Chapter 11. Dense Oscillations for the Compressible Euler Equations

In this chapter it is proved that the compressible Euler equations have a cascade of resonant nonlinear interactions that can create waves moving in a dense set of directions from three incoming waves.

§11.1. The $2-d$ isentropic Euler equations.

This system describes compressible, inviscid, fluid flow with negligible heat flow. The velocity and density are denoted $v = (v(t,x), v(t,x))$ and $\rho(t,x)$. The pressure is assumed to be a function, $p(\rho)$, of the density. The governing equations (away from shocks) are

$$\begin{aligned} &(\partial_t + v_1\partial_1 + v_2\partial_2)v_1 + (p'(\rho)/\rho)\partial_1\rho = 0\,, \\ &(\partial_t + v_1\partial_1 + v_2\partial_2)v_2 + (p'(\rho)/\rho)\partial_2\rho = 0\,, \\ &(\partial_t + v_1\partial_1 + v_2\partial_2)\rho + \rho(\partial_1 v + \partial_2 v_2) = 0\,. \end{aligned} \tag{11.1.1}$$

Here

$$x = (x_1, x_2), \qquad \text{and,} \qquad \partial_j := \partial/\partial x_j,$$

Denote by

$$u := (v_1, v_2, \rho), \qquad \text{and,} \qquad f(\rho) := p'(\rho)/\rho\,.$$

The system (11.1.1) is then of the form $L(u,\partial)u = 0$, with coefficient matrices,

$$A_0 = I\,, \qquad A_1(u) = \begin{pmatrix} v_1 & 0 & f(\rho) \\ 0 & v_1 & 0 \\ \rho & 0 & v_1 \end{pmatrix}, \qquad A_2(u) = \begin{pmatrix} v_2 & 0 & 0 \\ 0 & v_2 & f(\rho) \\ 0 & \rho & v_2 \end{pmatrix}. \tag{11.1.2}$$

The system is symmetrized by multiplying by

$$D(\rho) \ := \ \operatorname{diag}\big(\rho, \rho, f(\rho)\big) \ := \ \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & f(\rho) \end{pmatrix}.$$

At a constant background state, $\underline{u} := (\underline{v}, \underline{\rho})$, the linearized operator has symbol

$$L(\underline{u}, \tau, \xi) \ = \ \begin{pmatrix} \tau + \underline{v}.\xi & 0 & \underline{f}\,\xi_1 \\ 0 & \tau + \underline{v}.\xi & \underline{f}\,\xi_2 \\ \underline{\rho}\,\xi_1 & \underline{\rho}\,\xi_2 & \tau + \underline{v}.\xi \end{pmatrix}, \qquad \text{where} \qquad \underline{f} := f(\underline{\rho}).$$

The operator is symmetrized by multiplying by the constant matrix $D(\underline{\rho}) = \operatorname{diag}\big(\underline{\rho}, \underline{\rho}, \underline{f}\big)$. Compute,

$$\det\, L(\underline{u}\,,\, \tau, \xi) = (\tau + \underline{v}.\xi)\big[(\tau + \underline{v}.\xi)^2 - c^2|\xi|^2\big] \tag{11.1.3}$$

where the sound speed c is defined by

$$c^2 \ := \ p'(\underline{\rho})\,, \qquad c > 0\,. \tag{11.1.4}$$

Convention. *By a linear change of time variable $t' = c\,t$ we may assume without loss of generality that $c = 1$.*

The asymptotic relations

$$L(u^\epsilon, \partial) u^\epsilon \;\sim\; 0, \qquad \text{and,} \qquad D(\rho)\, L(u^\epsilon, \partial) u^\epsilon \;\sim\; 0,$$

are equivalent. The latter is symmetric. Solutions are constructed in Chapter 10. The construction was carried out after the change of variable $\tilde u = D^{1/2} u$ with new coefficients $D^{-1/2}\, D\, A_\mu\, D^{-1/2}$. As the formulae are somewhat simpler, we compute with the equations (11.1.1). The background state $\underline{u}$ is constant but non zero.

The method of §9.4 is to expand the symmetric hyperbolic expression

$$(D\,L)\Big(\underline{u} + \epsilon U^\epsilon(y,Y)\,,\; \frac{\partial}{\partial y} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y}\Big)\,\Big(\underline{u} + \epsilon\, U^\epsilon(y,Y)\Big)$$

in powers of ϵ and determine the profiles in the expansion of U^ϵ so that coefficients of each power of ϵ vanishes. Multiplying by D^{-1} shows that

$$L\Big(\underline{u} + \epsilon U^\epsilon(y,Y)\,,\; \frac{\partial}{\partial y} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y}\Big)\,\Big(\underline{u} + \epsilon\, U^\epsilon(y,Y)\Big) \;\sim\; \epsilon^\infty\,.$$

The operator on the left hand side is not symmetric. Expanding as in §9.4, the leading two terms yield

$$L(\underline{u}, \partial_Y)\, U_0(y,Y) \;=\; 0\,, \tag{11.1.5}$$

and,

$$L(\underline{u}, \partial_y)\, U_0 + \sum_\mu A'_\mu(\underline{u}) U_0\, \partial_{Y_\mu} U_0 + L(\underline{u}, \partial_Y)\, U_1 \;=\; 0 \tag{11.1.6}$$

On Fourier series,

$$L(\underline{u}, \partial_Y) \sum_\alpha a_\alpha(y)\, e^{i\alpha.Y} \;=\; i \sum_\alpha L(\underline{u}, \alpha)\, a_\alpha(y)\, e^{i\alpha.Y}\,.$$

The result of the next lemma is trivial in the symmetric case.

Lemma. *For $(\tau, \xi) \in \mathbb{R}^{1+d}$, $\mathbb{C}^N$ is the direct sum of the image and kernel of $L(\underline{u}, \tau, \xi)$. The the norms of the spectral projections $\pi(\tau,\xi)$ along the image onto the kernel are bounded independent of τ, ξ.*

Proof. Write $L(\tau,\xi) = D^{-1}\big(DL(\tau,\xi)\big)$. Since DL is hermitian and $D = D^* > 0$ it follows that L is hermitian in the scalar product

$$(u,v)_D \;:=\; (Du, v)\,.$$

In fact,

$$(Lu, v)_D := (DLu, v) = (u, DLv) = (Du, Lv) := (u, Lv)_D\,.$$

The image and kernel are orthogonal in this scalar product and therefore complementary.

The spectral projections are orthogonal with respect to the scalar product $(\ ,\ )_D$. They have norm equal to 1 in the corresponding matrix norm.

Since this norm is equivalent to the euclidean matrix norm, the $\pi(\tau,\xi)$ are uniformly bounded. ∎

As in the symmetric case, define the operator,

$$\mathbf{E}\Big(\sum a_\alpha(y)\, e^{i\alpha.Y}\Big) \;:=\; \sum \pi(\alpha)\, a_\alpha(y)\, e^{i\alpha.Y}\,.$$

The lemma shows that $\mathbf{E}$ is bounded on all $H^s([0,T]_t \times \mathbb{R}^d_x \times \mathbb{T}^{1+d}_Y)$. It projects onto the kernel of $L(\underline{u},\partial_Y)$, and

$$\mathbf{E}\, L(\underline{u},\partial_Y) \;=\; 0\,.$$

Thus (11.1.5) is equivalent to

$$\mathbf{E}\, U_0 \;=\; 0\,. \tag{11.1.7}$$

Multiplying (11.1.6) by $\mathbf{E}$ yields

$$\mathbf{E}\Big(L(\underline{u},\partial_y) \;+\; \sum_\mu A'_\mu(\underline{u})U_0\partial_{Y_\mu}\Big)U_0 \;=\; 0\,. \tag{11.1.8}$$

We use equations (11.1.7)-(11.1.8) for the principal profile. The advantage is that the coefficients of the non symmetric system L are simple.

The equations (1.17)-(1.18) have the exact same form as the equations derived for the symmetric operator DL, only the operator $\mathbf{E}$ has changed. Mulitplying the equations (1.17)-(1.18) by any operator whose restriction to $\ker L(\underline{u},\partial_Y)$ is equal to the identity, does not affect the equations. So, there are many equivalent versions.

§11.2. The dynamics of the Fourier coefficients of the leading profile.

For homogeneous oscillations, the equations for the leading profile, are equivalent to a system of coupled ordinary differential equations for its Fourier coefficients.

Consider profiles $U_0(t,x,Y)$ that are independent of x and $2\pi \times 2\pi \times 2\pi$ periodic in (Y_0,Y_1,Y_2). The approximate solution has the form

$$u^\epsilon(t,x) \;=\; u^\epsilon(t,x_1,x_2) \;\sim\; \epsilon\, U_0(t,t/\epsilon,x/\epsilon)\,.$$

The profile equations are

$$\mathbf{E}\, U_0 \;=\; U_0\,, \qquad \mathbf{E}\Big(\partial_t U_0 + \sum_{\mu=0}^{2} \big(A'_\mu(\underline{u})\, U_0\big)\,\frac{\partial U_0}{\partial Y_\mu}\Big)\,. \tag{11.2.1}$$

Define

$$B_\mu(V) \;:=\; \big(A'_\mu(\underline{u})\big)(V) \tag{11.2.2}$$

so that B_μ is a linear matrix valued function of the vector V. Since the leading coefficient of L is equal to the identity, $B_0 = 0$.

For convenience in denoting Fourier coefficients, suppress the subscript 0 and expand the leading profile,

$$U(t,Y) = U(t,Y_0,Y_1,Y_2) = \sum_{\alpha\in\mathbb{Z}\times\mathbb{Z}^2} U_\alpha(t)\, e^{i\alpha.Y}\,. \tag{11.2.3}$$

Inserting this in (11.2.1), the nonlinear term is equal to

$$\sum_\mu \Big(\sum_\alpha B_\mu(U_\alpha)\ e^{i\alpha.Y}\Big)\Big(\sum_\beta i\beta_\mu\ U_\beta\ e^{i\beta.Y}\Big)\,.$$

Setting the coefficient of $e^{i\gamma.Y}$ equal to zero yields

$$\frac{dU_\gamma}{dt}\ +\ \pi(\gamma)\sum_{\alpha+\beta=\gamma}\Big[\sum_{\mu=0}^{2} B_\mu(U_\alpha)(i\beta_\mu)U_\beta\Big]\ =\ 0\,. \tag{11.2.4}$$

The factor $\pi(\gamma)$ is from the operator $\mathbf{E}$.

When $\alpha+\beta=\gamma$, define

$$Q_\gamma^{\alpha\beta}(U_\alpha,U_\beta) := -\pi(\gamma)\Big[\sum_{\mu=0}^{2} B_\mu(U_\alpha)\,\beta_\mu\,U_\beta\Big]\,. \tag{11.2.5}$$

The map

$$U_\alpha, U_\beta \quad\longmapsto\quad Q_\gamma^{\alpha\beta}(U_\alpha,U_\beta)$$

defines a bilinear map

$$\ker(\pi(\alpha))\times\ker(\pi(\beta)) \quad\longmapsto\quad \ker(\pi(\gamma))\,.$$

In the important special case when the kernels are one dimensional, choose bases r_α and r_β for $\ker(\pi(\alpha))$ and $\ker(\pi(\beta))$ respectively. Choose the r_α to be homogeneous of degree zero in α. The bilinear form is then given by a scalar $\Gamma_\gamma^{\alpha\beta}$ according to,

$$Q_\gamma^{\alpha\beta}(r_\alpha,r_\beta)\ :=\ \Gamma_\gamma^{\alpha\beta}\,r_\gamma\,. \tag{11.2.6}$$

At the same time define scalar valued $a_\alpha(t)$ by

$$U_\alpha(t)\ =\ a_\alpha(t)\,r_\alpha\,. \tag{11.2.7}$$

In (11.2.4), U_α and U_β contribute to dU/dt from two terms corresponding to $\alpha+\beta=\gamma$ and $\beta+\alpha=\gamma$. Thus

$$\frac{da_\gamma(t)}{dt}\ =\ i\sum_{\alpha+\beta=\gamma}\Big(\Gamma_\gamma^{\alpha\beta}+\Gamma_\gamma^{\beta\alpha}\Big)\,a_\alpha(t)a_\beta(t)\,. \tag{11.2.8}$$

where the *interaction coefficient*, $\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha}$, is scalar valued, homogeneous of degree 1 in α, β, γ, and symmetric in α, β.

Formula 11.2.8 implies the following supplementary information to Theorem 9.1,

$$\forall\ t \in [0, T_*[\qquad \text{spec}\,(U_0(t,.,.))\ \subset\ \mathbb{Z} - \text{Span}\,\big(\text{spec}(U_0(0,.,.)\big)\,. \tag{11.2.9}$$

Here $\mathbb{Z} -$ Span means the $\mathbb{Z}$ module generated by. The information (11.2.9) is used in tandem with

$$\text{spec}\, U_0(t)\ \subset\ \text{Char}\, L(\underline{u}, \partial_Y)\,, \tag{11.2.10}$$

which follows from $\mathbf{E}U_0 = U_0$.

The main result of this chapter is the following. It asserts that three initial waves at $t = 0$ lead by nonlinear interaction to waves moving in a dense set of directions. It is an improvement of the main result of [JMR 1998].

Theorem 11.2.1. *There is a real smooth local in time solution $U_0(t, Y)$ of the leading profile equation which is $2\pi \times 2\pi \times 2\pi$ periodic in (Y_0, Y_1, Y_2) and satisfies*

$$U_\alpha(0) = 0 \ \text{ except for } \ \alpha \in \mathbb{Z}(1,1,0) \cup \mathbb{Z}(0,1,0) \cup \mathbb{Z}(0,3,4)\,,$$

and

$$\left\{ \frac{(\alpha_1, \alpha_2)}{\|(\alpha_2, \alpha_3)\|}\ :\ \exists (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3 \quad \text{s.t.} \quad \frac{d^2 U_\alpha(0)}{dt^2} \neq 0 \right\} \tag{11.2.11}$$

is dense in the unit circle. More precisely, S denotes the unit circle with its the four axis intercepts removed, then there is a discrete subset $D \subset S$ so that (11.2.11) contains all the rational points of $S \setminus D$.

The first assertion shows that at $\{t = 0\}$ there are only three oscillating wavetrains and the second asserts that in the future there are waves traveling in directions dense in the unit circle. In [JMR 1994] (which followed the article cited above in spite of the dates) it is shown that the $U_\alpha(t)$ are real analytic in time. The U_α with $d^2U_\alpha/dt^2 \neq 0$, therefore vanish at most at a discrete set of points. Thus the dense set of wavetrains are simultaneously illuminated with at most a countable set of exceptional times.

§11.3. Linear oscillations.

Consider background states with $\underline{v} = 0$. Thanks to Gallilean invariance, this not an essential restriction. Equation (11.1.3) simplifies to $\tau(\tau - |\xi|^2)$, and

$$\text{Char}\,(L(0, \partial)\ =\ \{\tau = 0\} \cup \{\tau^2 = |\xi|^2\}\,,$$

is the union of a horizontal plane and a light cone as for Maxwell's equations. In contrast to the case of electrodynamics, all the sheets of the characteristic variety are physical whereas for Maxwell, the divergence free constraints eliminate the plane.

Convention. *For the rest of the chapter the background state is $\underline{v} = 0$ and is suppressed when indicating the symbol $L(0, \partial)$ as $L(\partial)$.*

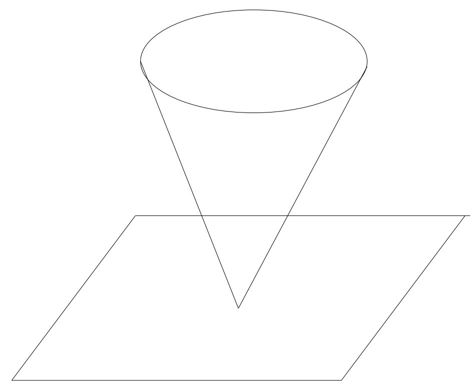

Figure 11.1. The characteristic variety of linearized Euler

Proposition 11.3.1. *The small divisor hypothesis of §9.5 is satisfied.*

Proof. Compute

$$\det\Big(L(\tau,\xi)-\sigma I\Big) \;=\; (\tau-\sigma)\,\big((\tau-\sigma)^2-|\xi|^2\big)\,.$$

For,

$$(\tau,\xi) \;=\; (n_0,n_1,n_2) \quad \text{with} \quad n\in\mathbb{Z}^3,$$

one has

$$\det\Big(L(\tau,\xi)-\sigma I\Big) \;=\; (n_0-\sigma)\,\big((n_0-\sigma)^2-n_1^2-n_2^2\big)\,.$$

If σ is a nonzero eigenvalue, then

$$\sigma=n_0\in\mathbb{Z}\setminus 0, \qquad \text{or,} \qquad (n_0-\sigma)^2=n_1^2+n_2^2\,.$$

In the first case one has the lower bound, $|\sigma|\ge 1$.
In the second case,

$$2n_0\sigma-\sigma^2 \;=\; n_0^2-n_1^2-n_2^2\,.$$

There is a further dichotomy. If $n_0=0$, then $|\sigma|=|n|\ge 1$. If $n_0\ne 0$, we prove that

$$\sigma \;\ge\; \frac{1}{4|n_0|}\,.$$

If not, one would have $|2\sigma n_0|<1/2$. Then,

$$\sigma<1/2, \qquad \text{so,} \qquad \sigma^2<1/4, \qquad \text{so} \qquad |2n_0\sigma-\sigma^2|<3/4\,.$$

On the other hand, $2n_0\sigma-\sigma^2\in\mathbb{Z}$, so $2n_0\sigma-\sigma^2=0$. Since $\sigma\ne 0$, one has $|\sigma|=2|n_0|\ge 2$, contradicting the assumed estimate $\sigma<1/4|n_0|$. ∎

Theorems 9.5.2, and 9.5.3 then yield approximate solutions with infinitely small residual, and Theorem 9.6.1 shows that these solutions are infinitely close to the exact solutions with the same initial data. We analyse the resonance relations and the profile equations in detail in order to prove Theorem 11.2.1.

It is sometimes confusing that $t, x,\ \tau, \xi,\ v, \rho$, and dual vectors to the v, ρ space, are all objects with three components. To maintain some distinction we use round brackets (t, x), (τ, ξ), for the first two and Dirac brackets $|v, \rho\rangle$ for the third. Dual vectors to the v, ρ are denoted, following Dirac, with reversed brackets $\langle . , . |$. The pairings of $\mathbb{R}^2_x$ and $\mathbb{R}^2_\xi$ and of $\mathbb{R}^3_{t,x}$ and $\mathbb{R}^3_{\tau,\xi}$ are indicated with a period, e.g. $(t, x).(\tau, \xi)$. The real pairing $\langle\ |\ \rangle$ is also the standard real euclidean pairing.

To study the profiles, we compute $\ker L(\tau, \xi)$ and the spectral projection $\pi(\tau, \xi)$ for each $(\tau, \xi) \in \mathrm{Char}\,(L)$. Since $L(\tau, \xi)$ is homogeneous of degree one with $\det L(1, 0, 0) \neq 0$, it suffices to consider $|\xi| = 1$. For ξ fixed there are three points in the characteristic variety, $\tau = 0$ and $\tau = \pm|\xi|$.

For $\tau = 0$ one has

$$L(0, \xi) = \begin{pmatrix} 0 & 0 & \underline{f}\xi_1 \\ 0 & 0 & \underline{f}\xi_2 \\ \underline{\rho}\xi_1 & \underline{\rho}\xi_2 & 0 \end{pmatrix}$$

$$\ker\,(L(0, \xi)) = \mathbb{R}\,| - \xi_2, -\xi_1, 0\rangle \tag{11.3.1}$$

$$\mathrm{range}\,(L(0, \xi)) = \mathrm{Span}\,\big\{|\xi_1, \xi_2, 0\rangle, |0, 0, 1\rangle\big\}. \tag{11.3.2}$$

Note that the range is orthogonal to the kernel so the spectral projection is the orthogonal projection

$$\pi(0, \xi)\ :=\ \Big|\xi_2, -\xi_1, 0\Big\rangle\Big\langle\xi_2, -\xi_1, 0\Big|. \tag{11.3.3}$$

Similarly

$$L(\pm 1, \xi) = \begin{pmatrix} \pm 1 & 0 & \underline{f}\xi_1 \\ 0 & \pm 1 & \underline{f}\xi_2 \\ \underline{\rho}\xi_1 & \underline{\rho}\xi_2 & \pm 1 \end{pmatrix}$$

$$\ker\,\big(L(\pm 1, \xi)\big)\ =\ \mathbb{R}\,|\xi_1, \xi_2, \mp 1/\underline{f}\rangle$$

$$\mathrm{range}\,\big(L(\pm 1, \xi)\big) = \mathrm{Span}\,\big\{| \pm 1, 0, \underline{\rho}\xi_1\rangle, |0, \pm 1, \underline{\rho}\xi_2\rangle\big\}$$

$$\pi(\pm 1, \xi)\ =\ \frac{1}{2}\,\Big|\xi_1, \xi_2, \mp 1/\underline{f}\Big\rangle\Big\langle\xi_1, \xi_2, \mp 1/\underline{\rho}\Big|\,.$$

These computations yield the plane wave solutions of the linearized equation;

$$|\xi_2, -\xi_1, 0\rangle\, F(\xi.x), \qquad |\xi_1, \xi_2, \mp 1/\underline{f}\rangle\, F(\pm|\xi|t + \xi.x)$$

where F is an arbitrary real valued function of one variable.

The first family of waves satisfy $\mathrm{div}\, v = 0$, $|\mathrm{curl}\, v| \sim |\xi|^2$, and has no variation in density. They are standing waves. Given the background state with velocity zero this means that they are convected with the background fluid velocity. They are called *vorticity waves.*

Waves of the second family have $\mathrm{curl}\, v = 0$ and $|\mathrm{div}\, v| \sim |\xi|^2$. The group velocity associated to $\tau = \pm|\xi|$ is equal to $-\nabla_\xi(\pm|\xi|) = \mp\xi/|\xi|$. These solutions are called are called *acoustic waves* or *compression waves.* They can move in any direction with speed one.

This prediction of the speed of sound, $c = p'(\rho)^{1/2}$, from the static measurement of $p(\rho)$ is an early success of continuum mechanics. It is also a model of what is found in science text analyses of a nonlinear hyperbolic model. That is, linearization at constant states, and computation of plane waves solutions and group velocities for the resulting constant coefficient equation governing small perturbations.

The solution of the linear oscillatory initial value problem

$$L(\partial_t, \partial_x) w = 0\,, \qquad w(0,x) = g\, e^{i\zeta . x}\,, \qquad g \in \mathbb{C}^3\,, \quad \zeta \in \mathbb{R}^2\,.$$

is equal to,

$$w \ = \ \big(\pi(0, \zeta/|\zeta|) g\big)\, e^{i\zeta . x} \ + \ \sum_{\pm} \big(\pi(\pm c, \zeta/|\zeta|) g\big) e^{\pm|\zeta| t + \zeta . x}\,.$$

§11.4. Resonance Relations.

Quadratic nonlinear interaction of oscillations $r_\alpha e^{i\alpha\cdot(t,x)}$ and $r_\beta e^{i\beta . (t,x)}$ with α and β belonging to Char (L) produce terms in $e^{i(\alpha+\beta).(t,x)}$ which will propagate as soon as $\alpha + \beta$ is characteristic.

Definition. A *(quadratic) resonance* is a linear relation $\alpha + \beta + \eta = 0$ between three nonzero elements of Char (L).

These are sometimes called resonances of order 3, as they involve three points of the characteristic variety. The corresponding interactions are called *three wave interactions*. The simplest resonances, always present for homogeneous systems L, are when α, β, η are colinear.

For semilinear problems one must consider linear relations among any number of characteristic covectors. Treating small amplitude oscillations for quasilinear problems, yields only quadratic nonlinearities in the equations for the profiles and thereby permits us to consider only triples.

Theorem 11.4.1. *Quadratic resonances for the Euler equations fall into three families:*

i. Colinear vectors satisfying $\tau^2 = |\xi|^2$.

ii. Triples α, β, η *which belong to* $\{\tau = 0\}$.

iii. Relations equivalent by $\mathbb{R}$*-dilation,* x*-rotation,* x*-reflection and permutation of the three covectors to*

$$(\pm 1, \alpha_1, \alpha_2) + (0, 0, -2\alpha_2) + (\mp 1, -\alpha_1, \alpha_2) \ = \ 0, \qquad \alpha_1^2 + \alpha_2^2 = 1, \quad \alpha_1 \geq 0\,.$$

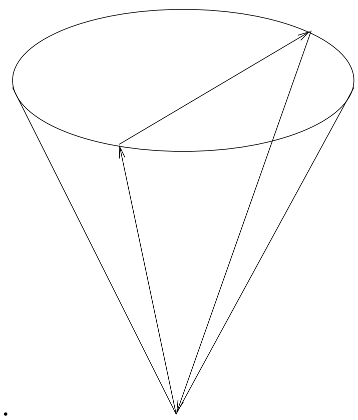

Figure 11.2 Resonance of type iii.

Proof. Seek α, β, η in $\{\tau(\tau^2 - |\xi|^2) = 0\}$ whose sum is zero. The classification above depends on counting how many of the α, β, η belong to $\{\tau = 0\}$.

If all three belong, it is case ii.

If exactly two belong, the relation $\alpha + \beta + \eta = 0$ is impossible since the τ component of the sum will equal the τ component of the covector which does not lie in $\tau = 0$.

If exactly one belongs, rotate axes so that the covector in $\tau = 0$ is parallel to $(0, 0, 1)$. The relation then is a multiple of case iii. except possibly for the sign of α_1 which can be adjusted by a reflection in x_1.

If none of the covectors belong to $\tau = 0$ we must show that the only possibility is colinear resonance.

A rotation followed by multiplication by a nonzero real reduces to the case $\alpha = (1, 1, 0)$.

Seek $\beta \in \operatorname{Char}(L)$ such that $\alpha + \beta \in \operatorname{Char}(L)$. If the τ component of β is positive then $\alpha + \beta$ belongs to the interior of the positive light cone unless α and β are colinear.

Thus it suffices to look for $\beta = (-|\xi|, \xi)$ with

$$(1, 1, 0) + (-|\xi|, \xi) \in \left\{\tau^2 = |\xi|^2\right\}.$$

This holds if and only if

$$(1 - |\xi|)^2 = [(1 + \xi_1)^2 + \xi_2^2].$$

Canceling common terms shows that this holds if and only if $-2|\xi| = 2\xi_1$, so we must have $\xi_2 = 0$ and $\xi_1 < 0$. Thus $(1, 1, 0)$ and $(-|\xi|, \xi)$ are colinear. ∎

§11.5. Interaction Coefficients.

To define interaction coefficients, basis vectors $r_{(\tau,\xi)}$ for $\ker L(\tau, , \xi)$ with $(\tau, \xi) \in \operatorname{Char} L$ are needed. Choose vectors which are the extensions of formulas (11.3.1), (11.3.2) homogeneous of degree zero,

$$r_{(0,\xi)} := |\xi|^{-1} \Big|\xi_2, -\xi_1, 0\Big\rangle \tag{11.5.1}$$

$$r_{(\pm c|\xi|,\xi)} := |\xi|^{-1} \Big|\xi_1, \xi_2, \mp|\xi|\underline{\rho}\Big\rangle. \tag{11.5.2}$$

Theorem 11.5.1. *Suppose that α, β, and $\alpha+\beta := \gamma$ are nonzero elements of* $\mathrm{Char}\, L$.
i. *If α and β are colinear elements of $\{\tau^2 = |\xi|^2\}$ with $\beta = a\alpha$, $a \neq 0, -1$, then the interaction coefficient $\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha}$ is given by*

$$\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} = \mathrm{sgn}(a)\,(\gamma_1^2+\gamma_2^2)^{1/2}\,(3 + h\underline{\rho}^2)/2\,. \tag{11.5.3}$$

where h is defined in (11.5.6).
ii. *If α and β belong to $\{\tau = 0\}$, the interaction coefficient is given by*

$$\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} = -(1/2)|\beta-\alpha|\,\sin\big(\angle(\alpha,\beta)\big)\,\cos\big(\angle(\alpha+\beta,\alpha-\beta)\big) \tag{11.5.4}$$

where $\angle(\alpha,\beta) \in \mathbb{R}/2\pi\mathbb{Z}$ denotes the angle between α and β measured in the positive sense.
iii. *If $\alpha = (\pm 1, \alpha_1, \alpha_2)$, $\alpha_1 > 0$, and $\beta = (0, 0, -2\alpha_2)$ as in Theorem 11.2.***iii***, then the interaction coefficient is given by*

$$\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} = \cos(\phi/2)\,\cos(\phi)\,\mathrm{sgn}(-\alpha_2)\,(\gamma_1^2+\gamma_2^2)^{1/2}, \quad \phi := \angle\big((\gamma_1,\gamma_2),(\alpha_1,\alpha_2)\big). \tag{11.5.5}$$

iiibis. *If $\alpha = (\pm 1, \alpha_1, \alpha_2)$, $\beta = (\mp 1, -\alpha_1, \alpha_2)$, $\gamma = (0, 0, 2\alpha_2)$ then the interaction coefficient $\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha}$ vanishes.*

The exceptional case, **iiibis**, asserts that for the creation of a vorticity wave by the interaction of two acoustic waves, the interaction coefficient vanishes.

Proof. First compute the matrices $B_j = D_u A_j(\underline{u})$. Define the constant h by

$$h := \frac{d}{d\rho}\left(\frac{p'(\rho)}{\rho}\right)\Bigg|_{\rho=\underline{\rho}}\,. \tag{11.5.6}$$

From (11.1.4) one finds

$$B_1\big(|\delta v_1, \delta v_2, \delta\rho\rangle\big) = \begin{pmatrix} \delta v_1 & 0 & h\delta\rho \\ 0 & \delta v_1 & 0 \\ \delta\rho & 0 & \delta v_1 \end{pmatrix} \tag{11.5.7}$$

$$B_2\big(|\delta v_1, \delta v_2, \delta\rho\rangle\big) = \begin{pmatrix} \delta v_2 & 0 & 0 \\ 0 & \delta v_2 & h\delta\rho \\ 0 & \delta\rho & \delta v_2 \end{pmatrix} \tag{11.5.8}$$

Case ii. When $\alpha = (0, \alpha_1, \alpha_2)$ and $\beta = (0, \beta_1, \beta_2)$ belong to $\{\tau = 0\}$, (11.5.1) shows that $\delta\rho = 0$ for r_α and r_β so

$$B_1(r_\alpha) = |\alpha|^{-1}\,\alpha_2\, I, \qquad B_2(r_\alpha) = -|\alpha|^{-1}\alpha_1 I\,,$$

and (11.3.3) yields $\pi(\gamma) = |r_\gamma\rangle\langle r_\gamma|$. Thus

$$\begin{aligned}\Gamma_\gamma^{\alpha\beta} =& \langle r_\gamma|[\beta_1 B_1(r_\alpha) + \beta_2 B_2(r_\alpha)] r_\beta\rangle = |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)\langle r_{\alpha+\beta}|r_\beta\rangle \\ =& |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1} < \alpha_2+\beta_2, -\alpha_1-\beta_1, 0\,\big|\,\beta_2, -\beta_1, 0\rangle\,.\end{aligned}$$

The last duality is equal to the scalar product $\langle \alpha+\beta|\beta\rangle$, so

$$\Gamma_\gamma^{\alpha\beta} = |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1}\,\langle\alpha+\beta|\beta\rangle\,.$$

Interchanging the role of α and β and summing yields

$$\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} = |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1}\,\langle\alpha+\beta|\beta-\alpha\rangle\,. \tag{11.5.9}$$

Formula (11.5.4) follows since

$$\begin{aligned} |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\beta|^{-1} &= -\sin\big(\angle(\alpha,\beta)\big) \\ |\alpha+\beta|^{-1}|\beta-\alpha|^{-1}\,\langle\alpha+\beta|\beta-\alpha\rangle &= \cos\big(\angle(\alpha+\beta,\beta-\alpha)\big)\,. \end{aligned}$$

Case i. By homogeneity and Euclidean invariance it suffices to compute the case

$$\alpha = (\pm1,1,0), \qquad \beta = a\alpha, \qquad a\in\mathbb{R}\setminus 0\,.$$

Then r_α, r_β, and $\pi(\gamma)$ are given by

$$r_\alpha = |1,0,\mp\underline{\rho}\rangle\,, \qquad r_\beta = \mathrm{sgn}(a) r_\alpha\,, \qquad \pi(\gamma) = \frac{1}{2}\,r_\alpha\langle 1,0,\mp1/\underline{\rho}|\,.$$

Since $\beta_2 = 0$, one has

$$\begin{aligned} \Gamma_\gamma^{\alpha\beta} r_\gamma &= \pi(\gamma)\Big[\beta_1 B_1(r_\alpha) r_\beta\Big] = a\,\mathrm{sgn}\,(a)\,\pi(\gamma)\begin{pmatrix} 1 & 0 & \mp h\underline{\rho} \\ 0 & 1 & 0 \\ \mp\underline{\rho} & 0 & 1\end{pmatrix} \\ &= |a|\,\frac{r_\alpha}{2}\,\Big\langle 1\,,\,0\,,\,\mp1/\underline{\rho}\ \Big|\ 1+h\underline{\rho}^2\,,\,0\,,\,\mp2\underline{\rho}\Big\rangle \;=\; |a|\,\big(3+h\underline{\rho}^2\big)\,\frac{r_\alpha}{2}\,. \end{aligned}$$

Now $r_\alpha = \pm r_\gamma$ the sign depending on whether $\gamma = (1+a)\alpha$ is a positive or negative multiple of α, that is by $\mathrm{sgn}\,(1+a)$. Thus

$$\Gamma_\gamma^{\alpha\beta} = \mathrm{sgn}(1+a)\,|a|\,(3+h\underline{\rho}^2)/2.$$

By homogeniety the case of general α is given by

$$\begin{aligned} \Gamma_\gamma^{\alpha\beta} =& (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a)\,|a|\,(3+h\underline{\rho}^2)/2 \\ =& (\beta_1^2+\beta_2^2)^{1/2}\,\mathrm{sgn}(1+a)\,(3+h\underline{\rho}^2)/2\,, \end{aligned}$$

since $|a|\|\alpha_1,\alpha_2\| = \|\beta_1,\beta_2\|$. Noting that $\alpha = a^{-1}\beta$, the reversed coefficient is given by

$$\Gamma_\gamma^{\beta\alpha} = (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a^{-1})\,(3+h\underline{\rho}^2)/2\,.$$

Adding yields

$$\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} = \Big[(\beta_1^2+\beta_2^2)^{1/2}\,\mathrm{sgn}(1+a) + (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a^{-1})\Big](3+h\underline{\rho}^2)/2\,.$$

In the three cases $a>0$, $-1<a<0$, and $a<-1$, the factor in square bracket is given by

$$\|\beta\|+\|\alpha\|\,,\qquad \|\beta\|-\|\alpha\|\,,\qquad \text{and,}\qquad -\|\beta\|+\|\alpha\|\,,$$

respectively. In all cases this is equal to $\mathrm{sgn}\,(a)\|\gamma\|$ which proves (11.5.3).

Case iii. It is sufficient to consider α with $\alpha_1^2+\alpha_2^2=1$ and $a>0$. Then

$$\alpha=(\pm1,\alpha_1,\alpha_2)\,,\quad \beta=(0,0,-2\alpha_2)\,,\quad \gamma=(\pm1,\alpha_1,-\alpha_2)\,,\quad r_\alpha=|\alpha_1,\alpha_2,\mp\underline{\rho}\rangle,$$
$$r_\beta=|\mathrm{sgn}\,(-\alpha_2),0,0\rangle\,,\quad r_\gamma=|\alpha_1,-\alpha_2,\mp\underline{\rho}\rangle\,,\quad \pi(\gamma)=\frac{r_\gamma}{2}\,\langle\alpha_1,-\alpha_2,\pm1/\underline{\rho}|\,.$$

Since $\beta_1=0$,

$$\Gamma_\gamma^{\alpha\beta} r_\gamma = \pi(\gamma)\,\Big[\beta_2 B_2(r_\alpha)r_\beta\Big] = \pi(\gamma)\Bigg[-2\alpha_2\begin{pmatrix}\alpha_2 & 0 & 0\\ 0 & \alpha_2 & \mp h\underline{\rho}\\ 0 & \mp\underline{\rho} & \alpha_2\end{pmatrix}\begin{pmatrix}\mathrm{sgn}\,(-\alpha_2)\\ 0\\ 0\end{pmatrix}\Bigg]$$
$$= \frac{r_\gamma}{2}\Big\langle\alpha_1\,,\,-\alpha_2\,,\,\mp1/\underline{\rho}\;\Big|\;-2\alpha_2^2\,\mathrm{sgn}(-\alpha_2)\,,\,0\,,\,0\Big\rangle\,.$$

Therefore, $\Gamma_\gamma^{\alpha\beta}=-\alpha_1\alpha_2^2\,\mathrm{sgn}(-a)$.

To compute the coefficient $\Gamma_\gamma^{\beta\alpha}$ note first that $B_2(r_\beta)=0$ and $B_1(r_\beta)=\mathrm{sgn}\,(-\alpha_2)I$, since for r_β , $\delta v_2=\delta\rho=0$ and $\delta v_1=\mathrm{sgn}\,(-\alpha_2)$. Therefore

$$\Gamma_\gamma^{\beta\alpha} r_\gamma = \pi(\gamma)\Big[\alpha_1 B_1(r_\beta)r_\alpha\Big] = \pi(\gamma)\Big[\alpha_1\mathrm{sgn}\,(-\alpha_2)r_\alpha\Big]$$
$$=\frac{\alpha_1}{2}\,\mathrm{sgn}\,(-\alpha_2)\,r_\gamma\,\Big\langle\alpha_1\,,\,-\alpha_2\,,\,\mp1/\underline{\rho}\;\Big|\;\alpha_1\,,\,\alpha_2\,,\,\mp\underline{\rho}\Big\rangle.$$

Therefore,

$$\Gamma_\gamma^{\beta\alpha} = \frac{\alpha_1}{2}\mathrm{sgn}(-\alpha_2)\,(\alpha_1^2-\alpha_2^2+1) = \alpha_1\,\mathrm{sgn}\,(-\alpha_2)\,\alpha_1^2\,.$$

Adding the previous results yields

$$\Gamma_\gamma^{\alpha\beta}+\Gamma_\gamma^{\beta\alpha} = \alpha_1\,(\alpha_1^2-\alpha_2^2)\,\mathrm{sgn}(-\alpha_2) = \cos(\phi/2)\,\big(\cos^2(\phi/2)-\sin^2(\phi/2)\big)\,\mathrm{sgn}(-\alpha_2),$$

since $(\alpha_1,\alpha_2)=(\cos(\phi/2),\sin(\phi/2))$. Formula (11.5.5) for the case $\alpha_1^2+\alpha_2^2=1$ follows. The general case follows by homogeneity.

Case iiibis. When $|\alpha_1,\alpha_2|=1$, one has

$$r_\alpha=|\alpha_1,\alpha_2,\mp\underline{\rho}\rangle\,,\qquad r_\beta=|-\alpha_1,\alpha_2,\mp\underline{\rho}\rangle\,,$$

$$r_\gamma=\mathrm{sgn}(\alpha_2)\,|1,0,0\rangle\,,\qquad \pi(\gamma)=|1,0,0\rangle\langle1,0,0|\,.$$

By definition, $\mathrm{sgn}(\alpha_2)\,\Gamma^{\alpha\beta}_\gamma$ is equal to the first component of the vector

$$\left[-\alpha_1\begin{pmatrix}\alpha_1 & 0 & \mp h\underline{\rho}\\ 0 & \alpha_1 & 0\\ \mp\underline{\rho} & 0 & \alpha_1\end{pmatrix}+\alpha_2\begin{pmatrix}\alpha_2 & 0 & 0\\ 0 & \alpha_2 & \mp h\,\underline{\rho}\\ 0 & \mp\underline{\rho} & \alpha_2\end{pmatrix}\right]\begin{pmatrix}-\alpha_1\\ \alpha_2\\ \mp\underline{\rho}\end{pmatrix}.$$

Therefore

$$\Gamma^{\alpha\beta}_\gamma = -\alpha_1\,\mathrm{sgn}(\alpha_2)\,\big(-\alpha_1^2+\alpha_2^2+h\underline{\rho}\big)\,.$$

To compute the coefficient $\Gamma^{\beta\alpha}_\gamma$ it suffices to change the sign of α_1 in the above computation which simply changes the sign of the result. Thus $\Gamma^{\beta\alpha}_\gamma=-\Gamma^{\alpha\beta}_\gamma$ which proves **iiibis**. ∎

§11.6. Dense oscillations for the Euler equations.

§11.6.1. The algebraic/geometric part.

The characteristic variety is given by the equation

$$\tau\big(\tau^2-(\xi_1^2+\xi_2^2)\,\big)=0.$$

Definition. In the τ,ξ space denote by Λ the lattice of integer linear combinations of the characteristic covectors

$$(1,1,0)\,,\quad (0,1,0)\,,\quad \text{and}\quad (0,3,4)\,. \tag{11.6.1}$$

The next result improves substantially the corresonding result of JMR 1998] with completely new proof.

Theorem 11.6.1. *For every rational r there is a point*

$$(\tau,\xi)\ \in\ \Lambda\cap\{\tau=|\xi|\}\,,$$

with $\xi_1/\xi_2=r$.

Proof. The integer linear combinations are

$$(\tau,\xi)\ =\ n_1(1,1,0)+n_2(0,1,0)+n_3(0,3,4)\ =\ \Big(n_1\,,\,n_1+n_2+3n_3\,,\,4n_3\Big)\,. \tag{11.6.2}$$

This (τ,ξ) belongs to $\Lambda\cap\{\tau=|\xi|\}$ if and only if

$$n_1^2\ =\ (n_1+n_2+3n_3)^2+(4n_3)^2,\qquad \text{and},\qquad n_1>0\,.$$

Dividing by n_1^2 and setting

$$q_2\ :=\ \frac{n_2}{n_1},\qquad q_3\ :=\ \frac{n_3}{n_1}\,, \tag{11.6.3}$$

we seek rational q_j satsifying

$$1 \ = \ (1+q_2+3q_3)^2 + (4q_3)^2 \,. \tag{11.6.4}$$

For any rational r, the line $q_2 = rq_3$ intersects the ellipse when

$$1 = (1+(r+3)q_3)^2 + (4q_3)^2, \quad \text{equivalently,} \quad q_3\Big(\big(4^2+(r+3)^3\big)q_3 + 2(r+3)\Big) \ = \ 0\,.$$

Where the linear factor vanishes gives a solution.

Mulitplying by the greatest common multiple of the denominators of the q_j gives an integer solution n. Multiplying by ± 1 gives the desired solution with $n_1 > 0$. ∎

§11.6.2. Construction of the profiles.

We construct a solution of the profile equation (11.2.1) satisfying the conditions at the end of Theorem 11.2.1.

Introduce $g \in C^\infty(S^1)$ all of whose Fourier coefficients are nonzero, for example

$$g(\theta) := \sum_{n\in\mathbb{Z}} g_n \, e^{in\theta}\,, \tag{11.6.5}$$

with $g_n \neq 0$ rapidly decreasing in n.

Let $U^0(T, X_1, X_2)$ be the solution of the linearized Euler equation given by

$$\begin{aligned} U^0(T,X) \ &:= |1,0,-1/\underline{f}\rangle \, g\big((1,1,0).(T,X_1,X_2)\big) \\ &\quad + |0,1,0\rangle \, g\big((0,1,0).(T,X_1,X_2)\big) + |4,-3,0\rangle \, g\big((0,3,4).(T,X_1,X_2)\big)\,. \end{aligned}$$

The spectrum of $U^0(T,X)$ is exactly equal to the integer multiples of the covectors in (11.6.1).

Theorem 9.1 implies that there is a unique local solution $U = \sum U_\alpha(t) e^{i\alpha.(T,X)}$ of the profile equations (11.2.1) (equivalently 11.2.8) for the Euler equations with initial value $U(0,T,X) = U^0(T,X)$. In addition (11.2.9) and (11.2.10) show that $\operatorname{spec} U(t)$ is contained in the set of charactersitic points of Λ. In particular $U(t)$ is $2\pi \times 2\pi \times 2\pi$ periodic in T, X_1, X_2 . As in §11.1, define $a_\alpha(t)$ by $U_\alpha(t) = a_\alpha(t) r_\alpha$.

The next result is stronger than that proved in [JMR 1998]. The new proof also smooths a rough part of the earlier demonstration.

Theorem 11.6.2. *There is a discrete set E of $\mathbb{Q}\setminus 0$ so that if $(\tau,\xi) \in \Lambda \cap \{\tau = |\xi|\}$ with $\xi_2/\xi_3 \in \mathbb{Q}\setminus\{0 \cup E\}$ then,*

$$\frac{d^2 U_{(\tau,\xi)}(0)}{dt^2} \neq 0\,, \qquad U_{(\tau,\xi)}(0) = \frac{dU_{(\tau,\xi)}(0)}{dt} = 0\,.$$

Proof. First derive a compact formula for d^2a_δ/dt^2. Define

$$K(\alpha,\beta) \ := \ i\Big(\Gamma^{\alpha\beta}_\gamma + \Gamma^{\beta\alpha}_\gamma\Big)\,.$$

Since $\gamma=\alpha+\beta$ this is in fact a function of (α,β). K is homogeneous of degree one and symmetric. The dynamics is given by

$$\frac{da_\delta}{dt} \ = \ \sum_{\alpha+\beta=\delta} K(\alpha,\beta)\, a_\alpha\, a_\beta\,.$$

Compute,

$$\begin{aligned}\frac{d^2a_\delta}{dt^2} \ &= \ \sum_{\alpha+\beta=\delta} K(\alpha,\beta)\big((\partial_t a_\alpha)a_\beta + a_\alpha\partial_t a_\beta\big)\\ &= \ \sum_{\alpha+\beta=\delta} K(\alpha,\beta)\Big(a_\beta \sum_{\mu+\nu=\alpha} K(\mu,\nu)a_\mu\, a_\nu\big) + a_\alpha \sum_{\kappa+\lambda=\beta} K(\kappa,\lambda)a_\kappa\, a_\lambda\Big)\end{aligned}$$

Therefore

$$\frac{d^2a_\delta}{dt^2} \ = \ \sum_{\alpha+\mu+\nu=\delta} J(\alpha,\mu,\nu)\, a_\alpha\, a_\mu\, a_\nu\,,$$

where $J(\alpha,\mu,\nu)$ is homogeneous of degree two and symmetric.

Consider $\delta \ = \ (\tau,\xi)$ as in (11.6.2) with $n_1n_2n_3\ne 0$. This is the unique, up to permutation, of δ as a sum of three vectors chosen from the spectrum of $U(0,Y)=U^0(Y)$. Compute the second time derivative at $t=0$. Up to permutation the only nonvanishing triple product whose indices sum to δ is is

$$a_{n_1(1,1,0)}\, a_{n_2(0,1,0)}\, a_{n_3(0,3,4)}\,.$$

The interaction coefficient

$$J\Big(n_1(1,1,0)\,,\, n_2(0,1,0)\,,\, n_3(0,3,4)\Big) \ = \ n_1^2\, J\Big((1,1,0)\,,\, q_2(0,1,0)\,,\, q_3(0,3,4)\Big) \qquad (11.6.6)$$

with q_j as in (11.6.3). To complete the proof of the theorem it suffices to show that (11.6.6) is nonvanishing with at most a discrete set of exceptions (q_2,q_3) on the ellipse (11.6.4) minus its axis intercepts.

Let

$$\alpha := (1,1,0), \qquad \beta := q_2(0,1,0), \qquad \text{and}, \qquad \gamma := q_3(0,3,4)\,.$$

Up to a nonvanishing combinatorial factor the interaction coefficient is the sum of three terms.

$$K(\alpha,\beta+\gamma)K(\beta,\gamma) \ + \ K(\beta,\gamma+\alpha)K(\gamma,\alpha) \ + \ K(\gamma,\alpha+\beta)K(\alpha,\beta)\,.$$

But,

$$K\big(\beta\,,\,\gamma\big)\ =\ 0\,.$$

With a discrete set of exceptions we must show that

$$\begin{aligned}K\Big((1,1,0)\,,q_2(0,1,0)+q_3(0,3,4)\Big)\,K\Big(q_2(0,1,0)\,,\,q_3(0,3,4)\Big)\\ +\ K\Big(q_2(0,1,0)\,,\,(1,1,0)+q_3(0,3,4)\Big)\,K\Big((1,1,0)\,,\,q_3(0,3,4)\Big)\end{aligned} \tag{11.6.7}$$

is nonvanishing.

The formulas for the interaction coefficients from Theorem 11.5.1 are real analytic functions with a few exceptional points. For example for case **i.** the exceptional case is $\gamma_1^2+\gamma_2^2=0$. This is ruled out by $q_3\neq 0$. For case **ii.** the exceptional cases are when (in the notation of Theorem 11.5.1) one of the vectors $\alpha,\ \beta,\ \alpha+\beta,\ \alpha-\beta$ vanishes. These are all ruled out by $n_1n_2n_3\neq 0$. There are no exceptions for case **iii.** Therefore (11.6.7) is real analytic on the ellipse (11.6.4) with the possible exception of the intersections with the axes.

Therefore on each component of the complement of the axes, (11.6.7) is either identically zero or has discrete zeros. Thus, it suffices to find a point (q_2,q_3) on each connected component of the ellipse minus its axis intercepts, so that (11.6.7) is non vanishing at (q_2,q_3).

When $\|q_2(0,1,0)=\|q_3(0,3,4)\|$,

$$q_2(0,1,0)+q_3(0,3,4)\quad\perp\quad q_2(0,1,0)-q_3(0,3,4),$$

so the formula (11.5.4) implies that

$$K\big(q_2(0,1,0)\,,\,q_3(0,3,4)\big)\ =\ 0\,.$$

For these values the other summand in (11.6.7) is the product of two factors each the interaction coefficient of a point on $\tau=|\xi|$ with a point of $\tau=0$. Formula (11.5.5) shows that such a coefficient vanishes only when ξ parts of the two characteristic covectors are opposite. One of the covectors has third component equal to zero and other has third component equal to $4q_3\neq 0$. They cannot be opposite. Therefore, (11.6.7) $\neq 0$ at such points. The constraint of equal length is equivalent to $q_2=\pm 5q_3$.

Next study the ellipse (11.6.4). It passes through the origin. At the origin,

$$\nabla_{q_2,q_3}\Big((1+q_2+3q_3)^2\ +\ (4q_3)^2\Big)\ =\ \Big(2(1+q_2+3q_3)\,,\,8q_3\Big)\ =\ (2\,,\,0)\,.$$

Therefore, the ellipse is tangent to the axis $q_2=0$ at that point. Since the ellipse is strictly convex it lies on one of the two halfspaces $\pm q_2\geq 0$. The ellipse meets the $q_3=0$ axis at the points $q_2=0$ and $q_2=-2$. Thus the ellipse lies in $q_2\leq 0$ touching $q_2=0$ uniquely at the origin. There are exactly two axis intercepts, $(0,0)$ and $(-2,0)$. They cut the ellipse into two connected components one in the quadrant $q_2<0\,,\,q_3>0$ and the other in the quadrant $q_2<0\,,\,q_3<0$.

There line $q_2=5q_3$ meets the component of the ellipse in $q_3<0$ and the line $q_2=-5q_3$ meets the ellipse the component in $q_3>0$. Thus (11.6.7) is real analytic and not identically equal to zero on each component. This completes the proof. ∎