p(u)| dx \;\le\; \|u_t(t)\|_{L^2_x}\; \|F_p(u(t)\|_{L^2_x}\,.$$

The latter is the product of a bounded and an integrable function so

$$\forall T, \quad u_t\, F_p(u) \;\in\; L^1([0,T]\times \mathbb{R}^d)\,.$$

Let

$$w \;:=\; \frac{|u|^{p+1}}{p+1}\,.$$

Since p is subcritical, one has for some $0 < \epsilon$,

$$\|w(t)\|_{L^1_x} \;\le\; C\|u(t)\|_{H^{1-\epsilon}(\mathbb{R}^d)} \;\in\; L^\infty([0,T])\,.$$

In particular $w \in L^1([0,T] \times \mathbb{R}^d)$ and the family $\{w(t)\}_{t\in[0,T]}$ is precompact in $L^1_{\rm loc}$.
Formally differentiating yields

$$w_t \;=\; u_t F_p(u) \;\in\; L^1([0,T] \times \mathbb{R}^d)\,. \tag{6.8.24}$$

Using the above estimates, it is not hard to justify (6.8.24).
It then follows that $w \in C([0,T]\,;\,L^1(\mathbb{R}^d))$ and

$$\int w(t,x)\;dx\Big|_{t=0}^{t=T} \;=\; \int_0^T \int u_t\;F_p(u)dx\,dt\,.$$

Together with (6.8.23) this proves the energy identity.
Once the energy law is known, one concludes global solvability in the attractive case since the blow up criterion (6.8.11) is ruled out by energy conservation. ∎

Chapter 3. Appendix I. Perturbation theory for semisimple eigenvalues.

The computation of the form of the operator $\pi\, L\, \pi$ requires formulas from the perturbation theory of eigenvalues. These results for multiple eigenvalues which are semisimple is not that well known. The key idea is that one should NOT make a choice of basis of eigenfunctions, but work systematically with the spectral projections.

Definition. *An eigenvalue λ of a matrix A is semisimple when the kernel and range of $A-\lambda I$ are complementary subspaces. In this case denote by π the spectral projection onto the kernel of $A-\lambda I$ along its range and by Q the partial inverse defined by*

$$Q\,\pi \;=\; 0\,, \qquad Q\,\big(A - \lambda\, I\big) = I - \pi\,. \tag{1}$$

Theorem. *Suppose that $]a,b[\ni s \to A(s)$ is a smooth family of complex matrices with an isolated smooth semisimple eigenvalue $\lambda(s)$. Then $\lambda(s)$ and $\pi(s)$ are smooth functions of s whose first derivatives satisfy*

$$\lambda'(s)\;\pi(s) \;=\; \pi(s)\;A'(s)\;\pi(s)\,, \tag{2}$$

$$\lambda''\;\pi \;=\; \pi\,A''\,\pi \;-\; 2\,\pi\,A'\,Q\,A'\,\pi\,, \tag{3}$$

$$\pi' \;=\; -\pi\,A'\,Q - QA'\pi\,. \tag{4}$$

Proof. For $\underline{s}$ fixed, choose $r>0$ so that $A(\underline{s})$ has only the eigenvalue $\lambda(\underline{s})$ in the disc $|z-\lambda(\underline{s})|\le 2r$. The smoothness of $\pi(s)$ near $\underline{s}$ follows from the contour integral representation

$$\pi(s) \;:=\; \frac{1}{2\pi i}\oint_{|z-\lambda(\underline{s})|=r} \Big(z - A(s)\Big)^{-1}\, dz\,.$$

The identity

$$Q(s) \;=\; (I-\pi(s))\Big(\pi(s)+A(s)\Big)^{-1} \;\in\; C^\infty\,.$$

The identity $A(s)\pi(s)=\lambda(s)\pi(s)$ implies that

$$\lambda(s) \;=\; \frac{\mathrm{Trace}\,\big(A(s)\pi(s)\big)}{\mathrm{Trace}\,\pi(s)} \;\in\; C^\infty\,.$$

The formulas (2-4) are proved by differentiating the identity $(A-\lambda)\pi = \pi(A-\lambda)=0$ with respect to s. The equation for each d^j/ds^j is analysed by considering its projections π and $I-\pi$. Equivalently, each equation is multiplied first by π, then by Q.

Denoting d/ds with a $'$. Differentiating $(A-\lambda)\pi$ yields

$$(A-\lambda)'\;\pi \;+\; (A-\lambda)\;\pi' \;=\; 0\,. \tag{5}$$

Mulitplying on the left by π eliminates the second term to yield

$$\pi\ (A-\lambda)'\ \pi \ \ = \ 0\,, \tag{6}$$

which is equivalent to (2).
Multiply equation (5) on the left by Q to find

$$(I-\pi)\pi' \ = \ -Q\,\big(A-\lambda\big)'\,\pi\,.$$

Since $Q\,\pi = 0$ this simplifies to

$$(I-\pi)\pi' \ = \ -QA'\pi\,. \tag{7}$$

Equation (5) is exhausted and we take a second derivative,

$$(A-\lambda)''\pi + 2(A-\lambda)'\pi' + (A-\lambda)\pi'' = 0\,.$$

Mutiply on the left by π to eliminate the last term,

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'\pi' \ = \ 0\,.$$

Subtract $2(\pi(A-\lambda)'\pi)\pi' = 0\,\pi' = 0$ to find

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'(I-\pi)\pi' \ = \ 0\,.$$

Then (7) yields

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'(-QA'\pi) \ = \ 0\,. \tag{8}$$

Since $\pi\,Q = 0$ one has

$$2\,\pi\lambda'(-QA'\pi) \ = \ 0\,. \tag{9}$$

Adding (8) and (9) yields (3).
To prove (4) knowing (7), what is needed is $\pi\,\pi'$. Differentiate $\pi^2 = \pi$ to find

$$\pi\,\pi' + \pi'\,\pi \ = \ \pi'\,, \qquad \text{whence} \qquad \pi\,\pi' \ = \ \pi'(I-\pi)\,. \tag{10}$$

Differentiate $\pi\,(A-\lambda) = 0$ to find

$$\pi'(A-\lambda) + \pi\,(A-\lambda)' \ = \ 0\,.$$

Mulitply on the right by Q to find

$$\pi'(I-\pi) \ = \ -\pi\,(A-\lambda)'\,Q\,.$$

Use (10) and simplify using $\pi\,Q = 0$ to find

$$\pi\,\pi' \ = \ -\pi\,A'\,Q\,.$$

Adding this to (7) completes the proof. ∎

Chapter 3, Appendix II. The stationary phase inequality.

Definition. A point $\underline{x}$ in an open subset $\Omega \subset \mathbb{R}^d$ is a stationary point of $\phi \in C^\infty(\Omega\,;\mathbb{R})$ when $\nabla_x \phi(\underline{x}) = 0$. It is nondegenerate when the matrix of second derivatives at $\underline{x}$ is nonsingular..

When $\underline{x}$ is a nondegenerate stationary point the map $x \mapsto \nabla_x\phi(x)$ has nonsingular Jacobian at $\underline{x}$. It follows that the map is a local diffeomorphism and in particular the stationary point is isolated.

Taylor's Theorem shows that

$$\nabla_x\phi(x) \;=\; \frac{1}{2}\,\nabla_x^2\phi(\underline{x})\,(x-\underline{x}) \;+\; O(|x-\underline{x}|^2)\,.$$

Therefore if $\omega \subset\subset \Omega$ contains $\underline{x}$ and no other stationary point, nondegeneracy implies that there is a constant $C>0$ so,

$$\forall \underline{x} \in \omega\,, \qquad \big|\nabla_x\phi(x)\big| \;\ge\; C\,|x-\underline{x}|\,. \tag{1}$$

We estimate the size of oscillatory integrals whose phase has a single nondegenerate stationary point. These integrals have a complete asymptotic expansion. Proving the estimate is easier than deriving the expansion. The estimate is proved by the method of nonstationary phase. I learned the dyadic proof below from G. Métivier. See [Stein, Harmonic Analysis, Real Variable Methods] for an alternate proof.

Theorem. *Suppose that $\phi \in C^\infty(\Omega\,;\mathbb{R})$ has a unique stationary point $\underline{x} \in \Omega$. Suppose that $\underline{x}$ is nondegenerate and let m denote the smallest integer strictly larger than $d/2$. Then for any $\omega \subset\subset \Omega$ there is a constant C so that for all $f \in C_0^\infty(\omega)$, and $0 < \epsilon < 1$,*

$$\Big|\int e^{i\phi/\epsilon} f(x)\,dx\Big| \;\le\; C\,\epsilon^{d/2}\,\sup_{|\alpha|\le m}\,\|\partial^\alpha f(x)\|_{L^\infty(\omega)}\,. \tag{2}$$

Lemma. *There is a nonnegative $\chi \in C_0^\infty(\mathbb{R}^d\backslash 0)$ so that for all $x \ne 0$, $\sum_{k=-\infty}^{\infty}\chi(2^k\,x) = 1$.*

Proof of Lemma. Choose nonnegative $g \in C_0^\infty(\mathbb{R}^d\setminus 0)$ so that $g \ge 1$ on $\{1 \le |x| \le 2\}$. Define the locally finite sum

$$G(x) \;:=\; \sum_{k=-\infty}^{\infty} g(2^k x)\,, \qquad G(2^k x) = G(x)\,.$$

Then $G \in C^\infty(\mathbb{R}^d\setminus 0)$, and $G \ge 1$. The function $\chi := g/G$ has the desired properties. ∎

Proof of Theorem. Translating coordinates we may suppose that $\underline{x} = 0$. Choose χ as in the lemma and write

$$\int e^{i\phi/\epsilon} f(x)\ dx \ = \ \sum_{k=-\infty}^{\infty} \int \chi(2^k\,x)\ e^{i\phi/\epsilon}\ f(x)\ dx \ := \ \sum_{k=-\infty}^{\infty} I(k)\,.$$

The half sum $\sum_{k<0} \chi(2^k x)$ is a smooth function on $\mathbb{R}^d$ which vanishes on a neighbhorhood of the origin and is identically equal to 1 outside a large ball. The nonstationary phase Lemma 1.2.2 implies that

$$\Big| \int e^{i\phi/\epsilon} \Big(\sum_{k<0} \chi(2^k\,x) \Big)\ f(x)\ dx \Big| \ \le \ C\,\epsilon^m \sup_{|\alpha|\le m} \|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

The sum $\sum_{2^k \epsilon^{1/2}\ge 1} \chi(2^k\,x)$ is a bounded function supported in a ball $|x| \le C\epsilon^{1/2}$ so

$$\Big| \int e^{i\phi/\epsilon} \Big(\sum_{2^k \epsilon^{1/2}\ge 1} \chi(2^k\,x) \Big)\ f(x)\ dx \Big| \ \le \ C\,\epsilon^{d/2}\, \|f(x)\|_{L^\infty(\omega)}\,.$$

There remains the sum over $1 \le 2^k < \epsilon^{-1/2}$. The change of variable $y = 2^k\,x$ yields

$$I(k) \ = \ 2^{-kd} \int \chi(y)\ e^{i\phi_k(y)/(2^{2k}\epsilon)}\ f(2^{-k}y)\ dy\,, \qquad \phi_k(y) \ := \ 2^{2k}\,\phi(2^{-k}\,y)\,.$$

It follows from (1) that there is a constant $c > 0$ so that on the support of χ,

$$c^{-1} \le \big|\nabla \phi_k\big| \le c\,.$$

In addition there is are constants $C(\alpha)$ independent of $k \ge 0$ so that $|\partial^\alpha \phi_k| \ \le \ C_\alpha$. The method of nonstationary phase shows that there is a constant independent of $k \ge 0$ so that

$$\Big| \int \chi(y)\ e^{i\phi_k(y)/(2^{2k}\epsilon)}\ f(2^{-k}y)\ dy \Big| \ \le \ C\,\big(2^{2k}\,\epsilon\big)^m \sup_{|\alpha|\le m} \|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

Therefore

$$\sum_{1\le 2^k < \epsilon^{-1/2}} |I(k)| \ \le \ C\,\epsilon^m \sum_{1\le 2^k < \epsilon^{-1/2}} 2^{-kd}\ 2^{2km} \sup_{|\alpha|\le m} \|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

The finite geometric sum has ratio $r = 2^{2m-d} > 1$. If K is the largest index,

$$r^K \ \le \ 1 + r + r^2 \ldots + r^K \ = \ \frac{r^{K+1} - 1}{r - 1} \ < \ \frac{r}{r-1}\, r^K \ := \ C(r)\, r^K\,.$$

The sum is comparable to the last term. Therefore, with $C = C(m,d) = r/(r-1)$,

$$\epsilon^m \sum_{1 \leq 2^k < \epsilon^{-1/2}} 2^{-kd}\, 2^{2km} \;\leq\; C\,\epsilon^m\, \big(2^K\big)^{2m-d} \;\leq\; C\, \epsilon^m\, \big(\epsilon^{-1/2}\big)^{2m-d} \;=\; C\,\epsilon^{d/2}\,.$$

This completes the proof. ∎

Corollary. *Suppose that $\phi(x,\zeta)$ is a family of phases depending smoothly on ζ on a neighborhood of $0 \in \mathbb{R}^q$ and that $\phi(x,0)$ satisfies the hypotheses of the preceding Theorem. Then there is a neighborhood $0 \in \mathcal{O}$ so that the hypotheses are satisfied for $\zeta \in \mathcal{O}$ and the estimate (1) holds with a constant independent of $\zeta \in \mathcal{O}$.*

Proof. The first assertion follows from the implicit function theorem applied to the system of equations $\nabla_x \phi(x,\zeta) = 0$. The estimates of the proof are all locally uniform which proves the second assertion. ∎

Chapter 3, Appendix III. Hadamard's Ovaloid Theorem.

Theorem (Hadamard). *If $d \geq 3$ and $\mathcal{M}$ is an oriented connected compact immersed hypersurface of $\mathbb{R}^d$ whose Gaussian curvature is nonzero at all points, then $\mathcal{M}$ is the boundary of a strictly convex set.*

Proof. Consider the Gauss map $\mathcal{N}$ from $\mathcal{M}$ to S^{d-1} which takes a point to its unit normal consistent with the orientation.

The nonvanishing curvature is equivalent to the differential of $\mathcal{N}$ being invertible at all points. The inverse function theorem shows that this is equivalent to $\mathcal{N}$ being a local diffeomorphism.

For any $\xi \in S^{d-1}$ the point(s) $\underline{x} \in \mathcal{M}$ where $x.\xi$ is maximal have normal equal to ξ so $\mathcal{N}$ is surjective.

The number of preimages of points is finite and locally constant, hence constant. Therefore $\mathcal{N}$ is a covering map.

Since S^{d-1} is simply connected, it follows that $\mathcal{N}$ is a homeomorphism and therefore a diffeomorphism. We recall the proof.

It suffices to show that $\mathcal{N}$ is injective. If $\mathcal{N}(m_1) = \mathcal{N}(m_2) = p \in S^{d-1}$ choose a curve $\gamma_0 : [a, b] \to \mathcal{M}$ connecting m_1 to m_2. The image $\mathcal{N} \circ \gamma$ is a closed curve μ_0 in S^{d-1} beginning and ending at p.

Simple connectivity implies that there is a homotopy of closed curves μ_t for $0 \leq t \leq 1$ beginning and ending at p with μ_1 reducing to the constant path p.

Since $\mathcal{N}$ is a covering, the homotopy lifting lemma shows that there is a homotopy γ_t, $0 \leq t \leq 1$ so that $\mathcal{N} \circ \gamma_t = \mu_t$.

The point $\gamma_t(a)$ is a point of $\mathcal{M}$ depending continuously on t with $\mathcal{N}(\gamma_t(a)) = p$. It follows that $\gamma_t(a)$ is constant and therefore equal to m_1. Similarly $\gamma_t(b) = m_2$. In particular this holds for $t = 1$.

But γ_1 is a lifting of the constant map μ_1 and is therefore constant. Therefore

$$m_1 \;=\; \gamma_1(a) \;=\; \gamma_1(b) \;=\; m_2$$

proving injectivity.

Thus each vector in S^{d-1} is normal to $\mathcal{M}$ at exactly one point. This shows that $\mathcal{M}$ is strictly convex in the sense that it intersects each tangent plane in exactly one point.

That it is strictly convex in the stronger sense of osculating ellipsoids, then follows from the nonvanishing Gaussian curvature. ∎

Example. A curve in $\mathbb{R}^2$ with positive curvature and looping twice about the origin shows that the result is not true when $d = 2$.

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