Chapter 10. Examples of Resonance in One Dimensional Space

§10.1. Resonance relations.

The examples in this chapter share a common spectral structure. The semilinear examples have

$$A_0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_1 = \operatorname{diag}\big\{1, 0, -1\big\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The quasilinear examples have $A_0 = I$ and $A_1'(0) = \operatorname{diag}\big\{1, 0, -1\big\}$. The operator

$$L \ := \ \partial_t \ + \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x \tag{10.1.1}$$

is equal to $L(\partial)$ in the first case and to $L(0, \partial)$ in the second.

The profiles are 2π periodic in Y

$$U(y, Y) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha \, e^{i\alpha . Y} \, . \tag{10.1.2}$$

The formal trigonometric series in §9.4 are Fourier series here. In the language of quasiperiodic profiles with reduced profile $\mathcal{U}$ described in §9.5, this corresponds to taking $m = 2$ and phases $\phi_\mu(y) := y_\mu \, , \mu = 0, 1$. With the assumption of periodicity there is no need to pass through the quasiperiodic detour. The profiles U already have a well defined sense. If one were to consider $\partial_t + \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)\partial_x$, the quasiperiodic setting would be necessary in order to capture the triad of resonant phases.

Proposition 10.1.1. *The small divisor hypothesis is satisfied.*

Proof. The matrix

$$L(\alpha . d\phi) = L(\alpha_0, \alpha_1) = \begin{pmatrix} \alpha_0 + \alpha_1 & 0 & 0 \\ 0 & \alpha_0 & 0 \\ 0 & 0 & \alpha_0 - \alpha_1 \end{pmatrix}$$

has eigenvalues $\alpha_0 + \alpha_1, \alpha_0, \alpha_0 - \alpha_1$. For $\alpha \in \mathbb{Z}^2$ the eigenvalues are integers.

When an eigenvalue is nonzero, it is bounded below by 1 in modulus. This proves that the inequality of the small divisor hypothesis with $N = 0$ and $C = 1$. ∎

Denote the standard basis elements of $\mathbb{C}^3$ by

$$r_1 := (1, 0, 0) \, , \qquad r_2 := (0, 1, 0) \, , \qquad r_3 := (0, 0, 1) \, . \tag{10.1.3}$$

The corresponding projectors are,

$$\pi_1 \ := \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi_2 \ := \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi_3 \ := \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $\alpha \neq 0$, $\pi(\alpha_0, \alpha_1)$ is nonzero in exactly three circumstances

$$\begin{aligned}
\alpha_0 + \alpha_1 = 0\,, &\qquad \text{in which case} \qquad \pi(\alpha) = \pi_1\,, \\
\alpha_0 = 0\,, &\qquad \text{in which case} \qquad \pi(\alpha) = \pi_2\,, \\
\alpha_0 - \alpha_1 = 0\,, &\qquad \text{in which case} \qquad \pi(\alpha) = \pi_3\,.
\end{aligned} \tag{10.1.4}$$

When $\alpha = 0$, $\pi(0) = I$. Let

$$\lambda_1 := +1\,, \qquad \lambda_2 := 0\,, \qquad \lambda_3 := -1\,.$$

The characteristic variety of L is the union of the three lines

$$\ell_j \;:=\; \Big\{\alpha = (\alpha_0, \alpha_1)\;:\;\alpha_0 + \lambda_j \alpha_1 = 0\Big\}\,, \qquad j = 1, 2, 3\,.$$

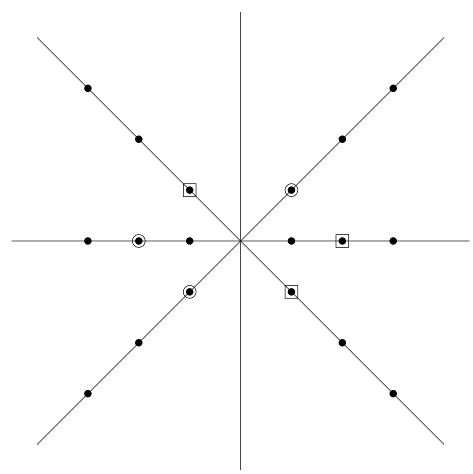

Figure 10.1 Char L and two resonant triads.

Since $\mathbf{E}U_0 = U_0$, the Fourier coefficients $\widehat{U_0}(y, \alpha_0, \alpha_1)$ vanish unless $\alpha \in \cup_j \ell_j$. The coefficients are polarized,

$$\alpha \in \ell_j \setminus 0 \qquad \Longrightarrow \qquad \pi(\alpha) = \pi_j \quad \text{and} \quad \pi_j\, \widehat{U_0}(y, \alpha) = \widehat{U_0}(y, \alpha)\,. \tag{10.1.5}$$

Since the π_j sum to I, one has

$$\mathbf{E} = \sum_1^3 \mathbf{E}_j\,, \qquad \text{where,} \qquad \mathbf{E}_j := \pi_j\, \mathbf{E}\,.$$

The definition of $\mathbf{E}$ yields,

$$\mathbf{E}_j \sum_{\alpha \in \mathbb{Z}^2} a_\alpha(y)\, e^{i\alpha.Y} \;=\; \sum_{\alpha \in \ell_j \cap \mathbb{Z}^2} \pi_j\, a_\alpha(y)\; e^{i\alpha.Y}.$$

For $n \in \mathbb{Z}$, define the scalar Fourier coefficients $\widehat{\sigma_j}$ encoding the spectra of $\widehat{U}_0$ from ℓ_j

$$\begin{aligned}
\widehat{\sigma_1}(y,n) &:= \big\langle \hat{U}_0(y,(n,-n),r_1\big\rangle\,, \\
\widehat{\sigma_2}(y,n) &:= \big\langle \hat{U}_0(y,(0,n),r_2\big\rangle\,, \\
\widehat{\sigma_3}(y,n) &:= \big\langle \hat{U}_0(y,(n,n),r_3\big\rangle\,.
\end{aligned}$$

The corresponding 2π periodic functions are

$$\sigma_j(y,\phi) \;:=\; \sum_{n\in\mathbb{Z}} \widehat{\sigma_j}(y,n)\, e^{in\phi}\,, \qquad j=1,2,3\,. \tag{10.1.6}$$

Then,

$$U_0(y,Y_0,Y_1) \;=\; \Big(\sigma_1(y,Y_0-Y_1)\,,\,\sigma_2(y,Y_1)\,,\,\sigma_3(y,Y_0+Y_1)\Big)\,, \tag{10.1.7}$$

and,

$$\begin{aligned}
\mathbf{E}_1\,U_0 &= r_1 \sum_{n\in\mathbb{Z}} \widehat{\sigma_1}(y,n)\, e^{in(Y_0-Y_1)}\,, \\
\mathbf{E}_2\,U_0 &= r_2 \sum_{n\in\mathbb{Z}} \widehat{\sigma_2}(y,n)\, e^{inY_1}\,, \\
\mathbf{E}_3\,U_0 &= r_3 \sum_{n\in\mathbb{Z}} \widehat{\sigma_1}(y,n)\, e^{in(Y_0+Y_1)}\,,
\end{aligned}$$

In general, the projection operators $\mathbf{E}$ have relatively simple integral forms. The next proposition treats the special cases of this section.

Proposition 10.1.2. *For $g(Y) \in \cap_s H^s(\mathbb{T}^2)$, the operators $\mathbf{E}_j$ are given by the formulas*

$$\begin{aligned}
(\mathbf{E}_1 g)(Y)\big) &= \int_0^{2\pi} \pi_1 g(\psi+(Y_0-Y_1),\psi)\;\frac{d\psi}{2\pi} \\
(\mathbf{E}_2 g)(Y) &= \int \pi_2 g(Y_0,Y_1)\;\frac{dY_0}{2\pi} \\
(\mathbf{E}_3 g)(Y)\big) &= \int_0^{2\pi} \pi_3 g(-\psi+(Y_0-Y_1),\psi)\;\frac{d\psi}{2\pi}\,.
\end{aligned} \tag{10.1.8}$$

The expressions show that the integrals depend only on Y_0-Y_1, Y_1 and Y_0+Y_1 respectively.

Proof. The case $\mathbf{E}_2$ is the easiest. One has

$$\mathbf{E}_2(a\,e^{i\alpha.Y}) \;=\; \begin{cases} \pi_2 a\, e^{i\alpha.Y} & \text{when} \quad Y_0 = 0 \\[2ex] \quad 0 & \text{when} \quad Y_0 \neq 0. \end{cases}$$

On monomials $\mathbf{E_2}$ agrees with $\int \ldots dY_0/2\pi$. By linearity and density

$$\mathbf{E}_2 g(Y_0, Y_1) \;=\; \int \pi_2 g(Y_0, Y_1)\, \frac{dY_0}{2\pi}\,,$$

proving the middle formula.

Consider next $\mathbf{E}_1$ for which the preserved monomials are of the form $e^{in(Y_0-Y_1)}$ with integer n. These monomials are constant on the lines $Y_0 - Y_1 = c$. The general monomial is of the form $e^{imY_0}e^{in(Y_0-Y_1)}$. To kill those with $m \neq 0$ it is sufficient to integrate over $Y_0 - Y_1 = c$. Parameterize $\{Y_0 - Y_1 = c\}$ by Y_1 to obtain,

$$\mathbf{E}_1 g \;=\; \int_0^{2\pi} \pi_1 g(Y_1 + c, Y_1)\, \frac{dY_1}{2\pi}\,.$$

On the domain of integration, $c = Y_0 - Y_1$ and Y_1 is a dummy variable yielding

$$\mathbf{E}_1 g \;=\; \int_0^{2\pi} g(\psi + (Y_0 - Y_1), \psi)\, \frac{d\psi}{2\pi}\,.$$

For $\mathbf{E}_3$ the monomials $e^{imY_0}e^{in(Y_0+Y_1)}$ with $m = 0$ are the ones preserved. One singles them out by integrating over $Y_0 + Y_1 = c$ which can be parameterized by Y_1 to yield

$$\mathbf{E}_3 g \;=\; \int_0^{2\pi} \pi_3 g(-Y_1 + c, Y_1)\, \frac{dY_1}{2\pi}\,,$$

which is the third formula. ∎

§10.2. Semilinear examples.

For initial data $U_0(0, x, Y) = \mathbf{E}\, U_0 \in \cap_s H^s(\mathbb{R}^d \times \mathbb{T}^{1+1})$ the leading profile equation has a unique smooth solution locally in time. Since the small divisor hypothesis is satisfied, the corrector profiles U_j exist and are uniquely determined from the initial values of $\mathbf{E}U_j\big|_{t=0}$. The semilinear analogues of Theorems 9.5.3-9.5.4 imply that they yield infinitely accurate approximate solutions.

The profile equation (9.4.14) has three components. The $j^{\rm th}$ component asserts that

$$\pi_j\, \mathbf{E}\,\Big(L(\partial_y)U_0 + f(U_0)\Big) \;=\; 0\,.$$

It generates an evolution equation for σ_j.

Compute using the diagonal structure of L and $[\mathbf{E}, \partial_y] = [\mathbf{E}, \pi_j] = 0$,

$$\begin{aligned}\pi_1\, \mathbf{E}\, L(\partial_y)\, U_0 &= \mathbf{E}_1\, \pi_1 L(\partial_y)\, U_0 = \mathbf{E}_1\, (\partial_t + \partial_x)\, \pi_1\, U_0 \\ &= (\partial_t + \partial_x)\, \mathbf{E}_1\, \big(\sigma_1(y, Y_0 - Y_1)\, r_1\big) = (\partial_t + \partial_x)\, \sigma_j(y, Y_0 - Y_1)\, r_1\,.\end{aligned}$$

Thus,

$$(\partial_t + \partial_x)\, \sigma_j(y, Y_0 - Y_1) \; + \; \Big\langle \mathbf{E}_1\big(f(U_0)\big), r_1 \Big\rangle \; = \; 0\,.$$

Equivalently,

$$\big(\partial_t + \partial_x\big)\, \sigma_1(y, Y_0 - Y_1) + \Big\langle \mathbf{E}_1\Big(f_1\big(\sigma_1(y, Y_0 - Y_1), \sigma_2(y, Y_1), \sigma_3(y, Y_0 + Y_1)\big) r_1\Big), r_1 \Big\rangle = 0. \tag{10.2.1}$$

Similarly, the second and third equations are equivalent to

$$\partial_t \sigma_2(y, Y_1) + \Big\langle \mathbf{E}_2\Big(f_2\big(\sigma_1(y, Y_0 - Y_1)\,,\, \sigma_2(y, Y_1)\,,\, \sigma_3(y, Y_0 + Y_1)\big) r_2\Big)\,,\, r_2 \Big\rangle = 0, \tag{10.2.2}$$

$$\big(\partial_t - \partial_x\big)\, \sigma_3(y, Y_0 + Y_1) + \Big\langle \mathbf{E}_3\Big(f_3\big(\sigma_1(y, Y_0 - Y_1), \sigma_2(y, Y_1), \sigma_3(y, Y_0 + Y_1)\big) r_3\Big), r_3 \Big\rangle = 0. \tag{10.2.3}$$

Equations (10.2.1)-(10.2.3) form a coupled system of three integrodifferential equations. They are differential in the variables t, x and integral in the variables Y_0, Y_1 which lie on a torus. The system is easy to approximate numerically. In dimension $d = 1$ the highly oscillatory initial value problem is at the borderline of computable for times $t \sim 1$ and $\epsilon < 10^{-3}$. In higher dimensions, the border of computability comes at much larger ϵ.

Example 10.2.1. Consider the three wave interaction system (9.2.2). The transport equation for σ_2 is

$$\partial_t\, \sigma_2(y, \theta_1) \; = \; c_2 \Big\langle \mathbf{E}_2\, \Big(\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1)\, r_2\, \Big)\,,\, r_2 \Big\rangle\,. \tag{10.2.4}$$

The profile equations are best understood in Fourier. Exand to find

$$\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1) \; = \; \sum_{m,n} \widehat{\sigma_1}(y, n)\, e^{in(Y_0 - Y_1)}\; \widehat{\sigma_3}(y, m)\, e^{im(Y_0 + Y_1)}\,.$$

The operator $\mathbf{E}_2$ selects the phases $\alpha.Y$ with $\alpha_0 = 0$. As the phase is equal to $n(Y_0 - Y_1) + m(Y_0 + Y_1)$, this yields $m = -n$, so

$$\mathbf{E}_2\Big(\big(\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1)\, r_2\Big) \; = \; \sum_n \widehat{\sigma_1}(y, n)\; \widehat{\sigma_3}(y, -n)\; e^{-2inY_1}\, r_2\,.$$

The profile equation (10.2.4) for $\widehat{\sigma_2}(y, n)$ splits according to the parity of n,

$$\partial_t\, \widehat{\sigma_2}(y, -2n) = c_2\, \widehat{\sigma_1}(y, n)\; \widehat{\sigma_3}(y, -n)\,, \qquad \partial_t\, \widehat{\sigma_2}(y, -2n+1) = 0\,, \quad n \in \mathbb{Z}\,. \tag{10.2.5}$$

The dynamics for σ_1 is given by

$$\big(\partial_t + \partial_x\big)\, \sigma_1(y, Y_0 - Y_1) \; = \; c_1 \Big\langle \mathbf{E}_1\Big(\sigma_2(y, Y_1)\, \overline{\sigma_3}(y, Y_0 + Y_1)\, r_1\Big)\,,\, r_1 \Big\rangle\,. \tag{10.2.6}$$

The third profile equation is,

$$\big(\partial_t + \partial_x\big)\,\sigma_3(y, Y_0+Y_1) \;=\; c_3\,\Big\langle \mathbf{E}_3\Big(\sigma_2(y,Y_1)\,\overline{\sigma_1}(y, Y_0-Y_1) r_3\Big)\,,\, r_3\Big\rangle\,. \tag{10.2.7}$$

For (10.2.6) use,

$$\overline{\sigma_3}(y,\phi) \;=\; \Big(\sum_n \widehat{\sigma_3}(y,n)\; e^{in\phi}\Big)^* \;=\; \sum_n \widehat{\sigma_3}(y,n)^*\; e^{-in\phi}\,,$$

$$\sigma_2(y,Y_1)\,\overline{\sigma_3}(y,Y_0+Y_1) \;=\; \sum_{m,n}\widehat{\sigma_2}(y,m)\; e^{imY_1}\;\widehat{\sigma_3}(y,n)^*\; e^{-in(Y_0+Y_1)}\,.$$

The phase of the product of the exponentals is $-nY_0 + (m-n)Y_1$. The operator $\mathbf{E}_1$ selects only those phases $\alpha.Y$ with $\alpha_0+\alpha_1=0$. In this case,

$$-n \;+\; (m-n) \;=\; 0, \qquad \Longleftrightarrow \qquad m \;=\; 2n.$$

Therefore,

$$\big(\partial_t + \partial_x\big)\,\sigma_3(y, Y_0+Y_1) \;=\; c_3\,\sum_n \widehat{\sigma_2}(y,2n)\;\widehat{\sigma_3}(y,n)\; e^{-in(Y_0-Y_1)}\,.$$

In terms of the Fourier coefficients this is equivalent to,

$$(\partial_t + \partial_x)\widehat{\sigma_1}(y,n) \;=\; c_3\,\widehat{\sigma_2}(y,-2n)\;\widehat{\sigma_3}(y,-n)^*\,. \tag{10.2.8}$$

An analgous computation shows that the third profile equation is equivalent to

$$(\partial_t - \partial_x)\widehat{\sigma_3}(y,-n) \;=\; c_3\,\widehat{\sigma_1}(y,n)^*\;\widehat{\sigma_2}(y,-2n)\,. \tag{10.2.9}$$

Exercise. Verify (10.2.9).

The equations (10.2.5), (10.2.8), (10.2.9) show that the nonlinear interactions are localized in the triads

$$\Big\{\widehat{\sigma_1}(y,n)\,,\,\widehat{\sigma_2}(y,-2n)\,,\,\widehat{\sigma_3}(y,-n)\Big\}\,. \tag{10.2.10}$$

The corresponding Fourier coefficients of U_0 are

$$\widehat{U_0}(y,n,-n), \qquad \widehat{U_0}(y,0,-2n) \qquad \widehat{U_0}(y,-n,-n)\,.$$

Two such triads are indicated in Figure 10.1. The interaction comes about through the resonance relation

$$-2x \;=\; (t-x) \;-\; (t+x)\,, \qquad (0,-2n) = (n,-n) + (-n,-n)\,.$$

For each n, the triple (10.2.10) satisfies the three wave interaction pde decoupled from the other Fourier coefficients. The initial data for the triad of Fourier coefficients are indpendent of ϵ and not rapidly oscillating. The fact that the triads are isolated shows that there is no possibility of interactions moving far in the scale of wave numbers.

Consider three special cases. For the initial value problem (9.1.1), $c_1 = c_3 = 0$, and the intial data are

$$\sigma_1(0,x,\phi) = a_1(x)\, e^{i\phi}\,, \qquad \sigma_2(0,x,\phi) = 0\,, \qquad \sigma_3(0,x,\phi) = a_3(x)\, e^{-i\phi}\,.$$

The initial data ignite the single resonant triad $\{1,-2,-1\}$. The function $\sigma(y,\phi)$ is given by

$$\sigma_1 = a_1(t-x)e^{i\phi}\,, \quad \sigma_2 = e^{-2i\phi}\int_0^t a_1(t-x)\, a_3(t+x)\, dt\,, \quad \sigma_3 = a_1(t+x)e^{-i\phi}\,.$$

In this particular case, the approximation of nonlinear geometric optics gives the exact solution.

Modify the third initial datum to

$$u_3(0,x) = a_3(x)e^{inx/\epsilon}\,, \qquad n \in \mathbb{Z}\setminus\{-1\}\,, \tag{10.2.11}$$

to find $\sigma_3(t,x,\phi) = a_3(t+x)\, e^{in\phi}$ and,

$$\mathbf{E}_2(\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_0) r_2 = \mathbf{E}_2\Big(a_1(t-x)\, a_3(t+x)\, e^{i\{(t+x)+n(t-x)\}} r_2\Big) = 0\,.$$

The product inside $\mathbf{E}_2$ always oscillates in time so is annihilated by $\mathbf{E}_2$ to give $\partial_t \sigma_2 = 0$. The oscillations in the second component of U_0 do not change in time and there is no interaction with the oscillations in the other components. This agrees with the nonstationary phase analysis in in §9.1.

Consider the real initial data

$$\sigma_1(0,x,\phi) = a_1(x)\,\sin\phi\,, \qquad \sigma_2(0,x,\phi) = 0\,, \qquad \sigma_3(0,x,\phi) = a_3(x)\,\sin(-\phi)\,.$$

In this case the initial data ignite two resonant triads

$$\Big\{(0,-2n),(n,n),(-n,-n)\Big\}, \qquad \text{and,} \qquad \Big\{(0,2n),(-n,-n),(n,n)\}.$$

Each triad of coefficients,

$$\widehat{\sigma_1}(t,x,1)\,,\, \widehat{\sigma_1}(t,x,-2)\,,\, \widehat{\sigma_1}(t,x,-1)\,, \qquad \text{and} \qquad \widehat{\sigma_1}(t,x,-1)\,,\, \widehat{\sigma_1}(t,x,2)\,,\, \widehat{\sigma_1}(t,x,1)\,,$$

solves the three wave interaction pde. All other coefficients vanish identically.

In the last two cases, the approximate solution is not an exact solution.

Proposition 10.2.11. *Consider the system of profile equations for the three wave interaction system with $c_j \in \mathbb{R} \setminus 0$. The following are equivalent.*

i. *For arbitrary initial data $\sigma(0,x,\phi) \in \cap_s H^s(\mathbb{R}\times\mathbb{T})$ there is a unique global solution $\sigma(t,x,\phi) \in \cap_s C^s\big(\mathbb{R}\,;\,H^s(\mathbb{R}\times\mathbb{T})\big)$.*

ii. *The coefficients c_j do not have the same sign.*

Proof. The explosive behavior is proved by considering a single resonant triad which blows up in finite time T_*.

For existence it suffices to observe that the $L^\infty([0,T]\times\mathbb{R})$ bound for solutions of the three wave system with c_j not all of the same sign proves an estimate

$$\|\widehat{\sigma}(t,x,n)\|_{L^\infty([0,T]\times\mathbb{R})} \ \le\ C\Big(\|\widehat{\sigma}(0,x,n)\|_{L^2(\mathbb{R})}\,,\,T\Big)\ \big\|\widehat{\sigma}(0,x,n)\big\|_{L^\infty(\mathbb{R})}\,,$$

with the function $C(\cdot,\cdot)$ independent of n. Summing on n, this suffices to establish an *apriori* estimate

$$\|\sigma(t,x,\phi)\|_{L^\infty([0,T]\times\mathbb{R}\times\mathbb{T})} \ \le\ C\Big(\|\sigma(0,x,\phi)\|_{H^s(\mathbb{R}\times\mathbb{T})}\ ,\,s\,,\,T\Big)\,,\qquad s>1\,.$$

This implies global solvability using Moser's inequality as in §6.4. ∎

When the profiles exist globally in time, Theorem 9.4 shows that the approximation of resonant nonlinear geometric optics is accurate on arbitrary long time intervals $0\le t\le T$. In particular the interval of existence of the exact solution grows infinitely long in the limit $\epsilon\to 0$. In the present case we know more, namely that the solutions exist globally. Note that the approximation is not justified on the infinite time interval $0<t<\infty$. One must exercise care in drawing conclusions about the large time behavior of exact solutions from the large time behavior of the profiles.

There is similar caution for the case of explosive profiles. It is tempting to conclude from profile blowup that there is a parallel blowup of exact solutions. This is not justified. Thoerem 9.4 justifies the approximation on arbitrary intervals of smoothness, $0<T<T_*$. One can draw some conclusions which have the flavor of explosion. Denote by v^ϵ the exact solution with the same intial data as the approximate solution u^ϵ. Choosing T very close to T_* on shows that

$$\lim_{T\to T_*}\ \liminf_{\epsilon\to 0}\ \big\|v^\epsilon(T,x)\big\|_{L^\infty(\mathbb{R}_x)}\ =\ \infty$$

This asserts that the family of exact solutions v^ϵ is unbounded, but it does not assert that any given member of the family explodes.

Example 10.2.2. Consider the modification of equation (9.1.1) where the equation for u_2 is changed to a general real quadratic interaction

$$\partial_t u_2 = \sum_{1\le i\le j\le 3} A_{i,j}\,u_i\,u_j \qquad (10.2.12)$$

The profile equation for σ_2 is

$$\partial_t \sigma_2(y, Y_1) = \Big\langle \mathbf{E}_2\Big(\sum_{1\le i\le j\le 3} A_{i,j}\ \sigma_i(y, h_i(Y))\ \sigma_j(y, h_j(Y)) r_2\Big)\,,\ r_2 \Big\rangle\,, \tag{10.2.13}$$

with

$$h_1(Y) := Y_0 - Y_1, \qquad h_2(Y) := Y_1, \qquad h_3(Y) := Y_0 + Y_1\,.$$

Write U_0 as in (10.1.7). The contribution of the term $A_{1,3}\,\sigma_1\,\sigma_3$ to the profile equation is computed exactly as before and yields

$$\begin{aligned} \mathbf{E}_2\big(A_{1,3}\,\sigma_1(y, Y_0 - Y_1)\,\sigma_3(y, Y_0 + Y_1)\,r_2\,\big) \ &= \ = \ A_{1,3}\sum_n \widehat{\sigma_1}(y,n)\ \widehat{\sigma_3}(y,-n)\ e^{-2inY_1}\,r_2 \\ &= \ A_{1,3}\big(\sigma_1 * \check{\sigma}_3\big)(y, -2Y_1)\,r_2\,. \end{aligned}$$

Denoting with an underline the mean value of a 2π periodic function one then computes the formulas

$$\begin{aligned} \mathbf{E}_2\big(A_{1,2}\,\sigma_1(y, Y_0 - Y_1)\,\sigma_2(y, Y_1)\,r_2\,\big) \ &= \ A_{1,2}\,\underline{\sigma_1}\,\sigma_2(y, Y_1)\,r_2\,, \\ \mathbf{E}_2\big(A_{2,3}\,\sigma_2(y, Y_1)\,\sigma_3(y, Y_0 + Y_1)\,r_2\,\big) \ &= \ A_{2,3}\,\sigma_2(y, Y_1)\,\underline{\sigma_3}\,r_2\,, \\ \mathbf{E}_2\big(A_{2,2}\,\sigma_2(y, Y_1)\,\sigma_2(y, Y_1)\,r_2\,\big) \ &= \ A_{2,2}\,\sigma_2(y, Y_1)^2\,r_2\,. \end{aligned}$$

Combining yields the profile equation

$$\partial_t \sigma_2 = A_{2,2}\,\sigma_2^2(y,\phi) + A_{1,2}\,\sigma_2(y,\phi)\,\underline{\sigma_1} + A_{2,3}\,\sigma_2(y,\phi)\,\underline{\sigma_3} + A_{1,3}\,(\sigma_1 * \check{\sigma_3})(y, -2\phi)\,. \tag{10.2.14}$$

Notice that the first three terms are local in y, ϕ while the quadratic convolution interaction term which comes from the resonance is local in Fourier and not in ϕ. For the initial data from (9.1.1), $\underline{\sigma_1} = \underline{\sigma_3} = 0$ and the profile equation simplifies to

$$\partial_t \sigma_2(y,\phi) = A_{2,2}\,\sigma_2^2 + A_{1,3}\,a_1(t-x)\,a_3(t+x)\,e^{-2i\phi}\,. \tag{10.2.15}$$

Only one Fourier component of σ_2 is affected by the resonant term. There is only one resonant triad active in this particular example. The $A_{2,2}\,\sigma_2^2$ broadens the spectrum of σ_2.

With general quadratic interactions in all the equations, one finds coupled integrodifferential equations with quadratic self interaction terms for all j. The resulting three by three systems are analogous to

$$\partial_t \sigma = a\,\sigma^2 + b\,\sigma * \sigma\,, \qquad \sigma = \sigma(t,\phi)\,. \tag{10.2.16}$$

It would be interesting to understand well the competition between the two quadratic terms on the right of (10.2.16). Note that the term that is local in ϕ is a convolution in n while the convolution in ϕ is local in n.

§10.3. Quasilinear examples.

The next examples resemble the semilinear examples. An important difference is that the amplitudes of the approximate solutions are smaller. One has

$$u^\epsilon(t,x) = \epsilon\, U_0(t,x,t/\epsilon,x/\epsilon)\,.$$

The prefactor of ϵ was absent in the semilinear case. For profiles periodic in Y, equation (9.5.1) simplifies to,

$$\mathbf{E}\, U_0 = \mathcal{U}_0\,, \qquad \mathbf{E}\Big(L(0,\partial_y)\,U_0 + \sum_{\mu=0}^{1} A'_\mu(0)\,U_0\,\frac{\partial U_0}{\partial Y_\mu}\Big) = 0\,. \tag{10.3.1}$$

Example 10.3.1 Consider the 3×3 sytem of quasilinear conservation laws

$$\begin{aligned}(\partial_t+\partial_x)u_1 &= 0\\ \partial_t u_2 + \partial_x(u_1u_3) &= 0\\ (\partial_t-\partial_x)u_3 &= 0\end{aligned} \tag{10.3.2}$$

The small divisor hypothesis is satisfied and equations (10.1.1) through (10.1.7) are unchanged. And, $A_0(u)=I$. The second component of the profile satisfies

$$\partial_t\,\sigma_2 + \Big\langle \mathbf{E}_2\Big(A_1'(0)U_0\,\frac{\partial U_0}{\partial Y_1}\Big)\,,\, r_2\Big\rangle = 0\,. \tag{10.3.3}$$

Equation (10.1.7) yields

$$\frac{\partial U_0}{\partial Y_1} = \Big(-\sigma_1'(y,Y_0-Y_1)\,,\,\sigma_2'(y,Y_1)\,,\,\sigma_3'(y,Y_0+Y_1)\Big)\,. \tag{10.3.4}$$

For (10.3.2),

$$A_1(U) \;=\; \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} + \begin{pmatrix}0&0&0\\U_3&0&U_1\\0&0&0\end{pmatrix} \;=\; A_1(0)+A'(0)U$$

so

$$A_1'(0)U_0 = \begin{pmatrix}0&0&0\\ \sigma_3(y,Y_0+Y_1)&0&\sigma_1(y,Y_0-Y_1)\\0&0&0\end{pmatrix}\,.$$

Suppressing the y dependence,

$$\begin{aligned}A_1'(0)\,U_0\,\frac{\partial U_0}{\partial Y_1} &= \Big(-\sigma_3(Y_0+Y_1)\,\sigma_1'(Y_0-Y_1)+\sigma_1(Y_0-Y_1)\sigma_3'(Y_0+Y_1)\Big)\,r_2\\ &= r_2\,\frac{\partial}{\partial Y_1}\Big(\sigma_1(Y_0-Y_1)\sigma_3(Y_0+Y_1)\Big)\,.\end{aligned}$$

$\mathbf{E}_2$ commutes with $\partial/\partial_{Y_\mu}$ and $\mathbf{E}_2$ applied to the product is computed as earlier to find,

$$\partial_t \sigma_2(t,x,Y_1) \;=\; \frac{\partial}{\partial Y_1}\Big(\sum_n e^{-2inY_1}\,\widehat{\sigma_1}(t,x,n)\,\widehat{\sigma_3}(t,x,-n)\Big) \tag{10.3.5}$$

The odd Fourier coefficients of σ_2 are stationary and the even ones evolve according to

$$\partial_t \hat\sigma_2(y,-2n) = -2in\,\hat\sigma_1(y,n)\,\hat\sigma_3(y,-n)\,. \tag{10.3.6}$$

The profile equations read

$$\begin{aligned}(\partial_t+\partial_x)\sigma_1 \;&=\; 0\,,\\ \partial_t\sigma_2 \;&=\; \partial_\phi\Big(\big(\sigma_1 * \check\sigma_3\big)(t,x,-2\phi)\Big)\,,\\ (\partial_t-\partial_x)\sigma_3 \;&=\; 0\,.\end{aligned} \tag{10.3.7}$$

The interaction equations (10.3.5) are in conservation form. This is a general phenomenon. If the original system is in conservation form,

$$\sum_{\mu=0}^{d}\partial_\mu A_\mu(u) \;=\; 0\,, \tag{10.3.8}$$

then the terms of the equation are $A'_\mu(u)\partial_\mu u$. The coefficients, $A'_\mu(u)$, have the special structure of being derivatives.

Exercise. *If the original system is real and in conservation form (10.30) then the profile equation (9.4.14) can be written in the conservation form*

$$\partial_t U_0 + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}\Big(\mathbf{E}\big(A_j(0)U_0\big)\Big) + \sum_{\mu=0}^{d}\sum_{j,k=1}^{N}\frac{\partial}{\partial\theta_k}\Big(\mathbf{E}\,\frac{\partial^2 A_\mu(0)}{\partial u_j\partial u_k}\,U_jU_k\Big) = 0\,. \tag{10.3.9}$$

For complex equations there are more terms because of the derivatives with respect to the conjugate variables but the conservation form persists.

Equation (10.3.9) implies that in the case of conservation laws a profile as in Theorem 9.1 that has mean zero with respect to θ at $\{t=0\}$ remains mean zero throughout its maximal interval of existence. As in Example 10.2.2, the profile equations in the mean zero case simplify.

Proposition 10.3.1. *Consider a real* 3×3 *system of conservation laws in* $1-d$*,*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}A(u) \;=\; 0\,,\qquad A(u) = \big(A_1(u),A_2(u),A_3(u)\big)\,, \tag{10.3.10}$$

satisfying $A'(0) = \operatorname{diag}\{1, 0, -1\}$. Introduce σ_j as in (10.1.7) and six interaction constants

$$b_j := \frac{\partial^2 A_j(0)}{\partial u_j^2}\,, \quad j = 1, 2, 3\,, \qquad c_1 := \frac{\partial^2 A_1(0)}{\partial u_2 \partial u_3}\,, \quad c_2 := \frac{\partial^2 A_2(0)}{\partial u_1 \partial u_3}\,, \quad c_3 := \frac{\partial^2 A_3(0)}{\partial u_1 \partial u_2}\,. \tag{10.3.11}$$

The profile equation (9.5.1) for periodic profiles (10.1.1) of mean zero is equivalent to the system of equations for the Fourier coefficients,

$$\begin{aligned}
\big(\partial_t + \partial_x\big)\, \hat\sigma_1(t,x,m) &= b_1\, im\, \widehat{(\sigma_1^2)}(t,x,m) + c_1\, im\, \hat\sigma_2(t,x,2m)\, \hat\sigma_3(t,x,-m)\,, \\
\partial_t\, \sigma_2(t,x,2m) &= b_2\, 2im\, \widehat{(\sigma_2^2)}(t,x,2m) + c_2\, 2im\, \hat\sigma_1(t,x,m)\, \hat\sigma_3(t,x,m)\,, \\
\big(\partial_t - \partial_x\big)\, \hat\sigma_3(t,x,m) &= b_3\, im\, \widehat{(\sigma_3^2)}(t,x,m) + c_3\, im\, \hat\sigma_2(t,x,2m)\, \hat\sigma_1(t,x,-m)\,, \\
\partial_t \hat\sigma_2(t,x,2m+1) &= b_3\, 2im\, \widehat{(\sigma_2^2)}(t,x,2m+1)\,.
\end{aligned} \tag{10.3.12}$$

Exercise. *Prove Proposition 10.3.1.*

The next goal is to analyse more closely the resonance terms. First consider the case where all the b_j vanish so the profile equations have only resonant interaction terms.

Example 10.3.2. Consider the case where $b_1 = b_2 = b_3 = 0$. Then for each $m \in \mathbb{Z}$, the three Fourier components $\{\hat\sigma_1(y,m), \hat\sigma_2(y,-2m), \hat\sigma_3(y,-m)\}$ evolve independent of the other Fourier components according to the laws

$$\begin{aligned}
\big(\partial_t + \partial_x\big)\, \hat\sigma_1(t,x,m) &= -c_1\, im\, \hat\sigma_2(t,x,-2m)\, \hat\sigma_3(t,x,\mp m)\,, \\
\partial_t\, \hat\sigma_2(t,x,2m) \ &= \ -c_2\, 2im\, \hat\sigma_1(t,x,m)\, \hat\sigma_3(t,x,-m)\,, \\
\big(\partial_t - \partial_x\big)\, \hat\sigma_3(t,x,-m) &= -c_3\, im\, \hat\sigma_2(t,x,-2m)\, \hat\sigma_1(t,x,m)\,.
\end{aligned} \tag{10.3.13}$$

The odd components of σ_2 belong to no such triad and are stationary,

$$\partial_t \widehat{\sigma_2}(t,x,2m+1) = 0\,. \tag{10.3.14}$$

For fixed $m \neq 0$, the triple $(i\sigma_1, i\sigma_2, i\sigma_3)$ satisfies the three wave interaction pde which we understand well. In addition to the information already gleaned, one has the following invariance properties.

Proposition 10.3.2. *The profile equations (10.3.13) have the following properties.*

1. *The set of σ so that for a fixed $m \in \mathbb{Z}$, and $\forall x, \phi$,*

$$\begin{aligned}
&\sigma_1(t,x,m) = -\sigma_1(t,x,-m)\,, \qquad \sigma_2(t,x,2m) = -\sigma_2(t,x,-2m)\,, \\
&\sigma_3(t,x,m) = -\sigma_3(t,x,-m)\,,
\end{aligned}$$

is invariant. Imposing this condition for all m shows that the set of σ so that $\hat\sigma(y,m)$ is odd in m is invariant under the dynamics. These are exactly the functions $\sigma(y,\phi)$ which are odd in ϕ.

2. *The set of σ so that for a fixed $m \in \mathbb{Z}$, $\{\hat\sigma_1(y,m), \hat\sigma_2(y,-2m), \hat\sigma_3(y,-m)\}$ are purely imaginary for all x, ϕ is invariant. Therefore the set of σ so that $\hat\sigma(y,m)$ is purely imaginary for all $m, x \in \mathbb{Z}\times\mathbb{R}$ is invariant.*

3. *The set of σ so that for a fixed $m \in \mathbb{Z}$, $\{\hat\sigma_1(t,x,m), \hat\sigma_2(t,x,-2m), \hat\sigma_3(t,x,-m)\}$ do not depend on x is invariant. Imposing this for all m shows that the set of σ which do not depend on x is invariant.*

Global solvability of the profile equations when $b = 0$ is completely resolved by our analysis of the three wave interaction pde.

Proposition 10.3.3. *If $b_1 = b_2 = b_3 = 0$ and the three constants c_1, c_2, and c_3 do not have the same sign, then the profile equations (10.3.13) are globally solvable in the sense that for arbitrary initial data $\sigma(0,x,\phi) \in \cap_s \mathrm{Re}\, H^s(\mathbb{R}\times\mathbb{T})$ there is a unique global solution $\sigma(t,x,\phi) \in \cap_s C^s\big(\mathbb{R}\,;\, H^s(\mathbb{R}\times\mathbb{T})\big)$. The norms $\|\sigma(t)\|_{H^s(\mathbb{R}\times\mathbb{T})}$ are bounded independent of $t\in\mathbb{R}$. In contrast, if the b_j vanish, and c_1, c_2, and c_3 have the same sign, then the profile equations (10.3.13) have solutions with finite blowup time $0 < T_* < \infty$.*

This blowup is quite striking. Consider for example the case of a profile whose Fourier series is supported on a single pair of resonant triads as in Figure 10.1,

$$U_0(t,x,Y) \;=\; -\Big(\,\zeta_1(t)\,\sin m(Y_0 - Y_1)\,,\, \zeta_2(t)\,\sin(-2mY_1)\,,\, \zeta_3(t)\,\sin(-m(Y_0+Y_1))\,\Big)\,. \tag{10.3.15}$$

The exact solution is described by

$$u^\epsilon(t,x) \;\sim\; -\epsilon\,\Big(\,\zeta_1(t)\,\sin\frac{m(t-x)}{\epsilon}\,,\, \zeta_2(t)\,\sin\frac{-2mt}{\epsilon}\,,\, \zeta_3(t)\,\sin\frac{-m(t+x)}{\epsilon}\,\Big)\,. \tag{10.3.16}$$

Suppose that $\zeta(t)$ is a solution of the three wave interaction ode. whose components have the same sign and blow up at time $T_* < \infty$ so that

$$\lim_{t\to T*_-} \big|\zeta(t)\big| \;=\; \infty\,. \tag{10.3.17}$$

The initial data and solutions are periodic in x. The data are bounded in $BV(I)$ for any bounded interval, and are $O(\epsilon)$ in $L^\infty(\mathbb{R})$. For the exact solutions, Theorem 9.4 together with finite speed of propagation yields the following result of unbounded amplification.

Proposition 10.3.4 *Suppose that the system (10.3.12) satifies $b = 0$ and that c_1, c_2, and c_3 have the same sign. Choose $\zeta(t)$ an real solution of the profile equation which explodes at time $0 < T_* < \infty$ and define the profile U_0 by (10.3.15). Let u^ϵ be the exact solution with the initial data $U_0(0,0,x/\epsilon) = U_0(t,t/\epsilon,x/\epsilon)|_{\{t=0\}}$. Then for any $T \in]0,T_*]$, u^ϵ smooth on $[0,T]\times\mathbb{R}$ for ϵ small. The data is bounded in the sense that*

$$\|u^\epsilon(0)\|_{L^\infty(\{|x|\le T^*+1\})} \;\le\; C\,\epsilon\,, \qquad \|u^\epsilon(0)\|_{BV(\{|x|\le T^*+1\})} \;\le\; C\,. \tag{10.3.18}$$

The family of solutions explodes in BV in the sense that

$$\lim_{T \to T^*_-} \ \lim_{\epsilon \to 0+} \Big| \int_{\{|x| \le T^*+1-T\}} u^\epsilon(T,x) \, \sin \frac{m(x+t)}{\epsilon} \ dx \Big| \ = \ \infty \, . \tag{10.3.19}$$

The solutions are small in L^∞ with data bounded in BV. The BV norm is amplified by as large a constant as one likes in the following sense. For any large $M > 0$ and small $\delta > 0$ one can chose $T \in [0, T^*[$ and $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$, u^ϵ is smooth on $[0,T] \times \mathbb{R}$,

$$\big\| u^\epsilon \big\|_{L^\infty([0,T]\times\mathbb{R})} < \delta \, , \tag{10.3.20}$$

and

$$\big\| u^\epsilon(T) \big\|_{BV\{|x| \le T^*+1-T\}} \ \ge \ M \, \big\| u^\epsilon(0) \big\|_{BV\{|x| \le T^*+1-T\}} \big\| \, . \tag{10.3.21}$$

Proof. Theorem 9.4 implies that

$$\lim_{\epsilon \to 0+} \int_{\{|x| \le T^*+1\}} u_3^\epsilon(T,x) \, \sin \frac{m(x+t)}{\epsilon} \ dx \ = \ \frac{T^*+1}{\pi} \, \zeta_3(T,m) \, .$$

An exercise in Chapter 9 showed that each component of ζ must explode as $T \to T^*$, and (10.41) follows.

To prove the last assertion of the proposition, choose $T < T^*$ and then ϵ_0 so that

$$|\zeta(T)| > M \, , \qquad \text{and} \qquad \sup_{t \in [0,T]} \ \epsilon_0 \, \big|\zeta(t)\big| < \delta \, .$$

Theorem 9.4 does the rest. ∎

A weakness of this result demonstrating unbounded amplification of the BV norm of a family of solutions with *sup* norm tending to zero and initial BV norms bounded is that the hypothesis $b = 0$ implies that the system is *not* genuinely nonlinear. In [JMR 1994], it is verified that for b sufficiently small, the profile equations have explosive solutions near those just constructed. In this way one has examples of families of solutions of a fixed genuinely nonlinear system which are uniformly small in L^∞, uniformly bounded in BV and for which the BV norm at time $t = 1$ is as large a multiple of the BV norm at $t = 0$ as one likes. This shows that desirable estimates of the form

$$\big\| u(1) \big\|_{BV} \ \le \ C \, \big\| u(0) \big\|_{BV}$$

are not true for L^∞ small solutions of genuinely nonlinear 3×3 systems. Such estimates for the scalar case were proved by Conway-Smoller and Kruzskov while Glimm and Lax proved such estimates for 2×2 systems when $d = 1$. The above examples show that the Glimm-Lax result cannot be extended to general genuinely nonlinear 3×3 systems. After its discovery using nonlinear geometric optics, alternate constructions of such amplification were found ([Bressan], [Temple]).