Infinitesimal Mean Value Property

Suppose that $u \in C(\mathbb{R}^d)$. The mean value of u on $|x| = r$ is

$$(Mu)(r) \ := \ \frac{\int_{|x|=r} u \, d\sigma}{\int_{|x|=r} 1 \, d\sigma}$$

Example 1 i. *$M(1) = 1$.*

ii. *Suppose that $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and one of the α_j is odd. For that j, x^α is odd in x_j so $M(x^\alpha) = 0$.*

iii. *By rotational invariance of $d\sigma$, $M(x_j^2)$ is independent of $1 \le j \le d$.*

iv. *$M(x_j^2) = r^2/d$.*

Proof.

$$\sum_{j=1}^d M(x_j^2) \ = \ M\Big(\sum_{j=1}^d x_j^2\Big) \ = \ M(|x|^2) \ = \ r^2 . \qquad \square$$

Proposition 1 *If $u \in C^4(\mathbb{R}^d)$ then as $r \to 0$,*

$$Mu \ = \ u(0) \ + \ \frac{r^2}{2d} \, \Delta u(0) \ + \ O(r^4) .$$

Proof.

$$u(x) \ = \ \sum_{|\alpha| \le 3} \frac{\partial^\alpha u(0)}{\alpha!} \, x^\alpha \ + \ O(r^4) .$$

Taking the mean, using the evaluations in the Example, proves the Proposition. $\square$

For a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ the rotate $u \circ R$ makes sense. Average over R yields a well defined Mu.

Proposition 2 *If $u \in \mathcal{D}'(\mathbb{R}^d)$ satisfies $Mu = u$ then u is harmonic.*

Proof. For $u \in C^4(\mathbb{R}^d)$ this follows from the Proposition. For distributions refer to the supplementary materials Weak Laplacian in the Encounters course materials on my web page. $\square$

A proof of the Mean Value Property of harmonic functions relying on their real analyticity can be given by extending the present argument to infinite order. It is contained in the LaplacianTD handout in the Encounters course materials on my web page.