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SO(d) and Haar Measure

1 SO(d).

Orthogonal transformations are characterized as follows.

Theorem 1 *For* $\mathcal{O} \in \mathrm{Hom}\,\mathbb{R}^d$, *TFAE.*
i. $\forall x \in \mathbb{R}^d, \quad \|\mathcal{O}x\| = \|x\|$.
ii. $\forall x, y \in \mathbb{R}^d, \quad \langle \mathcal{O}x, \mathcal{O}y\rangle = \langle x, y\rangle$.
iii. $\mathcal{O}^t\mathcal{O} = I$.
iv. $\mathcal{O}\mathcal{O}^t = I$.
v. *The rows of* $\mathcal{O}$ *form an orthonormal basis of* $\mathbb{R}^d$.
vi. *The columns of* $\mathcal{O}$ *form an orhonormal basis of* $\mathbb{R}^d$.

The only nontrivial step is **i**. $\Rightarrow$ **ii**.. It uses the polarization identity then conservation of length then polarization to find,

$$\langle x, y\rangle = \frac{\langle x+y\,,\, x+y\rangle - \langle x-y\,,\, x-y\rangle}{4}$$
$$= \frac{\langle A(x+y)\,,\, A(x+y)\rangle - \langle A(x-y)\,,\, A(x-y)\rangle}{4} = \langle Ax, Ay\rangle\,.$$

Definition 1 $SO(d) \in \mathrm{Hom}\,(\mathbb{R}^d)$ *is the algebra of orthogonal matrices* T *with* $\det T = 1$.

The one parameter groups in $SO(s)$ are characterized as folllows.

Theorem 2 *For* $A \in \mathrm{Hom}\,\mathbb{R}^d$, *TFAE.*
i. $\forall t \in \mathbb{R}^d, \quad e^{tA} \in SO(d)$.
ii. A *is antisymmetric, that is* $A^t = -A$.

2 SO(d) as a manifold.

$SO(d)$ is a smooth manifold. It is the subset of $d \times d$ matrices so that $Q^tQ = I$. Since Q^tQ is symmetric this is only $d(d+1)/2$ equations, those coming for example from the upper triangular part. Denote by $\mathcal{M}(d)$ the set of $d \times d$ real matrices. Define $F : \mathcal{M} \to \mathbb{R}^{d(d+1)/2}$ by

$$F_{ij} := (M^tM)_{ij}\,, \qquad i \le j \le d\,.$$

Exercise. *Verify that the equations $F = 0$ defining $SO(d)$ are everywhere independent. That is, their Jacobian matrix evalutated at a point of $SO(d)$ has rank $d(d+1)/2$.*

It follows that the dimension of $SO(d)$ is equal to $d^2 - d(d+1)/2 = d(d-1)/2$. Since the equations are real and polynomial, $SO(d)$ is a real algebraic subvariety of $\mathbb{R}^{d^2}$.

The dimension can be computed independently by finding local coordinate systems. In case of $d = 3$ to determine an element of $SO(3)$ first find its first row which is an arbitrary element of S^2. Then its second row must be chosen orthogonal to the first which is a circle of freedom. The third row is then detemined. The dimension count is the sum of the dimensions of S^2 and S^1 so $2 + 1 = 3 = d(d-1)/2$. In the general case one gets

$$(d-1) + (d-2) + \cdots + 1 = \frac{d(d-1)}{2}\,.$$

This argument shows that there are at least this many free parameters at any point. The polynomial equations show that there cannot be less than this, so this parameter count suffices to determine the dimension without checking the Jacobian as in the exercise.

3 Volume forms on vector spaces.

A volume form on $\mathbb{R}^d$ is a map which assigns a nonnegative real number to each parrallelogram. A parallelgram is defined by giving its sides which are d vectors $v_j \in \mathbb{R}^d$, $1 \le j \le d$. The parallelogram is then the set

$$\Big\{ w \in \mathbb{R}^d \ : \ \exists (t_1, \ldots, t_d) \in [0,1]^d, \quad w = \sum t_j v_j \Big\}\,.$$

The parallelogram is unchanged if one permutes the v_j. It is subjected to a shear transformation if a multiple of one of the v_j is added to the other. By Fubini, shears must preserve volumes.

Definition 2 *A volume form V is a mapping from d-tuples of d-vectors, $v_1, \ldots, v_d$ to nonegative numbers that satisfis,*
i. *For all $a \in \mathbb{R}$*

$$V(av_1, v_2, \cdots v_d) = |a|\, V(v_1, \ldots, v_d)\,.$$

ii. $V(v_1 + av_2, v_2, \ldots, v_d) \ = \ V(v_1, \ldots, v_d)$.
ii. *The value of V is unchanged if two of the v_j are transposed.*

Exercise. *Prove that the only volume forms are constant multiples of the absolute value of the determinant.*

Remark. Instead of $\mathbb{R}^d$ on can take any finite dimensional vector space.

4 Haar measure on Lie groups.

Definition 3 *A* **volume form** *on a manifold is a smooth assignment of a volume form to each tangent space.*

Definition 4 *If M, N are manifolds of the same dimension, V is a volume form on M and $\Psi : M \to N$ is a diffeomorphism then there is a natural* **push forward** *volume form $\Psi_* V$ on N defined by*

$$(\Psi_* V)(\Psi_* v_1, \ldots, \Psi_* v_d) \ = \ V(v_1, \ldots, v_d)\,.$$

Example If one views $\mathbb{R}^d$ as a manifold it is natural to take only volume forms that are translation invariant in the sense that its push forward under translation is equal to itself. With this constraint the volume forms on $\mathbb{R}^d$ are uniquely determined up to a constant multiple.

The construction on $\mathbb{R}^d$ generalizes immediately to $SO(d)$ or more generally to any Lie Group (a group which is a manifold and on which multiplication is smooth). If G is a lie group and $\gamma \in G$ denote by Λ_γ the diffeomorphism of G to itself defined by left multiplication,

$$\Lambda_\gamma(g) := \gamma g\,.$$

Definition 5 *A volume form V on a Lie Group is left invariant if and only for all γ, $(\Lambda\gamma)_* V = V$.*

A left invariant volume form is uniquely determined by its value at the identity element since the value at g is equal to $(\Lambda_g)_* V(e)$. Thus there is a unique up to constant multiple left invariant volume form. It is called **Haar measure**.

In the discussion of the mean value property, this volume form is the one that one uses to average over $SO(d)$. The indeterminate constant is not important because when taking an average, the constant cancels.