Chapter 6. The Nonlinear Cauchy Problem

§6.1. Introduction.

Nonlinear equations are classified according to the strength of the nonlinearity. The key criterion is what order terms in the equation are nonlinear.

A secondary condition is the growth of the nonlinear terms at infinity. When the functions which enter the nonlinear terms are uniformly bounded in absolute value, the behavior at infinity is not important.

Among the nonlinear equations in applications two sorts are most common. *Semilinear equations* are linear in their principal part. First order semilinear symmetric hyperbolic systems take the form

$$L(y,\partial_y)\,u + F(y,u) = f(y)\,, \qquad F(y,0) \;=\; 0 \tag{6.1.1}$$

where L is a symmetric hyperbolic operator, and the nonlinear function is a smooth map from $\mathbb{R}^{1+d}\times\mathbb{C}^N \to \mathbb{C}^N$ whose partial derivatives of all orders are uniformly bounded on sets of the form $\mathbb{R}^{1+d}\times K$, with compact $K\subset\mathbb{C}^N$. The derivatives are standard partial derivatives and not derivatives in the sense of complex analysis. A translation invariant semilinear equation with principal part equal to the d'alembertian is of the form

$$\Box u + F(u, u_t, \nabla_x u) \;=\; 0 \qquad F(0,0,0) \;=\; 0\,.$$

More strongly nonlinear, and typical of compressible inviscid fluid dynamics, are the quasi-linear systems,

$$L(y,u,\partial_y)\,u \;=\; f(y)\,, \tag{6.1.2}$$

where

$$L(y,u,\partial_y) = \sum_{j=0}^{d} A_j(y,u)\,\partial_j \tag{6.1.3}$$

has coefficients which are smooth hermitian symmetric matrix valued functions with derivatives bounded on $\mathbb{R}^{1+d}\times K$ as above. A_0 is assumed uniformly positive on such sets.

For semilinear equations there is a natural local existence theorem requiring data in $H^s(\mathbb{R}^d)$ for some $s > d/2$. The theorem gives solutions which are continuous functions of time with values in $H^s(\mathbb{R}^d)$. This shows that the spaces $H^s(\mathbb{R}^d)$ with $s > d/2$, are natural configuration spaces for the dynamics. Once a solution belongs to such a space, it is bounded and continuous so that $F(y,u)$ is well defined, bounded, and continuous. Nonlinear ordinary differential equations are a special case, so for general problems one expects at most a local existence theorem.

For quasilinear equations, the local existence theorem requires an extra derivative, that is initial data in $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$. Again the solution is a continuous functions of time with values in $H^s(\mathbb{R}^d)$. The classic example is Burgers' equation

$$u_t + u\,u_x \;=\; 0\,.$$

We treat first the semilinear case. The quasilinear case is treated in §6.6. The key step in the proof uses Schauder's Lemma, which bounds the $H^s(\mathbb{R}^d)$ norm of the composition $F(y,u)$ in terms of the $H^s(\mathbb{R}^d)$ norm of u.

§6.2. Schauder's Lemma and Sobolev embedding.

Consider the proof that u^2 belongs to $H^2(\mathbb{R}^2)$ as soon as u does. One must show that u^2, $\partial(u^2)$, and, $\partial^2(u^2)$ are square integrable.

Since $u \in L^\infty$ and $u \in L^2$, it follows that $u^2 \in L^2$. For the first derivative, write $\partial(u^2) = 2u\partial u$, which is the product of a bounded function and a square integrable function and so is in L^2.

The second derivative is more interesting. Write

$$\partial^2(u^2) \;=\; u\partial^2 u + 2(\partial u)^2\,.$$

The first is a product $L^\infty \times L^2$ so is L^2. For the second, one needs to know that $\partial u \in L^4$. The simplest such L^p estimate for Sobolev spaces is the Sobolev embedding Theorem.

Theorem 6.2.1 Sobolev. *If $s > d/2$, $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and*

$$\|w\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,\|w\|_{H^s(\mathbb{R}^d)}\,. \tag{6.2.1}$$

Proof. Inequality (6.2.1) for elements of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is an immediate consequence of the Fourier Inversion Formula,

$$w(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix.\xi}\ \hat w(\xi)\ d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{e^{-ix.\xi}}{<\xi>^s}\ <\xi>^s \hat w(\xi)\ d\xi\,.$$

The Schwarz inequality yields

$$|w(x)| \;\le\; \Big|\Big|\frac{1}{<\xi>^s}\Big|\Big|_{L^2(\mathbb{R}^d)} \|w\|_{H^s(\mathbb{R}^d)}.$$

The first factor on the right is finite if and only if $s > d/2$.

For $w \in H^s$, choose $w^n \in \mathcal{S}$ with

$$w^n \to w \quad \text{in} \quad H^s\,, \qquad \|w_n\|_{H^s(\mathbb{R}^d)} \;\le\; \|w\|_{H^s(\mathbb{R}^d)}\,.$$

Inequality (6.2.1), yields $\|w^n - w^m\|_{L^\infty(\mathbb{R}^d)} \le C\,\|w^n - w^m\|_{H^s(\mathbb{R}^d)}$. Therefore the w^n converge uniformly on $\mathbb{R}^d$ to a continuous limit γ. Therefore $w^n \to \gamma$ in $\mathcal{D}'(\mathbb{R}^d)$ with $\|\gamma\|_{L^\infty} \le C\,\|w\|_{H^s}$

However, $w^n \to w$ in H^s and therefore in $\mathcal{D}'$, so $w = \gamma$. This proves the continuity of w and the estimate (6.2.1). ∎

For the more delicate L^p estimates we will give two proofs.

Theorem 6.2.2 Schauder's Lemma. *Suppose that $G(x,u) \in C^\infty(\mathbb{R}^d \times \mathbb{C}^N\,;\,\mathbb{C}^N)$ such that $G(x,0)=0$, and for all $|\alpha| \le s+1$ and compact $K \subset \mathbb{C}^N$, $\partial^\alpha_{x,u} G \in L^\infty(\mathbb{R}^d \times K)$. Then the map $w \mapsto G(x,w)$ sends $H^s(\mathbb{R}^d)$ to itself provided $s > d/2$. The map is uniformly lipschitzian on bounded subsets of $H^s(\mathbb{R}^d)$.*

Proof of Schauder's Lemma for integer s. Consider $G = G(w)$. The case of G depending on x is uglier but requires no additional ideas. The key step is to estimate the H^s norm of $G(w)$ assuming that $w \in \mathcal{S}$, We prove that

$$\forall\, R, \quad \exists\, C = C(R), \quad \forall\, w \in \mathcal{S}(\mathbb{R}^d), \quad \|w\|_{H^s(\mathbb{R}^d)} \le R \quad \Longrightarrow \quad \|G(w)\|_{H^s(\mathbb{R}^d)} \le C(R)\,.$$

This suffices to prove the first assertion of the theorem since for $w \in H^s$, choose $w^n \in \mathcal{S}$ with

$$w^n \to w \quad \text{in} \quad H^s\,, \qquad \|w_n\|_{H^s(\mathbb{R}^d)} \ \le\ \|w\|_{H^s(\mathbb{R}^d)}\,.$$

Then Sobolev's Theorem implies that w^n converges uniformly on $\mathbb{R}^d$ to w, so $G(w^n)$ converges uniformly to $G(w)$. In particular, $G(w^n) \to G(w)$ in $\mathcal{D}'(\mathbb{R}^d)$.

However, $G(w^n)$ is bounded in H^s so passing to a subsequence we may suppose that $G(w^n) \to v$ weakly in H^s. Therefore $G(w^n) \to v$ in $\mathcal{D}'(\mathbb{R}^d)$. Equating the $\mathcal{D}'$ limits proves that $G(w) \in H^s$.

For $w \in \mathcal{S}$, consider a derivative $\partial_x^\alpha G(w(x))$ with $|\alpha| \le s$. Leibniz' rule implies that it is a finite sum of terms of the form

$$G^{(\gamma)}(w)\ \Pi_{j=1}^J\, \partial_x^{\alpha_j} w \tag{6.2.2}$$

where $|\gamma| = J \le s$, and $\alpha_1 + \cdots + \alpha_J = \alpha$. This is proved by induction on $|\alpha|$. Increasing the order by one the additional derivative either falls on the G term yielding an expression of the desired form with $|\gamma| = J$ increased by one, or on one of the factors in $\Pi\,\partial_x^{\alpha_j} w$ yielding an expression of the desired form with the same value of γ.

Following [Rauch 1983] we use the Fourier transform to show that

$$\|G^{(\gamma)}(\omega)\ \Pi_{j=1}^J\, \partial_x^{\alpha_j} w\|_{L^2} \ \le\ C(R)\,. \tag{6.2.3}$$

Sobolev's Theorem implies that

$$\|G^{(\gamma)}(w)\|_{L^\infty(\mathbb{R}^d)} \ \le\ C(R)\,.$$

The key is the following estimate which is applied to (6.2.3).

Lemma 6.2.3. *If $s > d/2$ there is a constant $C = C(s,d)$ so that for all $w_j \in \mathcal{S}(\mathbb{R}^d)$ and all multiindices α_j with $s' := \sum |\alpha_j| \le s$,*

$$\|\,\Pi_{j=1}^J \partial_x^{\alpha_j} w_j\,\|_{L^2(\mathbb{R}^d)} \ \le\ C\,\Pi_{j=1}^J\, \|w_j\|_{H^s(\mathbb{R}^d)}\,.$$

Proof. By Plancherel's theorem, it suffices to estimate the L^2 norm of the Fourier transform of the product. Set

$$g_i \;:= <\xi>^{s-|\alpha_i|}\, \mathcal{F}\big(\partial_x^{\alpha_i} w_i\big)\,,$$

where $<\xi>\equiv (1+|\xi|^2)^{1/2}$, so $\|g_i\|_{L^2} \le c\|w_i\|_{H^s}$. Denoting by $\mathcal{F}$, the Fourier transform,

$$\begin{aligned}\mathcal{F}\big(\Pi_{j=1}^J \partial_x^{\alpha_j} w_{l_j}\big)(\xi_1) \;&=\; \frac{g_1}{<\xi>^{s-|\alpha_1|}} * \frac{g_2}{<\xi>^{s-|\alpha_2|}} * \cdots * \frac{g_J}{<\xi>^{s-|\alpha_J|}}(\xi_1) \\ =\int_{\mathbb{R}^{d(J-1)}} &\frac{g_1(\xi_1-\xi_2)}{<\xi_1-\xi_2>^{s-|\alpha_1|}}\;\frac{g_2(\xi_2-\xi_3)}{<\xi_2-\xi_3>^{s-|\alpha_2|}}\cdots\frac{g_J(\xi_J)}{<\xi_j>^{s-|\alpha_J|}}\;d\xi_2\ldots d\xi_J\,.\end{aligned} \tag{6.2.4}$$

For each ξ, at least one of the J numbers $<\xi_1-\xi_2>,\ldots,<\xi_{J-1}-\xi_J>,<\xi_J>$ is maximal. Suppose it's the b^{th} number $<\xi_b-\xi_{b+1}>$ with the convention that $\xi_{J+1}\equiv 0$. Then since $\sum|\alpha_i|\le s$,

$$\langle\xi_b-\xi_{b+1}\rangle^{s-|\alpha_b|} \;\ge\; \langle\xi_b-\xi_{b+1}\rangle^{\sum_{j\ne b}|\alpha_j|} \;\ge\; \Pi_{j\ne b}\langle\,\xi_j-\xi_{j+1}\rangle^{|\alpha_j|}$$

which implies that

$$\Pi_{j=1}^J\langle\xi_j-\xi_{j+1}\rangle^{s-|\alpha_j|} \;=\; \langle\xi_b-\xi_{b+1}\rangle^{s-|\alpha_b|}\;\Pi_{j\ne b}\,\langle\xi_j-\xi_{j+1}\rangle^{s-|\alpha_j|} \;\ge\; \Pi_{j\ne b}\langle\xi_j-\xi_{j+1}\rangle^{s}\,.$$

Thus the integrand on the right side of (6.2.4) is dominated by

$$\Big|\, g_b(\xi_b-\xi_{b+1})\;\Pi_{j\ne b}\;\frac{g_j(\xi_j-\xi_{j+1})}{\langle\xi_j-\xi_{j+1}\rangle^s}\,\Big|\,. \tag{6.2.5}$$

Thus for any $\xi_1\in\mathbb{R}^d$, the integrand in (6.2.4) is dominated by the sum over b of the terms (6.2.5). Hence

$$\begin{aligned}\|\mathcal{F}\big(\Pi_{j=1}^J\partial_x^{\alpha_j}w_{l_j}\big)\|_{L^2(\mathbb{R}^d)} \;&\le\; \|\sum_{b=1}^J \frac{|g_1|}{<\xi>^s}*\cdots*|g_b|*\frac{|g_{b+1}|}{<\xi>^s}*\cdots*\frac{|g_J|}{<\xi>^s}\|_{L^2} \\ \le\; &\sum_{b=1}^J \Big\|\frac{g_1}{<\xi>^s}\Big\|_{L^1}\cdots\Big\|\frac{g_{b-1}}{<\xi>^s}\Big\|_{L^1}\,\|g_b\|_{L^2}\,\Big\|\frac{g_{b+1}}{<\xi>^s}\Big\|_{L^1}\cdots\Big\|\frac{g_J}{<\xi>^s}\Big\|_{L^1}\end{aligned}$$

where the last step uses Young's inequality.

As in Sobolev's Theorem, $s>d/2 \;\Rightarrow <\xi>^{-s}\in L^2(\mathbb{R}^d)$ and the Schwarz inequality yields

$$\|\,\frac{g_j}{<\xi>^s}\,\|_{L^1} \;\le\; C_1\,\|g_j\|_{L^2} \;\le\; C_2\,\|w_j\|_{H^s}.$$

Plugging this in the previous estimate proves the lemma. ∎

To prove the Lipschitz continuity asserted in Schauder's Lemma it suffices to show that for all R there is a constant $C(R)$ so that

$$w_j \in \mathcal{S}(\mathbb{R}^d) \quad \text{for } j = 1, 2 \qquad \text{and} \qquad \|w_j\|_{H^s(\mathbb{R}^d)} \leq R$$

imply

$$\|G(w_1) - G(w_2)\|_{H^s(\mathbb{R}^d)} \ \leq \ C\, \|w_1 - w_2\|_{H^s(\mathbb{R}^d)}\,.$$

Taylor's Theorem expresses

$$G(w_1) - G(w_2) = \int_0^1 G'(w_2 + \theta(w_1 - w_2))\, d\theta\ (w_1 - w_2)\,.$$

The estimates of the first part show that the family of functions $G'(w_2 + \theta(w_1 - w_2))$ parametrized by θ is bounded in $H^s(\mathbb{R}^d)$. Thus

$$\Big\| \int_0^1 G'(w_2 + \theta(w_1 - w_2))\ d\theta \Big\|_{H^s(\mathbb{R}^d)} \ \leq \ C(R)\,.$$

Applying the Lemma to the expression for $G(w_1) - G(w_2)$ as a product of two terms completes the proof. ∎

The standard proof of Schauder's Lemma for integer s uses the L^p version of the Sobolev Embedding Theorem. The general result of this sort is the following. Proofs can be found in [Hörmander I.4.5, Taylor III.13.6.4] for example.

Sobolev Embedding Theorem 6.2.4. *If $1 \leq s \in \mathbb{R}$ and $\alpha \in \mathbb{N}^d$ is a multiindex with $0 < s - |\alpha| < d/2$, there is a constant $C = C(\alpha, s, d)$ independent of $u \in H^s(\mathbb{R}^d)$ so that*

$$\| \partial_y^\alpha u \|_{L^{p(\alpha)}} \ \leq \ C \|\ |\xi|^s\, \hat{u}(\xi)\ \|_{L^2(\mathbb{R}^d)}\,, \tag{6.2.6}$$

where

$$p(\alpha) \ := \ \frac{2d}{d - 2s + 2|\alpha|}\,. \tag{6.2.7}$$

For $s - |\alpha| > d/2$, $\partial_y^\alpha u$ is bounded and continuous and

$$\|\partial_y^\alpha u\|_{L^\infty} \ \leq \ C\|u\|_{H^s(\mathbb{R}^d)}\,.$$

For $s - |\alpha| = d/2$, one has

$$\|\partial_y^\alpha u\|_{L^p(\mathbb{R}^d)} \ \leq \ C(p, s, \alpha)\, \|u\|_{H^s(\mathbb{R}^d)}$$

for all $2 \leq p < \infty$.

The formula for $p(\alpha)$ is forced by dimensional analysis. For a fixed nonzero $\psi \in C_0^\infty$, consider $u_\lambda(x) := \psi(\lambda x)$. The left hand side of (6.2.6) then is of the form $c\lambda^a$ for some a.

Similarly the right hand side is of the form $c'\lambda^b$ for some b. In order for the inequality to hold, one must have $\lambda^a \le c''\lambda^b$ for all positive λ so it is necessary that $a = b$.

Exercise. *Show that $a = b$ if and only if p is given by (6.2.7).*

Another way to look at the scaling argument is that for dimensionless u the left hand side of (6.2.6) has dimensions $length^{(d-p|\alpha|)/p}$ while the right hand side has dimensions $length^{(d-2s)/2}$. The formula for p results from equating these two expressions.

Standard proof of Schauder's Lemma. The usual proof for integer s uses the Sobolev estimates. together with Hölder's inequality. Hölder's inequality yields

$$\sum_{k=1}^{J} \frac{1}{p_k} \;=\; \frac{1}{2} \qquad \Longrightarrow \qquad \big\| \, \partial_x^{\alpha_1} w_{l_1} \cdots \partial_x^{\alpha_J} w_{l_J} \, \big\|_{L^2} \;\le\; \Pi_{k=1}^J \big\| \partial_x^{\alpha_k} w_{l_k} \big\|_{L^{p_k}} \,.$$

Since each factor $\partial_x^{\alpha_k} w_{l_k}$ belongs to L^2 it suffices to find q_k so that

$$\partial_x^{\alpha_k} w_{l_k} \in L^{q_k} \quad \text{and} \quad \sum \frac{1}{q_k} \le \frac{1}{2} \,.$$

Let $\mathcal{B}$ denote the set of $k \in \{1, \dots, J\}$ so that $s - |\alpha_k| > d/2$. For these indices the factor in our product is bounded, and so for $k \in \mathcal{B}$ set $q_k := \infty$.

Let $\mathcal{A} \subset \{1, \dots, J\}$ denote those indices i for which $s - |\alpha_i| < \frac{d}{2}$. For $k \in \mathcal{A}$, q_k is chosen as in Sobolev's Theorem,

$$q_k \;:=\; \frac{2d}{d - 2s + 2|\alpha_k|} \,.$$

If $s - |\alpha_k| = \frac{d}{2}$, the factor in the product belongs to L^p for all $2 \le p < \infty$ and the choice of q_k in this range is postponed.

With these choices, the Sobolev embedding theorem estimates

$$\|\partial_x^{\alpha_k} w_{l_k}\|_{L^{q_k}} \;\le\; C \|w\|_{H^s(\mathbb{R}^d)} \,.$$

Then since $\sum |\alpha_i| \le s$, and $s > d/2$,

$$\sum_{i \in \mathcal{A} \cup \mathcal{B}} \frac{1}{q_i} \;=\; \sum_{i \in \mathcal{A}} \frac{1}{q_i} \;=\; \sum_{i \in \mathcal{A}} \frac{d - 2s + 2|\alpha_i|}{2d} \;\le\; \frac{Jd - 2Js + 2s}{2d}$$

$$=\; \frac{Jd - 2(J-1)s}{2d} \;<\; \frac{Jd - (J-1)d}{2d} \;=\; \frac{1}{2} \,.$$

This shows there is room to pick large q_k corresponding to the case $s - |\alpha_k| = d/2$ so that $\sum 1/q_k < 1/2$, and the proof is complete. ∎

Another nice proof of Schauder's Lemma can be found in [Beals, pp 11-12]. Other arguments can be built on Littlewood-Paley decomposition of $G(w)$ as in [Bony, Meyer] and presented in [Alinhac-Gerard, Taylor III.13.10], or, on the representation

$$G(u) \;=\; \int \hat{G}(\xi)\,(e^{iu\xi}-1)\;d\xi\,.$$

The latter requires that one prove a bound on the norm of $e^{iu\xi}-1$ (see [Rauch-Reed 1982]) which grows at most polynomially in ξ. The last two arguments have the advantage of working when s is not an integer.

§6.3. Basic existence theorem.

The basic local existence theorem follows from Schauder's Lemma and the linear existence theorem.

Theorem 6.3.1 Schauder. *If $s > d/2$ and $f \in L^1_{\rm loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$, then there is a $T\, \in]0,1]$ and a unique solution $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ to the semilinear initial value problem defined by the partial differential equation (6.2) together with the initial condition*

$$u(0,x) \;=\; g(x) \in H^s(\mathbb{R}^d). \tag{6.3.1}$$

The time T can be chosen uniformly for f and g from bounded subsets of $L^1([0,1]\,;\,H^s(\mathbb{R}^d))$ and $H^s(\mathbb{R}^d)$ respectively. Consequently, there is a $T^*\, \in]0,\infty]$ and a maximal solution $u \in C\big([0,T^*[\,;\,H^s(\mathbb{R})^d)\big)$. If $T^* < \infty$ then

$$\lim_{t\to T^*}\ \|u(t)\|_{H^s(\mathbb{R}^d)} \;=\; \infty\,. \tag{6.3.2}$$

Schauder proved a quasilinear second order scalar version, but his argument, which is recalled in §10 of chapter 6 in [Courant], works without essential modification once you add the linear energy inequalities of Friedrichs. The following existence proof is inspired by Picard's argument for ordinary differential equations. Note how Picard's elegant bounds (6.3.8) replace the usual contraction argument which is less precise.

Proof. The solution is constructed as the limit of Picard iterates. The first approximation is not really important. Set

$$\forall\, t, x\,, \qquad u^1(t,x) \;:=\; g(x)\,.$$

For $\nu > 1$, the basic linear existence theorem implies that the Picard iterates defined as solutions of the linear initial value problems

$$L(y,\partial_y)\,u^{\nu+1} + F(y,u^\nu) \;=\; f(y)\,, \qquad u^{\nu+1}(0) = g$$

are well defined elements of $C([0,\infty[\,;\,H^s(\mathbb{R}^d)\,)\,.$

Let C denote the constant in the linear energy estimate (2.2.2). Choose a real number

$$R \ > \ 2C\, \|g\|_{H^s(\mathbb{R}^d)}\,. \tag{6.3.3}$$

Schauder's lemma implies that there is a constant $B(R) > 0$ so that

$$\|w(t,\cdot)\|_{H^s(\mathbb{R}^d)} \le R \qquad \Longrightarrow \qquad \|F(t,\cdot,w(\cdot))\|_{H^s(\mathbb{R}^d)} \ \le \ B\,.$$

Thanks to (6.3.3) one can choose $T > 0$ so that

$$C\,\Big(e^{CT}\|g\|_{H^s(\mathbb{R}^d)} + \int_0^T e^{C(T-\sigma)}\big(B + \|f(\sigma)\|_{H^s(\mathbb{R}^d)}\big)d\sigma\Big) \le R\,. \tag{6.3.4}$$

Using (2.2.2) shows that for all $\nu \ge 1$ and all $0 \le t \le T$

$$\|u^\nu(t)\|_{H^s(\mathbb{R}^d)} \ \le \ R\,. \tag{6.3.5}$$

Schauder's Lemma implies that there is a constant Λ so that for all t,

$$\|w_j\|_{H^s(\mathbb{R}^d)} \le R \Rightarrow \|F(t,x,w_1(x)) - F(t,x,w_2(x))\|_{H^s(\mathbb{R}^d_x)} \le \Lambda\|w_1 - w_2\|_{H^s(\mathbb{R}^d_x)}\,. \tag{6.3.6}$$

Then for $\nu \ge 2$, (2.2.2) applied to the difference $u^{\nu+1} - u^\nu$ implies that

$$\|u^{\nu+1}(t) - u^\nu(t)\|_{H^s(\mathbb{R}^d)} \ \le \ C\,\Lambda \int_0^t e^{C(t-\sigma)}\, \|u^\nu(\sigma) - u^{\nu-1}(\sigma)\|_{H^s(\mathbb{R}^d)}\; d\sigma\,. \tag{6.3.7}$$

Define

$$M_1 \ := \ \sup_{0\le t\le T} \|u^1(t) - u^2(t)\|_{H^s(\mathbb{R}^d)} \qquad \text{and} \quad M_2 \ := \ C\,\Lambda\, e^{CT}.$$

An induction on ν using (6.3.7) shows that for all $\nu \ge 2$

$$\|u^{\nu+1}(t) - u^\nu(t)\|_{H^s(\mathbb{R}^d)} \ \le \ M_1\frac{(M_2 t)^{\nu-1}}{(\nu-1)!}. \tag{6.3.8}$$

Exercise. *Prove (6.3.8).*

Estimate (6.3.8) shows that the sequence $\{u^\nu\}$ is Cauchy in $C([0,T]\,;\, H^s(\mathbb{R}^d))$. Let u be the limit.

Exercise. *Prove u satisfies the initial value problem (6.1.1), (6.3.1).*

This completes the proof of existence.

Uniqueness is a consequence of the integral inequality

$$\|u_1(t) - u_2(t)\|_{H^s(\mathbb{R}^d)} \;\le\; C_1 \int_0^t e^{C(t-\sigma)}\, \|u_1(\sigma) - u_2(\sigma)\|_{H^s(\mathbb{R}^d)}\, d\sigma\,, \tag{6.3.9}$$

which is proved exactly as (6.3.7). Gronwall's inequality implies that $\|u_1 - u_2\|$ vanishes identically. ∎

Remarks. 1. Similar estimates show that there is continuous dependence of the solutions when the data f and g converge in $L^1_{loc}(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$ and $H^s(\mathbb{R}^d)$ respectively.

2. Approximating the data by smooth data, and therefore the solutions by smooth solutions of approximating problems, the finite speed of propagation from §2.3 extends to the solutions just constructed.

Exercise. *Prove these two assertions.* **Discussion.** Concerning the first, more precise results are presented in §6.6.

Exercise. *Show that if the source term f satisfies $\partial_t^k f \in L^1_{\rm loc}([0,T^*[\,;\, H^{s-k}(\mathbb{R}^d))$ for $k = 1, 2, \ldots, s$ as in Theorem 2.2.2, then $u \in \cap_k\, C^k\big([0,T^*[\,;\, H^{s-k}(\mathbb{R}^d)\big)$.*

§6.4. Moser's inequality and the nature of the breakdown.

The breakdown (6.3.2) could in principal occur in a variety of ways. For example, the function might stay bounded and become more and more rapidly oscillatory. In fact this does not occur. Where the domain of existence ends the maximal amplitude of the solution must diverge to infinity. To prove this requires more refined inequalities than those of Sobolev and Schauder.

The Schauder Lemma implies that

$$\|G(y,w)\|_{H^s(\mathbb{R}^d_x)} \;\le\; h(\|w\|_{H^s(\mathbb{R}^d_x)})$$

with a nonlinear function h which depends on G.

Theorem 6.4.1 Moser's Inequality. *With the same hypotheses as Schauder's Lemma, there is a smooth function $h : [0,\infty[\to [0,\infty[$ so that for all $w \in H^s(\mathbb{R}^d)$ and t,*

$$\|G(x,w)\|_{H^s(\mathbb{R}^d_x)} \le h(\|w\|_{L^\infty(\mathbb{R}^d_x)})\, \|w\|_{H^s(\mathbb{R}^d_x)}\,. \tag{6.4.1}$$

This is proved by using Leibniz' rule and Hölder's inequality as in the standard proof of Schauder's Lemma. However in place of the Sobolev inequalities one uses the Galiardo-Nirenberg interpolation inequalities.

Theorem 6.4.2 Gagliardo-Nirenberg Inequalities. *If $w \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $0 < |\alpha| < s$ then*

$$\partial_x^\alpha\, w \in L^{2s/|\alpha|}(\mathbb{R}^d)$$

In addition, there is a constant $C = C(|\alpha|, s, d)$ so that

$$\|\partial^\alpha w\|_{L^{2s/|\alpha|}(\mathbb{R}^d)} \;\le\; C\,\|w\|_{L^\infty(\mathbb{R}^d)}^{1-|\alpha|/s} \Big(\sum_{|\beta|=s} \|\partial^\beta w\|_{L^2(\mathbb{R}^d)} \Big)^{|\alpha|/s} \tag{6.4.2}$$

Remarks. 1. The second factor on the right in (6.4.2) is equivalent to the L^2 norm of the operator $|\partial_x|^s$ applied to u where $|\partial_x|^s$ is defined to be the Fourier multiplier by $|\xi|^s$. This gives the correct extension to non integer s.

2. The indices in (6.4.2) are nearly forced. Consider which inequalities

$$\|\partial^\alpha w\|_{L^p(\mathbb{R}^d)} \;\le\; C\,\|w\|_{L^\infty(\mathbb{R}^d)}^{1-\theta} \Big(\sum_{|\beta|=s} \|\partial^\beta w\|_{L^2(\mathbb{R}^d)} \Big)^{\theta}$$

homogeneous of degree one in w might be true. The test functions $w = e^{ix\cdot\xi/\epsilon}\psi(x)$ with $\epsilon \to 0$ show that a necessary condition is $|\alpha| \le s\theta$. The idea is to use the L^∞ norm as much as possible and the s-norm as little as possible, which yields $|\alpha| = s\theta$. Considering $w = \psi(\epsilon x)$, or equivalently comparing the dimensions of the two sides forces $p = 2s/\alpha$.

Proof. The following paragraphs lead you through the proof. The motor is a clever use of integration by parts. To illustrate that, begin by proving the special case $s = 2,\ p = 4$, which for real valued $u \in C_0^\infty(\mathbb{R}^d)$, is

$$\|Du\|_{L^4} \;\le\; C\|u\|_{L^\infty}^{1/2}\,\|D^2u\|_{L^2}^{1/2}.$$

The centerpiece of the proof is the following case of the product rule for derivatives

$$D\Big(u\,(Du)^3\Big) \;=\; (Du)^4 + 3\,(u)\,(Du)^2\,(D^2u)\,.$$

Since u is compactly supported,

$$0 \;=\; \int_{\mathbb{R}^d} D\Big(u\,(Du)\,(Du)^2\Big)\;dx\,.$$

Combining these two observations yields

$$\int_{\mathbb{R}^d} |Du|^4\,dx \;=\; -3\int_{\mathbb{R}^d} |u\,(D^2u)\,(Du)^2|\;dx\,.$$

Applying Hölder's inequality to estimate the right hand side yields

$$\int_{\mathbb{R}^d} |Du|^4\,dx \;\le\; 3\,\|u\|_{L^\infty}\Big(\int_{\mathbb{R}^d} |D^2u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^d} |Du|^4\,dx\Big)^{1/2}.$$

If $u \neq 0$ the last factor is nonzero so dividing both sides by this term yields,

$$\Big(\int_{\mathbb{R}^d} |Du|^4\,dx\Big)^{1/2} \;\leq\; 3\,\|u\|_{L^\infty}\Big(\int_{\mathbb{R}^d} |D^2u|^2\,dx\Big)^{1/2}.$$

Equivalently,

$$\|Du\|_{L^4} \;\leq\; \sqrt{3}\;\|u\|_{L^\infty}^{1/2}\,\|D^2u\|_{L^2}^{1/2}\,,$$

which is the desired estimate.

Exercise. *Prove the general $s = 2$, $|\alpha| = 1$ estimate by a similar argument.* **Hint.** Use the primitive $c_p t|t|^{p-1}$ of t^p in a product rule argument.

To see how to go to higher derivatives, consider the case $s = 3$ in which case (6.4.2) asserts that $D^2u \in L^3$ and $Du \in L^6$. This is proved by appealing twice to Gagliardo-Nirenberg estimates with $s = 2$. Namely, one estimates

$$\|Du\|_{L^6} \;\leq\; C\|u\|_{L^\infty}^{1/2}\,\|D^2u\|_{L^3}^{1/2} \qquad \text{and} \qquad \|D^2u\|_{L^3} \;\leq\; C\|Du\|_{L^6}^{1/2}\,\|D^3u\|_{L^2}^{1/2}\,.$$

Using the second in the first yields the desired estimate for Du. Using that in the second, yields the desired estimate for D^2u. The second inequality is *not* a special case of (6.4.2). One needs the more general inequality

$$\|Du\|_{L^p} \;\leq\; C\,\|u\|_{L^q}^{1/2}\,\|D^2u\|_{L^r}^{1/2}\,, \qquad \text{where} \qquad \frac{2}{p} := \frac{1}{q} + \frac{1}{r}\,.$$

Exercise. *Prove this Gagliardo-Nirenberg estimate*

Exercise. *Prove that (6.4.2) follows from these estimates.* **Hint.** This takes artful index juggling. An alternative way of using the basic integrations by parts is given in [Taylor III.10.3].

Proof of Moser's Inequality. For $w \in \mathcal{S}(\mathbb{R}^d)$, G independent of x, and $\sigma := |\alpha| \leq s$, the quantity $\partial_x^\alpha(G(w))$ is a sum of terms of the form

$$G^{(\gamma)}(w)\;\Pi_{j=1}^J\,\partial_x^{\alpha_j} w \tag{6.4.3}$$

where $|\gamma| = J$, and $\alpha_1 + \cdots + \alpha_J = \alpha$. The first factor in (6.4.3) is bounded with L^∞ norm bounded by a nonlinear function of the L^∞ norm of w.

For the second factor, Hölder's inequality yields

$$\|\,\partial_x^{\alpha_1} w \cdots \partial_x^{\alpha_J} w\,\|_{L^2} \;\leq\; \Pi_{k=1}^J\;\|\partial_x^{\alpha_k} w\|_{L^{2/\lambda_k}}$$

provided the nonnegative λ_k satisfy $\sum \lambda_k = 1$.

The Gagliardo-Nirenberg inequalities yield

$$\|\partial_x^{\alpha_k} w\|_{L^{2\sigma/|\alpha_k|}} \;\le\; \|w\|_{L^\infty}^{(\sigma-|\alpha_k|)/\sigma}\,\|w\|_{H^\sigma}^{|\alpha_k|/\sigma}\,.$$

With these choices

$$\sum \lambda_k = \sum \frac{|\alpha_k|}{\sigma} = 1\,,$$

and one has

$$\|\,\partial_x^{\alpha_1} w \cdots \partial_x^{\alpha_J} w\,\|_{L^2} \;\le\; C(R)\,\|w\|_{H^s}\,.$$

∎

Exercise. *Carry out the proof for G which depend on x.*

Theorem 6.4.3. *If $T^* < \infty$ in Theorem 6.3.1, then*

$$\limsup_{t\to T^*} \|u(t)\|_{L^\infty} \;=\; \infty\,. \tag{6.4.4}$$

Proof. It suffices to show that it is impossible to have $T^* < \infty$ and $|u| \le R < \infty$ on $[0,T^*[\times\mathbb{R}^d$. The strategy is to show that if $|u(t,x)| \le R < \infty$ on $[0,T^*[\times\mathbb{R}^d$, then (6.3.2) is violated.

Use the linear inequality for $0 \le t < T$,

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \;\le\; C\left(\,\|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t \,\|(Lu)(\sigma)\|_{H^s(\mathbb{R}^d)}\; d\sigma\right). \tag{6.4.5}$$

Then use Moser's inequality to give

$$\|(Lu)(\sigma)\|_{H^s(\mathbb{R}^d)} = \|F(\sigma,x,u(\sigma,x)) - f(\sigma,x)\|_{H^s(\mathbb{R}^d)} \;\le\; C(R)\left(\,\|u(\sigma)\|_{H^s(\mathbb{R}^d)} + 1\right). \tag{6.4.6}$$

Insert (6.4.5) in (6.4.6) to find

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le C\left(\,\|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t \;\left(\,\|u(\sigma)\|_{H^s(\mathbb{R}^d)} + 1\,\right)\; d\sigma\right). \tag{6.4.7}$$

Gronwall's inequality shows that there is a constant $C'' < \infty$ so that for $t \in [0,T^*[$

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le C''\,. \tag{6.4.8}$$

This violates (6.3.2), and the proof is complete. ∎

A mild sharpening of this argument (due to Yudovich) shows that weaker norms than L^∞, for example the BMO norm, must also blow up at T^*.

Corollary 6.4.4. *If the data f and g in Theorem 6.4 belong to $\cap_s L^1_{loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$ and $\cap_s H^s(\mathbb{R}^d)$ respectively, then the maximal solution belongs to $C\big([0,T^*[\,;\,H^s(\mathbb{R}^d)\big)$ for all s.*

Proof. For $s > d/2$ denote by $T^*(s)$ the time of exisitence of the maximal solution constructed in Theorem 6.4 and by $u_s(t,x)$ the corresponding maximal solution. Suppose that $d/2 < s < \tilde s$. By uniqueness of H^s valued solutions, one must have

$$u_s = u_{\tilde s} \qquad \text{for} \quad 0 \le t \le \min\{T^*(s), T^*(\tilde s)\}\,. \tag{6.4.9}$$

It follows that $T^*(s)$ is nonincreasing in s.

On the other hand if $T^*(s) > T$ it follows that $u_s \in L^\infty([0,T]\times\mathbb{R}^d)$ so by (6.4.9), $u_{\tilde s} \in L^\infty([0,T]\times\mathbb{R}^d)$. Then Theorem 6.4.1 implies that $T^*(\tilde s) \ge T$. This proves that $T^*(s)$ is nondecreasing in s and is therefore independent of s. Equation (6.4.9) completes the proof. ∎

§6.5. Perturbation theory and smooth dependence.

In this section the dependence of solutions on data is investigated. The first result yields two versions of lipschitz dependence.

Theorem 6.5.1. i. *If u and v are two solution in $C([0,T]\,;\,H^s(\mathbb{R}^d))$, then there is a constant C depending only on $\sup_{[0,T]}\max\{\|u(t)\|_s, \|v(t)\|_s\}$ so that*

$$\forall\, 0 \le t \le T, \quad \|u(t)-v(t)\|_s \;\le\; C\,\|u(0)-v(0)\|_s\,. \tag{6.5.1}$$

ii. *If $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ is a solution then there are constants $C,\delta > 0$ so that if $\|u(0)-h\|_s < \delta$ then the solution v with $v(0) = h$ belongs to $C([0,T]\,;\,H^s(\mathbb{R}^d))$ and $\sup_{0\le t]\le T}\|v(t)-u(t)\|_s < C\,\delta$.*

Proof. i. Choose Λ so that for w_1 and w_2 in H^s with

$$\|w_j\|_s \;\le\; \sup_{[0,T]}\max\{\|u(t)\|_s, \|v(t)\|_s\}\,,$$

and $0 \le t \le T$

$$\|F(t,x,w_1(x)) - F(t,x,w_2(x))\|_{H^s(\mathbb{R}^d_x)} \le \Lambda\|w_1-w_2\|_{H^s(\mathbb{R}^d_x)}\,.$$

Then subtracting the equations for u and v yields

$$\|u(t)-v(t)\|_{H^s(\mathbb{R}^d)} \;\le\; \|u(0)-v(0)\|_s \;+\; \Lambda\int_0^t e^{C(t-\sigma)}\,\|u(\sigma)-v(\sigma)\|_{H^s(\mathbb{R}^d)}\,d\sigma\,.$$

Gronwall's inequality completes the proof of **i.**.

To prove **ii** it suffices to consder $\delta < 1$. Write $v = u + w$ so the initial value problem is equivalent to

$$L\, w + F(u+w) - F(u) = 0\,, \qquad w(0) = h\,.$$

So long as

$$\sup_{[0,t]}\ \|w(s)\|\ \le\ 2\,,$$

one estimates $\|F(w+u) - F(u)\|_s \le K\|w\|_s$ to find

$$\|w(t)\|_s\ \le\ \|h\|_s\ +\ \int_0^t K\,\|w(\sigma)\|_s\, d\sigma\,.$$

Gronwall implies that

$$\|w(t)\|_s\ \le\ \|h\|_s\, e^{Kt}\,.$$

Choose $C := e^{KT}$ and consider only δ so small that $\delta C < 2$. It follows that a local solution $w \in C([0,\underline{t}]\,,\, H^s)$ with $\underline{t} < T$ satisfies

$$\sup_{[0,\underline{t}]} \|w(t)\|_s\ <\ \min\ \big\{C\delta\,,\, 2\big\}\,.$$

Therefore the maximal solution is defined at least on $[0,T]$, and on that interval satisfies $\|w(t)\|_s \le 2\,\|h\|_s$ which completes the proof of **ii.** ∎

Given a solution u we compute a perturbation expansion for the solution with initial data $u(0)+g$ with small g. To simplify the notation, consider the semilinear equation

$$L(y,\partial)\, u\ +\ F(u)\ =\ 0\,, \qquad F(0) = 0\,.$$

Consider the map, $\mathcal{N} : u(0) \mapsto u$ from H^s to $C([0,T]\,;\, H^s(\mathbb{R}^d))$. At the end we will show that this map is smooth. For the moment we simply compute the Taylor expansion, assuming that it exists. Assuming smoothness, the solution with data $u(0)+g$ has expansion

$$\mathcal{N}(u(0)+g)\ \sim\ u + M_1(g) + M_2(g) + \ldots\ \sim\ \sum_{j=1}^{\infty} M_j(g)\,, \tag{6.5.1}$$

where the M_j are continuous symmetric j-linear operators from H^s to $C([0,T]\,;\, H^s(\mathbb{R}^d))$. To compute them, fix g and consider the initial data equal to $u(0)+\delta g$. The resulting solution has an expansion in δ

$$\mathcal{N}(u+\delta g)\ \sim\ u + \delta u_1 + \delta^2 u_2 + \cdots\,. \tag{6.5.2}$$

However,

$$\mathcal{N}(u+\delta g)\ \sim\ u + M_1(\delta g) + M_2(\delta g) + \ldots\ \sim\ u + \sum_{j=1}^{\infty} \delta^j\, M_j(g)\,.$$

Comparing with (6.5.2) one sees that

$$u_j \;=\; M_j(g, g, \cdots, g)\,, \qquad j \;\text{ copies of }\; g\,.$$

To compute u_j plug the expansion (6.5.2) into the equation

$$L(y,\partial)\Big(u + \sum_{j\ge 1}\delta^j u_j\Big) \;+\; F\Big(u + \sum_{j\ge 1}\delta^j u_j\Big) \;\sim\; 0\,.$$

The initial condition yields

$$u_1(0) \;=\; g\,, \qquad u_j(0) \;=\; 0\,, \;\; j \ge 2\,. \tag{6.5.3}$$

Expanding the left hand side in powers of δ, the terms u_j are determined by setting the coefficients of the successive powers of δ equal to zero. Introduce the compact notation for the Taylor expansion

$$F(v+h) \;\sim\; F(v) + F_1(v;h) + F_2(v;h,h) + \ldots\,,$$

where for $v \in \mathbb{C}^N$, $F_j(v\,;\,\cdot)$ is a symmetric j linear map from $(\mathbb{C}^N)^j \to \mathbb{C}^N$.
Also introduce the *linearized operator*

$$\mathcal{L}\,w \;:=\; L(y,\partial)w \;+\; F_1(u;w)$$

at the solution u. One has

$$L(u+\delta w) + F(u+\delta w) \;=\; \delta\,\mathcal{L}\,w + O(\delta^2)\,,$$

so $L(u+\delta w) + F(u+\delta w) \;=\; O(\delta^2)$, if and only if $\mathcal{L}\,w = 0$. Setting the coefficients of δ^j equal to zero for $j = 1,2,3$ yields the initial value problems

$$\mathcal{L}\,u_1 \;=\; 0\,, \qquad u_1(0,x) \;=\; g\,, \tag{6.5.4}$$

$$\mathcal{L}\,u_2 \;+\; F_2(u;u_1,u_1) \;=\; 0\,, \qquad u_2(0,x) \;=\; 0\,, \tag{6.5.5}$$

$$\mathcal{L}\,u_3 + F_2(u;u_2,u_1) + F_2(u;u_1,u_2) + F_3(u;u_1,u_1,u_1) \;=\; 0\,, \quad u_3(0,x) = 0\,, \tag{6.5.6}$$

which determine u_j for $j = 1,2,3$. The pattern is clear. The initial value problem determining u_j is linear in u_j with source terms which are nonlinear functions of $u_1, \ldots, u_{j-1}$.

Exercise. *Suppose that the u_j are determined by solving these initial value problems. Then define $u_{\rm approx}(\delta)$ using Borel's theorem so that*

$$u_{\rm approx}(\delta) \;\sim\; \sum_{j\ge 1}\delta^j\,u_j\,, \quad \text{in} \quad C([0,T]\,;\,H^s(\mathbb{R}^d))\,.$$

Prove that for δ sufficiently small the exact solution of the initial value problem exists on $[0,T]$ and $u_{\rm exact)}(\delta) - u_{\rm approx}(\delta) \sim 0$ in $C([0,T]\,;\,H^s(\mathbb{R}^d))$. **Hint.** Compute a nonlinear equation for the error which has source terms $O(\delta^\infty)$. Use the method of Theorem 6.5.1.**ii.** **Discussion.** The key element is the stability argument at the end which shows that a nonlinear problem with infinitely small sources has a solution which is infinitely small. In science texts it is routine to overlook the need for such stability arguments.

The next result is stronger than that of the exercise.

Theorem 6.5.2. *If $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ is a solution then the map $\mathcal{N}$ from initial data to solution is smooth from a neighborhood of $u(0)$ to $C([0,T]\,;\,H^s(\mathbb{R}^d))$. The derivative is given by $\mathcal{N}_1(u(0),g) = u_1$ from (6.5.4). Derivatives of each order are uniformly bounded on the neighborhood.*

Proof. The preceding computations show that if $\mathcal{N}$ is differentiable then $\mathcal{N}_1(u(0),g) = u_1$ from (6.5.4). It suffices to show that this is the derivative of $\mathcal{N}$ and that the map from $u(0)$ to $\mathcal{N}_1(u(0),\cdot)$ is locally bounded and smooth with values in the linear maps from $H^s(\mathbb{R}^d)$ to $C([0,T]\,;\,H^s(\mathbb{R}^d))$.

To prove differentiability let $u := \mathcal{N}(u(0))$ be the base solution and $v := u + u_1$ be the first approximation. Then

$$L\,u + F(u) \;=\; 0\,, \qquad L\,v + F(v) \;=\; F(u+u_1) - F_1(u,u_1)\,.$$

Schauder's lemma together with Taylor's theorem shows that

$$\big\|F(u+u_1) - F_1(u,u_1)\big\|_{C([0,T]\,;\,H^s)} \;\le\; C\,\big\|u_1\big\|^2_{C([0,T]\,;\,H^s)}$$

Since the initial values of u and v are equal, the basic linear energy estimate proves that

$$\big\|u - v\big\|_{C([0,T]\,;\,H^s)} \;\le\; C\,\big\|u_1\big\|^2_{C([0,T]\,;\,H^s)}\,.$$

This proves that $\mathcal{N}$ is differentiable and the formula for the derivative. The formula implies that the derivative is locally bounded.

The derivative is computed by solving (6.5.4). Since u is a differentiable function of $u(0)$ with locally bounded derivative. Then $F_1(u,\cdot)$ is differentiable with locally bounded derivative. As in the proof of differentiability, it follows that $\mathcal{N}_1(u(0),\cdot)$ is a differentiable function of $u(0)$ with locally bounded derivative. The higher differentiability follows by an inductive argument. ∎

§6.6 The Cauchy problem for quasilinear symmetric hyperbolic systems.

Since most quasilinear systems which occur in practice are real, we will present only that case. The equations have the form

$$L(u,\partial)u \;:=\; \sum_{\mu=0}^{d} A_\mu(u)\,\partial_\mu u \;=\; f\,, \tag{6.6.1}$$

where the coefficient matrices A_μ are smooth symmetric matrix valued functions of u defined on an open subset of $\mathbb{R}^d$. The leading coefficient, $A_0(u)$, is assumed to be strictly positive. The leading coefficient, $A_0(u)$, is assumed to be strictly positive. One can almost as easily treat coefficients which are function of y and u.

The existence theorem is local in time, and for small times the values of u are close to values of the initial data. Thus for convenience we can modify the coefficients outside a neighborhood of the values taken by the initial data to arrive at a system with everywhere defined smooth matrix valued coefficients. Even more we may suppose that the coefficients take constant values outside a compact subset of u space.

In contrast to the linear case, one cannot reduce to the case $A_0 = I$. However, if one is interested only in solutions which take values near a constant value $\underline{u}$, changing variable to $v := A_0(\underline{u})^{1/2} u$ one can reduce to the case $A_0(\underline{u}) = I$. This is useful for quasilinear geometric optics.

§6.6.1. Existence of solutions.

Local existence is analogous to Theorem 6.4, except that it is important that the coefficients $A_\mu(u(x))$ be lipschitz continuous functions of y. For this reason we work in Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > 1+d/2$. The importance of the Lipschitz condition is seen from the basic $L^2(\mathbb{R}^d)$ energy law when $f = 0$,

$$\frac{d}{dt}\Big(A_0(u)\,u(t)\,,\,u(t)\Big) = \Big(\big(\sum_\mu \partial_\mu(A_\mu(u))\big)\;u(t)\,,\,u(t)\Big), \quad \operatorname{div} A := \sum_\mu \partial_\mu(A\mu(u)). \tag{6.6.2}$$

To control the growth of the L^2 norm uses the lipschitz bound. It is not obvious but is true, that the same bound suffices to control the growth of higher derivatives. The existence part of the following Theorem is essentially due to Schauder [Sch].

Theorem 6.6.1. *If $\mathbb{N} \ni s > 1 + d/2$, $f \in L^1_{loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$, and $g \in H^s(\mathbb{R}^d)$, then there is a $T > 0$ and a unique solution*

$$u \;\in\; \cap_{j=0}^s\; C^j([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$$

to the initial value problem

$$L(u,\partial)u \;=\; f\,, \qquad u(0,x) \;=\; g(x)\,. \tag{6.6.3}$$

The time T can be chosen uniformly for f, g belonging to bounded subsets of $L^1_{loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$ and $H^s(\mathbb{R}^d)$ respectively. Therefore, there is a $T_* \in]0,\infty]$ and a maximal solution in $\cap_j C^j([0,T_*[\,;\,H^{s-j}(\mathbb{R}^d))$. If $T_* < \infty$ then $\lim_{t\nearrow T_*}\|u(t)\|_{H^s(\mathbb{R}^d)} = \infty$. A more precise result is

$$\limsup_{t\nearrow T_*}\; \|u(t), \nabla_y u(t)\|_{L^\infty(\mathbb{R}^d)} \;=\; \infty\,. \tag{6.6.4}$$

Remark. If we had not modified the coefficients to be everwhere defined and smooth, the blow up criterion would be that either (6.6.4) occurs or, the values of u approach the boundary of the domain where the coefficients are defined.

This is so since if one has a solution of the original system whose values are taken in a compact subset K of the domain of definition of the coefficients, one can modify the coefficients outside a compact neighborhood of K. Theorem 6.6.2 implies that there is a solution on a larger time interval.

The standard proof of Theorem 6.6.1 proceeds by considering the sequence of approximate solutions satisfying

$$L(u^\nu,\partial)u^{\nu+1} \;=\; f\,, \qquad u^{\nu+1}\big|_{t=0} \;=\; g\,.$$

The linear equation satisfied by $u^{\nu+1}$ has coefficients $A_\mu(u^\nu)$ depending on u^ν for which one has only H^s control. The key to the proof is to derive *a priori* estimates for solutions of linear symmetric hyperbolic intitial value problems with coefficient matrices having only H^s regularity (see [Metivier, Lax]).

Schauder's approach was to approximate the functions A_μ by polynomials in u and the data f, g by real analytic functions and to use the Cauchy-Kowalsekaya Theorem (see [Courant-Hilbert]). *A priori* estimates are used to control the approximate solutions on an fixed, possibly small, time interval. We solve the equation by the method of finite differences. A disadvantage of this method is that it reproves the linear existence theorem. An advantage is that the basic *a priori* estimate for the difference scheme allows one to prove existence and the sharp blowup criterion at the same time.

Proof. For ease of reading, we present the case $f = 0$. The approximate solution u^h is the unique local solution of the ordinary differential equation in $H^s(\mathbb{R}^d)$

$$A_0(u)\partial_t u^h \;+\; \sum_j A_j(u^h)\,\delta_j^h u^h \;=\; 0\,, \qquad u^h(0,x) \;=\; g(x)\,. \tag{6.6.5}$$

For each fixed $h > 0$, the map

$$w \;\mapsto\; A_0(u)^{-1}\sum_j A_j(u^h)\,\delta_j^h w^h$$

from $H^s(\mathbb{R}^d)$ to itself is uniformly lipschitzean on bounded subsets. It follows that there is a unique maximal solution

$$u^h \;\in\; C^1([0,T_*^h[\,;\,H^s(\mathbb{R}^d))\,, \qquad T_*^h \in]0,\infty]\,.$$

If $T_*^h < \infty$, then $\lim_{t\nearrow T_*^h}\|u^h(t)\|_{H^s(\mathbb{R}^d)} = \infty$.

The heart of the existence proof are uniform estimates for u^h on an h independent interval. The starting point is an $L^2(\mathbb{R}^d)$ estimate,

$$\frac{d}{dt}\Big(A_0(u^h)\,u^h(t)\,,\,u^h(t)\Big) \;=\; \Big(\big(\partial_t(A_0(u^h)\big)\,u^h\,,\,u^h\Big) \;+\; \sum_j \Big(\big(A_j\delta_j^h + (A_j\delta_j^h)^*\big)u^h\,,\,u^h\Big)\,.$$

Thanks to the symmetry of A_j and the antisymmetry of δ_j^h,

$$A_j\delta_j^h + (A_j\delta_j^h)^* \;=\; [A_j(u^h)\,,\,\delta_j^h]\,.$$

There is a constant, $C = C(A_\mu)$, so that

$$\|\partial_t A_0(u^h)\|_{\mathrm{Hom}(L^2(\mathbb{R}^d))} \;+\; \|A_j\delta_j^h + (A_j\delta_j^h)^*\|_{\mathrm{Hom}(L^2(\mathbb{R}^d))} \;\leq\; C\,\|\nabla_y u^h(t)\|_{L^\infty(\mathbb{R}^d)}\,. \tag{6.6.6}$$

For $|\alpha| \leq s$ and $\partial = \partial_x$, compute

$$\begin{aligned}\frac{d}{dt}\Big(A_0(u^h)\,\partial^\alpha u^h(t)\,,\,\partial^\alpha u^h(t)\Big) \;&=\; \Big(A_0(u^h)\partial^\alpha\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \\ &\Big(A_0(u^h)\partial^\alpha u^h\,,\,\partial^\alpha\partial_t u^h\Big) + \Big(\big(\partial_t A_0(u^h)\big)\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \\ &:=\; \Big(A_0(u^h)\partial^\alpha\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \Big(A_0(u^h)\partial^\alpha u^h\,,\,\partial^\alpha\partial_t u^h\Big) + E_1\,,\end{aligned} \tag{6.6.7}$$

beginning the collection of terms which we will prove are acceptably large.
The first term on the right of (6.6.7) is equal to

$$\Big(\partial^\alpha A_0(u^h)\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \Big([A_0(u^h),\partial^\alpha]\partial_t u^h,\partial^\alpha u^h\Big) \;:=\; \Big(\partial^\alpha A_0(u^h)\partial_t u^h\,,\,\partial^\alpha u^h\Big) + E_2\,. \tag{6.6.8}$$

Analogously, the symmetry of A_0 shows that the second term in (6.6.7) is equal to

$$\begin{aligned}\Big(\partial^\alpha u^h\,,\,A_0(u^h)\partial^\alpha\partial_t u^h\Big) \;&=\; \Big(\partial^\alpha u^h\,,\,\partial^\alpha A_0(u^h)\partial_t u^h\Big) \;+\; \Big(\partial^\alpha u^h,[A_0(u^h),\partial^\alpha]\partial_t u^h\Big) \\ &:=\; \Big(\partial^\alpha u^h\,,\,\partial^\alpha A_0(u^h)\partial_t u^h\Big) \;+\; E_3\,.\end{aligned} \tag{6.6.9}$$

Using the differential equation, the sum of the nonerror terms in (6.6.8-9) is equal to the sum on j of

$$\begin{aligned}&\Big(\partial^\alpha u^h\,,\,\partial^\alpha A_j(u^h)\,\delta_j^h u^h\Big) \;+\; \Big(\partial^\alpha A_j(u^h)\,\delta_j^h u^h\,,\,\partial^\alpha u^h\Big) \\ &\quad=\; \Big(\partial^\alpha u^h\,,\,A_j(u^h)\,\delta_j^h\partial^\alpha u^h\Big) \;+\; \Big(A_j(u^h)\,\delta_j^h\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \;+\; E_4 \\ &\quad=\; \Big(\big(A_j(u^h)\,\delta_j^h + (A_j(u^h)\delta_j^h)^*\big)\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \;+\; E_4 \\ &\quad:=\; E_5 \;+\; E_4\,,\end{aligned} \tag{6.6.10}$$

where

$$E_4 \;:=\; \Big(\partial^\alpha u^h\,,\,[\partial^\alpha, A_j(u^h)]\,\delta_j^h u^h\Big) \;+\; \Big([\partial^\alpha, A_j(u^h)]\,\delta_j^h u^h\,,\,\partial^\alpha u^h\Big)\,.$$

Denote by

$$\mathcal{E}(w) \;:=\; \sum_{|\alpha|\leq s}\Big(A_0(w)\partial_x^\alpha w\,,\,\partial_x^\alpha w\Big)\,.$$

Since A_0 is strictly positive, there is a constant C independent of w so that

$$\frac{1}{C} \sum_{|\alpha| \le s} \|\partial_x^\alpha w(t)\|^2_{L^2(\mathbb{R}^d)} \ \le\ \mathcal{E}(w) \ \le\ C \sum_{|\alpha| \le s} \|\partial_x^\alpha w\|^2_{L^2(\mathbb{R}^d)} \,. \tag{6.6.11}$$

Summing over all $|\alpha| \le s$ yields

$$\frac{d\mathcal{E}(u^h(t))}{dt} \ =\ \sum_{j=1}^{5} E_j \,. \tag{6.6.12}$$

Lemma 6.6.2. *For all $R > 0$, $1 \le j \le 5$, and $0 < h < 1$, there is a constant $C(R)$ depending only on L and R so that*

$$\big\|u^h(t), \nabla_y u^h(t)\big\|_{L^\infty(\mathbb{R}^d)} \ \le\ R \quad \Longrightarrow \quad |E_j| \ \le\ C(R) \sum_{|\alpha| \le s} \|\partial_x^\alpha u^h(t)\|^2_{H^s(\mathbb{R}^d)} \,.$$

Proof of Lemma. The cases $j = 1$ and $j = 5$ follow from (6.6.6). The remaining three cases are similar and we present only $j = 3$ which is the worst. It suffices to show that

$$\|[\partial^\alpha, A_j(u^h)]\, \partial_t u^h\|^2_{L^2(\mathbb{R}^d)} \ \le\ C(R) \sum_{|\alpha| \le s} \|\partial_x^\alpha u^h(t)\|^2_{H^s(\mathbb{R}^d)} \,. \tag{6.6.13}$$

The quantity on the left of (6.6.13) is a linear combination of terms

$$\partial^\beta(A_j(u^h))\ \partial^\gamma\big((A_0^{-1}A_j)(u^h)\delta_j^h\, u^h\big)\,, \qquad \beta + \gamma = \alpha, \quad \beta \ne 0\,.$$

Since $\beta \ne 0$ this is equal to

$$\begin{aligned}(\partial^\beta A_j(u^h) - A_j(0))\, \partial^\gamma\big((A_0^{-1}A_j(u^h) - A_0^{-1}A_j(0))\delta_j^h\, u^h\big) \\ +\ \partial^\beta(A_j(u^h) - A_j(0))\, (A_0^{-1}A_j)(0)\partial^\gamma \delta_j^h\, u^h \,.\end{aligned}$$

Estimate

$$\|A_j(u^h) - A_j(0)\|_{L^\infty} \ +\ \|(A_0^{-1}A_j)(u^h) \ -\ (A_0^{-1}A_j)(0)\|_{L^\infty} \ \le\ C(R)\,,$$

and from Moser's inequality,

$$\|A_j(u^h) - A_j(0)\|_{H^s} \ +\ \|(A_0^{-1}A_j)(u^h) \ -\ (A_0^{-1}A_j)(0)\|_{H^s} \ \le\ C(R)\, \|u^h\|^{1/2}_{H^s}\,.$$

The Gagliardo-Nirenberg estimates then imply (6.6.13). ∎

The local solution is constructed so as to take values in the set

$$\mathcal{W} \;:=\; \Big\{ w \in H^s(\mathbb{R}^d) \; ; \; \mathcal{E}(w) \le \mathcal{E}(g) + 1 \Big\} .$$

Choose $R > 0$ so that

$$w \in \mathcal{W} \quad \Longrightarrow \quad \|w\|_{L^\infty} + \|\nabla_x w\|_{L^\infty} + \| \sum_j A_j(w) \delta_j^h w \|_{L^\infty} \; < \; R \, .$$

So long as $u^h(t)$ stays in $\mathcal{W}$, one has

$$\frac{d\mathcal{E}(u^h(t))}{dt} \;\le\; C(R)\, C\, \mathcal{E}(u^h(t)) \;\le\; C(R)\, C \Big(\mathcal{E}(g) + 1 \Big) .$$

with C from (6.6.11) Therefore,

$$\mathcal{E}(u^h(t)) - \mathcal{E}(g) \;\le\; T\, C(R)\, C \Big(\mathcal{E}(g) + 1 \Big) .$$

Define T by

$$T\, C(R)\, C \Big(\mathcal{E}(g) + 1 \Big) \;=\; \frac{1}{2} \, .$$

If follows that for all h, u^h takes values in $\mathcal{W}$ for $0 \le t \le T$.

Using this uniform bound one can pass to a subsequence which converges weakly in $L^\infty([0,T]\,;\, H^s(\mathbb{R}^d))$ and stongly in $C^j([0,T]\,;\, H^{s-j}(\mathbb{R}^d))$ for $1 \le j \le s$.

The limit satisfies the initial value problem and also

$$\frac{d\mathcal{E}(u(t))}{dt} \;\le\; C(R)\, \mathcal{E}(u(t)) \, . \tag{6.6.14}$$

This together with the uniform continuity of u implies that $\|u(t)\|_{H(\mathbb{R}^d)}$ is continuous. It follows that $u \in C([0,T]\,;\, H^s(\mathbb{R}^d))$. That $\partial_t^j u \in C([0,T]\,;\, H^{s-j}(\mathbb{R}^d))$ follows by using the differential equation to express these derivatives in terms of spatial derivatives as in the semilinear case.

Uniqueness is proved by deriving a linear equation for the difference $w := u - v$ of two solutions u and v. Toward that end compute

$$A_\mu(u)\partial_\mu u - A_\mu(v)\partial_\mu v \;=\; A_\mu(u)\partial_\mu(u-v) \;+\; (A_\mu(u) - A_\mu(v))\partial_\mu v \, .$$

Write $A_\mu(u) - A_\mu(v) \;=\; \mathcal{G}_\mu(u,v)\,(u-v)$, to find

$$A_\mu(u)\partial_\mu u - A_\mu(v)\partial_\mu v = \mathcal{A}_\mu \partial w + \mathcal{B}_\mu w,$$

$$\mathcal{A}_\mu(y) := A_\mu(u(y)), \quad \mathcal{B}_\mu(y) := \mathcal{G}_\mu(u(y), v(y))\, \partial_\mu v(y).$$

Therefore

$$\mathcal{L}(y,\partial)\,w \;=\; 0\,, \qquad \mathcal{L}(y,\partial_y) := \sum \Big(\mathcal{A}_\mu \partial_\mu + \mathcal{B}_\mu\Big)\,. \tag{6.6.15}$$

The energy method yields

$$\frac{d}{dt}\big(\mathcal{A}_0\, w(t), w(t)\big) \;\le\; C\,\big(\mathcal{A}_0\, w(t)\,,\, w(t)\big)\,, \tag{6.6.16}$$

Since $w|_{t=0} = 0$, it follows that $w = 0$ which is the desired uniqueness.

All that remains is the proof of the precise blow up criterion (6.6.4). This is immediate since if the lipschitz norm does not blow up, then (6.6.14) implies that the $H^s(\mathbb{R}^d)$ norm does not blow up. This completes the proof of Theorem 6.6.1. ∎

§6.6.2. Examples of breakdown.

In this section we exhibit a simple mechanism, wave breaking, for the breakdown of solutions with u bounded and $\nabla_x u$ diverging to infinity as $t \nearrow T^*$. The method of proof leads to two Liouville type theorems.

The classic example is Burgers' equation

$$u_t + u\, u_x \;=\; 0\,. \tag{6.6.17}$$

For a smooth solution on $[0,T]\times\mathbb{R}^d$ the equation shows that u is constant on the integral curves of $\partial_t + u\,\partial_x$. Therefore those integral curves are straight lines.

For the solution of the initial value problem with

$$u(0,x) \;=\; g(x) \in C_0^\infty(\mathbb{R})\,, \tag{6.6.18}$$

the value of u on the line $(t, x+g(x)t)$ must be equal to $g(x)$. This is an implicit equation,

$$u(t, x+tg(x)) \;=\; g(x)\,, \tag{6.6.19}$$

uniquely determining a smooth solutions for t small.

However, if g is not monotone increasing, consider the lines starting from two points $x_1 < x_2$ where $g(x_1) > g(x_2)$. The lines intersect in $t > 0$ at which point the conditions that u take value $g(x_1)$ and $g(x_2)$ contradict. Thus the solution must break down before this time. While the solution is smooth, $u(t)$ is a rearrangement of $u(0)$ so the sup norm of u does not blow up. The existence theorem shows that the gradient must explode.

That the gradient explodes can also be proved by differentiating the equation to show that $v := \partial_x u$ satisfies

$$v_t + u\,\partial_x v + v^2 \;=\; 0\,.$$

This equation is exactly solvable since

$$\frac{d}{dt}\, v(t, x+g(x)t) \;=\; v_t + u\,\partial_x v \;=\; -\,v^2\,.$$

Therefore,

$$v(t, x + g(x)t) \;=\; \frac{g'(x)}{1 - g'(x)t}\,.$$

Proposition 6.6.2. *The maximal solution of the initial value problem (6.6.4-5) satisfies*

$$T_* \;=\; \frac{1}{-\min g'(x)}\,. \tag{6.6.20}$$

Proof. The preceding computations shows that T_* can be no larger than the right hand side of (6.6.20). On the other hand, the implicit function theorem provides a smooth solution of (6.6.18) so long as the map $x \mapsto x + tg(x)$ is a diffeomorphism from $\mathbb{R}$ to itself. This holds exactly for t smaller than the right hand side of (6.6.20). ∎

The method of proof yields the following results of Liouville type.

Theorem 6.6.3. i. *The only global solutions $u \in C^1(\mathbb{R}^{1+d})$ of Burgers' equation 6.6.5 are the constants.*
ii. *The only global solutions $\psi(x) \in C^3(\mathbb{R}^d)$ of the eikonal equation $|\nabla_x \psi| = 1$ are affine functions.*

Proof. i. Denote $g(x) := u(0,x)$. If there is a point with $g'(\underline{x}) < 0$ the above proof shows that $u_x(t, \underline{x} + g(\underline{x})t)$ diverges as $t \nearrow T^*$. If there is a point with $g'(\underline{x}) > 0$ then an analogous argument shows that $u_x(t, \underline{x} + g(\underline{x})t)$ diverges as $t \searrow -1/g'(\underline{x})$. Therefore g is constant and the result follows.
ii. Denote by

$$V \;:=\; 2 \sum \partial_j \psi \, \partial_j\,,$$

a C^1 vector field. Differentiating $\sum (\partial_j \psi)^2 = 1$, yields for each partial derivative $\partial \psi$,

$$V\,\partial\psi \;=\; 0\,, \qquad 0 \;=\; V\,\partial^2 \psi \;+\; 2\sum_j (\partial_j \partial \psi)^2 \;\geq\; V\,\partial^2\psi \;+\; (\partial^2 \psi)^2\,. \tag{6.6.21}$$

The first implies that $\nabla_x \psi$ is constant on the integral curves of V. Therefore the integral curves are stationary points or straight lines $\underline{x} + s\nabla_x \psi(\underline{x})$.
If ψ is not linear, there is a point $\underline{x}$ at which the matrix of second derivatives at $\underline{x}$ is not equal to zero. The same holds on a neiborhood of $\underline{x}$ so we can choose $\underline{x}$ so that $\nabla_x \psi(\underline{x}) \neq 0$. A linear change of coordinates yields $\partial_1^2 \psi(\underline{x}) \neq 0$.
Then

$$h(s) \;:=\; \partial_1^2 \psi(\underline{x} + 2s\nabla_x \psi(\underline{x}))\,, \quad \text{satisfies} \quad \frac{dh}{ds} \;\leq\; -h(s)^2\,.$$

If $h(0) < 0$ then h diverges to $-\infty$ at a finite positive value of s. Similarly if $h(0) > 0$ then h diverges to $+\infty$ at a finite negative value of s. Thus ψ cannot be globally C^2. ∎

§6.6.3. Dependence on initial data.

Theorem 6.6.1 shows that the map from $u(0)$ to $u(t)$ maps $H^s(\mathbb{R}^d)$ to itself and takes bounded sets to bounded sets. In contrast to the case of semilinear equations, this mapping is not smooth. It is not even lipschitzean. It is lipschitzean as a mapping from $H^s(\mathbb{R}^d)$ to $H^{s-1}(\mathbb{R}^d)$.

Suppose that $v \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ with $s > 1 + d/2$ solves (6.6.1). Denote by $\mathcal{N}$ the map $u(0) \mapsto u(\cdot)$ from initial data to solution. It is defined on a neighborhood, $\mathcal{U}$, of $v(0)$ in $H^s(\mathbb{R}^d)$ to $\cap_j C^j([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$.

Theorem 6.6.4. *Decreasing the neighborhood $\mathcal{U} \subset H^s(\mathbb{R}^d)$ if necessary, the map*

$$\mathcal{U} \ni u(0) \;\mapsto\; u(\cdot) \;\in\; \cap_{\{j\,:\,s-j-1>d/2\}}\, C^j([0,T]\,;\,H^{s-1-j}(\mathbb{R}^d))$$

is uniformly lipschitzean.

Proof. The assertion follows from the linear equation (6.6.15) for the difference of two solutions. The coefficients $\mathcal{A}_\mu$ belong to $C^j([0,T]\,:\,H^{s-j}(\mathbb{R}^d))$ for $0 \le j \le s$. On the other hand, the coefficients $\mathcal{B}_\mu \in C^j([0,T]\,:\,H^{s-j-1}(\mathbb{R}^d))$ for $0 \le j \le s-1$ have one less derivative. For this linear equation, the change of variable $\tilde w = \mathcal{A}_0^{-1/2} w$ reduces to the case $\mathcal{A}_0 = I$.

The estimate is proved by computing

$$\frac{d}{dt} \sum_{|\alpha| \le s-1} (\partial^\alpha \tilde w(t)\,,\,\partial^\alpha \tilde w)\,.$$

The restriction to $s-1$ comes from the fact that $\mathcal{B}$ is only $s-1$ times differentiable.

Exercise. *Carry out this proof using the proof of Theorem 6.6.1 as model.* ∎

We next prove differentiable dependence by the perturbation theory method of §6.5. Suppose that

$$L(v,\partial)\,v \;=\; 0\,,$$

and consider the perturbed problem

$$L(u,\partial)u \;=\; 0\,, \qquad u|_{t=0} \;=\; v(0) \;+\; g\,, \tag{6.6.22}$$

with g small. To compute the Taylor expansion, introduce the auxiliary problems

$$L(\tilde u,\partial)\tilde u \;=\; 0\,, \quad \tilde u|_{t=0} \;=\; v(0) \;+\; \delta g\,, \qquad \tilde u \;\sim\; u_0 + \delta u_1 + \delta^2 u_2 + \cdots\,. \tag{6.6.23}$$

Then $L(\tilde u,\partial)\tilde u$ has expansion in powers of δ computed from the expression

$$0 = \sum_\mu \Big(A_\mu(u_0) + \delta A'_\mu(u_0)(u_1) + \delta^2 A''_\mu(u_0)(u_1,u_1) + \cdots\Big)\,\partial_\mu\Big(u_0 + \delta u_1 + \delta^2 u_2 + \cdots\Big).$$

The $O(\delta^0)$ term yields

$$L(u_0, \partial)u_0 \ = \ 0\,, \qquad u_0|_{t=0} \ = \ u(0)\,, \tag{6.6.24}$$

yielding, $u_0 = v$, is the unperturbed solution.
The $O(\delta)$ term yields

$$\sum_\mu A_\mu(v)\partial_\mu u_1 \ + \ \sum_\mu \big[A'_\mu(v)u_1\big]\partial_\mu v \ = \ 0\,, \qquad u_1|_{t=0} \ = \ g\,. \tag{6.6.25}$$

Introduce the linearization of L at the solution v by

$$\mathbf{L}\, w \ := \ \sum_\mu A_\mu(v)\partial_\mu w \ + \ \sum_\mu \big[A'_\mu(v)(w)\big]\partial_\mu v\,. \tag{6.6.26}$$

The equation of first order perturbation theory becomes

$$\mathbf{L}\, u_1 \ = \ 0\,, \qquad u_1|_{t=0} \ = \ g\,. \tag{6.6.27}$$

In the zero order term of $\mathbf{L}$, the coefficient depends on ∂v so in general u_1 will be one derivative less regular than v.
The $O(\delta^2)$ terms yield

$$\mathbf{L}\, u_2 \ + \ \sum_\mu \big[A'_\mu(v)(u_1)\big]\partial_\mu u_1 \ + \ \sum_\mu \big[A''_\mu(v)(u_1, u_1)\big]\partial_\mu v \ = \ 0\,, \qquad u_2|_{t=0} \ = \ 0\,. \tag{6.6.28}$$

There is a source term depending on ∂u_1 so typically, u_2 will be one derivative less regular than u_1 and therefore two derivatives less regular than v.
Continuing in this fashion yields initial value problems determining u_j as symmetric j-multilinear functionals of g provided that v is sufficiently smooth.

Theorem 6.6.5. *Suppose that $s > 1 + d/2$, and $v \in C([0,T] \,:\, H^s(\mathbb{R}^d))$ satisfies (6.6.1). Then the map, $\mathcal{N}$, from initial data to solution is a differentiable function from a neighborhood of $v(0)$ in $H^s(\mathbb{R}^d)$ to $C([0,T]\,;\, H^{s-1}(\mathbb{R}^d))$. The derivative is locally bounded. If $s - j > d/2$ then $\mathcal{N}$ is j times differentiable as a map with values in $C([0,T]\,;\, H^{s-j}(\mathbb{R}^d))$. The derivatives are locally bounded.*

Sketch of Proof. The linear equation determining u_1 has coefficient which involve the first derivative of v. As a result u_1 will in general be one derivative less regular than v. That is as bad as it gets. It is not difficult to show using an estimate as in Theorems 6.5.2, 6.6.4 that

$$\Big\|\mathcal{N}(u(0)+g) - \Big(\mathcal{N}(u(0)) + u_1\Big)\Big\|_{C\left([0,T]\,;\,H^{s-1}(\mathbb{R}^d)\right)} \ \le \ C\,\|g\|^2_{H^s(\mathbb{R}^d)}\,.$$

This yields differentiability, the formula for the derivative, and local boundedness.

Similarly, the calculations before the Theorem show that if $\mathcal{N}$ is twice differentiable then one must have

$$\mathcal{N}_2(v(0), g, g) \;=\; u_2\,,$$

where u_2 is the solution of (6.6.28). It is straight forward to show that $\mathcal{N}_2$ so defined is a continuous quadratic map from $H^s \mapsto C\big([0,T]\,;\,H^{s-2}(\mathbb{R}^d)\big)$.

A calculation like that in Theorem 6.5.2 shows that

$$\Big\| \mathcal{N}(u(0)+g) - \Big(\mathcal{N}(u(0)) + u_1 + u_2\Big)\Big\|_{C\big([0,T]\,;\,H^{s-2}(\mathbb{R}^d)\big)} \;\le\; C\,\big\|g\big\|^3_{H^s(\mathbb{R}^d)}\,.$$

This is not enough to conclude that $\mathcal{N}$ is twice differentiable. What is needed is a formula for the variation of $\mathcal{N}_1(v(0), g)$ when $v(0)$ is varied. The derivative $\mathcal{N}_1(v(0), g) = u_1$ is determined by solving the linear Cauchy problem (6.6.27) which has the form

$$L(v, \partial)\, u_1 + B(v, \partial v)\, u_1 \;=\; 0\,, \qquad u_1(0) \;=\; g\,.$$

The map from $v(0)$ to the coefficients in (6.6.26) is differentiable and locally bounded from $H^s \to C([0,T]\,;\,H^{s-1})$. Provided that $s-1 > d/2+1$ it follows from a calculation like that used to show that $\mathcal{N}$ is differentiable, that the map from $v(0)$ to u_1 is differentiable from H^s to $C([0.T]\,;\,H^{s-2}(\mathbb{R}^d))$, that $\mathcal{N}$ is twice differentiable, and the second derivative is locally bounded. The straight forward but notationally challenging computations are left to the reader.

The inductive argument for higher derivatives is similarly passed to the reader. ∎

We next show by example that the loss of one derivative expressed in Theorems 6.6.4 and 6.6.5 is sharp.

The example is Burgers' equation, $v_t + v\, v_x = 0$, with initial data

$$v(0,x) \;=\; (x_+)^2\,, \qquad x_+ \;:=\; \max\{x, 0\}\,.$$

Since the initial data is nondecreasing and C^1 it follows that there is a solution $v \in C^1([0,\infty[\times\mathbb{R})$. The data is piecewise smooth with jump in second derivative at $x = 0$.

The characteristic through the initial singularity is the t axis. It follows from the description using characteristics, that $v = 0$ in $\{t \ge 0\,,\, x < 0\}$. And v is smooth up the the boundary of the positive octant $\{t \ge 0\,,\, x > 0\}$. Thus, v is piecewise smooth in $t \ge 0$ with singularities confined to $\{x = 0\}$ where v_{xx} jumps.

In this case, equation (6.6.27) is

$$\partial_t u_1 \;+\; v\,\partial_x u_1 \;+\; v_x u_1 \;=\; 0, \qquad u_1(0) \;=\; g\,. \tag{6.6.29}$$

Consider a smooth initial function, for example a function g which is equal to one on a neighborhood of $x = 0$. Since $v \in C^1$ it follows that the solution is continuous. The method of characteristics shows that u_1 is equal to one on the left side of the t axis.

On the other hand, differentiating with respect to x and letting x tend to zero from above shows that the function

$$J(t) \ := \ \partial_x u_1(t, 0+) \ := \ \lim_{x \searrow 0} \ \partial_x u_1(t, x)$$

satisfies

$$\frac{dJ}{dt} \ = \ v_{xx}(0+) \ = \ 1\,.$$

Therefore the first derivative of u_1 jumps across the t axis. While the solution is piecewise smooth with jumps in second derivatives, u_1 is piecewise smooth with jumps in the first derivative. This is exactly the loss of one derivative in Theorems 6.6.4-6.6.5.

§6.7. Global small solutions for maximally dispersive nonlinear systems.

In dimensions greater than one, solutions of linear constant coefficient hyperbolic systems with constant coefficients and no hyperplanes in their characteristic variety tend to zero as $t \to \infty$. The maximally dispersive systems decay as fast as is possible, consistent with L^2 conservation. Consider a nonlinear system

$$L(\partial)\, u + G(u) \ = \ 0\,, \qquad G(0) = 0\,, \quad \nabla_u G(0) = 0\,.$$

Solutions with small initial data, say $u\big|_{t=0} = \epsilon\, f$ are approximated by solutions of the linearized equation

$$L(0, \partial) u \ = \ 0\,,$$

with the same initial data. On bounded time intervals, the error is $O(\epsilon^2)$. When solutions of $L\, u = 0$ decay in L^∞, $G(u)$ is even smaller. There is a tendency to approach linear behavior for large times. For $G = O(|u|^p)$ at the origin, the higher is p the stronger is the tendency. The higher is the dimension, the more dispersion is possible and the stronger can be the effect.

We prove that for maximally dispersive systems in dimension $d \geq 4$ with $p \geq 3$, the Cauchy problem is globally solvable for small data. This line of investigation has been the subject of much research. The CBMS lectures of Strauss present a nice selection of topics. The important special case of perturbations of the wave equation was the central object of a program of F. John in which the contributions of S. Klainerman were capital. I recommend the books of Sogge, Hörmander, Shatah-Struwe, and Strauss for more information. The analysis we present follows ideas predating the John-Klainerman revolution. A quasilinear version including refined estimates for scattering operators can be found in [Satoh, Kajitani-Satoh] . The sharp result in the spirit of John-Klainerman is that there is global existence of small solution in $d \geq 4$ and $p \geq 2$. Estimates sufficient for the sharp result are proved in [Georgiev, Lucente, Ziliotti].

The global existence result is in sharp contrast to the example

$$\frac{\partial u}{\partial t} \ + \ \frac{\partial u}{\partial x_1} \ - \ u^2 \ = \ 0\,, \qquad u(0, x) \ = \ \epsilon\, \phi(x)\,, \quad 0 \leq \phi \in C_0^\infty(\mathbb{R}^d) \setminus 0\,,$$

for which solutions blow up in time $O(\epsilon^{-1})$ independent of dimension. The associated linear problem is completely nondispersive.

Assumtion 1. $L(\partial)$ is a maximally dispersive symmetric hyperbolic system with constant coefficients as in §3.4.

Assumption 2. $G(u)$ is a smooth nonlinear function whose leading Taylor polynomial at the origin is homogeneous of degree $p \geq 3$.

Theorem 6.7.1. *Suppose that $d \geq 4$, $p \geq 3$, and σ is an integer greater than $(d+1)/2$. For each $\delta_1 > 0$, there is a $\delta_0 > 0$ so that if*

$$\|f\|_{H^\sigma(\mathbb{R}^d)} + \|f\|_{W^{\sigma,1}(\mathbb{R}^d)} \ \leq\ \delta_0\,, \qquad \Big(\|f\|_{W^{\sigma,1}(\mathbb{R}^d)} := \sum_{|\alpha|\leq\sigma} \|\partial_x^\alpha f\|_{L^1(\mathbb{R}^d)}\Big)\,, \tag{6.7.1}$$

then the solution of the Cauchy problem

$$L\,u \ +\ G(u) \ =\ 0\,, \qquad u\big|_{t=0} \ =\ f\,, \tag{6.7.2}$$

exists globally and satisifes for all $t \in \mathbb{R}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \ \leq\ \langle t\rangle^{-(d-1)/2}\,\delta_1\,, \quad \text{and} \quad \|u(t)\|_{H^\sigma(\mathbb{R}^d)} \ \leq\ \delta_1\,. \tag{6.7.3}$$

There is a $c > 0$, so that for δ_1 small one can take $\delta_0 = c\,\delta_1$.

Proof. We treat the case of $t \geq 0$. For simplicity we treat only the case of G equal to a homogeneous polynomial. The modifications for the general case are outlined in an exercise after the proof.

Decreasing δ_1 makes the task more difficult. If $\delta_1 \leq 1$ is given, choosing δ_0 sufficiently small, the solution satisfies (6.7.3) on some maximal interval $[0,T[$, $T \in]0,\infty]$. The proof relies on *a priori* estimates for the solution on this maximal interval.

Denote by $S(t) := e^{-it\sum_j A_j\partial_j}$ the unitary operator on $H^s(\mathbb{R}^d)$ giving the time evolution for the linear equation $Lu = 0$,

$$\|S(t)\|_{H^s(\mathbb{R}^d)} \ =\ \|f\|_{H^s(\mathbb{R}^d)}\,. \tag{6.7.4}$$

The Theorem in §3.4.2 yields the estimate

$$\begin{aligned} \|S(t)\,f\|_{L^\infty(\mathbb{R}^d)} \ &\leq\ C_0\,\langle t\rangle^{-(d-1)/2}\,\Big(\|f\|_{H^s(\mathbb{R}^d)} + \sum_{j=-\infty}^{\infty} \big\|\,|D|^{(d+1)/2} f_j\big\|_{L^1}\Big) \\ &\leq\ C_1\langle t\rangle^{-(d-1)/2}\ \delta_0\,. \end{aligned} \tag{6.7.5}$$

Duhamel's formula reads

$$u(t) \ =\ S(t)\,f \ +\ \int_0^t S(t-s)\ G(u(s))\ ds\,. \tag{6.7.6}$$

For the homogeneous polynomial G we have Moser's inequality (**Exercise.** *Verify.*)

$$\|G(u)\|_{H^\sigma(\mathbb{R}^d)} \ \leq\ C_2\, \|u\|_{L^\infty(\mathbb{R}^d)}^{p-1}\ \|u\|_{H^\sigma(\mathbb{R}^d)}\,. \tag{6.7.7}$$

Use this to estimate

$$\begin{aligned} \|u(t)\|_{H^\sigma} \ &\leq\ \delta_0\ +\ \int_0^t C_2\,\big(\langle t-s\rangle^{-(d-1)/2}\,\delta_1\big)^{p-1}\ \delta_1\ ds \\ &\leq\ \delta_0 + C_3\,\delta_1^2\,, \qquad C_3 := C_2 \int_0^\infty \langle t\rangle^{-(d-1)/2}\,dt\,. \end{aligned} \tag{6.7.8}$$

The L^∞ norm satisfies,

$$\|u(t)\|_{L^\infty} \ \leq\ C_1\,\langle t\rangle^{(d-1)/2}\,\delta_0\ +\ \int_0^t \big\|S(t-s)\ G(u(s))\big\|_{L^\infty}\ ds\,. \tag{6.7.9}$$

Use the dispersive estimate (3.4.8-3.4.9) to find

$$\big\|S(t-s)\,G(u(s))\big\|_{L^\infty} \ \leq\ C_6\,\langle t-s\rangle^{-(d-1)/2}\,\|G(u(s))\|_{W^{\sigma,1}}\,. \tag{6.7.10}$$

Lemma 6.7.2. *There is a constant C so that for all u one has*

$$\|G(u)\|_{W^{\sigma,1}} \ \leq\ C\,\|u\|_{L^\infty}^{p-2}\ \|u\|_{H^\sigma}^2\,. \tag{6.7.11}$$

Proof of Lemma. Leibniz' rule shows that it suffices to show that if $|\alpha_1+\ldots+\alpha_p| = s \leq \sigma$ then

$$\big\|\partial^{\alpha_1}u\,\partial^{\alpha_2}u\ \cdots\ \partial^{\alpha_p}u\big\|_{L^1} \ \leq\ C\,\|u\|_{L^\infty}^{p-2}\ \|u\|_{L^2}\|\,|D|^s u\|_{L^2}\,.$$

Both sides have the dimensions ℓ^{d-s}.

Define $\theta_i := |\alpha_i|/s$ so $\sum\theta_i = 1$. The Gagliardo-Nirenberg estimate interpolating between $u\in L^\infty$ and $|D|^s u \in L^2$ is

$$\|\partial^{\alpha_i}u\|_{L^{p_i}} \ \leq\ C\,\|u\|_{L^\infty}^{1-\theta_i}\ \|\,|D|^s u\|_{L^2}^{\theta_i}\,, \qquad \frac{1}{p_i} = \frac{1-\theta_i}{\infty} + \frac{\theta_i}{2} = \frac{\theta_i}{2}\,.$$

Define $\theta := 1/(p-1)$ and interpolate between $\partial^{\alpha_i}u\in L^{p_i}$ and $\partial^{\alpha_i}u\in L^2$ to find

$$\|\partial^{\alpha_i}u\|_{L^{r_i}} \ \leq\ \|\partial^{\alpha_i}u\|_{L^{p_i}}^{1-\theta}\ \|\partial^{\alpha_i}u\|_{L^2}^{\theta}\,, \qquad \frac{1}{r_1} \ =\ \frac{1-\theta}{p_i} + \frac{\theta}{2}\,.$$

Therefore,

$$\|\partial^{\alpha_i}u\|_{L^{r_i}} \ \leq\ \|u\|_{L^\infty}^{(1-\theta_i)(1-\theta)}\ \|u\|_{L^2}^{(1-\theta_i)\theta}\ \|\,|D|^s u\|_{L^2}^{\theta_i}\,, \qquad 1 = \sum 1/r_i\,.$$

Hölder's inequality implies

$$\big\|\partial^{\alpha_1} u\, \partial^{\alpha_2} u \ \cdots\ \partial^{\alpha_p} u\,\big\|_{L^1} \ \le\ \Pi_{i=1}^p \|\partial^{\alpha_i} u\|_{L^{r_i}} \ \le\ C\, \|u\|_{L^\infty}^{p-2}\ \|u\|_{L^2}\ \|\,|D|^s u\|_{L^2}\,,$$

which completes the proof. ∎

Estimates (6.7.10-11) together with the hypothesis that $p \ge 3$ yield

$$\begin{aligned}\int_0^t \big\|S(t-s)\, G(u(s))\big\|_{L^\infty}\, ds \ &\le\ C_7 \int_0^{t/2} \langle t-s\rangle^{-(d-1)/2}\, \langle s\rangle^{-(p-2)(d-1)/2}\, \delta_1^2\, ds \\ &\le\ C_8\, \langle t\rangle^{-(d-1)/2}\, \delta_1^2\,.\end{aligned} \tag{6.7.12}$$

Combining yields

$$\|u(t)\|_{L^\infty} \ \le\ \Big(C_1\, \delta_0 + C_8\, \delta_1^2\Big) \langle t\rangle^{-(d-1)/2}$$

Decreasing δ_1 if necessary we may suppose that

$$C_3\, \delta_1^2 \ <\ \frac{\delta_1}{2}\,, \qquad \text{and} \qquad C_8\, \delta_1^2 \ <\ \frac{\delta_1}{2}\,.$$

Then, choose $\delta_0 > 0$ so that

$$\delta_0 \ +\ C_3\, \delta_1^2 \ <\ \frac{\delta_1}{2}\,, \qquad \text{and} \qquad C_1 \delta_0 \ +\ C_8\, \delta_1^2 \ <\ \frac{\delta_1}{2}\,.$$

With these choices, the estimates show that on the maximal interval $[0, T[$, one has

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \ \le\ \langle t\rangle^{-(d-1)/2}\, \frac{\delta_1}{2}\,, \quad \text{and} \quad \|u(t)\|_{H^s(\mathbb{R}^d)} \ \le\ \frac{\delta_1}{2}\,. \tag{6.7.13}$$

If T were finite, the solution would satisfy (6.7.3) on the interval $[0, T+\epsilon]$ for small positive ϵ violating the maximality of T. Therefore $T = \infty$. The estimate (6.7.13) on $[0, T[$ completes the proof. ∎

Exercise. *For the case of G which are not homogeneous show that there are smooth functions H_α and functions G_α homogeneous of degree p so that*

$$G(u) \ =\ \sum\ G_\alpha(u)\ H_\alpha(u)\,,$$

the sum being finite. Modify the Moser inequality arguments appropriately to prove the general result.

§6.8. The subcritical nonlinear Klein-Gordon equation in the energy space.

§6.8.1. Introductory remarks.

The mass zero nonlinear Klein-Gordon equation is

$$\Box_{1+d} u \ + \ F(u) \ = \ 0 \,. \tag{6.8.1}$$

where

$$F \in C^1(\mathbb{R})\,, \qquad F(0) = 0\,, \qquad F'(0) = 0\,. \tag{6.8.2}$$

The classic examples from quantum field theory are the equations with $F(u) = u^p$ with $p \geq 3$ an odd integer. For ease of reading we consider only real solutions.
The equation (6.8.1) is Lorentz invariant and if

$$G'(s) \ = \ F(s)\,, \qquad G(0) \ = \ 0\,, \tag{6.8.3}$$

The local energy density is defined as

$$e(u) \ := \ \frac{u_t^2 + |\nabla_x u|^2}{2} \ + \ G(u)\,. \tag{6.8.4}$$

Solutions $u \in H^2_{\rm loc}(\mathbb{R}^{1+d})$ satisfy the differential energy law,

$$\partial_t e - {\rm div}\big(u_t\, \nabla_x u\big) \ = \ u_t \Big(\Box u \ + \ F(u)\Big) = 0\,. \tag{6.8.5}$$

The corresponding integral conservation law for solutions suitably small at infinity is,

$$\partial_t \int_{\mathbb{R}^d} \frac{u_t^2 + |\nabla_x u|^2}{2} \ + \ G(u)\ dx \ = \ 0\,, \tag{6.8.6}$$

is one of the fundamental estimates in this section. Solutions are stationary for the Lagrangian,

$$\int_0^T \int_{\mathbb{R}^d} \frac{u_t^2 - |\nabla_x u|^2}{2} - G(u)\ dt\, dx\,.$$

When F is smooth, the methods of §6.3-6.4 yield local smooth existence.

Theorem 6.8.1. *If $F \in C^\infty$, $s > d/2$, $f \in H^s(\mathbb{R}^d)$, and $g \in H^{s-1}(\mathbb{R}^d)$, then there is a unique maximal solution*

$$u \ \in \ C\big([0, T_*[\ ;\ H^s(\mathbb{R}^d)\big) \ \cap \ C^1\big([0, T_*[\ ;\ H^{s-1}(\mathbb{R}^d)\big)\,.$$

satisfying

$$u(0,x) = f\,, \quad u_t(0,x) = g\,.$$

If $T_* < \infty$ then

$$\limsup_{t \to T_*} \ \|u(t)\|_{L^\infty(\mathbb{R}^d)} \ = \ \infty\,.$$

In favorable cases, the energy law (6.8.6) gives good control of the norm of $u, u_t \in H^1 \times L^2$. Controling the norm of the difference of two solutions is, in contrast, a very difficult problem for which many fundamental questions remain unresolved.

An easy first case is nonlinearities F which are uniformly lipschitzean. In this case, there is global existence in the energy space.

Theorem 6.8.2. *If F satisfies $F' \in L^\infty(\mathbb{R})$, then for all Cauchy data $f, g \in H^1 \times L^2$ there is a unique solution*

$$u \in C\big(\mathbb{R}\,;\; H^1(\mathbb{R}^d)\big) \;\cap\; C^1\big(\mathbb{R}\,;\; L^2(\mathbb{R}^d)\big)\,.$$

For any finite T, the map from data to solution is uniformly lipschitzean from $H^1 \times L^2$ to $C([-T,T\,;\,H^1) \cap C^1([-T,T]\,;\,L^2)$. If $f, g \in H^2 \times H^1$ then

$$u \;\in\; L^\infty\big(\mathbb{R}\,;\; H^2(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; L^\infty\big(\mathbb{R}\,;\; H^1(\mathbb{R}^d)\big)\,.$$

If $f, g \in H^s \times H^{s-1}$ with $1 \le s < 2$, then

$$u \;\in\; C\big(\mathbb{R}\,;\; H^s(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; C\big(\mathbb{R}\,;\; H^{s-1}(\mathbb{R}^d)\big)\,.$$

Sketch of Proof. The key estimate is the following. If u and v are solutions then

$$\Box(u-v) \;=\; F(v) - F(u), \qquad |F(u) - F(v)| \;\le\; C|u-v|\,.$$

Multiplying by $u_t - v_t$ yields

$$\frac{d}{dt}\int (u_t - v_t)^2 + |\nabla_x(u-v)|^2 dx \;=\; 2\int (u_t - v_t)\,\big(F(v) - F(u)\big)\,dx \;\le\; C\,\|u_t - v_t\|_{L^2}^2\,\|u-v\|_{L^2}^2\,.$$

It follows that for any T there is an *a priori* estimate

$$\sup_{|t|\le T}\;\Big(\|u(t) - v(t)\|_{H^1} \;+\; \|u_t - v_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0) - v(0)\|_{H^1} \;+\; \|u_t(0) - v_t(0)\|_{L^2}\Big)\,.$$

This estimate exactly corresponds to the asserted Lipschitz continuity of the map from data to solutions.

Applying the estimate to $v = u(x+h)$ and taking the supremum over small vectors h, yields an *a priori* estimate

$$\sup_{|t|\le T}\;\Big(\|u(t)\|_{H^2} \;+\; \|u_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0)\|_{H^2} \;+\; \|u_t(0)\|_{H^1}\Big)\,,$$

which is the estimate correponding to the H^2 regularity. ∎

Higher regularity for dimensions $d \ge 10$ is an outstanding open problem. For example, for $d \ge 10$, smooth compactly supported initial data, and $F \in C_0^\infty$ or $F = \sin u$, it is not known if the above global unique solutions are smooth. For $d \le 9$ the result can be found in [Brenner-vonWahl 1982]. Smoothness would follow if one could prove that $u \in L^\infty_{\rm loc}$. What is needed is to show that the solutions do not get large in the pointwise sense. Compared to the analogous regularity problem for Navier-Stokes this problem has the advantage that solutions are known to be unique and depend continuously on the data.

§6.8.2. The ordinary differential equation and nonlipshitzean F.

Concerning global existence for functions $F(u)$ which may grow more rapidly than linearly as $u \to \infty$, the first considerations concern solutions which are independent of x and therefore satisfy the ordinary differential equation,

$$u_{tt} \ + \ F(u) \ = \ 0\,. \tag{6.8.7}$$

Global solvability of the ordinary differential equation is analysed using the energy conservation law

$$\Big(\frac{u_t^2}{2} \ + \ G(u)\Big)' \ = \ u_t\Big(u_{tt} + F(u)\Big) \ = \ 0\,.$$

Think of the equation as modeling a nonlinear spring. The spring force is attractive, that is pulls the spring toward the origin when

$$F(u) > 0 \quad \text{when} \quad u > 0 \qquad \text{and,} \qquad F(u) < 0 \quad \text{when} \quad u < 0\,.$$

In this case one has $G(u) > 0$ for all $u \neq 0$. Conservation of energy then gives a pointwise bound on u_t uniform in time

$$u_t^2(t) \ \le \ u_t^2(0) + 2G(u(0))\,, \qquad |u_t(t)| \ \le \ \big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

This gives a pointwise bound

$$|u(t)| \ \le \ |u(0)| + |t|\big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

In particular the ordinary differential equation has global solutions.

In the extreme opposite case consider the replusive spring force $F(u) = -u^2$ and $G(u) = -u^3/3$. The energy law asserts that $u_t^2/2 - u^3/3 := E$ is independent of time. Consider solutions with

$$u(0) > 0, \quad u_t(0) > 0 \qquad \text{so} \quad E > -\frac{u^3(0)}{3}\,.$$

For all $t > 0$,

$$|u_t| \ = \ \Big|\frac{u^3}{3} + E\Big|^{1/2}\,,$$

At $t = 0$ one has

$$u_t(0) \ = \ \Big(\frac{u^3(0)}{3} + E\Big)^{1/2} > 0\,.$$

Therefore u increases and $u^3/3 + E$ stays positive and one has for $t \geq 0$

$$u_t(t) \;=\; \Big(\frac{u^3(t)}{3} + E\Big)^{1/2} > 0\,.$$

Both u and u_t are strictly increasing.
Since

$$\frac{du}{\big(\frac{u^3}{3} + E\big)^{1/2}} \;=\; dt\,,$$

$u(t)$ approaches ∞ at time

$$T \;:=\; \int_{u(0)}^{\infty} \frac{du}{\big(\frac{u^3}{3} + E\big)^{1/2}}\,.$$

Exercise. *Show that if there is an $M > 0$ so that $G(s) < 0$ for $s \geq M$ and*

$$\int_M^\infty \frac{1}{\sqrt{|G(s)|}}\, ds \;<\; \infty$$

then there are solutions of the ordinary differential equation which blow up in finite time.

Proposition 6.8.3 [J.B. Keller 1957]. *If*

$$a, \delta > 0, \qquad d \leq 3, \qquad E := \delta^2/2 - a^3/3, \qquad T \;:=\; \int_a^\infty \big|\frac{u^3}{3} + E\big|^{-1/2}\, du\,,$$

and $\phi, \psi \in C^\infty(\mathbb{R}^d)$ satisfy

$$\phi \geq a \quad \text{and} \quad \psi \geq \delta \qquad \text{for} \quad |x| \leq T\,,$$

the the smooth solution of

$$\Box_{1+d} u \;-\; u^2\,, \qquad u(0) = \phi, \quad u_t(0) = \psi$$

blows up on or before time T.

Proof. Denote by $\underline{u}$ the solution of the ordinary differential equation with initial data $\underline{u}(0) = a$, $\underline{u}_t(0) = \delta$.
If $u \in C^\infty\big([0, \underline{t}] \times \mathbb{R}^d\big)$, then finite speed of propagation and positivity of the fundamental solution of $\Box_{1+d}$ imply that

$$u \;\geq\; \underline{u} \qquad \text{on} \quad \big\{|x| \leq T - \underline{t}\big\}\,.$$

Since $\underline{u}$ diverges as $t \to T$ it follows that $\underline{t} \leq T$ ∎

In the case of attractive forces where $G \geq 0$ one can hope that there is global smooth solvability for smooth initial data. This question has received much attention and is very far from being understood. For example even in the uniformly lipschitzean case where solutions H^2 in x exist globally, higher regularity is unknown in high dimensions.

§6.8.3. Subcritical nonlinearities.

In the remainder of this section we will study solvability in the energy space defined by $u, u_t \in H^1 \times L^2$. This regularity is suggested by the basic energy law. For uniformly lipschitzean nonllinearities the global solvability is given by Theorem 6.8.2. The interest is in attractive nonlinearities with superlinear growth at infinity.

A crucial role is played by the rate of growth of F at infinity. There is a critical growth rate so that for nonlinearities which are subcritical and critical there is a good theory based on Strichartz estimates. The analysis is valid in all dimensions.

To concentrate on essentials, we present the family of attractive (repulsive) nonlinearities $F = u|u|^{p-1}$ ($F = -u|u|^{p-1}$) with potential energies given by $\pm \int |u|^{p+1}/(p+1)dx$. Start with four natural notions of subcriticality. They are in increasing order of strength. One could expect to call p subcritical when

1. $H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ so the nonlinear term makes sense for elements of H^1.

2. $H^1(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$ so the potential energy makes sense for elements of H^1.

3. $H^1(\mathbb{R}^d)$ is compact in $L^{p+1}_{\text{loc}}(\mathbb{R}^d)$ so the potential energy is in a sense small compared to the kinetic energy.

4. $H^1(\mathbb{R}^d) \subset L^{2p}(\mathbb{R}^d)$ so the nonlinear term belongs to $L^2(\mathbb{R}^d)$ for elements of H^1.

The Sobolev embedding is

$$H^1(\mathbb{R}^d) \ \subset \ L^q(\mathbb{R}^d)\,, \quad \text{for,} \quad q \ = \ \frac{2d}{d-2}\,. \tag{6.8.8}$$

The above conditions then read (with the values for $d = 3$ given in parentheses),

1. $p \leq 2d/(d-2)$, $\quad (p \leq 6)$,

2. $p + 1 \leq 2d/(d-2)$, $\quad \big(\text{equiv.} \quad p \leq (d+2)/(d-2)\big)$, $\quad (p \leq 5)$,

3. $p < (d+2)/(d-2)$, $\quad (p < 5)$,

4. $p \leq d/(d-2)$, $\quad (p \leq 3)$.

The correct answer is **3**. Much that will follow can be extended to the critical case $p = (d+2)/(d-2)$. The case **1** in contrast is supercritical and comparatively little is known. It is known that in the supercritical case, solutions are very sensitive to initial data. The dependence is not lipschitzean, and it is lipschitzean in the subcritical and critical cases. The books of Sogge, and Shatah-Struwe and the orignal 1985 article of Ginibre and Velo are good references. The sensitive dependence is a recent result of Lebeau.

Notation. *Denote by* $L^q_tL^r_x([0,T])$ *the space* $L^q_tL^r_x([0,T]\times\mathbb{R}^d)$, *Denote with an open interval*

$$L^q_tL^r_x([0,T[)\ :=\ \cup_{0<\underline{T}<T}\ L^q_tL^r_x([0,\underline{T}])\,.$$

Theorem 6.8.4. **i.** *If* p *is subcritical for* H^1, *that is* $p<(d+2)/(d-2)$, *then for any* $f\in H^1(\mathbb{R}^d)$ *and* $g\in L^2(\mathbb{R}^d)$ *there is* $T_*>0$ *and a unique solution*

$$u\in C([0,T_*[\ \ H^1(\mathbb{R}^d))\ \cap\ C^1([0,T_*[\,;\,L^2(\mathbb{R}^d))\ \cap\ L^p_tL^{2p}_x([0,T_*[) \tag{6.8.9}$$

of the repulsive problem

$$\Box u\ -\ u|u|^{p-1}\ =\ 0\,,\qquad u(0)=f,\quad u_t(0)=g\,. \tag{6.8.10}$$

If $T_*<\infty$ *then*

$$\liminf_{t\nearrow T_*}\ \|\nabla_{t,x}u\|_{L^2(\mathbb{R}^d)}\ =\ \infty\,. \tag{6.8.11}$$

The energy conservation law (6.8.6) is satisfied.

ii. *For the attractive problem*

$$\Box u\ +\ u|u|^{p-1}\ =\ 0\,,\qquad u(0)=f,\quad u_t(0)=g\,. \tag{6.8.12}$$

one has the same result with $T_*=\infty$ *and with* $u\in L^p_tL^{2p}_x(\mathbb{R})$. *For any* $T>0$, *the map from Cauchy data to solution is uniformly lipschitzean*

$$H^1\times L^2\quad\to\quad C([-T,T]\,;\,H^1)\ \cap\ C([-T,T]\,;\,L^2)\ \cap\ L^p_tL^{2p}_x([0,T]))\,.$$

In the proof of this result and all that follows a central role is played by the linear wave equation and its solution for which we recall the basic energy estimate

$$\|\nabla_{t,x}u(t)\|_{L^2(\mathbb{R}^d)}\ \leq\ \|\nabla_{t,x}u(0)\|_{L^2(\mathbb{R}^d)}\ +\ \int_0^t\|\Box u(t)\|_{L^2(\mathbb{R}^d)}\,dt\,.$$

This is completed by the L^2 estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)}\ \leq\ \int_0^t\|u_t(t)\|_{L^2(\mathbb{R}^d)}\,dt\,.$$

In particular, for $h\in L^1_{\rm loc}\big(\mathbb{R}\,;\,L^2(\mathbb{R}^d)\big)$ there is a unique solution

$$u\ \in\ C\big(\mathbb{R}\,;\,H^1(\mathbb{R}^d)\big)\ \cap\ C^1\big(\mathbb{R}\,;\,L^2(\mathbb{R}^d)\big)\,,$$

to

$$\Box u \;=\; h\,, \qquad u(0) \;=\; 0\,, \quad u_t(0) \;=\; 0\,.$$

This solution is denoted

$$\Box^{-1} h\,.$$

In order to take advantage of this we seek solutions so that

$$F_p(u) \;:=\; \pm\, u|u|^{p-1} \;\in\; L^1_t L^2_x\,.$$

Compute

$$\|F_p(u)\|_{L^1_t L^2_x} \;=\; \int_0^T \Big(\int |u^p|^2\ dx\Big)^{1/2}\ dt\,,$$

where

$$\Big(\int |u|^{2p}\ dx\Big)^{1/2} \;=\; \Big[\Big(\int |u|^{2p}\Big)^{1/2p}\Big]^p \;=\; \|u\|^p_{L^{2p}(\mathbb{R}^d)}\,,$$

so

$$\|F_p(u)\|_{L^1_t L^2_x} \;=\; \int_0^T \|u\|^p_{L^{2p}_t \mathbb{R}^d_x}\ dt \;=\; \|u\|^p_{L^p_t L^{2p}_x}\,. \tag{6.8.13}$$

The above calculation proves the first part of the next lemma.

Lemma 6.8.5. *The mapping $u \mapsto F_p(u)$ takes $L^p_t L^{2p}_x([0,T]$ to $L^1_t L^2_x([0,T])$. It is uniformly Lipshitzean on bounded subsets.*

Proof. Write

$$F_p(v) \;-\; F_p(w) \;=\; G(v,w)\big(v-w\big)\,, \qquad |G(v,w)| \;\le\; C\Big(|v|^{p-1} \;+\; |w|^{p-1}\Big)\,.$$

Write

$$\big\|G(v,w)(v-w)\big\|^2_{L^2_x} \;=\; \int |G|^2\ |v-w|^2\ dx\,.$$

Use Hölder's inequality for $L^{p/(p-1)}_x \times L^p_x$ to estimate by

$$\le\; \Big(\int |G(v,w)|^{2p/(p-1)} dx\Big)^{\frac{p-1}{p}} \Big(\int |v-w|^{2p} dx\Big)^{\frac{1}{p}}\,.$$

Then

$$\|F_p(v) \;-\; F_p(w)\|_{L^2} \;\le\; C\,\|v,w\|^{p-1}_{L^{2p}_x}\ \|v-w\|_{L^{2p}_x}\,.$$

Finally estimate the integral in time using Hölder's inequality for $L^{p/(p-1)}_t \times L^p_t$. ∎

It is natural to seek solutions $u \in L^p_t L^{2p}_x([0,T])$. With that as a goal we ask when it is true that

$$\Box^{-1}\big(L^1_t L^2_x\big) \;\subset\; L^p_t L^{2p}_x\,.$$

This is exactly in the family of questions addressed by the Strichartz inequalities. The next Lemma gives the inequalities adapted to the present situation.

Lemma 6.8.6. *If*

$$q > 2\,, \qquad \text{and} \qquad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1\,, \tag{6.8.14}$$

then there is a constant $C > 0$ so that for all $T > 0$, $h, f, g \in L^1_t(L^2_x) \times H^1 \times L^2$ the solution of

$$\Box u = h\,, \qquad u(0) = f\,, \qquad u_t(0) = g\,,$$

satisfies

$$\|u\|_{L^q_t L^r_x([0,T])} \le C\Big(\|h\|_{L^1_t L^2_x([0,T])} + \|\nabla_x f\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)}\Big). \tag{6.8.15}$$

Proof. 1. Rewrite the wave equation as a symmetric hyperbolic pseudodifferential system motivated by D'Alembert's solution of the $1-d$ wave equation. Factor,

$$\partial_t^2 - \Delta = (\partial_t + i|D|)\,(\partial_t - i|D|) = (\partial_t + i|D|)\,(\partial_t - i|D|)\,.$$

Introduce

$$v_\pm := (\partial_t \mp i|D|)u\,, \qquad V := (v_+, v_-)\,,$$

so

$$V_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i|D|\,V = \begin{pmatrix} h \\ h \end{pmatrix}.$$

Lemma 3.4.8 implies that for $\sigma = d-1$, $q > 2$, (q,r) σ- admissible, and h, f, g with spectrum in $\{R_1 \le |\xi| \le R_2\}$ one has

$$\|u\|_{L^q_t L^r_x} \le C\,\|\nabla_{t,x} u\|_{L^q_t L^r_x} \le C\,\|V\|_{L^q_t L^r_x} \le C\Big(\|h\|_{L^1_t L^2_x} + \||D|f\|_{L^2} + \|g\|_{L^2}\Big).$$

2. Denote by ℓ the dimensions of t and x. With dimensionless u , the terms on right of this inequality have dimension $\ell^{d/2\,-1}$.
The dimension of the term on the left is equal to

$$\big(\ell^{dq/r}\,\ell\big)^{1/q} = \ell^{\frac{d}{r} + \frac{1}{q}}\,.$$

The two sides have the same dimensions if and only if

$$\frac{d}{r} + \frac{1}{q} = \frac{d}{2} - 1\,. \tag{6.8.16}$$

Under this hypothesis it follows that the same inequality holds, with the same constant C for data with support in $\lambda R_1 \le |\xi| \le \lambda R_2$.

Comparing (6.8.16) with σ-admissibility which is equivalent to

$$\frac{d}{r} + \frac{1}{q} \leq \frac{d}{2} - \frac{1}{2} - \frac{1}{r},$$

shows that (6.8.16) implies admissibility since $r \geq 2$.

3. Lemma 6.8.6 follows using Littlewood-Paley theory as at the end of §3.4.3. ∎

We now answer the question of when $\Box^{-1}$ maps $L^1_t L^2_x$ to $L^p_t L^{2p}_x$. This is the crucial calculation. In Lemma 6.8.6, take $r = 2p$ to find

$$\frac{1}{q} + \frac{d}{2p} = \frac{d-2}{2},$$

so,

$$\frac{1}{q} = \frac{d-2}{2} - \frac{d}{2p} = \frac{p(d-2)-d}{2p}, \qquad q = p\left(\frac{2}{p(d-2)-d}\right).$$

We want $q \geq p$, that is

$$\frac{2}{p(d-2)-d} \geq 1, \quad \Leftrightarrow \quad p(d-2) - d \leq 2 \quad \Leftrightarrow \quad p \leq \frac{d+2}{d-2}.$$

The critical case is that of equality, and the subcritical case that we treat is the one with strict inequality. For $d = 3$ the critical power is $p = 5$ and for $d = 4$ it is $p = 3$. In the subcritical case the operator has small norm for $T << 1$.

The strategy of the proof is to write the solution u as a perturbation of the solution of the linear problem, at least for small times. Define u_0 to be the solution of

$$\Box u_0 = 0, \qquad u_0(0) = f, \quad \frac{\partial u_0}{\partial t}(0) = g. \tag{6.8.17}$$

Write

$$u = u_0 + v \tag{6.8.18}$$

with the hope that v will be small at least for t small.

Lemma 6.8.7. *If $u = u_0 + v$ with $v \in L^p_t L^{2p}_x\big([0,T]\big)$ satisfying*

$$v = \pm \Box^{-1} F_p(u_0 + v). \tag{6.8.20}$$

then

$$u \in C([0,T]\,;\, H^1(\mathbb{R}^d)) \cap C^1([0,T]\,;\, L^2(\mathbb{R}^d)) \cap L^p_t L^{2p}_x([0,T]) \tag{6.8.21}$$

satisfies

$$\Box u \pm F_p(u) = 0, \qquad u(0) = f, \quad u_t(0) = g, \tag{6.8.22}$$

Conversely, if u satisfies (6.8.21)-(6.8.22) then $v := u - u_0 \in L^p_t L^{2p}_x([0,T])$ and satisfies (6.8.21)

Proof. The Strichartz inequality implies that $u_0 \in L^p_t L^{2p}_x$ and by hypothesis the same is true of v. Therefore $u_0 + v$ belongs to $L^p_t L^{2p}_x$ so $F_p(u_0 + v) \in L^1_t L^2_x$.

Therefore $v = \pm \Box^{-1} F_p$ is $C(H^1) \cap C^1(L^2)$. The differential equation and initial condition for v are immediate.

The converse is similar, not used below, and left to the reader. ∎

Proof of Theorem 6.8.4. For $K > 0$ arbitrary but fixed, we prove unique local solvability with continuous dependence for $0 \le t \le T$ with T uniform for all data f, g with

$$\|f\|_{H^1} \; + \; \|g\|_{L^2} \; \le \; K\,.$$

Choose $R = R(K)$ so that for such data,

$$\|u_0\|_{L^p_t L^{2p}_x([0,1])} \; \le \; \frac{R}{2}\,.$$

Define

$$B \; = \; B(T) \; := \; \Big\{ v \in L^p_t L^{2p}_x([0,T]) \; : \; \|v\|_{L^p_t L^{2p}_x([0,T])} \; \le R \Big\}\,.$$

We show that for $T = T(K)$ sufficiently small, the map $v \mapsto \Box^{-1} F_p(u)$ is a contraction from B to itself.

This is a consequence of three facts.

1. Lemma 6.8.5 shows that F_p is uniformly lipschitzean from B to $L^1_t L^2_x([0,T])$ uniformly for $0 < T \le 1$.

2. Lemma 6.8.6 together with subcriticality shows that there is an $r > p$ so that $\Box^{-1}$ is uniformly lipshitzean from $L^1_t L^2_x$ to $L^r_t L^{2p}_x$ uniformly for $0 < T < 1$.

3. The injection $L^r_t L^{2p}_x \mapsto L^p_t L^{2p}_x$ has norm which tends to zero as $T \to 0$.

This is enough to carry out the existence parts of Theorem 6.8.4.

If there are two solutions u, v with the same initial data, compute

$$\Box\,(u - v) \; = \; G(u,v)(u - v)\,.$$

Lemma 6.8.6 together with subcriticality shows that with r slightly larger than p,

$$\|u - v\|_{L^r_t L^{2p}_x} \; \le \; C\,\|G(u,v)(u - v)\|_{L^1_t L^2_x} \; \le \; C\,\|u - v\|_{L^p_t L^{2p}_x}\,.$$

Use this estimate for $0 \le t \le T << 1$ noting that Hölder's inequality shows that for $T \to 0$,

$$\|u - v\|_{L^p_t L^{2p}_x} \; \le \; C\,T^\rho\,\|u - v\|_{L^r_t L^{2p}_x} \; \le \; C\,T^\rho \|u - v\|_{L^p_t L^{2p}_x}\,, \qquad \rho > 0\,,$$

to show that the two solutions agree for small times. Thus the set of times where the solutions agree is open and closed proving uniqueness.

To prove the energy law note that $F_p(u) \in L^1_t L^2_x$ so the linear energy law shows that

$$\int \frac{|u_t|^2 + |\nabla_x u|^2}{2} \, dx \Big|_{t=0}^{t} = \mp \int_0^t \int u_t \ F_p(u) \ dx\, dt\,. \tag{6.8.23}$$

Now

$$u_t \ \in \ L^\infty_t L^2_x\,, \quad \text{and} \quad F_p(u) \in L^1_t L^2_x\,.$$

Hölder's inequality shows that

$$\int |u_t \ F_p(u)| dx \ \leq \ \|u_t(t)\|_{L^2_x} \ \|F_p(u(t)\|_{L^2_x}\,.$$

The latter is the product of a bounded and an integrable function so

$$\forall T, \quad u_t\, F_p(u) \ \in \ L^1([0,T] \times \mathbb{R}^d)\,.$$

Let

$$w \ := \ \frac{|u|^{p+1}}{p+1}\,.$$

Since p is subcritical, one has for some $0 < \epsilon$,

$$\|w(t)\|_{L^1_x} \ \leq \ C \|u(t)\|_{H^{1-\epsilon}(\mathbb{R}^d)} \ \in \ L^\infty([0,T])\,.$$

In particular $w \in L^1([0,T] \times \mathbb{R}^d)$ and the family $\{w(t)\}_{t\in[0,T]}$ is precompact in $L^1_{\rm loc}$. Formally differentiating yields

$$w_t \ = \ u_t F_p(u) \ \in \ L^1([0,T] \times \mathbb{R}^d)\,. \tag{6.8.24}$$

Using the above estimates, it is not hard to justify (6.8.24).
It then follows that $w \in C([0,T]\,;\, L^1(\mathbb{R}^d))$ and

$$\int w(t,x) \ dx \Big|_{t=0}^{t=T} \ = \ \int_0^T \int u_t \ F_p(u) dx\, dt\,.$$

Together with (6.8.23) this proves the energy identity.

Once the energy law is known, one concludes global solvability in the attractive case since the blow up criterion (6.8.11) is ruled out by energy conservation. ∎