Chapter 9. Resonant Interaction and Quasilinear Systems

This chapter describes two extensions. First, we describe the resonant interaction of wave trains with distinct phases. This is *multiphase nonlinear geometric optics.* Second, the semilinear analysis is extended to the quasilinear case with the goal of discussing compressible inviscid fluid dynamics.

§9.1. Introduction to resonance.

Even at the level of formal asymptotic expansions, resonance poses a challenge. It was Majda and Rosales [1986] who got it right. The approach presented in this section is that of Joly, Métivier, and Rauch in the Duke Math. J, 1994. The essence of the phenomenon is illustrated by the following simple example.

Example 1. Consider the oscillatory semilinear initial value problem

$$\begin{aligned} (\partial_t + \partial_x) u_1 &= 0 \qquad & u_1\big|_{t=0} &= a_1(x)\, e^{ix/\epsilon} \\ \partial_t u_2 &= u_1 u_3 \qquad & u_2\big|_{t=0} &= 0 \\ (\partial_t - \partial_x) u_3 &= 0 \qquad & u_3\big|_{t=0} &= a_3(x)\, e^{ix/\epsilon} \end{aligned} \tag{9.1}$$

with initial amplitudes $a_j \in C_0^\infty(\mathbb{R})$. The exact solution is given by

$$u_1(t,x) = a_1(x-t)\, e^{i(x-t)/\epsilon}\,, \quad u_3(t,x) = a_3(x+t)\, e^{i(x+t)/\epsilon}\,, \quad u_2 = \int_0^t u_1(t,x)\, u_3(t,x)\; dt\,.$$

The key observation is that the phases, $(x \pm t)/\epsilon$ that appear in the integrand for u_2 add to $2ix/\epsilon$ which is independent of t. The formula for u_2 is,

$$u_2 = e^{i2x/\epsilon} \int_0^t a_1(x-t)\, a_3(x+t)\, dt\,. \tag{9.2}$$

The oscillatory wave trains, u_1 and u_3 with phases

$$\phi_1 := (x-t)/\epsilon, \qquad \phi_2 := (x+t)/\epsilon,$$

respectively, interact to generate a third wave u_2 with phase

$$\phi_3 := 2x/\epsilon.$$

The phases satisfy the resonance relation

$$\phi_1 \ + \ \phi_2 \ = \ \phi_3\,.$$

The amplitude of the new wave is of the same order, ϵ^0, as the waves from which it is formed.

The linear operator

$$L(\partial_t, \partial_x) \ := \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_t \ + \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x,$$

in the background has principal symbol,

$$L(i\tau, i\xi) \ := \ i \begin{pmatrix} \tau + \xi & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau - \xi \end{pmatrix}.$$

The characteristic variety of L has equation $0 = \det L(\tau, \xi) = \tau(\tau + \xi)(\tau - \xi)$. The three phases satisfy the eikonal equation

$$\phi_t(\phi_t + \phi_x)(\phi_t - \phi_x) = 0\,.$$

For the solutions with $\nabla_{t,x}\phi \neq 0$, this is equivalent to exactly one of the equations

$$\phi_t = 0, \qquad \phi_t + \phi_x = 0, \qquad \text{or} \qquad \phi_t - \phi_x = 0,$$

at all points (assuming the domain of definition is connected).

Variants of this example illustrate two properties of resonance.

Example 2. Suppose that the initial condition for u_3 is changed to $u_3\big|_{t=0} = a_3(x)e^{i\psi(x)}$ with $d\psi(x)/dx$ nowhere equal to 1. Then the integral defining u_2 is an oscillatory integral in time with phase $(x - t + \psi(x + t))/\epsilon$. The time derivative of the phase is $O(1/\epsilon)$ so the method of nonstationary phase shows that $u_2 = O(\epsilon)$. The resonant interaction is destroyed.

Exercise. More generally, if $\{x : \psi'(x) = 1\}$ has measure zero, then $u_2 = o(1)$ as $\epsilon \to 0$. Again the offspring wave is smaller than the parents.

For those who know about Young measures, it is interesting to note that the Young measures of the intitial data are independent of the function ψ so long as $\psi' \neq 0$. Thus *data with the same Young measures yield solutions with different Young measures.*

Introduce the symmetric form $\sum \phi_j = 0$ for resonance relations. If ψ_k satisfy $\sum n_k\, \psi_k = 0$, then the phases $\phi_k := n_k\psi_k$ satisfy the symmetric form. The symmetric form is often easier to manipulate.

Example. We find all triples of resonant linear eikonal phases with pairwise independent differentials for $L = \partial_t + \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)\partial_x$ with λ_j distinct real numbers. Seek such ϕ_j satisfying the resonance relation $\sum \phi_j = 0$. The independent differentials together with the eikonal relation force (up to permutation),

$$\phi_j(t, x) \ = \ \alpha_j(x - \lambda_j\, t), \qquad (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus 0.$$

The α are determined up to scalar mulitplication by the resonance relation which is equivalent to the pair of equations,

$$\sum_j \alpha_j \;=\; 0, \qquad \text{and,} \qquad \sum_j \alpha_j\,\lambda_j \;=\; 0\,.$$

Exercise. Show that if $f, g, h \in C^\infty(\mathbb{R})$ each has nonvanishing derivative at the origin, and at least one of them has nonvanishing second derivate, then the three phases

$$f(t), \qquad g(t-x), \qquad \text{and} \qquad h(t+x)$$

cannot be resonant on a neighborhood of the origin. **Discussion.** This shows that linear phases are the only possibilities for resonant triples with pairwise independent differentials.

The example and exercises show that *the phenomenon of resonance is both rare and sensitive when viewed from the perspective of perturbing the phases.*

§9.2. The three wave interaction PDE.

Consider the system

$$\begin{aligned}(\partial_t + \partial_x)u_1 &= c_1\,u_3\,u_2\,,\\ \partial_t u_2 &= c_2\,u_1 u_3\,,\\ (\partial_t - \partial_x)u_3 &= c_3\,u_1\,u_2\,.\end{aligned} \tag{9.3}$$

with real $c_j \in \mathbb{R}\setminus 0$. This equation and its relatives maximizes the intermode interaction. The absence of a term in u_1^2 in the first equation has a consequence that harmonics are not generated in the first mode. Harmonic generation is absent in the other equations too.

Multiplying the first equation by $a_1\,u_1$, the second by $a_2\,u_2$, and the third by $a_3\,u_3$, shows that if a_1, a_2, and a_3 are real numbers so that $\sum a_j c_j = 0$ then for solutions one has the differential conservation laws

$$\frac{\partial}{\partial t}\big(a_1\,u_1^2 + a_2\,u_2^2 + a_3\,u_3^2\big) \;+\; \frac{\partial}{\partial x}\big(a_1\,u_1^2 \;-\; a_3\,u_3^2\big) \;=\; \big(2\sum_j\; a_j\,c_j\big)u_1u_2u_3 \;=\; 0\,.$$

Integrating dx yields the integral conservation laws,

$$\frac{d}{dt}\int a_1\,u_1^2 + a_2\,u_2^2 + a_3\,u_3^2\;\;dx \;=\; 0\,.$$

This is a two dimensional space of conservation laws parameterized by the a.

We are interested in complex solutions. For complex solutions, conservation laws involving $|u_j|^2$ are more interesting than those involving u_j^2 as they yield L^2 bounds. The complex analogue of (9.3) with such conservation laws is,

$$\begin{aligned}(\partial_t + \partial_x)u_1 &= c_1\,u_2\,u_3^*\\ \partial_t u_2 &= c_2\,u_1 u_3\\ (\partial_t - \partial_x)u_3 &= c_3\,u_1^*\,u_2\,.\end{aligned} \tag{9.4}$$

with $c_j \in \mathbb{R} \setminus 0$.

Mutiplying the $j^{\rm th}$ equation by $a_j u_j^*$ and taking the real part shows that if $a_j \in \mathbb{R}$ satisfy $\sum a_j\, c_j = 0$, then solutions satisfy

$$\partial_t\Big(\sum_j a_j |u_j|^2\Big) \;+\; \partial_x\Big(a_1\,|u_1|^2 \;-\; a_3\,|u_3|^2\Big) \;=\; 2\,\Big(\sum_j a_j\, c_j\Big)\,\mathrm{Re}\Big(u_1^*\,u_2\,u_3^*\Big) \;=\; 0\,.$$

If the c_j do not all have the same sign, then there are conservation laws of this type with all the $a_j > 0$. This yields an L^2 bound on the solution. On the other hand, if the c_j are all positive then initial data with $u_j(0,x)$ real and positive yield real solutions such that for all j, u_j is nondecreasing along j characteristics. For sufficiently positive data there is finite time blow up.

Proposition. *Suppose that $c_j \ge c > 0$ and the real valued initial data satisfy*

$$\forall j, \quad \forall |x| \le R, \quad u_j(0,x) \;\ge A > 0.$$

i. *If $u(t,x) \in C^\infty([0,t_*[\times\mathbb{R})$ is a solution, then $u_j(t,x) \ge y(t)$ for $t_* > t \ge 0$ and $|x| \le R-t$, where $y = A/(1-cAt)$ is the solution of $y' = c\,y^2,\ y(0) = A$.*

ii. *If $T_* := (cA)^{-1}$ is the blow up time for y, and $R > T_*$, then u blows up on or before time T_* in the sense that one must have $t_* \le T_*$.*

Proof. The second assertion follows from the first.

Since the speed of propagation is no larger than 1, the values of u in $|x| \le R - t$ are unaffected by the values of the Cauchy data for $|x| > R$. Therefore, it suffices to prove that $u_j \ge y(t)$ when the data satisfy $u_j(0,x) \ge A$ for all x.

Define

$$m(t) \;:=\; \min_{x\in\mathbb{R}\,,\,j}\ u_j(t,x)\,.$$

Since the u_j are nondecreasing on j-characteritics, it follows that $m(t)$ is nondecreasing. And, $m(0) \ge A > 0$. In addition one has the lower bound obtained by integration along j characteristics,

$$u_j(t,x) \;\ge\; m(0) + c\int_0^t m(t)^2\ dt\,.$$

Taking the infinum on x yields

$$m(t) \;\ge\; m(0) + c\int_0^t m(t)^2\ d \;\ge\; A + c\int_0^t m(t)^2\,dt\,.$$

The function y is characterized as the solution of

$$y(t) \;=\; A + c\int_0^t y(t)^2\,dt\,.$$

For $\epsilon > 0$ small, let y^ϵ be the solution of $(y^\epsilon)' = c(y^\epsilon)^2$ with $y^\epsilon(0) = A - \epsilon$ so

$$y^\epsilon(t) \;=\; A - \epsilon + c\int_0^t y^\epsilon(t)^2\,dt\,.$$

It follows that $m(t) > y^\epsilon(t)$ for all $0 \le t < t_*$. For, if this were not so there would be a smallest $\underline{t} \in]0, t[_*$ where $m(\underline{t}) = y^\epsilon(\underline{t})$. Then

$$y^\epsilon(\underline{t}) \;=\; m(\underline{t}) \;\ge\; A + c\int_0^{\underline{t}} m(t)^2\,dt \;>\; A - \epsilon + c\int_0^t (y^\epsilon(t))^2\,dt \;=\; y^\epsilon(\underline{t})\,.$$

This contradiction establishes $m > y^\epsilon$. Passing to the limit $\epsilon \to 0$ proves $m \ge y$ which is the desired conclusion. ∎

Only the signs of the c_j play a roll in the qualitative behavior of the equation (9.4).

Proposition. *There is exactly one positive diagonal linear transformation*

$$u \;:=\; (d_1\,v_1\,,\,d_2\,v_2\,,\,d_3\,v_3)\,, \qquad d_j > 0\,,$$

which transform the the system to the analogous system with interaction coefficients $\{c_1, c_2, c_3\}$ *replaced by*

$$\Big\{\frac{c_1}{|c_1|}\,,\,\frac{c_2}{|c_2|}\,,\,\frac{c_3}{|c_3|}\Big\}\,.$$

Proof. The change of variables yields an anlogous system for v with the interaction coefficients replaced by

$$\Big\{\frac{d_2\,d_3}{d_1}\,c_1\,,\,\frac{d_3\,d_1}{d_2}\,c_2\,,\,\frac{d_1\,d_2}{d_3}\,c_3\Big\}\,.$$

Need d_j so that

$$\frac{d_2\,d_3}{d_1}\,c_1 = \frac{c_1}{|c_1|}\;, \qquad \frac{d_3\,d_1}{d_2}\,c_2 = \frac{c_2}{|c_2|}\,, \qquad \frac{d_1\,d_2}{d_3}\,c_3 = \frac{c_3}{|c_3|}\,.$$

Multiplying the j^{the} equation by d_j^2 yields the equivalent system,

$$\frac{d_1^2}{|c_1|} \;=\; \frac{d_2^2}{|c_2|} \;=\; \frac{d_3^2}{|c_3|} \;=\; d_1 d_2 d_3\,.$$

The first two equalities hold if and only if,

$$(d_1^2, d_2^2, d_3^2) = a(|c_1|, |c_2|, |c_3|) \qquad \text{with} \quad a > 0\,.$$

Then, the last equation holds if and only if

$$a \;=\; a^3\,|c_1c_2c_3|\,.$$

This uniquely determines a, and therefore d. ∎

Remark. For general $d_j \neq 0$, the three quantities $d_1d_2/d_3\,,\, d_2d_3/d_1\,,\, d_3d_1/d_2$ have the same sign. Using $d_j \neq 0$ allows us to multiply the three interactiion coefficients by -1 if desired. Thus every system is transformed to one with interaction coefficients all equal to $+1$ or two equal to $+1$. There are four equivalence classes, the last three depending on the location of the coefficient -1.

Proposition. i. *If the real interaction coefficients $c_j \neq 0$ do not all have the same sign, then the Cauchy problem for (9.4) has a unique global solution $u \in \cap_s H^s([0,s]\times\mathbb{R})$ for arbitrary Caucy data in $\cap_s H^s(\mathbb{R})$.*

ii. *If the c_j have the same sign there are smooth compactly supported data so that the solution of the Cauchy problem explodes in finite time.*

Proof. For real data, this equation reduces to the previous one and the explosive behavior has already been treated.

To prove **i.**, the results of section 6.4 show that it suffices to prove for every $T>0$, an *a priori* bound for the $L^\infty([0,T]\times\mathbb{R})$ norm.

From the conservation law, one has

$$\sup_{t\in[0,T]}\ \int \sum_j |u_j|^2\;dx \;\leq\; K \;<\; \infty\,.$$

The equation for u_2 yields,

$$|u_2(t,\underline{x})| \;\leq\; |u_2(0,\underline{x})\;| + \int_0^t |u_1\,u_3(t,\underline{x})|\;dt\,. \tag{9.5}$$

The key idea is to estimate the integral on the right hand side using energy estimates for u_1 and u_3.

For any $\underline{x}\in\mathbb{R}$ integrate the identity

$$(\partial_t+\partial_x)|u_1|^2 \;=\; 2\,\mathrm{Re}\Big(u_1^*(\partial_t+\partial_x)u_1\Big) \;=\; 2\,\mathrm{Re}\,c_1u_1^*u_2u_3\,,$$

over the strip $[0,t]\times\,]-\infty,\underline{x}]$ to find that

$$\int_0^t |u_1(t,\underline{x})|^2\,dt \;\leq\; K + 2|c_1|\int_{[0,t]\times\mathbb{R}} |u_1u_2u_3|\;dt\,dx\,.$$

Estimate the integral dx on the right using the $L^\infty \times L^2 \times L^2$ Hölder inequality to find

$$\int_0^t |u_1(t,\underline{x})|^2\,dt \ \le\ K \ +\ 2|c_1| \int_0^t K\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\ dt\,.$$

By symmetry,

$$\int_0^t |u_3(t,\underline{x})|^2\,dt \ \le\ K \ +\ 2|c_3| \int_0^t K\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\ dt\,.$$

The Cauchy-Schwarz inequality implies that

$$\int_0^t |u_1(t,\underline{x}))\,u_2(t,\underline{x})|\ dt \ \le\ K \ +\ 2\,\max\{|c_1|,|c_3|\} \int_0^t \|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,. \tag{9.6}$$

Estimate the integral on the right in (9.5) using (9.6) to find

$$|u_2(t,\underline{x})| \ \le\ C \ +\ \int_0^t C\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,,$$

with C independent of $(t,\underline{x}) \in [0,T]\times\mathbb{R}$. Taking the supremum of the left hand side over $\underline{x}$ yields

$$\|u_2(t)\|_{L^\infty(\mathbb{R})} \ \le\ C \ +\ \int_0^t C\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,, \qquad 0\le t\le T\,.$$

Gronwall's inequality bounds the sup norm of u_2 over bounded time intervals.

To estimate u_1 one needs L^2 estimates for u_2 and u_3 on the speed one characteristics $x=\underline{x}+t$. These are obtained by integrating $\partial_t|u_2|^2$ and $(\partial_t-\partial_x)|u_3|^3$ over $\{(s,x)\,:\,0\le s\le t,\ \text{and } x\ge\underline{x}+s\}$.

A similar argument works for u_3. ∎

§9.3. The three wave interaction ODE.

For the three wave PDE, (9.4), and phases equal to the resonant triplet, waves of each pair of families influence, by resonant interaction, the wave of the third. The simplest examples showing this are solutions of the special form

$$u_1 = A_1(t)\,e^{i(t-x)/\epsilon}\,, \qquad u_2 = A_2(t)\,e^{-i2x/\epsilon} \qquad u_3 := A_3(t)e^{-i(t+x)/\epsilon}\,, \tag{9.7}$$

with amplitudes A_j independent of x. The oscillatory structure evolves in time, but is uniform in space. Equation (9.4) is satisfied if and only if the amplitudes A_j satisfy the *three wave interaction equations*

$$A_1' = c_1\,A_2\,A_3^*\,, \qquad A_2' = c_2\,A_1\,A_3\,, \qquad A_3' = c_3\,A_1^*\,A_2\,. \tag{9.8}$$

This is a nonlinear system of ordinary differential equation for three complex quantities A_j. The phase space is $\mathbb{C}^3$, hence six dimensional as a real vector space.

The equilibria are the points where (at least) two of the three $\{A_j\}$ vanish. There are three linear subspaces of equilibria, each with real dimension equal to 2,

$$\big\{A_2 = A_3 = 0\big\}\,, \qquad \big\{A_3 = A_1 = 0\big\}\,, \qquad \text{and,} \qquad \big\{A_1 = A_2 = 0\big\}\,.$$

Each pair of planes intersect at the origin. The system (9.8) is highly symmetric.

Proposition. i. *The quantity* $\mathrm{Im}\big(A_1(t)\,A_2^*(t)\,A_3(t)\big)$ *, is constant on solutions of (9.8).*
ii. *If a_j are real numbers so that $\sum a_j\,c_j = 0$ then the quantity $\sum_j a_j\,|A_j(t)|^2$ is constant on solutions of (9.8).*
iii. *If A is a solution and $\theta \in \mathbb{R}$, then $\tilde{A}$ obtained by each of the three gauge transformations*

$$\tilde{A} := \big(e^{i\theta}\,A_1\,,\,A_2\,,\,e^{-i\theta}\,A_3\big), \quad \tilde{A} := \big(A_1\,,\,e^{i\theta}\,A_2\,,\,e^{i\theta}\,A_3\big), \quad \tilde{A} := \big(e^{i\theta}\,A_1\,,\;e^{i\theta}\,A_2\,,\,A_3\big),$$

is also a solution. The conserved quantities in **i,ii** *are invariant under the gauge transformations.*
iv. *If A is a solution and $\sigma \in \mathbb{R}\setminus 0$, then $\tilde{A}$ obtained by the scaling*

$$\tilde{A}_j(t) \;=\; \sigma A_j(\sigma t)\,,$$

is also a solution.

Proof. i. Compute

$$\begin{aligned}
(A_1A_2^*A_3)_t \;&=\; (A_1)_tA_2^*A_3 + A_1(A_2^*)_tA_3 + A_1A_2^*(A_3)_t \\
&=\; c_1A_2A_3^*A_2^*A_3 + c_2A_1A_1^*A_3^*A_3 + c_3A_1A_2^*A_1^*A_2 \\
&=\; c_1|A_2A_3|^2 \;+\; c_1|A_1A_3|^2 \;+\; c_1|A_2A_1|^2 \;\in\; \mathbb{R}\,.
\end{aligned}$$

ii. Compute

$$\begin{aligned}
\frac{d}{dt}|A_1|^2 \;&=\; 2\mathrm{Re}\,A_1^*\frac{d}{dt}A_1 \;=\; 2c_1\mathrm{Re}(A_1^*A_2A_3^*), \\
\frac{d}{dt}|A_2|^2 \;&=\; 2\mathrm{Re}\,A_1^*\frac{d}{dt}A_1 \;=\; 2c_2\mathrm{Re}(A_2^*A_1A_3), \\
\frac{d}{dt}|A_3|^2 \;&=\; 2\mathrm{Re}\,A_1^*\frac{d}{dt}A_1 \;=\; 2c_3\mathrm{Re}(A_1^*A_2A_3^*).
\end{aligned}$$

The real parts are of $A_1A_2^*A_3$ or its complex conjugate, so are equal. Therefore one has

$$\frac{d}{dt}\Big(\sum a_j\,|A_j(t)|^2\Big) \;=\; \Big(2\sum_j a_j\,c_j\Big)\mathrm{Re}(A_2^*A_1A_3) \;=\; 0\,.$$

The assertions **iii,iv** are immediate. ∎

Remarks. 1. When the $c_j \neq 0$ do not all have the same sign one can choose the $a_j > 0$. In this case, the three wave interaction system is globally solvable.

2. When the three c_j have the same sign, there exist solutions which blow up in finite time. This is proved by comparison with an explosive Ricatti equation as for the three wave interaction PDE.

Exercise. Suppose that the c_j are strictly positive and the initial data of the A_j are strictly positive. Denote by $T_* < \infty$ the blow up time. Prove that all three components $A_j(t)$ blow up as $t \to T_*$.

3. The gauge transformations commute. The third gauge transformation is the product of the preceding two. The abelian group of gauge transformations is a two dimensional torus of mappings

$$A \quad \mapsto \quad \left(e^{i\theta_1} A_1\,,\, e^{i\theta_2} A_2\,,\, e^{i\theta_2}\, e^{-i\theta_1}\, A_3\right).$$

Proposition. *For i, j, k distinct, the equilibrium $A_i = A_j = 0\,,\ \underline{A}_k \neq 0$ of (9.8), is unstable if the interaction coefficients c_i and c_j have the same sign. The stable and unstable manifolds have real dimension equal to 2.*

When c_i and c_j have opposite signs, orbits of the linearized equation are bounded. For initial data starting close to the equilibrium, the solutions of the nonlinear system exist for all time and $A_i(t), A_j(t)$ and $|A_k(t)|$ stay close to their initial values uniformly in time. If $\underline{A}_k$ is real then the equilibrium is stable for the restriction of the dynamics to $A \in \mathbb{R}^3$.

The equilibrium $(0,0,0)$ is unstable if and only if the three c_j have the same sign.

Proof. For ease of reading consider the equilibrium $(0,0,\underline{A}_3) \neq 0$. The linearized equation at this equilibrium is

$$B' \;=\; \begin{pmatrix} 0 & c_1\,\underline{A}_3^* & 0 \\ c_2\,\underline{A}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} B\,.$$

The eigenvalues of the coefficient matrix are the solutions λ of

$$\lambda(\lambda^2 - c_1 c_2 |\underline{A}_3|^2) \;=\; 0\,.$$

If c_1 and c_2 have the same sign, then the roots are $0, \pm|c_1 c_2|^{1/2}|\underline{A}_3|$. The positive eigenvalue implies that the equilibrium is unstable. The stable and unstable manifolds have complex dimension equal to 1 and real dimension equal to 2.

If c_1, c_2 have opposite signs then $|c_2||B_1|^2 + |c_1||B_2|^2$ is constant on orbits of the linearized equation. It follows that each orbit of the linearized equation is uniformly bounded in time.

In the case of opposite signs, the functional $|c_2|^2|A_1|^2 + |c_1||A_2|^2$ is constant on orbits of (9.8). For initial data which start near $(0,0,\underline{A}_3)$ the components $A_1(t), A_2(t)$ stay close to zero for all time.

The conserved quantity $a_1|A_1|^2 + a_2|A_2|^2 + |A_3|^2$ together with the control of $A_1(t), A_2(t)$ implies that $|A_3(t)|$ stays close to $|A_3(0)|$. In particular, the orbit exists for all time.

The stability of the origin when the c_j do not have the same sign follows from the conservation of $\sum a_j |A_j|^2$ with positive a_j. The instability is proved using explosive positive (resp. negative) solutions when the c_j are positive (resp. negative). ∎

This Proposition falls short of determining the stability in $\mathbb{C}^3$ of the equilbrium $(0, 0, \underline{A}_k)$ when c_i and c_j have opposite signs.

For any triple of interaction coefficients, there exists $i \neq j$ so that c_i and c_j have the same sign. Then the equilibria defined by $A_i = A_j = 0$ is unstable. The unstable equilibrium exists even in the globally solvable case where the c_j do not all have the same sign. For example, if c_1 and c_2 have the same sign and c_3 the opposite, then their is a conserved Euclidean norm $\sum a_j |A_j|^2$. On the other hand, most orbits starting near $A_1 = 0, A_2 = 0, A_3 = 1$ stray far from this state. This situation is described as saying oscillations on the third mode generate *frequency conversion* to modes one and two. The solution cannot grow, but it can wander far from its initial state. The energy originally localized nearly entirely on mode 3, moves substantially away. An appreciable portion of the energy passes to modes 1,2.

The analysis of the interactions in the highly oscillatory family (9.4) reduces to the analysis of a system of nonlinear ordinary differential equations This is a special case of a general phenomenon for *homogeneous oscillations*, that is oscillations which in a sense are the same at all positiions of space. Proving general results of this sort is one of our goals. Another is to extend our semilinear analysis of Chapters 7 and 8 to the quasilinear case. We take up the construction of high frequency asymptotic solutions, this time with several phases and in the quasilinear case.

§9.4. Formal asymptotic solutions for resonant quasilinear geometric optics.

We give a self contained, but rapid derivation of the equations of quasilinear geometric optics. Consider the quasilinear system of partial differential operators,

$$L(u, \partial)u \;:=\; \sum_{\mu=0}^{d} A_\mu(u)\, \partial_\mu u\,.$$

Suppose that the system is symmetric in the sense of the first paragraph of §6.6. We will study solutions whose values are close to 0. As explained in the last paragraph of §6.6, one can, by a change dependent variable, $u := A_0(0)^{1/2} v$, reduce to the system

$$\sum_{\mu=0}^{d} \tilde{A}_\mu(v)\; \partial_\mu v \;=\; 0, \qquad \tilde{A}_\mu(v) \;:=\; A_0(0)^{-1/2}\; A_\mu(A_0(0)^{1/2} v)\; A_0(0)^{-1/2}\,,$$

with

$$\tilde{A}_\mu \;=\; \tilde{A}_\mu^*\,, \qquad \text{and,} \qquad \tilde{A}_0(0) \;=\; I\,.$$

We suppose that such a change has been performed and suppress the tildes.

For $u \approx 0$ approximate

$$A_\mu(u) \approx A_\mu(0) + A_\mu'(0)u$$

to find that the nonlinear terms are of the form

$$\big(A'_\mu(0)u\big)\,\partial_\mu u \ +\ \text{higher order terms.}$$

We assess the time of nonlinear interaction for an oscillatory wave of the form $\epsilon^p\, e^{i\phi(x)/\epsilon}$ with the goal of choosing p. Assume that $A'_\mu(0) \neq 0$ for some μ, so that the leading nonlinear terms are quadratic. The analysis when the leading Taylor polynomial is higher order can be carried out as in earlier sections. For the important examples from inviscid fluid dynamics, the hypothesis of quadratic nonlinearity is usually verified. The nonlinear terms have amplitude $O(\epsilon^{2p-1})$. For phases satisfying the eikonal equation these should yield a response which is of size $O(\epsilon^{2p-1})$ for $t \sim O(1)$. We choose the amplitudes so that the time of nonlinear interaction is $O(1)$ so we want $\epsilon^{2p-1} \sim \epsilon^p$ which yields the critical power

$$p = 1\,,$$

Consider the interaction of waves with linear phases $\phi_j(y)$ which by nonlinear interaction yield possible phases $\sum n_j\phi_j$ with $n_j \in \mathbb{Z}$. Each of these candidate phases is a linear function $\alpha.y$ with $\alpha \in \mathbb{R}^{1+d}$. Thus an expression

$$\epsilon\, U(y, y/\epsilon) \qquad \text{with} \qquad U(y,Y) \sim \sum_{\alpha\in\mathbb{R}^d} U_\alpha(y)\, e^{i\alpha.Y}$$

is of the critical amplitude and includes all the anticipated terms. In a formal trigonometric sum over α it is understood that there are at most a countable number of nonvanishing coefficients U_α.

Each term in the sum has amplitude $\sim \epsilon$. The $L^\rho_{\rm loc}$ norms of first derivatives of each term are ~ 1.

For the scalar Burgers equation $u_t + uu_x = 0$, solutions with compactly supported initial data with $\|\partial_x u(0,x)\|_{L^1} \sim 1$ break down at times $t \sim 1$. This shows that the estimate for the time of nonlinear interaction is not too large. It shows that expression $\epsilon\, U(y, y/\epsilon)$ should not be viewed as small since initial data of this size can lead to blow up in finite time for quasilinear problems. Our analysis will show that nonlinear effects are present for times $t \sim 1$. This is also shown by the fact that initial data with of this structure lead to blow in time $O(1)$ for Burgers equation.

The exact structure of the function of Y given by $U(y,Y)$ is left unspecified for the moment. Equivalently, consider $U(y,Y)$ as a formal trigonometric series in Y with coefficients which are smooth functions of y. To solve the profile equations and prove accuracy will require supplementary hypotheses. These hypotheses do not play a role in the derivation of the profile equations.

Pose the *ansatz*

$$u^\epsilon = \epsilon\, U^\epsilon(y, y/\epsilon)\,, \tag{9.9}$$

$$U^\epsilon(y,Y) \ \sim\ \sum_{j=0}^\infty \epsilon^j\, U_j(y,Y) \ \sim\ U_0(y,Y) + \epsilon\, U_1(y,Y) + \cdots\,, \tag{9.10}$$

$$U_j(y,Y) \ \sim\ \sum_{\alpha\in\mathbb{R}^d} U_{j,\alpha}(y)\, e^{i\alpha.Y}\,. \tag{9.11}$$

Write

$$L(u^\epsilon,\partial)u^\epsilon \;=\; L(\epsilon U^\epsilon,\,\partial)\Big(\epsilon\, U^\epsilon(y,y/\epsilon)\Big)\,. \tag{9.12}$$

Expanding in a Taylor series shows that

$$A_\mu\Big(\epsilon U^\epsilon(y,Y)\Big) \;\sim\; A(0)+\epsilon A'_\mu(0)U_0+\cdots\,, \tag{9.13}$$

to find

$$L(\epsilon U(y,Y)\,,\,\partial_y) \;\sim\; L_0 \;+\; \epsilon L_1 \;+\; \cdots \;=\; \sum_{j=0}^{\infty}\epsilon^j\, L_j(y,Y,\partial_y)\,.$$

The L_j are operators whose coefficients are functions of y,Y involving the derivatives $\partial_u^\beta A_\mu(0)$ and the profiles U_k with $k\le j-1$. The most important come from the two terms in (9.13),

$$L_0 = L(0,\partial_y)\,, \qquad \text{and,} \qquad L_1 = \sum_\mu A'_\mu(0)U_0(y,Y)\,\partial_\mu\,.$$

The chain rule shows that

$$\frac{\partial}{\partial y_\mu}U^\epsilon(y,y/\epsilon) \;=\; \Big(\frac{\partial}{\partial y_\mu}+\frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big)U(y,Y)^\epsilon\Big|_{Y=y/\epsilon}\,. \tag{9.14}$$

So

$$L(u^\epsilon,\partial)u^\epsilon \;=\; W(\epsilon,y,Y)\Big|_{Y=y/\epsilon}\,, \tag{9.15}$$

where

$$W(\epsilon,y,Y) \;=\; L\Big(\epsilon U(y,Y)\,,\,\frac{\partial}{\partial y_\mu}+\frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big)\,\epsilon\, U(y,Y)$$

Expand to find

$$\begin{aligned}
W(\epsilon,y,Y) \;&\sim\; \Big(\sum_{j=0}^{\infty}\epsilon^j L_j\Big(y,Y,\frac{\partial}{\partial y_\mu}+\frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big)\Big)\Big(\epsilon\sum_{k=0}^{\infty}\epsilon^k U_k(y,Y)\Big)\\
&\sim\; \Big(\sum_{j=0}^{\infty}\Big[\epsilon^j L_j\Big(y,Y,\frac{\partial}{\partial y_\mu}\Big)+\epsilon^{j-1}L_j\Big(y,Y,\frac{\partial}{\partial Y_\mu}\Big)\Big]\Big)\Big(\epsilon\sum_{k=0}^{\infty}\epsilon^k U_k(y,Y)\Big)\\
&\sim\; \sum_{j=0}^{\infty}\epsilon^j\, W_j(y,Y)\,.
\end{aligned}$$

In particular,

$$W_0(y,Y) = L(0,\partial_Y)\,U_0(y,Y)\,, \tag{9.16}$$

and

$$\begin{aligned}
W_1(y,Y) &= L(0,\partial_y)\,U_0 + L_1(y,Y,\partial_Y)U_0 + L(0,\partial_Y)U_1\\
&= L(0,\partial_y)\,U_0 + \sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0 + L(0,\partial_Y)\,U_1\,.
\end{aligned} \tag{9.17}$$

This expression involves both U_0 and U_1 which is typical of multiscale methods. Note the quadratically quasilinear terms $A'_\mu(0)U_0\,\partial_Y U_0$ which involve derivatives in the fast variables Y. For $j\geq 2$ the formula for W_j is

$$\begin{aligned} W_j &= \sum_{k+\ell=j}\Big(L_k(y,Y,\partial_y)+L_{k+1}(y,Y,\partial_Y)\Big)U_\ell \\ &= L(0,\partial_y)\,U_j + L_1(y,Y,\partial_Y)U_j + L(0,\partial_Y)\,U_{j+1} + \text{terms in } U_0,U_1,\dots U_{j-1}\,. \end{aligned}$$

The strategy is to choose profiles U_j so that $W_j(y,Y)=0$ for all $y,Y,$ not just on the $d+2$ dimensional subset $\{Y=y/\epsilon\}$ parameterized by $(\epsilon,y)=(\epsilon,t,x)$.

Setting W_0 in (9.16) shows that W_0 must lie in the kernel of $L(0,\partial_Y)$. For (9.17), the profile U_1 is as yet undetermined. However, in order that it is possible to choose a U_1 so that (9.17) holds, requires the second of the equations,

$$U_0\in \operatorname{Kernel} L(0,\partial_Y)\,,\quad \text{and,}\quad L(0,\partial_y)\,U_0+\sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0 \in \operatorname{Image} L(0,\partial_Y)\,. \tag{9.18}$$

To understand (9.18) requires a study of the operator $L(0,\partial_Y)$. This is straight forward using the Fourier representation,

$$L(0,\partial_Y)\,U \;=\; L(0,\partial_Y)\sum_\alpha U_\alpha(y)\,e^{i\alpha.Y} \;=\; i\sum_\alpha L(0,\alpha)\,U_\alpha(y)\,e^{i\alpha.Y}\,. \tag{9.19}$$

As an operator acting on formal trigonometric series, $L(0,\partial_Y)$ has kernel consisting of those series whose α^{th} coefficient belongs to the kernel of $L(0,\alpha)$. Recall the definition of $\pi(\alpha)$ as the projection onto the kernel of $L(0,\alpha)$ along its range. The kernel of $L(0,\partial_Y)$ is then the set of trigonometric series such that $\pi(\alpha)\,U_\alpha=U_\alpha$. The image is the set of series with U_α belonging to the image of $L(0,\alpha)$. Equivalently, $\pi(\alpha)\,U_\alpha=0$.

Define an operator $\mathbf{E}$ from formal trigonometric series to themselves by

$$\mathbf{E}\sum_\alpha U_\alpha(y)\,e^{i\alpha.Y} := \sum_\alpha \pi(\alpha)\,U_\alpha(y)\,e^{i\alpha.Y}\,. \tag{9.20}$$

The previous remarks show that on formal trigonometic series the operator $\mathbf{E}$ projects along the image of $L(0,\partial_Y)$ onto its kernel. Therefore, the two conditions in (9.18) are equivalent to the pair of equations

$$\mathbf{E}\,U_0 = U_0\,, \tag{9.21}$$

and

$$\mathbf{E}\Big(\,L(0,\partial_y)\,U_0+\sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0\Big)=0\,. \tag{9.22}$$

These are the fundamental equations of resonant quasilinear geometric optics. They are analogues of (7.26) and (7.27).

Since $A(0) = I$ equation (9.22) is equivalent to

$$\partial_t U_0 + \mathbf{E}\Big(\sum_j A_j(0)\,\partial_j U_0 + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_0\Big) = 0\,. \tag{9.23}$$

Written this way, the equation looks like an evolution equation for U_0. Since the operator $\mathbf{E}$ does not depend on t one has, at least formally,

$$\partial_t\,\big(I - \mathbf{E})U_0 = 0$$

so the constraint (9.21) is satisfied as soon as it is satisfied at $t = 0$. It is reasonable to expect that U_0 can be determined from its initial data required to satisfy $\mathbf{E}U_0(0, x, Y) = U_(0, x, Y)$.

The equation $W_1 = 0$ is equivalent to the pair of equations $\mathbf{E}\,W_1 = 0$ and $(I - \mathbf{E})\,W_1 = 0$. The first, $\mathbf{E}\,W_1 = 0$, is the second equations in (9.22).

Introduce the operator $\mathbf{Q}$ on trigonometric series by

$$\mathbf{Q}\sum_\alpha U_\alpha(y)\,e^{i\alpha.Y} \;:=\; \sum_\alpha Q(\alpha)\,U_\alpha(y)\,e^{i\alpha.Y}\,, \tag{9.24}$$

where $Q(\alpha)$ is the partial inverse of $L(0, \alpha)$ defined by

$$Q(\alpha)\,\pi(\alpha) \;=\; 0, \qquad Q(\alpha)\,L(0, \alpha) \;=\; I - \pi(\alpha)\,.$$

$\mathbf{Q}$ is a partial inverse to $L(0, \partial_Y)$. It is determined by

$$\mathbf{Q}\,\mathbf{E} \;=\; 0, \qquad \mathbf{Q}\,L(0, \partial_Y) \;=\; I - \mathbf{E}\,. \tag{9.25}$$

Since $Q(\alpha)$ commutes with $L(0, \alpha)$, it follows that $\mathbf{Q}$ commutes with $L(0, \partial_Y)$.

The second part, $(I - \mathbf{E})W_1 = 0$, of the equation $W_1 = 0$ is equivalent to $\mathbf{Q}W_1 = 0$. Multiplying (9.17) by $\mathbf{Q}$ shows this is equivalent to

$$\big(I - \mathbf{E}\big)U_1 \;=\; -\mathbf{Q}\,\Big(\,L(0, \partial_y)\,U_0 + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_0\Big)\,. \tag{9.26}$$

Once U_0 is determined, his determines $(I - \mathbf{E})U_1$.

Multiplying $W_{j-1} = 0$ by $\mathbf{Q}$ and $W_j = 0$ by $\mathbf{E}$ shows that the equations $(I - \mathbf{E})W_{j-1} = 0$ together with $\mathbf{E}W_j = 0$ are equivalent to a pair of equations

$$\big(\,I - \mathbf{E}\,\big)\,U_j = \mathbf{Q}\,\big(\text{terms in } U_0, U_1, \ldots U_{j-1}\big)\,, \tag{9.27}$$

and

$$\mathbf{E}\Big(\,L(0, \partial_y)\,U_j + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_j\Big) = \mathbf{E}\,\big(\text{terms in } U_0, U_1, \ldots U_{j-1}\big)\,. \tag{9.28}$$

Note that once $U_0, \ldots, U_{j-1}$ are determined, equations (9.27) and (9.28) will serve to determine U_j from initial values $\mathbf{E}U_j(0, x, Y)$.

§9.5. Solvability of quasiperiodic profiles equations and small divisors.

The present multidimensional quasilinear case is a special case of the analysis in Duke Math J. 1994. An essential step is to pass from formal trigonometric series in Y to a more manageable class.

One class which will serve us well is to consider profiles $U_0(y, Y)$ which are periodic in Y. Though this suffices for the most interesting examples we construct, it is inadequate. Consider, for example, the one dimensional problem with leading part $\partial_t + \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)\partial_x$ considered in the last example of §9.1. It is important to be able to treat functions oscillating with resonant trio of phases $\alpha_j(x - \lambda_j t)$ as in that example. For the phases $n.y/\epsilon$ which appear for periodic profiles the ratio of the coefficients of t and x are rational. Thus one could only treat the case of $\lambda \in \mathbb{Q}$. to be a rational multiple of α. Quasiperiodic functions as in the next definition are sufficient to treat a wide class of problems including general λ.

Notation. *Suppose that the real linear functions $\{\phi_j(Y)\}_{j=1}^m$ are linearly independent over the rationals. To a function $\mathcal{U}(y, \theta_1, \ldots, \theta_m)$ smooth and 2π multiply periodic in θ, associate the quasiperiodic profile $U(y, Y) := \mathcal{U}\big(y, \phi_1(Y), \ldots, \phi_m(Y)\big)$. An induced operator $\mathcal{E}$ mapping periodic functions to themselves is defined by*

$$\mathcal{E}\Big(\sum_{n\in\mathbb{Z}^m} \mathcal{U}_n(y) e^{in.\theta}\Big) \;:=\; \sum_{n\in\mathbb{Z}^m} \pi\big(\sum_k n_k d\phi_k\big)\, \mathcal{U}_n(y) e^{in.\theta}\,,$$

so that $(\mathcal{E}\mathcal{U})(y, \phi(Y)) := \mathbf{E}\, U(y, Y)$. Similarly define the partial inverse,

$$\mathcal{Q}\mathcal{U} \;:=\; \sum_{n\in\mathbb{Z}^m} Q\big(\sum_k n_k d\phi_k\big)\, \mathcal{U}_n(y) e^{in.\theta}\,.$$

Introduce the shorthand, $n.d\phi := \sum_k n_k d\phi_k$.

To write (9.21)-(9.22) as an equation for $\mathcal{U}$ note that

$$\frac{\partial}{\partial Y_\mu} \mathcal{U}(y, \phi_1(Y) \ldots, \phi_m(Y)) = \sum_{k=1}^m \frac{\partial \phi_k}{\partial Y_\mu} \frac{\partial \mathcal{U}_0}{\partial \theta_k}\,.$$

The profile equation for U_0 are equivalent to,

$$\mathcal{E}\,\mathcal{U}_0 = \mathcal{U}_0\,, \qquad \mathcal{E}\,\Big(\, L(0, \partial_y)\, \mathcal{U}_0 + \sum_\mu A'_\mu(0) \mathcal{U}_0 \sum_k \frac{\partial \phi_k}{\partial Y_\mu} \frac{\partial \mathcal{U}_0}{\partial \theta_k}\,\Big) = 0\,, \qquad (9.29)$$

for $\mathcal{U}_0$. This equation has the form

$$\partial_t \mathcal{U}_0 \;+\; G(\mathcal{U}_0, \partial_{y,\theta})\, \mathcal{U}_0 \;=\; 0\,,$$

where

$$G(\mathcal{U}, \partial_{y,\theta}) \ := \ \mathcal{E}\Big(\sum_{j=0}^{d} A_j(0)\, \partial_{y_j} \ + \ \sum_\mu A'_\mu(0)\mathcal{U} \sum_k \frac{\partial \phi_k}{\partial Y_\mu} \frac{\partial}{\partial \theta_k} \ \Big)\mathcal{U} \ := \ \mathcal{E}\, K(\mathcal{U}, \partial_{y,\theta})\mathcal{U}\,.$$

The notation is chosen to suggest a quasilinear hyperbolic system. But, the operator $\mathcal{E}$ is nonlocal in θ. However, $\mathcal{E}$ is an orthogonal projection operator in $L^2(\mathbb{R}^d_x \times \mathbb{T}^m_\theta)$ which commutes with $\partial_{y,\theta}$.

Theorem 9.2 (JMR, Duke J. 1994). *Suppose that $H_0(y,\theta) \in \cap_s(H^s(\mathbb{R}^d \times \mathbb{T}^m)$ satisfies the constraint $\mathcal{E} H_0 = H_0$. Then there is $T_* > 0$ and a unique maximal local solution*

$$\mathcal{U}_0 \in \cap_s\, C^s\big([0, T_*[\,;\ H^s(\mathbb{R}^d \times \mathbb{T}^m)\big)$$

satisfying (9.29) together with the intitial condition $\mathcal{U}\big|_{t=0} = H_0$. If $T_* < \infty$ then

$$\limsup_{t \nearrow T_*}\ \big\| \mathcal{U}_0(t) \big\|_{\mathrm{Lip}(\mathbb{R}^d \times \mathbb{T}^m)} = \infty\,.$$

Sketch of Proof. The key idea is to derive *a priori* estimates as in the case of quasilinear hyperbolic systems. One differentiates the equation applying $\partial^\beta_{x,\theta}$, and takes the real part of the $L^2(\mathbb{R}^d \times \mathbb{T}^m)$ scalar product with $\partial^\beta_{x,\theta}\mathcal{U}_0$ (suppressing the subscript 0) to find

$$\frac{d}{dt}\Big(\frac{1}{2} \|\partial^\beta_{x,\theta}\mathcal{U}\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^m)} \Big) \ = \ \mathrm{Re}\, \Big(\partial^\beta_{x,\theta}\mathcal{U}\,,\ \partial^\beta_{x,\theta}\mathcal{E} K(\mathcal{U}, \partial_{y,\theta})\mathcal{U} \Big)_{L^2(\mathbb{R}^d \times \mathbb{T}^m)}\,.$$

Using the commutation and symmetry properties of **E** yields

$$\begin{aligned} \big(\partial^\beta_{x,\theta}\mathcal{U}\,,\ \partial^\beta_{x,\theta}\mathcal{E} K(\mathcal{U}, \partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d \times \mathbb{T}^m)} \ &= \ \big(\mathcal{E}\, \partial^\beta_{x,\theta}\mathcal{U}\,,\ \partial^\beta_{x,\theta} K(\mathcal{U}, \partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d \times \mathbb{T}^m)} \\ &= \ \big(\partial^\beta_{x,\theta}\mathcal{U}\,,\ \partial^\beta_{x,\theta} K(\mathcal{U}, \partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d \times \mathbb{T}^m)}\,, \end{aligned}$$

the last equality using $\mathcal{E}\,\mathcal{U} = \mathcal{U}$. The last *is* a quasilinear hyperbolic expression. Using Gagliardo-Nirenberg estimates as in the treatment of the quasilinear Cauchy problem, one has

$$\mathrm{Re}\, \big(\partial^\beta_{x,\theta}\mathcal{U}\,,\ \partial^\beta_{x,\theta} K(\mathcal{U}, \partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d \times \mathbb{T}^m)} \ \le \ C\big(\|\mathcal{U}\|_{\mathrm{Lip}(\mathbb{R}^\mathrm{d} \times \mathbb{T}^\mathrm{m})}\big) \|\mathcal{U}\|^2_{H^{|\beta|}(\mathbb{R}^d \times \mathbb{T}^m)}\,.$$

Summing on $|\beta| \le s \in \mathbb{N}$ yields

$$\frac{d}{dt} \big\|\mathcal{U}(t)\big\|^2_{H^s(\mathbb{R}^d \times \mathbb{T}^m)} \ \le \ C(\|\mathcal{U}\|_{\mathrm{Lip}(\mathbb{R}^\mathrm{d} \times \mathbb{T}^\mathrm{m})}) \|\mathcal{U}\|^2_{H^s(\mathbb{R}^d \times \mathbb{T}^m)}\,.$$

Local well-posedness in H^s for $\mathbb{N} \ni s > 1 + (d+m)/2$ so that $H^s \subset \mathrm{Lip}$ is then proved as for quasilinear hyperbolic systems. ∎

To find the higher order profiles $\mathcal{U}_j$ with $j \geq 1$ the equations involve the operator $\mathcal{Q}$, for example (9.26). *Without further hypotheses, $\mathcal{Q}$ may be very ill behaved.* The matrices $Q(\alpha)$ may grow very rapidly as α grows. This has two consequence. First, $\mathbf{Q}$ may not even map smooth profiles in θ to distributions in θ. In that case the equations for the higher profiles do not make sense. Second, there are known examples where the error of approximation by the leading term is $o(\epsilon^p)$ but is not $O(\epsilon^{p+\delta})$ for any $\delta > 0$ (see [JMR 1992]).

What is needed in order to get a reasonably well behaved operator $\mathcal{Q}$ is that the matrices $Q(n.d\phi)$ grow no faster than polynomially in $|n|$. The trouble spots for $\mathcal{Q}$ are eigenvalues of $L(0, n.d\phi)$ which though not equal to zero, are very close to zero.

Small divisor hypothesis. *There is a $C > 0$ and an integer N so that for all $n \in \mathbb{Z}^m \setminus 0$, if $\lambda \neq 0$ is an eigenvalue of $L(0, \sum_k n_k d\phi_k) \neq 0$ then*

$$|\lambda| \;\geq\; \frac{C}{|n|^N}\,.$$

In [JMR, Duke M. J], it is proved that this hypothesis is generically satisfied. It is often not difficult to verify this hypothesis as examples in the next sections show.

Proposition. *If the small divisor hypothesis is satisfied then there is a constant $C > 0$ and an integer M so that for all $n \in \mathbb{Z}^m$,*

$$\|Q(n.d\phi)\| \;\leq\; C\,\langle n\rangle^M\,.$$

Proof. From the small divisor hypothesis, one knows that the nonzero eigenvalues of $Q(n.d\phi)$ lie in an annulus $2c/\langle n\rangle^N \leq |z| \leq \langle n\rangle^N/2$. Define a larger annulus containing the eigenvalues strictly in its interior by

$$D(n) \;:=\; \left\{ z \,:\, c/\langle n\rangle^N \leq |z| \leq \langle n\rangle^N \right\}.$$

Then

$$Q(n.d\phi) \;=\; \frac{1}{2\pi i}\oint_{\partial D(n)} \;\frac{1}{z}\,\big(zI - L(0, n.d\phi)\big)^{-1}\,dz\,.$$

For $z \in \partial D(n)$, $\|zI - L(0, n.d\phi)\| \leq C\langle n\rangle^N$. The nearest eigenvalue is no closer than $C\langle n\rangle^{-N}$. Therefore $\|(zI - L(0, n.d\phi))^{-1}\| \leq C\langle n\rangle^{N'}$, and the Proposition follows. ∎

The Proposition implies that when the small divisor hypothesis is satisfied, $\mathbf{Q}$ maps $\cap_s H^s(\mathbb{R}^d \times \mathbb{T}^m)$ continuously to itself. The next Theorem is linear and easier than the previous one.

Theorem 9.3. *Suppose that the small divisor hypothesis is satisfied and that $\mathcal{U}_0$ is as in Theorem 9.1 and for $j \geq 1$ initial profiles $H_j(y, \theta) \in \cap_s(H^s(\mathbb{R}^d \times \mathbb{T}^m)$ satisfy $\mathcal{E}H_j =$*

H_j. Then higher order profiles $\mathcal{U}_j \in \cap_s C^s([0,T_*[\,;\, H^s(\mathbb{R}^d \times \mathbb{T}^m))$ for $j \geq 1$ are uniquely determined by the initial conditions $\mathcal{E}\,\mathcal{U}_j = H_j$ and the transcriptions of (9.27) and (9.28) to the reduced profiles.

Suppose that the profiles $\mathcal{U}_j$ of all orders are determined as in Theorems 9.2 and 9.3. Borel's Theorem constructs

$$C^\infty\big([0,1] \times [0,T_*[\,:\, \cap_s H^s(\mathbb{R}^d)\big) \;\ni\; \mathcal{U}(\epsilon, y, \theta) \;\sim\; \sum_j \epsilon^j\, \mathcal{U}_j(y,\theta)\,. \tag{9.30}$$

Define approximate solutions

$$u^\epsilon(t,x) := \epsilon\,\mathcal{U}(\epsilon, t, x, \phi_1(t,x)/\epsilon, \ldots, \phi_m(t,x)/\epsilon) \in \cap_s C^s([0,T_*[\,;\, H^s(\mathbb{R}^d))\,. \tag{9.31}$$

Theorem 9.4. *With the above definitions, the residual*

$$r^\epsilon := L(u^\epsilon, \partial)\, u^\epsilon \tag{9.32}$$

is infinitely small in the sense that

$$\forall T \in [0,T_*[\,,\ \gamma \in \mathbb{N}^{d+1}\,,\ N \in \mathbb{N}\,,\ \exists c > 0\,,\ \forall \epsilon \in]0,1], \qquad \|\partial_y^\gamma r^\epsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \leq c\,\epsilon^N \tag{9.33}$$

§9.6. Stability and accuracy of the approximate solutions.

The approximate solutions are of size $O(\epsilon)$ but taking a derivative costs a power of ϵ. Thus $(\epsilon\partial_y)^\gamma$ applied to the approximate solutions which is $O(\epsilon)$. The next theorem implies that the approximate solutions are infinitely close to the exact solutions with the same initial values.

The result differs from Theorem 8.6 in two ways. First it is on $\mathbb{R}^d$ rather than local in Ω_T. Much more important it is quasilinear instead of semilinear and that requires some changes in the proof. The reader is referred to [JMR 1995-96] for a presentation of two related stability results where the changes required to pass from the simpler semilinear case to the quasilinear case are presented.

Stability Theorem 9.5. *Suppose that $T > 0$ and that u^ϵ is a family of smooth approximate solutions to $L(u,\partial)\, u = 0$ which are $O(\epsilon)$ in the sense that for all $\gamma \in \mathbb{N}^{1+d}$, $\exists c(\gamma) > 0\,,\ \forall \epsilon \in]0,1]$*

$$\|\,(\epsilon\partial_y)^\gamma u^\epsilon\,\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq c(\gamma)\,\epsilon\,. \tag{9.34}$$

Suppose that the residuals $r^\epsilon := L(u^\epsilon, \partial)\, u^\epsilon$ are infinitely small in the sense that

$$\forall \gamma \in \mathbb{N}^{d+1}\,,\ N \in \mathbb{N}\,,\ \exists c > 0\,,\ \forall \epsilon \in]0,1], \qquad \|\partial_y^\gamma r^\epsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \leq c\,\epsilon^N\,. \tag{9.35}$$

Define $v^\epsilon \in C^\infty([0,T_*(\epsilon)[\times\mathbb{R}^d)$ to be the maximal solution of the initial value problem

$$L(v^\epsilon,\partial)\, v^\epsilon = 0\,, \qquad v^\epsilon(0,x) = u^\epsilon(0,x)\,. \tag{9.36}$$

Then there is an $\epsilon_0>0$ so that for $\epsilon<\epsilon_0$, the time of existence satisfies $T_*(\epsilon)>T$, and the approximate solution u^ϵ is infinitely close to the exact solution v^ϵ in the sense that for all integers s and N

$$\|u^\epsilon - v^\epsilon\|_{H^s([0,T]\times\mathbb{R}^d)} \le c(s,N)\,\epsilon^N\,.$$

§9.7. Semilinear resonant nonlinear geometric optics.

The simplest examples, like those in §9.2, are semilinear. The first examples in §10 are semilinear. In this section we simply state the form of the *ansatz* and profile equations in the semilinear case. The precise theorem statements and proofs closely resemble the quasilinear case and can be found in the references.

For a semilinear system

$$L(\partial)\,u + f(u) = 0\,, \qquad L(\partial) := \sum_{\mu=0}^{d} A_\mu\,\partial_\mu\,,$$

recall that $\pi(\alpha)$ is orthogonal projection on the kernel of $L(\alpha)$ and $\mathbf{E}$ is the operator on formal trigonometric series $\mathbf{E}\sum a_\alpha(y)\,e^{i\alpha.\theta} := \sum \pi(\alpha)\,a_\alpha(y)\,e^{i\alpha.\theta}$. The critical size for semilinear problems is amplitudes $O(\epsilon^p)$ with $p=0$. The approximate solutions have the form

$$u^\epsilon \ \sim\ U_0^\epsilon(y, y/\epsilon)\,, \tag{9.37}$$

$$U_0(y,Y) \ \sim\ \sum_{\alpha\in\mathbb{R}^d} U_{0,\alpha}(y)\,e^{i\alpha.Y}\,. \tag{9.38}$$

The amplitudes are $O(1)$ as $\epsilon\to 0$ in contrast to the quasilinear case where the amplitudes where $O(\epsilon)$ but in agreement with the one phase semilinear theory.

The profile equations for U_0 are

$$\mathbf{E}\,U_0 = U_0\,, \tag{9.39}$$

$$\mathbf{E}\,\Big(L(\partial_y)\,U_0(y,\theta) + f\big(U_0(y,\theta)\big)\Big) = 0\,. \tag{9.40}$$

Solutions of the profile equation of the quasiperiodic form

$$U(\epsilon,y,Y) = \mathcal{U}(\epsilon,y,\phi_1(Y),\dots,\phi_m(Y)) \ \in\ C^\infty([0,\epsilon_0]\,;\,\cap_s H^s([0,T]\times\mathbb{R}^d\times\mathbb{T}^m)\,,$$

with

$$\mathcal{U}(\epsilon,y,\theta_1,\dots,\theta_m) \ \sim\ \sum_{j=0}^{\infty}\epsilon^j\,\mathcal{U}_j(y,\theta)\,,$$

in the sense of Taylor series exist provided the small divisor hypthesis of the preceding section holds with $L(n.d\phi)$ in place of $L(0,n.d\phi)$. They yields approximate solutions with infinitely small residual. The accuracy of these solutions follows from the stability result of Chapter 8.