

String Energy by Noether's Method

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This note presents Noether's Theorem by applying the method to derive the formula of conservation of energy for the vibrating spring equation,

$$m \ddot{x} + k x = 0, \quad m > 0, \quad k > 0. \quad (1)$$

The method constructs conservation laws from continuous symmetries of variational problems.

Definition 1 *The kinetic and potential energy are defined as*

$$K.E. := \frac{m \dot{x}^2}{2}, \quad P.E. := \frac{k x^2}{2}.$$

*The sum of the two is called the **energy** and is denoted $E(t)$. The **action** is defined as*

$$S(x) := \int_a^b \frac{m \dot{x}^2}{2} - \frac{k x^2}{2} dt = \int_a^b K.E. - P.E. dt. \quad (2)$$

*The integrand in (2) is called the **lagrangian** and is denoted \mathcal{L} .*

The family of curves $x + \epsilon\psi$ in the next theorem pass through the same endpoints as the curve $x(t)$.

Theorem 1 (Principal of stationary action.) *The function $x \in C^\infty([a, b])$ satisfies (1) if and only if for all $\psi \in C^\infty([a, b])$ with $\psi(a) = \psi(b) = 0$,*

$$\frac{d}{d\epsilon} S(x + \epsilon\psi) \Big|_{\epsilon=0} = 0.$$

Proof. Define $x_\epsilon := x + \epsilon\psi$. Differentiation under the integral yields,

$$\frac{d}{d\epsilon} \int_a^b \frac{m \dot{x}_\epsilon^2}{2} - \frac{k x_\epsilon^2}{2} dt = \int_a^b m \dot{x}_\epsilon \dot{\psi} - k x_\epsilon \psi dt.$$

Integrate by parts in the first term. The boundary terms vanish since $\psi = 0$ at the boundary. Therefore,

$$\frac{d}{d\epsilon} \int_a^b \frac{m \dot{x}_\epsilon^2}{2} - \frac{k x_\epsilon^2}{2} dt = \int_a^b (-m \ddot{x}_\epsilon - k x_\epsilon) \psi dt.$$

Setting $\epsilon = 0$ proves the Theorem. ■

Noether's theorem asserts that symmetries of the lagrangian lead to conserved quantities for the Euler-Lagrange equations. In physics the conservation law associated to time translation invariance is usually called the energy. Similarly spatial translations are associated to momentum and rotational symmetry to angular momentum. We derive the conservation of energy for the spring equation from the time translation invariance of the lagrangian.

Theorem 2 *If $x \in C^\infty(\mathbb{R})$ satisfies (1) then, $E(t)$ is independent of t .*

Proof. For any $-\infty < a < b < \infty$ we prove that $E(a) = E(b)$.

Define $x_\epsilon(t) := x(t - \epsilon)$ the translate of x . *The time translation invariance of \mathcal{L} implies that $\int_{a+\epsilon}^{b+\epsilon} \mathcal{L}(x_\epsilon) dt$ is independent of ϵ .* Computing the derivative with respect to ϵ at $\epsilon = 0$ will yield the desired conclusion. Decompose,

$$\int_{a+\epsilon}^{b+\epsilon} = \int_a^b + \int_b^{b+\epsilon} - \int_a^{a+\epsilon}.$$

For ϵ small,

$$\int_b^{b+\epsilon} - \int_a^{a+\epsilon} = \epsilon (K.E. - P.E.)|_a^b + O(\epsilon^2). \quad (3)$$

For the \int_a^b use

$$x(t - \epsilon) = x - \epsilon \dot{x}(t) + O(\epsilon^2).$$

Define $\psi := -\dot{x}$ so $x_\epsilon = x + \epsilon \psi + O(\epsilon^2)$. As in the preceding theorem,

$$\int_a^b = S(x) + \epsilon \left(\int_a^b m \dot{x} \dot{\psi} - k x \psi dt \right) + O(\epsilon^2). \quad (4)$$

In the middle term integrate by parts to find

$$\begin{aligned} \int_a^b m \dot{x} \dot{\psi} - k x \psi dt &= \int_a^b \psi (-m \ddot{x} - k x) dt + m \dot{x} \psi |_a^b \\ &= -\epsilon m \dot{x} \dot{x} |_a^b = -2\epsilon K.E. |_a^b. \end{aligned} \quad (5)$$

Combining (3), (4), and (5), yields,

$$\begin{aligned}\int_{a+\epsilon}^{b+\epsilon} &= S(x) + \epsilon \left((K.E. - P.E) - 2K.E \right) \Big|_a^b + O(\epsilon^2) \\ &= S(x) + \epsilon \left(-K.E. - P.E \right) \Big|_a^b + O(\epsilon^2).\end{aligned}$$

Since the left hand side is independent of ϵ the coefficient of ϵ vanishes, so, $E(b) = E(a)$. ■

Remark. In (5) the integral term is equal to,

$$\int_a^b \dot{x} (m \ddot{x} + kx) dt.$$

To derive the energy law without the variational argument the strategy is to multiply by \dot{x} and integrate from a to b .