LECTURE #1. FIVE PROBLEMS: AN INTRODUCTION TO THE QUALITATIVE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

by

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INTRODUCTION

The three lectures presented here have several goals. For emphasis we list them.

(I) It is often remarked that one of the justifications for proving existence and uniqueness theorems in partial differential equations is that the methods and ideas developed are also useful in the more interesting qualitative questions concerning solutions. The problems discussed illustrate this point.

(II) The five problems of the first lecture are quite simple and in my opinion physically interesting. Despite this fact they do not seem to be in any of the standard elementary P.D.E. treatises. It is my hope that they will find their way into introductory courses.

(III) It is an important fact that interesting physical questions often take a qualitative rather than quantitative form and as such can be treated by the methods mentioned in (I).

(IV) Physical intuition is a very valuable aid in formulating questions; however, it is often unreliable, especially when it comes to answering these questions. We give several examples where reasonable physical intuition is dead wrong. The instances where mathematics tells us our intuition is faulty are clearly of great value. Formulas are smarter than people.

(V) By studying the qualitative implications of mathematical models our intuition can be improved. After all, intuition only reflects our belief based on accumulated experience.

(VI) The last two lectures are intended to serve as an introduction to work with M. Taylor on scattering by unusual obstacles. The original paper is not easy reading. I hope these are.

1. A PROBLEM IN HEAT CONDUCTION

Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded, open set with smooth boundary (of course $n = 2,3$ are of special importance). We assume that the region Ω is filled with a homogeneous medium and is insulated at $\partial\Omega$. For example we could be considering the flow of heat in a metal plate or in jello inside a thermos bottle. The mathematical model for the temperature $u(t,x)$ at time t and place x is

(1) $$u_t = c\,\Delta u \quad \text{in} \quad \Omega ,$$

(2) $$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega.$$

Here c is a physical constant depending on the medium, $\Delta = \Sigma \frac{\partial^2}{\partial x_i^2}$, and $\frac{\partial}{\partial \nu}$ is differentiation in the direction of the normal to $\partial\Omega$. Given the initial temperature, $u(0,x)$, (1) and (2) suffice to determine the time evolution.

QUESTION. *If initially the medium is strongly heated in a neighborhood of a point* x_0 *(for example with a torch) so that* $u(0,x)$ *has a sharp maximum at* x_0, *is it true that for each* $t > 0$ *the maximum of* $u(t,x)$ *occurs at or near* x_0?

Intuitively, one feels that heat will flow from the hot spot to the cooler ones with the result that the temperature gradually decreases near x_0 and increases at other points. This vague idea which even sounds like the second law of thermodynamics indicates that the answer is yes.

ANALYSIS. We solve (1), (2) by eigenfunction expansion. It is well known that there are eigenvalues $0 = \lambda_0 > \lambda_1 \geq \ldots$ converging to $-\infty$ and eigenfunctions $\Phi_0, \Phi_1, \ldots$ such that

$$\Delta\Phi_j = \lambda_j\Phi_j \text{ , } j = 0,1,2,\ldots \text{ ,}$$

$$\frac{\partial}{\partial\nu}\Phi_j = 0 \text{ on } \partial\Omega \text{ ,}$$

$$\{\Phi_j\} \text{ is an orthonormal basis for } L_2(\Omega).$$

The solution $u(t,x)$ is then given by

$$u = \sum_j a_j e^{c\lambda_j t} \Phi_j$$

where $a_j = (u(0,x), \Phi_j)_{L_2(\Omega)}$. We choose $\Phi_0 = \frac{1}{|\Omega|^{1/2}} > 0$. Then since $\lambda_j < 0$ for $j > 0$ we have

$$u = \frac{1}{|\Omega|} \int_\Omega u + o(1).$$

To be precise, for any k

$$\|u(t) - \frac{1}{|\Omega|} \int_\Omega u\|_{H_k(\Omega)} = O(e^{c\lambda_1 t}) \quad \text{as} \quad t \to \infty.$$

EXERCISE. Prove this.

The conclusion is that u converges rapidly to its average value. This is a rigorous version of the "approach to equilibrium".

HYPOTHESIS. λ_1 *is a simple eigenvalue, that is,* $\lambda_1 > \lambda_2$.

This hypothesis is satisfied by "most" domains. Its failure is usually due to some sort of symmetry. For example, in $\mathbb{R}^2$ it is valid for all ellipses except the circle and all rectangles except the square.

EXERCISE. Verify the assertions of the last sentence.

With this assumption we now look at the next term in the expression for u.

$$u = a_0\Phi_0 + a_1 e^{c\lambda_1 t} \Phi_1 + O(e^{c\lambda_2 t}) .$$

EXERCISE. Formulate and prove a precise sense for $O(e^{c\lambda_2 t})$.

HYPOTHESIS. *The eigenfunction* Φ_1 *has a unique maximum (resp. minimum) at* P_+ *(resp.* P_-*).*

Again this behavior is expected in unexceptional cases. Since Φ_1 is orthogonal to constants we have $P_+ > 0 > P_-$.

ANSWER TO QUESTION. *If* $a_1 > 0$ *(resp.* < 0*) the point where* $u(t)$ *achieves its maximum approaches* P_+ *(resp.* P_-*) as* $t \to \infty$.

Thus, rather than staying in the neighborhood of the original hot spot the maximum always moves to one of two points. It is very instructive to consider the case $\Omega = [0,\pi] \subset \mathbb{R}^1$, $\Phi_1(x) = \sqrt{2/\pi}\cos x$. If the initial hot spot is to the left (resp. right) of $\pi/2$ then the maximum moves to 0 (resp. π).

EXERCISE. Verify the last assertion.

One can even get a feeling for why this happens. If the initially hot region is to the left of $\pi/2$ then there is more medium to the right that must be heated. Thus the heat is drawn off more rapidly on the right and the region of high temperature moves to the left because of "erosion on the right".

2. A PROBLEM FROM ELECTROSTATICS

Let Ω_1, Ω_2 in $\mathbb{R}^3$ be disjoint bounded open sets with smooth boundaries. We suppose that perfect conductors occupy these regions. Suppose a positive charge q is placed on Ω_1. The charges on the conductors distribute themselves rapidly and equilibrium is reached. What happens is that the positive charge on Ω_1 attracts negatives on Ω_2 and a charge distribution as in Figure 1 is established.

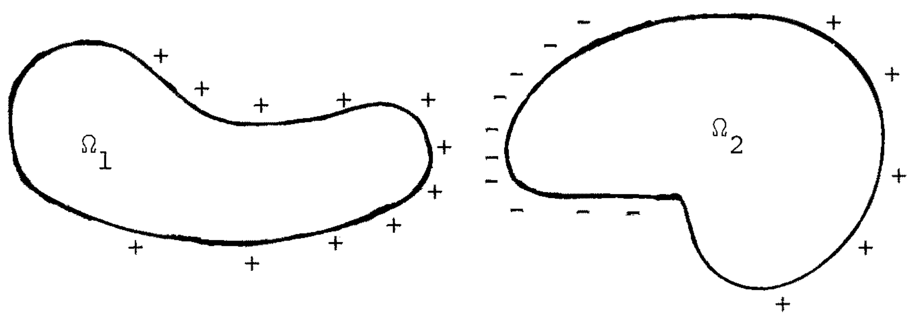

Figure 1

QUESTION. *Is it possible that the attractive force between charges is so strong that so much plus charge is drawn to one side of* Ω_1 *that a net negative charge is present at some point of* Ω_1*?*

ANALYSIS. We suppose that units are chosen so that the electrostatic potential due to a unit positive charge at the origin is $(4\pi|x|)^{-1}$. Recall that if ϕ is the electrostatic potential then the electric field is given by $E = -\text{grad}\ \phi$ and the force on a point charge q at x is qE. In the exterior of the conductors Ω_1, Ω_2 the potential $\phi(x)$ satisfies

$$\Delta\phi = 0, \quad \phi = O(1/|x|) \quad \text{as} \quad |x| \to \infty. \tag{3}$$

In addition the charge on the conductors is located entirely on the surface and

(4) $\frac{\partial\phi}{\partial\nu}$ = charge density per unit area at $\partial\Omega_1$, $\partial\Omega_2$.

Here ν is the normal direction pointing into Ω_i, i = 1,2. Thus the condition that Ω_1 carry charge q and Ω_2 be neutral is

$$\int_{\partial\Omega_1}\frac{\partial\phi}{\partial\nu} = q \ , \quad \int_{\partial\Omega_2}\frac{\partial\phi}{\partial\nu} = 0 . \tag{5}$$

Since charge is free to move in the tangential directions at $\partial\Omega_i$ the tangential force must vanish there. Thus we have $E_{tan} = 0$ at $\partial\Omega_i$, or in terms of ϕ

$$\phi = \text{constant on } \partial\Omega_i \ , \ i = 1,2,. \tag{6}$$

There is a unique function ϕ satisfying (3)-(6).

PROOF OF UNIQUENESS. Suppose there were two solutions ϕ_1 , ϕ_2. Let $\psi = \phi_1 - \phi_2$. If $\psi \neq 0$ it must assume a positive maximum or negative minimum since $\psi = O(1/|x|)$ as $|x| \to \infty$. By the maximum principle this extremum must occur on one of the conductors, which we call Ω_i. Since ψ is constant on this conductor, each point of Ω_i is an extremum of the same type. By the Hopf maximum principle $\frac{\partial\psi}{\partial\nu} \neq 0$ at every point of Ω_i so $\int_{\Omega_i}\frac{\partial\psi}{\partial\nu} \neq 0$, which cannot be. Therefore $\psi \equiv 0$. □

PROOF OF EXISTENCE. Let V_1 be the solution of the standard exterior Dirichlet problem

$$\Delta V_1 = 0 \quad \text{in} \quad \mathbb{R} \setminus (\Omega_1 \cup \Omega_2) \ ,$$

$$V_1 = O(1/|x|) \quad \text{as} \quad |x| \to \infty,$$

$$V_1 = 1 \quad \text{on} \quad \partial\Omega_1 \ , \ V_1 = 0 \quad \text{on} \quad \partial\Omega_2 \ .$$

Let V_2 be defined similarly except that the role of the conductors is reversed. If ϕ is a solution and $c_i = \phi|_{\Omega_i}$, $i = 1,2$ then $\phi = c_1V_1 + c_2V_2$. We will try to find numbers c_1 , c_2 such that $c_1V_1 + c_2V_2$ solves (3)-(6). Conditions (3), (4), (6) are automatic. On the other hand (5) yields a pair of simultaneous linear equations for c_1 and c_2. To prove existence it suffices to show that the corresponding homogeneous equations have only the trivial solution. This follows from uniqueness. □

PROPOSITION. $\phi > 0$ *on* $\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2)$ *and attains its maximum on* $\partial\Omega_1$ *where* $\frac{\partial\phi}{\partial\nu} > 0$.

PROOF. First we show that $\phi \geq 0$. If not ϕ would attain a negative minimum which by the maximum principle would occur on $\partial\Omega_i$ for $i = 1$ or 2. Also since ϕ is constant on $\partial\Omega_i$ each point is a minimum so $\frac{\partial\phi}{\partial\nu} < 0$ at each point of $\partial\Omega_i$, violating (5). Once we know $\phi \geq 0$ the maximum principle and (5) imply that $\phi > 0$.

EXERCISE. Prove the last assertion.

Since $\phi > 0$ it must attain a positive maximum at all points of $\partial\Omega_i$ for $i = 1$ or 2. Then $\frac{\partial\phi}{\partial\nu} > 0$ at $\partial\Omega_i$ so by (5), $i = 1$. □

ANSWER TO QUESTION. *Since* $\frac{\partial\phi}{\partial\nu} > 0$ *on* $\partial\Omega_1$ *we see that the answer is* *__NO__*.

3. ANOTHER PROBLEM FROM ELECTROSTATICS

Since we have developed these nice tools we'll do another problem. We have essentially the same situation as before except that conductor number 2 is grounded. Mathematically this means that the boundary condition on Ω_2 becomes

$$\phi = 0 \quad \text{on} \quad \partial\Omega_2 \ .$$

Practically this means that Ω_2 is connected to some very large object, for example, Lake Michigan. Physically a new phenomenon occurs. The attractive force of the positive charges on Ω_1 causes negative charge to flow from the "large object" to Ω_2 so that Ω_2 becomes negatively charged. The negative charge is called induced charge. As before we could ask whether any point of Ω_1 has a net negative charge or if any point of Ω_2 has a net positive charge. The answer to both questions is NO.

QUESTION. *Is it possible that the total negative charge induced on* Ω_2 *is greater (in absolute value) than the positive charge on* Ω_1*?*

If c is the value of ϕ on Ω_1 then it is easy to see that $\phi = cV_1$. It is also a simple matter to show that $0 \le V_1 \le 1$ so that $\frac{\partial V_1}{\partial\nu} > 0$ at all points of Ω_1.

It follows that $c > 0$ since $\int_{\partial\Omega_1} \frac{\partial\phi}{\partial\nu} = q > 0$. To get more information about ϕ we apply Green's identity in the region $R = \{x \in \mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2) \mid |x| \le r\}$ to obtain

$$0 = \int_R \Delta\phi = \int_{|x|=r} \frac{\partial\phi}{\partial r} + \int_{\partial\Omega_1} \frac{\partial\phi}{\partial\nu} + \int_{\partial\Omega_2} \frac{\partial\phi}{\partial\nu} .$$

Therefore

$$\int_{|x|=r} \frac{\partial\phi}{\partial r} = -q - \int_{\partial\Omega_2} \frac{\partial\phi}{\partial\nu} . \tag{7}$$

This proves that the left hand side is independent of r for r large. In addition, using the standard multipole expansion of harmonic functions (see [4; Ch. 5, §7, 8]*) one shows that (3) and $\phi \ge 0$ imply

$$\frac{\partial\phi}{\partial r} = \frac{\text{negative constant}}{|x|^2} + O(1/|x|^3). \tag{8}$$

Thus the left hand side of (7) is negative for r large, and hence it is negative. Therefore

$$\left|\text{Total charge on } \Omega_2\right| = -\int_{\Omega_2} \frac{\partial\phi}{\partial\nu} = q + \int_{|x|=r} \frac{\partial\phi}{\partial r} < q.$$

ANSWER TO QUESTION. *NO*.

For a heuristic proof using lines of force see [6; §89c]. In fact Chapter III of [6] is one of the nicest treatments of electrostatics available. A more

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* References are to the Bibliography at the end of Lecture #3.

mathematical approach can be found in [3] or [4; Ch. 9, §12].

4. A PROBLEM ABOUT WAVE MOTION WITH FRICTION

We consider damped wave motion in $\mathbb{R}^3$. A typical mathematical model is

$$u_{tt} = \Delta u - a(x)u_t$$

where $a(x) \geq 0$ represents a frictional resistance. Given the initial position and velocity $u(0,x)$, $u_t(0,x)$, the motion is uniquely determined. It is a simple matter to show that the energy

$$E(t) = \int_{\mathbb{R}^3} (u_t^2 + |\nabla u|^2)dx$$

is a decreasing function of time. In fact,

$$\frac{dE}{dt} = -\int_{\mathbb{R}^3} a(x)u^2(t,x)dx \leq 0.$$

It appears from this formula and from physical intuition that if the friction coefficient $a(x)$ is increased the energy dissipation is enhanced. For fixed initial data we can consider the solution as depending parametrically on $a(x)$, that is, we have $u(t,x;\, a(x))$ and $E(t;\, a(x))$.

QUESTION. *If* $\tilde{a}(x) \geq a(x)$ *for all* $x \in \mathbb{R}^3$ *it is true that* $E(t;\, a(x)) \geq E(t;\, \tilde{a}(x))$ *for* $t \geq 0$ *and all initial data?*

ANALYSIS. We attack the problem in case the functions

a, $\tilde{a}$ are constant. In this case we can effectively use the Fourier transform. Let

$$\hat{u}(t,\xi;\ a) = (2\pi)^{-3/2} \int u(t,x;\ a) e^{ix\cdot\xi}\, dx$$

be the partial Fourier transform of u. For $\hat{u}$ we have the ordinary differential equation in t depending parametrically on ξ,

$$\hat{u}_{tt} = -|\xi|^2 \hat{u} - a\hat{u}_t.$$

For fixed ξ this is just the equation of a damped spring. Furthermore, the energy is given by

$$E = \int (|\hat{u}_t|^2 + |\xi|^2 |\hat{u}|^2) d\xi$$

which is merely the "sum" of the separate spring energies. Let us concentrate on the spring equation

$$\ddot{y} + a\dot{y} + y = 0$$

with energy

$$\dot{y}^2 + y^2 = e(t) = e(t;\ a).$$

REDUCED QUESTION. *For the spring equation is* $e(t;\ a)$ *a monotone decreasing function of* a *for arbitrary initial data?*

EXERCISE. Show that a yes or a no answer for the reduced question implies the same answer for the original question.

ANSWER TO QUESTIONS. *NO.*

Since any question about constant coefficient second order ordinary differential equations must be trivial this is left as an exercise. However, as a hint I suggest you investigate the overdamped case $a >> 1$.

Once the answer has been found and the root cause identified as overdamping several observations can be made. First the answer is not entirely unreasonable since in the case of extreme damping everything happens very slowly including energy decay. Second, looking at the Fourier transform solution we see that overdamping corresponds to $|\xi|$ small compared to a. Thus after a while only the slowly decaying modes will be noticeable. That is, $\hat{u}(t,\xi)$ will tend to become concentrated near $\xi = 0$. This corresponds to a flattening out of u. Not only does dissipation decrease energy but it tends to iron out the "wrinkles" in u.

5. A PROBLEM ABOUT PERFECT SHADOWS

In this section we will study wave motion in the presence of a periodic (in time) driving force. This falls into the class of problems called radiation problems. We will investigate whether obstacles can form perfect shadows, that is, whether there can be an open set which is unaffected by the radiation. A typical mathematical model is

$$(9) \qquad u_{tt} = c^2 \Delta u + F(x) e^{i\alpha t}, \quad (t,x) \in \mathbb{R} \times (\mathbb{R}^3 \setminus \Omega),$$

where Ω represents an obstacle. In addition we have some condition which prescribes how the wave interacts with the obstacle. This usually takes the form of a boundary condition, for example, a Dirichlet or Neumann condition at $\partial\Omega$. In many situations it can be shown that the solution can be written as

$$v(x)e^{i\alpha t} + \text{transient wave motion.} \tag{10}$$

Here $v(x)e^{i\alpha t}$ is a motion at the same frequency as the driving term and the transient term tends to zero at each point $x \in \mathbb{R}^3 \setminus \Omega$ as $t \to \infty$. Thus after an initial adjustment an observer sees the steady state $ve^{i\alpha t}$, for the transient has died away. This is called the principle of limiting amplitude (see [5; Thm. 4.4] for a proof).

We suppose that the radiating term $Fe^{i\alpha t}$ is spatially localized in a region $R \equiv \text{supp } F$.

QUESTION. *Is it possible for there to be a perfect shadow? Precisely, can there be an open set* ω *in the exterior of* $\Omega \cup R$ *such that* $v = 0$ *on* ω*?*

Physically one might try to construct such a set as in Figure 2.

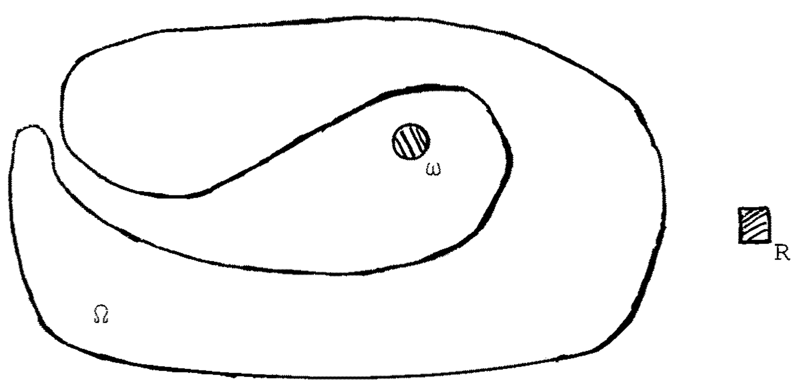

Figure 2

ANALYSIS. Plugging the expression (10) into the differential equation (9), we see that v must satisfy

$$(c^2\Delta + \alpha^2)v = -F.$$

Thus exterior to R we have $(c^2\Delta + \alpha^2)v = 0$, so v is real analytic outside R. Therefore if $v = 0$ on an open set ω exterior to R, then $v \equiv 0$ outside R.

ANSWER. *The only way for there to be any region that is not affected by the light is for the radiation to be confined entirely to* R, *that is, there is no radiation at all.*

There is one possibility that is overlooked here. This is illustrated by Figure 3.

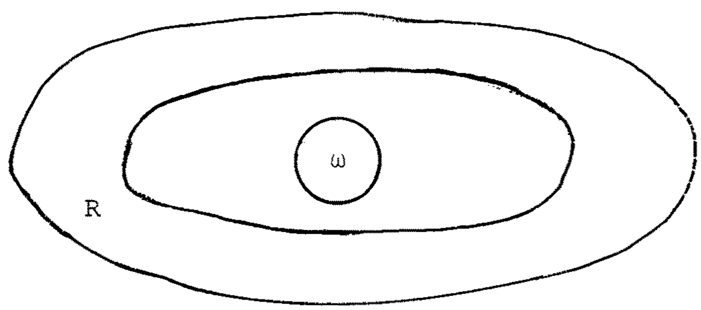

Figure 3

It is possible to have a perfect shadow inside the source. For example the source can radiate outgoing spherical waves which never affect the inside of the "antenna". For more detailed information see [8].

LECTURE #2. THE MATHEMATICAL THEORY OF CRUSHED ICE

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In this lecture we will investigate the cooling efficiency of crushed ice. With certain idealizations this becomes a problem of estimating the smallest eigenvalue for an elliptic boundary value problem. These eigenvalue estimates will be needed in the next lecture in order to study scattering by many small objects.

Consider a container filled with some homogeneous continuous medium and occupying an open region $\Omega \subset \mathbb{R}^3$. The boundary of Ω is assumed to be insulated and smooth. Ω contains spherical coolers $\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_n$ of radius r (depending on n) and centers $x_1, x_2, \ldots, x_n$.* Overlap of coolers is permitted. We view $K_n = \bigcup_{i=1}^{n} \kappa_i$ as one large cooler for the medium in $\Omega_n \equiv \Omega \setminus K_n$. The coolers are assumed to be stationary and to have internal mechanisms which maintain their boundaries at temperature zero. If $u(t,x)$ is the

* Other shapes can easily be treated. See the remark after the statement of the theorem.

temperature at time t and place x, then heat flow in Ω is modeled by the boundary value problem

$$u_t = c\,\Delta u \quad \text{for} \quad t \geq 0,\ x \in \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$
$$u = 0 \quad \text{on} \quad \partial K_n.$$

Here c is a constant depending on the conductivity and specific heat of the medium and Δ is the Laplacian in $\mathbb{R}^3$. Notice that only heat flow by conduction is considered here - the medium is stationary. To completely describe u, the initial temperature distribution $u(0,x)$ must be given. The boundary value problem is solved by eigenfunction expansion

$$u = \Sigma a_j e^{c\lambda_j t} \Phi_j$$

where $a_j = (u(0,x),\ \Phi_j(x))_{L^2(\Omega_n)}$ and the Φ_j are normalized eigenfunctions of Δ:

$$\Delta\Phi_j = \lambda_j \Phi_j\ ,$$
$$\Phi_j = 0 \quad \text{on} \quad \partial K_n\ ,$$
$$\frac{\partial \Phi_j}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\ .$$

The eigenvalues are ordered so that $0 > \lambda_1 > \lambda_2 \geq \cdots$. As in Lecture 1, u goes to zero like $e^{c\lambda_1 t}$, so λ_1 gives the rate of cooling. We are interested in the behavior of λ_1 as $n \to \infty$, $r \to 0$.

This problem is related to common experience with crushed ice. It is well known that a given volume of ice is more efficient as a cooler if it is divided into many small pieces. The idealization we have made is that the ice neither moves nor melts.

ORTHODOX EXPLANATION. The reason that crushed ice is an efficient coolant is that the surface area of the ice is large. After all, the cooling takes place by contact of the medium with the coolers, and increasing the surface area increases the amount of contact. Furthermore the total volume of coolers in our case goes like nr^3 and the total surface area like nr^2 so for fixed volume, the surface area $nr^2 = \frac{1}{r} \cdot (nr^3)$ goes to infinity as $r \to 0$. This analysis leads to

GUESS #1. If $n \to \infty$, $r \to 0$ in such a way that the total volume of the coolers stays constant then the cooling becomes infinitely efficient, that is $\lambda_1 \to -\infty$, provided the coolers are evenly spaced throughout Ω.

In addition there is a companion conjecture.

GUESS #2. If $n \to \infty$ and $r \to 0$ in such a way that the total surface area goes to zero, then the cooling disappears in the limit, i.e. $\lambda_1 \to 0$.

Reasonable as all this is, one of the above guesses is incorrect. If you think that is not surprising try to figure out which one without reading further. The correct answer is provided by the following

THEOREM. *For* nr *small we have* $-\lambda_1 \le \frac{2nr}{|\Omega|}(1 + O(nr))$.

If the coolers are evenly spaced in Ω *then there are positive constants* c_1, c_2 *such that* $-\lambda_1 \geq c_1 nr - c_2$ *for all* n.

The precise sense of even spacing will emerge in the proof. In addition the constants c_1, c_2 and the term $O(nr)$ can be crudely estimated from the data of the problem if necessary. If one is interested in coolers which are not spherical, one can get lower bounds by replacing the coolers by spherical ones contained inside them and upper bounds by larger spherical coolers. In this way the estimates can be carried over to more or less arbitrary shapes.

Applying the theorem we see that Guess #1 is correct, for if $nr^2 \to \infty$ then $nr = \frac{1}{r}(nr^2) \to \infty$, so $\lambda_1 \to -\infty$. Notice, however, that for r small $nr >> nr^2$ so the cooling efficiency is greater than predicted by surface area considerations. In particular, nr may grow infinitely large even though $nr^2 \to 0$ so that Guess #2 is wrong. The correct results replace nr^2 by nr in both guesses. I do not have any intuitive idea why this should be but in the next lecture we will gain some insight into the failure of our intuition.

PROOF OF THE UPPER BOUND FOR $-\lambda_1$

We use the variational characterization of λ_1.

$$-\lambda_1 = \inf \frac{\int_{\Omega_n} |\nabla\psi|^2}{\int_{\Omega_n} \psi^2} \quad ,$$

the infimum over all $\psi \in C^\infty(\overline{\Omega}_n)$ such that $\psi \not\equiv 0$ and

$\psi = 0$ on ∂K_n. The infimum is attained for multiples of Φ_1. It is important to notice that the condition $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$ is not imposed; this is a natural boundary condition (see [1; Ch. 6,§1]). To get an upper bound we will plug in a good trial function. To describe the function let me review the notion of capacity. For a reasonably well-behaved set $\Gamma \subset \mathbb{R}^3$ there is a unique solution to the Dirichlet problem

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma,$$
$$\phi = 1 \quad \text{on } \partial\Gamma\ ,$$
$$\phi = O(1/|x|) \quad \text{as } |x| \to \infty.$$

ϕ is called the capacitary potential of Γ. The capacity of Γ, $\operatorname{cap}(\Gamma)$, is defined as

$$\int_{\partial\Gamma} \frac{\partial\phi}{\partial\nu}$$

where ν is the outward normal to $\mathbb{R}^3 \setminus \Gamma$. Physically this is the amount of charge which must be placed on a conductor occupying the region Γ in order to raise its potential to 1. Thus capacity measures the ability to hold charge. For ϕ we have $|\nabla\phi| = O(1/|x|^2)$ so a straightforward application of Green's identity shows that

$$0 = \int_{\mathbb{R}^3\setminus\Gamma} \phi\,\Delta\,\phi = -\int_{\mathbb{R}^3\setminus\Gamma} |\nabla\phi|^2 + \int_{\partial\Gamma} \phi\frac{\partial\phi}{\partial\nu}\ ,$$

so

$$\int_{\partial\Gamma} \frac{\partial\phi}{\partial\nu} = \int_{\mathbb{R}^3\setminus\Gamma} |\nabla\phi|^2\ ,$$

an alternate expression for the capacity. We need two facts (see [7; p.125 ff.]):

(α) The capacity of a sphere of radius r is r.

(β) Capacity is a subadditive set function.

Using these we have

$$\operatorname{cap}(K_n) = \operatorname{cap}(\cup \kappa_i) \le \Sigma \operatorname{cap}(\kappa_i) = nr.$$

Let ϕ_n be the capacitary potential of K_n. The trial function will be $(1-\phi_n)|_{\Omega_n}$. We immediately have that

(i) $\phi_n = 1$ on ∂K_n,

(ii) $\displaystyle\int_{\Omega_n} |\nabla\phi_n|^2 < \int_{\mathbb{R}^3\setminus K_n} |\nabla\phi_n|^2 \le nr.$

Furthermore, for any $x \notin \Gamma$ we can perform the "ball of radius ϵ argument" on

$$\int_{\mathbb{R}^3\setminus K_n} \phi_n \cdot \Delta_y \frac{1}{|x-y|}$$

to conclude that

$$\phi_n(x) = \frac{1}{4\pi} \int_{\partial K_n} \frac{\partial\phi_n}{\partial\nu}(y) \frac{1}{|x-y|}\, dy.$$

Therefore

$$4\pi|\phi_n(x)| \le \operatorname{cap}(K_n) \cdot \operatorname{dist}(x, K_n) \le nr \cdot \operatorname{dist}(x, K_n).$$

Let $\mathcal{O}$ be a bounded open set at positive distance from Ω. Then there is a constant c such that

$$\|\Phi\|_{L^2(\Omega)} \le c(\|\Phi\|_{L^2(\mathcal{O})} + \|\nabla\Phi\|_{L^2(\mathbb{R}^3)}),$$

the inequality holding for all $\Phi \in H_1(\mathbb{R}^3)$. Applying this to

$$\Phi = \begin{cases} \phi_n & \text{in } \mathbb{R}^3 \setminus K_n \\ 1 & \text{in } K_n \end{cases}$$

we get

$$\|\phi_n\|_{L^2(\Omega_n)} \le c\,n\,r$$

with c independent of n. The function $\psi = 1 - \phi_n$ then satisfies

$$\int_{\Omega_n} |\nabla\psi|^2 = \int_{\Omega_n} |\nabla\phi_n|^2 \le c\,n\,r,$$

$$\int_{\Omega_n} \psi^2 = |\Omega_n| - 2\int_{\Omega_n} \phi_n + \int_{\Omega_n} \phi_n^2$$

$$\ge |\Omega_n| - 2|\Omega_n|^{\frac{1}{2}}\|\phi_n\| + \|\phi_n\|^2$$

$$\ge |\Omega_n| - (\tfrac{1}{2}|\Omega_n| + 2\|\phi_n\|^2) + \|\phi_n\|^2$$

$$\ge \frac{1}{2}|\Omega_n| - O(nr)$$

provided nr is small. Therefore

$$-\lambda_1 \le \int_{\Omega_n} |\nabla\psi|^2 \Big/ \int_{\Omega_n} \psi^2 \le \frac{c\,n\,r}{\frac{1}{2}|\Omega| - O(nr)}$$

which completes the proof of the upper bound.

PROOF OF THE LOWER BOUND FOR $-\lambda_1$. It is a well known general principle that lower bounds are harder to obtain than upper bounds. Ours is no exception to the rule. The notion of evenly spaced is that one can cover Ω by balls with centers at $x_1, x_2, x_3, \dots, x_n$ with the property that there is not a great deal of overlap. To be precise we assume that there are numbers $R(n) > 2r$ such that the balls $B_i = \{x \mid |x-x_i| < R(n)\}$, $i=1,2,\dots,n$ satisfy

(i) $\bigcup_{i=1}^{n} B_i \supset \Omega$,

(ii) there is a number M independent of n such that each point of Ω is in at most M of the B_i.

As a consequence of (ii) we see that $nR^3 \le c|\Omega|$ for some constant c independent of n. The number R serves as a measure of distance between adjacent coolers.

Suppose $\psi \in H_1(\Omega_n)$, $\psi = 0$ on ∂K_n. We extend ψ to $\psi_{ext} \in H_1(\mathbb{R}^3)$ by extending it as zero in K_n and then doing a Lions type reflection across $\partial\Omega$. This can be done so that

$$\|\psi_{ext}\|_{H_1(\mathbb{R}^3)} \le c\|\psi\|_{H_1(\Omega_n)}$$

provided $\frac{|K_n|}{|\Omega|} \le \bar{c} < 1$ with $\bar{c}$ independent of n. When this condition fails the problem becomes uninteresting. Let $\Omega_{ext} = \bigcup_{i=1}^{n} B_i$. First we get a lower bound for $\int_{\Omega_{ext}} |\nabla\psi_{ext}|^2$ by estimating $\int_{B_i} |\nabla\psi_{ext}|^2$ from below. We use the

LEMMA. *If* $r < \frac{1}{2}R$ *and* $A = \{x \mid r < |x| < R\}$ *then* $\int_A |\nabla\phi|^2 \geq \frac{cr}{R^3} \int_A \phi^2$ *for all* $\phi \in H_1(A)$ *with* $\phi = 0$ *on* $|x| = r$.

PROOF. The minimum of $\left(\int_A |\nabla\phi|^2\right) \big/ \int_A \phi^2$ occurs for the eigenfunction $\Delta\phi = \lambda\phi$, $\phi = 0$ for $|x| = r$, $\phi_r = 0$ for $|x| = R$ corresponding to the largest eigenvalue λ. This eigenfunction must be positive. (The proof of this well-known fact is not trivial.) It follows (exercise) that the eigenfunction is rotationally symmetric so $\phi(x) = f(|x|)$. Using this fact the Lemma is reduced to showing

SUBLEMMA. $\int_r^R f'(t)^2 t^2 \, dt \geq \frac{cr}{R^3} \int_r^R f^2(t) t^2 \, dt$ *for all* $f \in C^1[r,R]$ *satisfying* $f(r) = 0$.

The sublemma is a consequence of the inequality

$$(\dagger) \quad \int_r^\rho f^2(t)\phi(t)\,dt \leq \int_r^\rho \phi(t)\,dt \int_r^\rho \frac{1}{\phi(t)}\,dt \int_r^\rho (f'(t))^2 \phi(t)\,dt$$

with $\phi(t) = t^2$. One proof of this inequality can be found in [9; Lemma 4.5]. We present an argument due to Jim Ralston. Write $f(t) = \int_r^t g(s)\,ds$ with $g = f'$. Then

$$f^2(t) = \left(\int_r^t g(s)\sqrt{\phi(s)} \cdot \frac{1}{\sqrt{\phi(s)}}\,ds\right)^2 \leq \int_r^\rho g^2(s)\phi(s)\,ds \int_r^\rho \frac{1}{\phi(s)}\,ds$$

and $(\dagger)$ follows immediately. □

We then have

$$\int_{\Omega_{ext}} |\nabla\psi_{ext}|^2 \geq \frac{1}{M} \sum \int_{B_i} |\nabla\psi_{ext}|^2 \geq \frac{cr}{MR^3} \sum \int_{B_i} \psi^2$$

$$\geq \frac{c}{M} \frac{nr}{nR^3} \int_{\Omega_{ext}} \psi^2 \geq \frac{c'}{M|\Omega|} nr \int_{\Omega} \psi^2 .$$

From the inequality for the Lions reflection defining ψ_{ext} we have

$$\int_{\Omega_n} |\nabla\psi|^2 + c \int_{\Omega_n} \psi^2 \geq \int_{\mathbb{R}^3} |\nabla\psi_{ext}|^2$$

which yields the estimate

$$\int_{\Omega_n} |\nabla\psi|^2 \geq \left(\frac{c'}{M|\Omega|} nr - c\right) \int_{\Omega_n} \psi^2 ,$$

proving the lower bound for $-\lambda_1$. □

LECTURE #3. SCATTERING BY MANY TINY OBSTACLES

by

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The subject of scattering theory, as treated by mathematicians, is usually concerned with interactions with one target object. For example we have potential scattering in quantum mechanics and the problems of acoustic scattering by macroscopic obstacles.

However, one of the most commonly encountered situations is the scattering by many similar small targets. For example in the classical Rutherford experiment alpha particles (Helium nuclei) are scattered by a thin foil containing an enormous number of individual targets (atoms). In this case a very satisfactory treatment can be given by treating the scattering by a single atom and just adding up the results assuming that the various scattering events are independent (incoherent scattering). An adequate mathematical explanation for this success is still lacking. In this paper we focus attention on the opposite situation where the presence of many obstacles leads to qualitatively new phenomena. The most striking examples are when the many obstacles behave as if they were a solid object or in the opposite extreme case where because of their small size many tiny obstacles may have negligible effect on incident waves. We call the first solidification and the second fading.

EXAMPLE 1. Dust particles in the atmosphere have a negligible effect on the propagation of sound waves.

EXAMPLE 2. A cloud of small conductors sprayed into the air appears solid on a radar screen.

EXAMPLE 3. The water droplets in a cloud give a solid appearance while those in the atmosphere are essentially invisible. Here you can clearly see that it is a balance between number and size that determines whether there is fading or solidification.

EXAMPLE 4. The atomic nature of crystalline matter is not apparent in its interaction with macroscopic objects. This is perhaps the most common example of solidification.

EXAMPLE 5. It is well-known that a region enclosed by walls made of conductors will have no electric field on the inside. In practice it is observed that a screen made of conductors has essentially the same effect as a solid conductor. On the other hand if the wire is sufficiently thin it is clear that this screening effect will not be present. The conventional explanation (see, for example [2; Ch. 7, §10]) does not take this into account. The problem of electrostatic screening is discussed in a forthcoming paper by M. Taylor and myself.

BASIC PROBLEM. *What values of the relevant physical parameters correspond to solidifying obstacles and which correspond to fading?*

We will study one situation in which a substantial first step toward solving the basic problem can be made. The obstacles are assumed to lie in a nice bounded open set Ω. The obstacles will be n spheres $\kappa_1, \kappa_2, \ldots, \kappa_n$ of radius r (depending on n) and centers $x_1, x_2, \ldots, x_n$; and as before $K_n = \bigcup_{i=1}^{n} \kappa_i$ is the effective obstacle. We will consider two problems simultaneously:

$$(1) \qquad u_{tt} = \Delta u \quad \text{in} \quad \mathbb{R}^3 \setminus K_n \ , \ u = 0 \quad \text{on} \quad \partial K_n \ ,$$

$$(2) \qquad u_t = i\Delta u \quad \text{in} \quad \mathbb{R}^3 \setminus K_n \ , \ u = 0 \quad \text{on} \quad \partial K_n \ .$$

The first corresponds to acoustic scattering by soft spheres and the second quantum mechanical scattering by impenetrable spheres. The solutions are defined for $(t,x) \in \mathbb{R} \times (\mathbb{R}^3 \setminus K_n)$ and we consider them to be extended as zero inside K_n. We will consider the limit as $n \to \infty$, $r \to 0$. Solidifying obstacles means that the solutions (for fixed initial data) converge to solutions of

$$(1)_\infty \qquad u_{tt} = \Delta u \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega \ , \ u = 0 \quad \text{on} \quad \partial\Omega \ ,$$

$$(2)_\infty \qquad u_t = i\Delta u \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega \ , \ u = 0 \quad \text{on} \quad \partial\Omega \ .$$

Disappearance would mean that the solutions converge to the solutions of the wave equation or Schrödinger equations on all of $\mathbb{R}^3$, that is, without obstacles. If we view the solutions as functions of t with values in $L^2(\mathbb{R}^3)$ the notion of convergence that is natural is uniform convergence on compact time intervals, that is convergence in $C((-\infty,\infty) ; L^2(\mathbb{R}^3))$.

We have the following result.

If $nr \to 0$ *the obstacles fade.*
If $nr \to \infty$ *and the spheres are evenly spaced in* Ω *the obstacles solidify.*

Thus the critical combination of parameters is nr. In order to show the flavor of our method and the connection with crushed ice a proof of the solidification half of the above statement will be given in detail. This proof is extracted from [9] where one may also find a proof of the fading assertion (Theorem 4.2) and many other results in the spirit of this lecture.

NOTATIONS:

(i) Δ_n is the selfadjoint operator on $L^2(\mathbb{R}^3 \setminus K_n)$ defined by Dirichlet conditions on ∂K_n. That is, $D(\Delta_n) = \{u \in H_2(\Omega_n) \mid u = 0 \text{ on } \partial K_n\}$, and $\Delta_n u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}$

(ii) Δ_∞ is the selfadjoint operator on $L^2(\mathbb{R}^3 \setminus \Omega)$ with Dirichlet conditions on $\partial\Omega$.

(iii) If $v \in L^2(U)$ for some set $U \subset \mathbb{R}^3$ we consider $v \in L^2(\mathbb{R}^3)$ by setting v equal to zero on the complement of U. Thus for $v \in L^2(U)$, $\tilde{v} \in L^2(\tilde{U})$ we may consider $\|v-\tilde{v}\|_{L^2(\mathbb{R}^3)}$.

The solution u to problem (1) is given by

$$u(t) = \frac{\sin\sqrt{-\Delta_n}\, t}{\sqrt{-\Delta_n}} u_t(0) + (\cos\sqrt{-\Delta_n}\, t)u(0)$$

where $u(0), u_t(0) \in L^2(\mathbb{R}^3 \setminus K_n)$ are the initial position and velocity and objects like $\cos\sqrt{-\Delta_n}\, t$ are operators defined by the functional calculus for self-adjoint operators. (Note: $\Delta_n \le 0$ and $\frac{\sin t\sqrt{-x}}{\sqrt{-x}}$ is a smooth function on $(-\infty,0]$.) Similarly for problem (2) we have

$$u(t) = e^{it\Delta_n} u(0).$$

With these formulas in mind we see that the solidification result is a consequence of the

THEOREM. *Suppose* $nr \to \infty$ *and the obstacles are evenly distributed in* Ω. *For every bounded continuous function* F *on* $(-\infty,0]$ *we have*

$$F(\Delta_n)f \to F(\Delta_\infty)f \quad \text{in} \quad L^2(\mathbb{R}^3) \tag{3}$$

for all $f \in L^2(\mathbb{R}^3 \setminus \Omega)$.

Note that $f \in L^2(\mathbb{R}^3 \setminus \Omega)$ implies $f \in L^2(\mathbb{R}^3 \setminus K_n)$ so that the assertion makes sense.

EXERCISE. This theorem shows that the solutions of (1), (2) converge to the solutions of $(1)_\infty$, $(2)_\infty$ for fixed t. Show that the convergence is automatically uniform on compact time intervals. (Hint: Show that the convergence in (3) must be uniform over compact sets of F

(in $BC(-\infty,0]$).)

PROOF OF THE THEOREM. The idea of the proof is to infer "resolvent convergence" from some elementary estimates, a compactness argument and the lower bound for $-\lambda_1$ found in Lecture #2. A dose of soft analysis then finishes the job.

STEP #1. If $f \in L^2(\mathbb{R}^3 \setminus \Omega)$ then $(1-\Delta_n)^{-1}f \to (1-\Delta_\infty)^{-1}f$ in $H_1(\mathbb{R}^3)$ (only convergence in $L^2(\mathbb{R}^3)$ will be needed.) To prove this we show first that $\{(1-\Delta_n)^{-1}f \mid n=1,2,\ldots\}$ is a bounded subset of $H_1(\mathbb{R}^3)$. As usual we extend $(1-\Delta_n)^{-1}f$ to $\mathbb{R}^3$ by setting it equal to zero on K_n.

Since $(1-\Delta_n)^{-1}$ is an operator of norm 1 we have $\|(1-\Delta_n)^{-1}f\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}$. Let $v_n = (1-\Delta_n)^{-1}f$, $v_\infty = (1-\Delta_\infty)^{-1}f$. Then

$$\|v_n\|^2_{H_1(\mathbb{R}^3)} = \|v_n\|^2_{H_1(\mathbb{R}^3\setminus\Omega_n)} = ((1-\Delta_n)v_n\ ,\ v_n)_{L^2(\mathbb{R}^3\setminus\Omega_n)}$$

$$= (f,v_n)_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}\|v_n\|_{L^2(\mathbb{R}^3)}\ ,$$

which proves the desired boundedness. We next show that v_n converges weakly to v_∞ in $H_1(\mathbb{R}^3)$, symbolically $v_n \rightharpoonup v$. By the weak compactness of the unit ball in a Hilbert space we see that $\{v_n\}$ is weakly compact. Suppose that w is a weak limit point and that $v_{n_j} \rightharpoonup w$. We will show that $w = v_\infty$ which establishes the convergence $v_n \rightharpoonup w$.

First we show that in the sense of distributions $(1-\Delta)w = f$ in $\mathbb{R}^3 \setminus \Omega$. For $\phi \in C_0^\infty(\mathbb{R}^3 \setminus \Omega)$ we have

$$(w, (1-\Delta)\phi) = \lim(v_{n_j}, (1-\Delta)\phi).$$

Furthermore $(1-\Delta)v_{n_j} = f$ so the right hand side is (f,ϕ), proving the assertion.

Next we show that $w = 0$ in Ω. We know that $v_{n_j} = 0$ on ∂K_n so

$$\frac{\int_{\Omega\backslash K_n} |\nabla v_{n_j}|^2}{\int_{\Omega\backslash K_n} v_{n_j}^2} \leq -\lambda_1 .$$

We have a uniform bound on $\|v_n\|_{H_1(\mathbb{R}^3)}$, so the numerator is bounded as $j \to \infty$. Since $-\lambda_1 \to \infty$ we must have $\|v_{n_j}\|_{L^2(\Omega)} \to 0$ as $j \to \infty$ and therefore $w = 0$ in Ω. The three conditions

$$w \in H_1(\mathbb{R}^3), \ (1-\Delta)w = f \ \text{in} \ \mathbb{R}^3 \setminus \Omega, \ w = 0 \ \text{in} \ \Omega$$

identify w as $(1-\Delta_\infty)^{-1}f$.

Having established weak convergence the norm convergence is a consequence of $\|v_n\|_{H_1(\mathbb{R}^3)} \to \|v_\infty\|_{H_1(\mathbb{R}^3)}$. We have shown that $\|v_n\|^2_{H_1(\mathbb{R}^3)} = (f,v_n)_{L^2(\mathbb{R}^3)}$. Because of weak convergence this approaches $(f,v_\infty)_{L^2(\mathbb{R}^3)}$, which is equal to $\|v_\infty\|^2_{H_1(\mathbb{R}^3)}$, and Step #1 is complete.

STEP #2. (Proof of the theorem for bounded continuous functions on $(-\infty,0]$ which vanish at $-\infty$). Let A be the algebra of bounded continuous functions on $(-\infty,0]$ which vanish at $-\infty$. Let $\mathcal{A} \subset A$ be the set of functions for which the assertion of the theorem is true. A is a Banach space under the sup norm and $\mathcal{A}$ is easily seen to be a closed linear subspace of A. In addition $\mathcal{A}$ is a subalgebra. To see this suppose $F,G \in \mathcal{A}$; then for $f \in L^2(\mathbb{R}^3\backslash\Omega)$

$$\begin{aligned}(FG)(\Delta_n)f &= F(\Delta_n)G(\Delta_n)f \\ &= F(\Delta_n)G(\Delta_\infty)f + F(\Delta_n)[G(\Delta_n)f - G(\Delta_\infty)f].\end{aligned}$$

Since $F \in \mathcal{A}$ the first term converges to $F(\Delta_\infty)G(\Delta_\infty)f$. The second term has norm dominated by

$$\sup_{(-\infty,0]} |F| \cdot \|G(\Delta_n)f - G(\Delta_\infty)f\|_{L^2(\mathbb{R}^3)}$$

which goes to zero since $G \in \mathcal{A}$. Thus $FG \in \mathcal{A}$. The assertion of Step #1 is that $F(x) = (1-x)^{-1} \in \mathcal{A}$ and clearly F separates points. By the Stone-Weierstrauss Theorem, $\mathcal{A} = A$.

STEP #3. (Endgame) What we are proving is that $F(\Delta_n)$ converges strongly to $F(\Delta_\infty)$ as operators from $L^2(\mathbb{R}^3\backslash\Omega)$ to $L^2(\mathbb{R}^3)$. Since we have a uniform bound on $\|F(\Delta_n)\|$ it suffices to show that $F(\Delta_n)f \to F(\Delta_\infty)f$ for a dense set of f, for example for f of the form $e^{\eta\Delta_\infty}g$ for $g \in L^2(\mathbb{R}^3\backslash\Omega)$ and $\eta > 0$. (Note: $e^{\eta\Delta_\infty}g \to g$ as $\eta \to 0$.) Now

$$F(\Delta_n)e^{\eta\Delta_\infty}g - F(\Delta_\infty)e^{\eta\Delta_\infty}g$$
$$= [F(\Delta_n)e^{\eta\Delta_n}g - F(\Delta_\infty)e^{\eta\Delta_\infty}g] +$$
$$F(\Delta_n)[e^{\eta\Delta_\infty}g - e^{\eta\Delta_n}g] .$$

By the result of Step #1 applied to the functions $F(x)e^{\eta x}$ and $e^{\eta x}$ the vectors in brackets tend to zero. The proof is complete. □

A final remark is in order on the interpretation of this result. It says that for fixed initial data the solutions of (1), (2) converge to those of $(1)_\infty$, $(2)_\infty$ provided $nr \to \infty$. How large nr must be before the convergence is evident will depend on the initial data. For example, consider the acoustic equation. No matter how large n is we may pose initial data which is an incoming wave of extremely high frequency , λ. If the wavelength is high enough the geometrical optics approximation becomes valid and one will not observe solidification. The solidification is intimately related to the failure of geometrical optics. It is caused by an overdose of diffraction.

A quantitative guess of how large λ must be can be made by dimensional considerations. It is evident both physically and mathematically that solidification does not depend on the size of Ω or the absolute number n of obstacles but on the density $n|\Omega|^{-1}$. For evenly spaced spheres if R is a measure of the distance between obstacles then $nR^3 \approx |\Omega|$. Thus in terms of r, R the number $n|\Omega|^{-1}r$ is essentially rR^{-3}, which has the dimensions $(\text{length})^{-2}$. A reasonable dimensionless quantity to replace nr is $r\lambda^2 R^{-3}$, where λ is a measure of the wavelength of the incident wave. In practical considerations I believe that this is the absolute number which must be large. Perhaps in the future this idea will find expression in concrete estimates.

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