# Newton's Theorem

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## 1 Coulomb's law

Coulomb's law asserts that the electric field of a charge distribution $\rho(x)$ with compact support is given by

$$E(x) \ = \ \int \frac{x-y}{4\pi|x-y|^3}\ \rho(y)\ dy = -\,\text{grad} \int \frac{1}{4\pi|x-y|}\ \rho(y)\ dy$$

Coulomb's law is equivalent to the following characterization of the electrostatic field by partial differential equations. For a compactly supported distribution, $\rho \in \mathcal{E}'(\mathbb{R}^3)$, the electrostatic field is the unique solution $E$ to

$$\text{div}\, E \ = \ 4\pi\rho\,, \qquad \text{curl}\, E \ = \ 0\,, \qquad |E| \ = \ O(1/|x|^2) \ \text{ as } \ x \to \infty\,. \tag{1}$$

The decay condition is implied by the weaker condition $E \in H^s(\mathbb{R}^3)$ for some $s \in \mathbb{R}$. Since $\text{curl}\, E = 0$ there is an equivalent discription in terms of the electostatic potential $\phi$,

$$E \ = \ -\,\text{grad}\, \phi\,. \tag{2}$$

The electrostatic potential $\phi$ is the unique solution of

$$\Delta\phi \ = \ -4\pi\rho\,, \qquad |\phi| \ = \ O(1/|x|) \ \text{ as } \ x \to \infty\,. \tag{3}$$

When $\rho \in H^{-1}_{comp}(\mathbb{R}^3)$, the decay condition is implied by the weaker condition $\phi \in \dot H^1(\mathbb{R}^3)$ where $\dot H^1$ denotes the homogeneous Sobolev space.

## 2 Gauss' Law

From this perspective Gauss' Law in electrostatics and Gauss' Law in vector integral calculus (a.k.a. the Divergence Theorem) are identical.

**Theorem 1 (Gauss' law.) i.** *If $\rho$ is $C^\alpha$ on a neighborhood of a nice domain with boundary $\overline{\Omega}$ then $E \in C^1(\overline{\Omega})$ and the flux of $E$ through $\partial\Omega$, is equal to the total charge in $\Omega$*

$$\int_{\partial\Omega} E \cdot \mathbf{n}\, d\sigma \;=\; \int_{\Omega} \rho(x)\, dx\,. \tag{4}$$

**ii.** *If $\rho \in \mathcal{E}'(\mathbb{R}^3)$ and the support of $\rho$ intersects $\overline{\Omega}$ in a compact subset of $\Omega$, then $E$ is infinitely differentiable on a neighborhood of $\partial\Omega$ and (4) holds.*

**Proof. i.** That $\phi \in C^{1+\alpha}$ follows from the interior elliptic regularity theorem. Equation (4) follows from the Divergence Theorem.

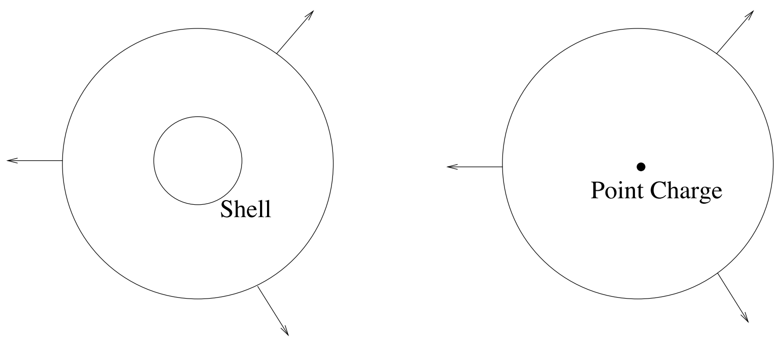
**ii.** The smoothness follows from the fact that $\phi$ is harmonic on a neighborhood of the boundary. Identity (4) follows on passing to the limit in the identity applied to $\phi^\epsilon$ that is the solution with charge distribution equal to $j^\epsilon * \rho$ with $j^\epsilon(x) = \epsilon^{-3} j(x/\epsilon)$ a smooth compactly supported approximation to $\delta(x)$. $\square$

# 3 Newton's Theorem

At the end of the *Principia* Newton proves that the gravitational field of a uniform spherical shell is the same outside the shell as the field of a point mass at the center with the same total mass. This is one of the hardest results in the *Principia*. With today's tools there are remarkably simple proofs. Newton did NOT have Gauss' Law!

**Theorem 2 (Newton.)** *The electric field of a uniform spherical surface charge distribution vanishes inside the shell. Outside it is equal to the field of a point charge at the center with charge equal to the total surface charge.*

**Proof. Outside.** Consider the two problems side by side.




Both fields are spherically symmetric by unicity. Gauss' Theorem implies that the fluxes though concentric spheres outside the shell are the same. Therefore they are identical.

**Inside.** Potential is hamonic and constant on shell hence constant. ∎