Meet Electrostatics

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Abstract

This article presents a fleshed out version of two electrostatics problems from my 1974 Tulane Lectures. They are problems interesting in themselves. They serve as an introduction to electrostatics. They illustrate the power of the language of Distribution Theory and standard results on the Dirichlet Problem.

1 Coulomb's law

Coulomb's law asserts that the electric field of a charge distribution $\rho(x)$ with compact support is given by

$$E(x) \;=\; \int \frac{x-y}{4\pi|x-y|^3}\,\rho(y)\,dy = -\,\mathrm{grad}\int \frac{1}{4\pi|x-y|}\,\rho(y)\,dy\,.$$

This formula makes sense as a convolution in the sense of distributions for $\rho \in \mathcal{E}'(\mathbb{R}^3)$.[1] Coulomb's law is equivalent to the following characterization of the electrostatic field by partial differential equations. For a compactly supported distribution, E is the unique solution to

$$\mathrm{div}\, E \;=\; 4\pi\rho\,, \qquad \mathrm{curl}\, E \;=\; 0\,, \qquad |E| \;=\; O(1/|x|^2) \;\text{ as }\; x\to\infty\,. \tag{1}$$

The decay condition is implied by the weaker condition $E \in H^s(\mathbb{R}^3)$ for some $s \in \mathbb{R}$. Since $\mathrm{curl}\, E = 0$ there is a unique electostatic potential ϕ,

$$E \;=\; -\,\mathrm{grad}\,\phi,, \qquad |\phi| \;=\; O(1/|x|) \;\text{ as }\; x\to\infty\,. \tag{2}$$

The potential ϕ is the unique solution of

$$\Delta\phi \;=\; -4\pi\rho\,, \qquad |\phi| \;=\; O(1/|x|) \;\text{ as }\; x\to\infty\,. \tag{3}$$

The decay condition is implied by the weaker condition $\phi \in \dot H^1(|x|>R)$ for $R \gg 1$ where $\dot H^1$ denotes the homogeneous Sobolev space.

[1] This is Laurent Schwartz' notation for distributions of compact support.

2 Gauss' Law

Gauss' Law in electrostatics and Gauss' Law in vector integral calculus (a.k.a. the Divergence Theorem) are identical.

Theorem 1 (Gauss' law.) *If* $\rho \in \mathcal{E}'(\mathbb{R}^3)$ *and* $\operatorname{supp}\rho \cap \partial\Omega = \emptyset$*, then* E *is infinitely differentiable on a neighborhood of* $\partial\Omega$ *and the outward flux of* E *through* $\partial\Omega$ *is equal to the total charge in* Ω*.*

$$\int_{\partial\Omega} E \cdot \mathbf{n} \; d\Sigma \;=\; \int_{\Omega} \rho(x)\; dx\,. \tag{4}$$

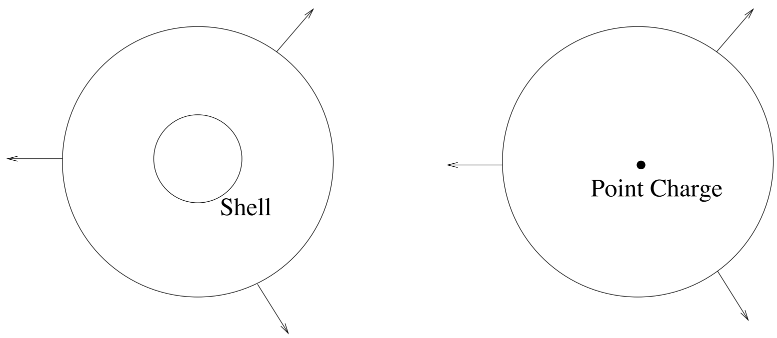
Proof. If $E \in H^1(\Omega)$ the result follows from the Divergence Theorem, $\int_{\partial\Omega} E \cdot \mathbf{n}\; d\Sigma = \int_\Omega \operatorname{div} E\, dx$.
In the general case, the smoothness at the boundary follows from the fact that ϕ is harmonic on a neighborhood of the boundary. Identity (4) follows on passing to the limit in the identity applied to ϕ^ϵ that is the solution with charge distribution equal to $j^\epsilon * \rho$ with $j^\epsilon(x) = \epsilon^{-3} j(x/\epsilon)$ a smooth compactly supported approximation to $\delta(x)$. □

3 Newton's Theorem

At the end of the *Principia*, Newton proves that the gravitational field of a uniform spherical shell is the same outside the shell as the field of a point mass at the center with the same total mass. This is one of the hardest results in the *Principia*. With today's tools there are remarkably simple proofs. Newton did NOT have Gauss' Law!

Theorem 2 (Newton.) *The electric field of a uniform spherical surface charge distribution vanishes inside the shell. Outside it is equal to the field of a point charge at the center with charge equal to the total surface charge.*

Proof. Outside. Consider the two problems side by side.



Both fields are spherically symmetric by unicity. Gauss' Theorem implies that the fluxes though concentric spheres outside the shell are the same. Therefore they are identical.

Inside. Potential is hamonic and constant on shell hence constant. ∎

4 Large $|x|$ behavior of potentials

Suppose that

$$\Delta\phi = \rho \in \mathcal{E}'(\mathbb{R}^3), \qquad \text{and} \qquad \phi = O(1/|x|) \ \text{ as } \ |x| \to \infty\,. \tag{5}$$

Then

$$\phi(x) \ = \ \frac{1}{4\pi}\int \frac{1}{|x-y|}\ \rho(y)\ dy\,.$$

For $|x| \gg 1$ and $y \in \operatorname{supp} g$, $|x-y|^{-1} = |x|^{-1} + O(|x|^{-2})$. Therefore

$$\phi(x) \ = \ \frac{1}{4\pi|x|}\int \rho(y)\ dy \ + \ O\Big(\frac{1}{|x|^2}\Big)\,. \tag{6}$$

This is the first of a series of increasingly more accurate approximations proved in Proposition 4. I do not like the physics text treatments of this topic so present a short self contained description. The approximation (6) of one harmonic function by another can be differentiated thanks to the harmonic function analogue of Cauchy's inequalities.

4.1 Moments

Definition 1 *The* **moments** ω_α *of the distribution* $\rho \in \mathcal{E}'(\mathbb{R}^3)$ *are defined for* $\alpha \in \mathbb{N}^3$ *by*

$$\omega_\alpha(\rho) \ := \ \big\langle \rho\,,\, x^\alpha \big\rangle\,.$$

In the sequel we often adopt the common abuse of notation using the short-hand $\int \rho(x) x^\alpha dx$ for the pairing.

Exercise 1 *Show that the distribution* $\partial^\alpha \delta$ *has moments given by*

$$\omega_\alpha \ = \ \alpha! \ := \ \Pi_{j=1}^3 (\alpha_j!)\,, \qquad \text{and,} \qquad \omega_\beta = 0 \quad \text{for} \quad \beta \neq \alpha\,. \tag{7}$$

It follows that the distribution

$$\sum_{|\alpha| \le m} \left(\frac{\omega_\alpha(\rho)}{\alpha!}\, \partial^\alpha \right) \delta \tag{8}$$

has the same moments of order $\le m$ as ρ.

Example 1 *If* $h \in \mathcal{E}'(\mathbb{R}^3)$ *and* $|\alpha| \ge m+1$*, then the moments of order* $\le m$ *of* $\partial^\alpha h$ *all vanish. The next result shows that these examples exhaust all distributions with moments of order* $\le m$ *vanishing.*

Proposition 3 **i.** *The moments of* ρ *of order less than or equal to* m *vanish.*
ii. *There exist* $h_\alpha \in \mathcal{E}'(\mathbb{R}^3)$ *for all* $|\alpha| = m+1$*, so that* $\rho \ = \ \sum_{|\alpha| \le m+1} \partial^\alpha h_\alpha\,.$

Proof. ii⇒i. The example.
i⇒ii. An inductive argument shows that is suffices to show that $\phi = \sum_j \partial_j h_j$ *with* $h_j \in \mathcal{E}'$ *and the moments of* h_j *up to order* $m-1$ *vanish.*
Reason on the Fourier side. The vanishing moment hypothesis is equivalent to $\widehat{\phi}$ vanishing together with its partial derivatives up to order m at the origin. Write

$$\widehat{\phi}(\xi) \ = \ \int_0^1 \frac{d}{ds}\, \widehat{\phi}(s\xi)\; ds \ = \ \sum \xi_j\, \gamma_j(\xi) \quad \text{with} \quad \gamma_j \ := \ \int_0^1 \frac{\partial \widehat{\phi}}{\partial \xi_j}\,(s\xi)\, d\xi\,.$$

Define $h_j = i\,\mathcal{F}^{-1}\gamma_j$ so $\phi = \sum \partial_j h_j$.

An explicit computation shows that

$$\mathcal{F}^{-1}\Big(\frac{\partial\widehat{\phi}}{\partial\xi_j}\,(s\xi)\Big) \;=\; c\,\frac{x_j}{s}\;\phi\Big(\frac{x}{s}\Big)\;s^3\,.$$

Therefore, if ϕ is supported in a closed ball B centered at the origin, then so is h_j.

A moment of h_j of order k is a linear combination of moments of ϕ of order $k+1$. Thus the moments of h_j of order $\leq m-1$ vanish. □

4.2 Multipoles

Definition 2 *A* **multipole of order** m *is a distribution* $P_m(\partial)\delta$ *where* $P_m(\partial)$ *is a non zero homogeneous real partial differential operator of degree* m*.*

Since the partial derivatives of order m of δ are linearly dependent, the multipoles corresponding to distinct P_m are distinct.

If $P_m(\partial)\delta$ is a multipole of order m, denote by ψ the corresponding electric potential characterised by

$$\Delta\psi = P_m(\partial)\,\delta\,,\qquad \psi = O(1/|x|)\ \ \text{as}\ \ |x|\to\infty\,.$$

Then

$$\psi \;=\; P_m(\partial)\,\frac{1}{4\pi|x|}\;\in\; C^\infty(\mathbb{R}^3\setminus 0)\,. \tag{9}$$

The potential ψ is homogeneous of degree $-1-m$.

4.3 Mulitpole approximations

The multipole approximations to ϕ for large $|x|$ are the following.

Proposition 4 *For* $\rho\in\mathcal{E}'(\mathbb{R}^3)$ *and* $0\leq m\in\mathbb{N}$ *the solution* ϕ *of* (5) *satisfies*

$$\Big|\phi\;-\;\sum_{|\alpha|\leq m}\Big(\frac{\omega_\alpha(\rho)}{\alpha!}\;\partial^\alpha\Big)\frac{1}{4\pi|x|}\Big| \;=\; O\Big(\frac{1}{|x|^{2+m}}\Big)\quad\text{as}\quad |x|\to\infty\,.$$

Proof. Define

$$\zeta \ := \ \phi \ - \sum_{|\alpha|\le m} \Big(\frac{\omega_\alpha(\rho)}{\alpha!}\ \partial^\alpha\Big)\frac{1}{4\pi|x|}\,.$$

Then $\zeta = O(|x|^{-1})$ and

$$\Delta\zeta \ = \ \rho(y) \ - \sum_{|\alpha|\le m} \Big(\frac{\omega_\alpha(\rho)}{\alpha!}\ \partial^\alpha\Big)\delta\,.$$

The right hand side has vanishing moments up to and including order m. Proposition 3 implies that the right hand side can be written as a finite sum of terms $\partial^\alpha h_\alpha$ with h_α of compact support and $|\alpha| = m+1$. This expresses $4\pi\zeta$ as a finite sum of terms

$$\int \frac{1}{|x-y|}\ \partial^\alpha h_\alpha(y)\,dy \ = \ \int \Big((-\partial)^\alpha \frac{1}{|x-y|}\Big)\, h_\alpha(y)\,dy \ = \ O\Big(\frac{1}{|x|^{2+m}}\Big).$$

This completes the proof. □

4.4 A cancellation property of multipole potentials

Proposition 5 *If $m \ge 1$, then the potential ψ from* (9) *satisfies*

$$\int_{|x|=1} \psi(x)\ d\Sigma \ = \ 0\,. \tag{10}$$

Proof. Gauss' Law in the unit ball implies that for $m \ge 1$,

$$\int_{|x|=1} \frac{\partial\psi(x)}{\partial r}\ d\Sigma \ = \ \int_{\mathbb{R}^3} P_m(\partial)\delta\ dx \ = \ 0\,. \tag{11}$$

The Euler homogeneity shows that on $|x| = 1$,

$$\partial_r\psi \ = \ x\cdot\partial_x\psi \ = \ (-1-m)\psi\,.$$

Equation (11) implies the desired result. □

5 The Capacitory Potential

Definition 3 *The* **capacitory potential** V *of the nice bounded open set* Ω *is defined as the unique solution of the Dirichlet problem*

$$\Delta V = 0 \ \text{ on } \ \mathbb{R}^3 \setminus \overline{\Omega}, \quad V = 1 \ \text{ on } \ \overline{\Omega}\,, \quad V = O(1/|x|) \ \text{ as } \ |x| \to \infty\,.$$

The potential V is a continuous function, piecewise smooth on $\mathbb{R}^3$.

Exercise 2 *Find the relation between the capacitory potential of a set* Ω *and that of* $R\Omega$*.*

Exercise 3 *Find the capacitory potential of a ball of radius* R*. Verify that the scaling with* R *is consistent with Exercise 2 .*

6 Perfect conductors and surface charges

Perfect conductiors are modelled as nice open sets with the property that the electrostatic field must be identically equal to zero on their interior. [2] The electrons in such a conductor are free to move and in the presence of an electric field redistribute themselves pushed about by the electric field. Equilibrium is attained when the field vanishes at all points inside the conductor. Such models are only reasonable when applied electric fields vary slowly with repect to the natural time scales for the motion of conduction electrons.

Since $E = 0$ in Ω it follows that $\operatorname{div} E = 0$ so there can be no volume charge distribution inside a perfect conductor. *If one places charges on a conductor they move to the boundaries.* One expects a surface charge distribution.

Suppose that the region Ω is occupied by a perfect conductor. Denote by $d\Sigma$ the surface area measure on $\partial\Omega$. Denote by $\mathbf{n}$ the outward unit normal vector to Ω. The electric field vanishes identically inside Ω. The electric field is usually discontinuous across the boundary.

Proposition 6 *Suppose that* Ω *is a perfect condutor and in a neighborhood of* $\overline{\Omega}$ *the only charges are surface charges on* $\partial\Omega$ *and that* ϕ *is piecewise smooth and continuous with singularities only on* $\partial\Omega$*.*

[2] There are three superb references for electrostatics. In chronological order they are [4], [1], [7]. More mathematical are [2], [3] Chapter 9 §12, and [5].

i. *Then the tangential component of* $E|_{\partial\Omega}$ *measured from* $\mathbb{R}^3 \setminus \overline{\Omega}$ *vanishes. It is continuous across the boundary.*

ii. *In a neighborhood of* $\partial\Omega$ *one has in the sense of distributions*

$$\operatorname{div} E \;=\; E\big|_{\mathbb{R}^3\setminus\overline{\Omega}} \cdot \mathbf{n}\; d\Sigma\,, \tag{12}$$

where the normal component is computed from the exterior of Ω.

Remark 1 1. *There is a surface charge distribution with density equal to the normal component* $E\big|_{\mathbb{R}^3\setminus\overline{\Omega}} \cdot \mathbf{n}$ *per unit area.*

2. *The total charge on* Ω *is equal to*

$$\int_{\partial\Omega} E\big|_{\mathbb{R}^3\setminus\overline{\Omega}} \cdot \mathbf{n}\; d\Sigma \;=\; -\int_{\partial\Omega} \nabla\phi \cdot \mathbf{n}\; d\Sigma \;=\; -\int_{\partial\Omega} \frac{\partial\phi}{\partial\mathbf{n}}\; d\Sigma\,.$$

Proof. The electrostatic potential must be constant on Ω and $O(1/|x|)$ at infinity so is equal to a constant multiple of the capacitory potential V. Therefore E is piecewise smooth near $\partial\Omega$.

The second assertion of the Proposition is a direct computation of $\operatorname{div} E$ in the sense of distributions.

The first is a consequence of $\operatorname{curl} E = 0$ and the analogous evaluation

$$\operatorname{curl} E = \big(E\big|_{\mathbb{R}^3\setminus\overline{\Omega}}\big)_{tan}\; d\Sigma\,. \qquad \square$$

Definition 4 *The* **capacity** $\operatorname{Cap}(\Omega)$ *is defined to be the total charge on* $\overline{\Omega}$ *for the capacitory potential* V,

$$\operatorname{Cap}(\Omega) \;:=\; -\int_{\partial\Omega} \frac{\partial V}{\partial\mathbf{n}}\; d\Sigma\,.$$

If charge q is placed on Ω then the electrostatic potential is equal to $qV/\operatorname{Cap}(\Omega)$.

7 Induced charge

Suppose that perfect conductors occupy disjoint nice open sets Ω_1 and Ω_2 Study the charges induced on the grounded conductor Ω_2 by charge q placed on the conductor Ω_1. In the figure, postive charge is placed on Ω_1.

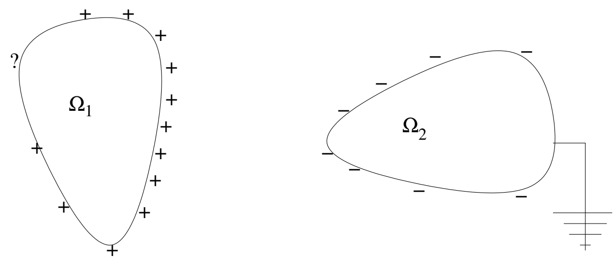


Figure 1: Charge $q > 0$ on Ω_1. Ω_2 grounded.

7.1 Existence and uniqueness

The equilibrium electrostatic potential is the unique solution of

$$\Delta\phi = 0 \quad \text{on} \quad \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2), \qquad \phi = O(|x|^{-1}) \ \text{ as } \ |x| \to \infty, \tag{13}$$

so that ϕ is constant on Ω_1 and equal to zero on Ω_2 and

$$-\int_{\partial\Omega_1} \frac{\partial\phi}{\partial\mathbf{n}}\, d\Sigma = q\,.$$

Proof of existence and uniqueness. Define V_1 to be the solution of the Dirichlet problem

$$\Delta V_1 = 0 \ \text{ on } \ \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2), \quad V_1 = 1 \ \text{ on } \ \Omega_1, \quad V_1 = 0 \ \text{ on } \ \Omega_2,$$

$$V_1 = O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty\,.$$

Then

$$0 < V_1 < 1 \quad \text{on} \quad \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)\,,$$

The maximum value of V_1 is attained on Ω_1 and minimum value on Ω_2. Hopf's Lemma (see [6]) implies that $\partial V_1/\partial\mathbf{n} < 0$ at all points of $\partial\Omega_1$ and also $\partial V_1/\partial\mathbf{n} > 0$ at all points of $\partial\Omega_2$.

ln particular, there is a strictly positive charge on $\overline{\Omega}_1$ The potential $\phi = cV_1$ for a uniquely determined constant c. □

The induced charge on Ω_2 is equal to

$$-\int_{\partial\Omega_2} \frac{\partial\phi}{\partial\mathbf{n}}\, d\Sigma\,.$$

7.2 A qualitative question

Question. *The induced charge has sign opposed to the sign of q. Is it possible for the induced charge to be larger in absolute value than $|q|$? Equal to $|q|$?*

Proposition 7 *If Ω_j are nice disjoint perfect conductors with Ω_2 grounded and Ω_1 carrying charge q, then the induced charge on Ω_2 is always strictly less than $|q|$ in absolute value.*

Remark 2 *A heuristic proof using lines of force is presented in Section 89c of Maxwell's remarkable treatise [4].*

Proof. For the potential V_1, the surface charge on Ω_1 is strictly postive and that on Ω_2 is strictly negative.

Determine the asymptotic behavior of the strictly positive function V_1. Since Ω_1 carries charge $q \neq 0$, the charge distribution on the conductors is not identically equal to zero. Thus, there is a smallest integer m so that the moments of order $< m$ of the charge distribution vanish and there is a nonvanishing moment of order m.[3] Then with non vanishing homogeneous P_m,

$$V_1 \;=\; P_m(\partial)\frac{1}{4\pi|x|} \;+\; O\bigg(\frac{1}{|x|^{m+2}}\bigg)\,. \tag{14}$$

Next show that this implies that $m = 0$.

Indeed, if $m \geq 1$, then Proposition 5 shows that the first term on the right of (14) is homogeneous of degree $-m-1$ and has both positive and negative values. Since $V_1 > 0$ this is not possible. Therefore, the only possible multipole approximation for V is

$$V_1 \;=\; \frac{c_1}{|x|} \;+\; O\bigg(\frac{1}{|x|^2}\bigg), \qquad c_1 > 0\,.$$

For the induced charge problem treat the case $q > 0$. The potential for the opposed charge is obtained by multiplying by -1. For $q > 0$, the electrostatic

[3] Otherwise, all moments would vanish. Then the Fourier Transform $\widehat{\rho} = 0$ since it is entire analytic with derivatives of all orders vanishing at the origin.

potential for the induced charge problem has solution $\phi = c_2 V_1$ with $c_2 > 0$. The corresponding differentiated multipole approximation is

$$\frac{\partial \phi}{\partial r} \;=\; \frac{-c_1 c_2}{|x|^2} \;+\; O\left(\frac{1}{|x|^3}\right), \qquad c_j > 0\,.$$

The flux of the electric field through the boundary of a ball of large radius R is strictly postivie. Gauss' Law implies that there is a net positive charge in the ball. Therefore the positive charge on Ω_1 is strictly larger than the negative induced charge on Ω_2. The proof is complete. □

8 Two charged conductors

Next analyse the problem with two conductors when neither one is grounded.

8.1 Existence and uniqueness

As in the preceding section a charge q_1 is placed on Ω_1. In this section charge q_2 is placed on Ω_2. The electrostatic potential must satisfy (13), and the boundary conditions

$$\phi \;=\; c_j \;\; \text{on} \;\; \Omega_j\,.$$

The surface charge constraints are

$$-\int_{\partial\Omega_j} \frac{\partial \phi}{\partial \mathbf{n}}\, d\Sigma \;=\; q_j \qquad j = 1, 2\,.$$

Proposition 8 *There is one and only one smooth solution on $\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2)$ satisfying these conditions.*

Proof of uniqueness. It suffices to show that $\phi = 0$ is the only solution with charges $q_1 = q_2 = 0$. Suppose that ϕ is such a solution. Then if ϕ is not identically zero it must assume nonzero values. Since ϕ tends to zero, ϕ must attain either a positive maximum or negative minimum at a boundary point. Suppose that point lies on $\partial\Omega_j$ and is a maximum. Then all points of $\partial\Omega_j$ are maxima and Hopf's Lemma (see [6]) implies that $\partial\phi/\partial\mathbf{n} < 0$ at all points of $\partial\Omega_j$. Therefore

$$\int_{\partial\Omega_j} \frac{\partial \phi}{\partial \mathbf{n}}\, d\Sigma \;<\; 0\,.$$

This contradicts the assumption that there is no charge on Ω_j. The case of a minimum is exactly analogous. This proves uniqueness.

Proof of existence. Define V_1 as in the preceding section and V_2 to be the solution of

$$\Delta V_2 = 0 \ \text{ on } \ \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2), \quad V_2 \ = \ O(|x|^{-1}) \ \text{ as } \ |x| \to \infty\,,$$

$$V_2 = 0 \ \text{ on } \ \Omega_1, \qquad V_2 = 1 \ \text{ on } \ \Omega_2.$$

A solution ϕ must be of the form $\phi = c_1 V_1 + c_2 V_2$ where the c_j are the constant values of the electrostatic potential on Ω_j.

The surface charge constraints yield two linear equations for the two unknowns c_j. The uniqueness result shows that the resulting equations cannot have two distinct solutions. It follows that they have exactly one solution. That solution yields a ϕ satisfying all conditions. $\square$

8.2 A qualitative question

Consider now the special case where charge $q > 0$ is placed on Ω_1 and Ω_2 is neutral, that is $q_2 = 0$. Then negative charges on Ω_2 are attracted toward Ω_1 and correspondingly the positive charges on Ω_1 are attracted toward Ω_2.

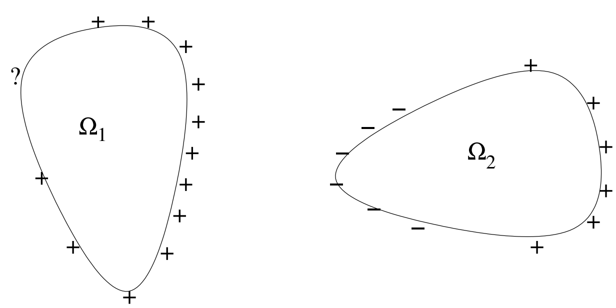


Figure 2: Charge $q > 0$ on Ω_1. Ω_2 neutral.

Question. *In this case with neutral Ω_2, is it possible that the charges on Ω_1 are so strongly attracted to Ω_2 that there are places on the surface of Ω_1 far from Ω_1 with a negative surface charge density?*

Answer. First show that $\phi \geq 0$. If not ϕ would assume negative values and therefore attain a strictly negative minimum. That minimum would have to be on one of the conductors Ω_j and each point of $\partial\Omega_j$ would be a minimum. Hopf's Lemma implies that $\partial\phi/\partial\mathbf{n} > 0$ at all points of $\partial\Omega_j$ so there is a net negative charge on Ω_j. However, Ω_1 carries postive charge and Ω_2 is neutral a contradiction.

Once we know that $\phi \geq 0$ on $\mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, the maximum principle implies that $\phi > 0$ on that set.

The function ϕ attains a positive maximum at all points of $\partial\Omega_1$ or $\partial\Omega_2$ or both. Whichever it is, Hopf's Lemma implies that the charge density $-\partial\phi/\partial\mathbf{n} > 0$ at all points of the corresponding $\partial\Omega$. Thus that Ω carries positive charge and the surface charge density at all points of $\partial\Omega$ is strictly positive. Since Ω_2 is neutral Ω must be Ω_1 and the surface density is strictly positive. *The answer to the question is NO.*

Question. *Is this result in the literature somewhere other than my Tulane Lectures [8].*

Example 2 *Now place positive charge q_1 and small positive charge $0 < q_2 \ll 1$ on Ω_2. By continuity, the surface charge density on $\partial\Omega_1$ will be everywhere positive.*

When $q_2 \gg q_1 > 0$, the same continuity argument shows that the surface charge densities will be close to those obtained when $q_1 = 0$ so there will be surface charge densities on $\partial\Omega_1$ of both signs.

References

[1] R.P. Feynman, Lectures on Physics Vol. II, Addison- Wesley, Palo Alto, 1964.

[2] K. O. Friedrichs, *Mathematical Methods of Electro- magnetic Theory*, Courant Institute Lecture Notes, 1974.

[3] O. D. Kellogg, *Foundations of Potential Theory*, Dover Publications, New York, 1953.

[4] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Dover Publications, 1954.

[5] C. Müller, *Foundations of the Mathematical Theory of Electromagnetic Waves* Grund. Math. Wiss. Einz. Band 155, Sringer-Verlag, 1969.

[6] M. H. Protter, and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Eng!ewood Cliffs, N. J., 1967.

[7] E. M. Purcell, *Electricity and Magnetism* 3^{rd} *ed.*, Cambridge University Press, 2013.

[8] J. Rauch, Five problems; an introduction to the qualitative theory of partial differential equations, in *Lecture Notes in Mathermatics no. 446*, Springer-Verlag, 1974.

[9] J. Rauch and M. Taylor, Penetration into shadow regions and unique continuation properties in hyperbolic mixed problems, Indiana U. Math. J. **22** (1973), 277-285.