Exercises on the Laplacian J. Rauch

A function u on $\mathbb{R}^d$ is **invariant under the orthogonal group** when for all $R \in SO(d)$, $u \circ R = u$. The value of such a function depends only on $|x|$. We call such functions **radial**.

The **Dirichlet integral** on the open set Ω is denoted

$$J(u) \ := \ \int_\Omega |\nabla u|^2 \ dx, \qquad u \in C_0^\infty(\mathbb{R}^d)\,.$$

The Laplacian is characterized by,

$$\frac{dJ(u+\epsilon\phi)}{d\epsilon}\bigg|_{\epsilon=0} \ = \ -2\int \Delta u \ \phi \ dx, \qquad \phi \in C_0^\infty(\Omega).$$

1. Use this characterization to prove that for radial functions

$$\Delta\, u \ = \ \frac{1}{r^{d-1}}\,\big(r^{d-1}u_r\big)_r\,, \qquad r \neq 0\,.$$

Hint. Write J in polar coordinates.

Denote by $\langle \cdot\,,\,\cdot\rangle$ the bilinear pairing of $\mathbb{C}^d \times \mathbb{C}^d$ defined by

$$\langle \zeta\,,\,x\rangle \ := \ \sum \zeta_i\, x_i\,.$$

A quadratic form q on $\mathbb{R}^d$ is associated with a unique symmetric matrix $B = B^t$ by $q(v) = \langle Bv, v\rangle$.

2. Prove that if q is an $SO(d)$ invariant quadratic form then q is a scalar mutliple of $\langle v, v\rangle$.

A differential operator $P(\partial)$ with constant coefficients is **invariant** under orothogonal transformations iff for all $R \in SO(d)$ and $u \in C^\infty(\mathbb{R}^d)$

$$P\,(u\circ R) \ = \ (Pu)\circ R\,.$$

For a constant coefficient partial differential operator replacing ∂ by $i\xi$ yields a polynomial $P(\xi)$ called the **symbol**. Then

$$P(\partial)\, e^{i\langle \xi, x\rangle} \ = \ P(\xi)\, e^{i\langle \xi, x\rangle}\,.$$

3. Show that if $P(\partial)$ is a constant coefficient scalar partial differential operator invariant under orthogonal transformations, then its symbol $P(\xi)$ is an invariant polynomial.[1] Conclude that if P is homogeneous of degree 2 then it is a scalar multiple of the the Laplacian.

If $p(x)$ is a polynomial it is uniquely expressed as a sum of homogeneous terms,

$$p \;=\; p_0 + p_1 + \cdots + p_m\,.$$

4. Show that p is orthogonally invariant if and only if each p_j is.

5. Show that there are no invariant polynomial of degree 1. Of degree 3. Of any odd degree.

6. Show that the invariant polynomials of degree $2k$ are equal to $c(x_1^2+\cdots+x_d^2)^k$.

7. Show that every homogeneous invariant scalar partial differential with constant coefficients is of even order and is equal to $c\,\Delta^k$ for some k.

8. Denote by $A(u,r)$ average of u over the ball of radius r centered at the origin. Show that there are constant coefficient homogeneous scalar partial differential operator $P_k(\partial)$ of degree k so that

$$A(u,r) \;=\; u(0) + \frac{1}{d+2}\Delta u(0)|x|^2 + (P_4(\partial)u(0))|x|^4 + (P_6(\partial)u(0))|x|^6 + \cdots\,.$$

9. Show that each operator P_k is invariant under orthogonal transformations. Conclude that $P_{2k} = c(k,d)\Delta^k$ for some constant $c(k,d)$.

Discussion. This proves that if u is harmonic, then the Taylor expansion of $A(u,r) - u(0)$ as a function of r vanishes identically. This is the infinitesimal mean value property. Since harmonic functions are real analytic the Taylor series converges to $A(u,r) - u(0)$ and one concludes that $A(u,r) = u(0)$, the **Mean Value Property**. In my talk I presented a shorter proof using the fact that $A(u,r)$ is a radial harmonic function. This is so since A is the average over $R \in SO$ of $u \circ R$. It did not use real analyticity.

[1] The converse is proved using the Fourier Transform.