FOCUSING OF SPHERICAL NONLINEAR PULSES IN $\mathbb{R}^{1+3}$

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ABSTRACT. This paper describes the behavior of spherical pulse solutions of semilinear wave equations in the limit of short wavelength. In three space dimensions we study the behavior of solutions which are described by nonlinear geometric optics away from the focal point. With a natural subcriticality hypothesis on the nonlinearity we prove that the possibly nonlinear effects at the focal point do not affect the usual description in terms of the Maslov index. That is one has nonlinear geometric optics before *and after* the focal point with only the usual phase shift of -1. The reason is that the nonlinear effects occur on too small a set. We obtain a global asymptotic description which includes an approximation near the caustic, which is a solution of the free wave equation.

1. INTRODUCTION

Many recent works have investigated the algorithms of nonlinear geometric optics (see [8] for a survey). These algorithms construct approximate solutions which are accurate when the wavelength denoted by ε tends to zero. Most of these articles are valid for wave trains, satisfying the so-called *slowly varying envelope approximation*. It has been known for a long time that the slowly varying envelope approximation can be violated when studying ultrashort laser pulses (see e.g. [11]). In that case, wave trains are replaced with pulses with length $O(\varepsilon)$. The mathematical study of such pulses is recent, and the construction of correctors and justification of the approximation are different from the analogous problem for the propagation of wave trains (see [1], [2]).

In the present article, we give a first result analyzing what happens when short pulses pass through a focal point. We study semilinear wave equations in three space dimensions, with a subcritical nonlinearity. For wavetrains it is known that the caustic does not change the leading order wave train asymptotics away from the caustic (see [7], [10], [9], [3]). We analyse spherical pulses (Figure1), and show that nonlinear geometric optics, as constructed in [2], is valid away from the caustic. At the focus, the approximation is not good, since the exact solution is a regular function, whereas the profile of geometric optics is singular. However, this phenomenon occurs in so small a region that the approximation is valid before and after the caustic. As in [6] or in [4], the leading term of the approximation satisfies a nonlinear transport equation away from the focal point. In a small neighborhood of the focal point, the exact solution is approximated by a solution of the linear wave equation. In this sense we have nonlinear propagation and a linear caustic. The phenomena in the case of pulses are similar to those which are encountered in the case of wave trains ([3]). The detailed description, Theorem 2, in the immediate neighborhood of the caustic is sharper than existing results for wave trains. The analysis is also different.

The case of subcritical nonlinearities is particularly pertinent for nonlinear optics. This is so because the subcriticality hypothesis bears on the growth at infinity of the

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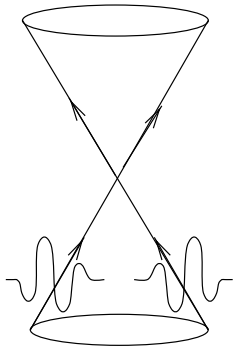

FIGURE 1. Focusing of pulses.

nonlinear function and the nonlinear optical response of most materials is saturated at large amplitudes, which means that not merely slow growth but even bounded nonlinearities are the most reasonable models.

Consider the initial value problem in three space dimensions,

$$\begin{cases} \Box u^\varepsilon + a|\partial_t u^\varepsilon|^{p-1}\partial_t u^\varepsilon = 0, & (t,x)\in[0,T]\times\mathbb{R}^3, \\ u^\varepsilon\big|_{t=0} = \varepsilon U_0\left(r,\dfrac{r-r_0}{\varepsilon}\right), & \partial_t u^\varepsilon\big|_{t=0} = U_1\left(r,\dfrac{r-r_0}{\varepsilon}\right), \end{cases} \tag{1.1}$$

where a is a complex number, $r=|x|$, $r_0>0$.

Hypothesis. *The nonlinearity is* **subcritical** *in the sense that* $1<p<2$.

The functions U_0 and U_1 are infinitely differentiable, supported in $r>0$, bounded, and, there is a $z_0>0$ so that for all $r\geq 0$,

$$\operatorname{supp} U_j(r,.)\subset[-z_0,z_0]. \tag{1.2}$$

The last assumption implies that the initial data are pulse like in the limit $\varepsilon\to 0$.

The nonlinearity need not be p-homogeneous, but only p-homogeneous at infinity or even bounded above by a p-homogeneous at infinity. This is so since we study the influence of the caustic $\{r=0\}$, where $\partial_t u^\varepsilon$ is large. The results are the same if the nonlinearity $|\partial_t u^\varepsilon|^{p-1}\partial_t u^\varepsilon$ is replaced with

$$\left(1+|\partial_t u^\varepsilon|^2\right)^{(p-1)/2}\ \partial_t u^\varepsilon\,, \qquad \text{or,} \qquad \frac{|\partial_t u^\varepsilon|^2}{1+|\partial_t u^\varepsilon|^{3-p}}\ \partial_t u^\varepsilon\,.$$

Since the initial data are spherical so is the solution so, with the usual abuse of notation,

$$u^\varepsilon(t,x) \;=\; u^\varepsilon(t,|x|), \qquad u^\varepsilon(t,|x|)\;\in\; C^\infty_{\text{even in } r}(\mathbb{R}_t\times\mathbb{R}_r)\,.$$

Introduce $v^\varepsilon := (v^\varepsilon_-, v^\varepsilon_+)$ where

$$\tilde u^\varepsilon(t,r) := r u^\varepsilon(t,r), \qquad v^\varepsilon_\mp := (\partial_t \pm \partial_r)\tilde u^\varepsilon\,, \qquad v^\varepsilon_\mp\in C^\infty(\mathbb{R}_t\times\mathbb{R}_r)\,. \tag{1.3}$$

Then (1.1) becomes

$$\begin{cases} (\partial_t\pm\partial_r)v^\varepsilon_\pm = r^{1-p}g(v^\varepsilon_- + v^\varepsilon_+), \qquad g(y) := b|y|^{p-1}y\,,\quad b := -a2^{-p}\,, \\ v^\varepsilon_- + v^\varepsilon_+\big|_{r=0} = 0\,, \\ v^\varepsilon_\mp\big|_{t=0} = P_0\left(r,\dfrac{r-r_0}{\varepsilon}\right) \pm \varepsilon P_1\left(r,\dfrac{r-r_0}{\varepsilon}\right), \end{cases} \tag{1.4}$$

The initial data P_j and Q_j are related to the U_j by,

$$\begin{aligned} P_0(r,z) &:= rU_1(r,z) + r\partial_z U_0(r,z)\,, \\ P_1(r,z) &:= U_0(r,z) + r\partial_r U_0(r,z), \end{aligned} \tag{1.5}$$

and inherit the smoothness and compact support of the U_j. Taking the focusing into account, a natural generalization for the *ansatz* given in [2] consists in seeking approximate solutions,

$$\begin{aligned} (v^\varepsilon)_{\rm app} &:= \big((v^\varepsilon_-)_{\rm app}\,,\,(v^\varepsilon_+)_{\rm app}\big)\,, \\ (v^\varepsilon_-)_{\rm app}(t,r) &:= V^{\rm in}(t,r,z_1)\Big|_{z_1=\frac{r-r_0+t}{\varepsilon}}\,, \\ (v^\varepsilon_+)_{\rm app}(t,r) &:= V^{\rm foc}(t,r,z_2)\Big|_{z_2=\frac{t-r-r_0}{\varepsilon}} + V^{\rm out}(t,r,z_3)\Big|_{z_3=\frac{r-t-r_0}{\varepsilon}}\,, \end{aligned} \tag{1.6}$$

where the profiles $V^{\rm in}$, $V^{\rm foc}$ and $V^{\rm out}$ are compactly supported in the z variable. As shown on Figure 2, $V^{\rm in}$ corresponds to an incoming spherical pulse and $V^{\rm out}$ to an outgoing spherical pulse both generated at time $t=0$. The latter pulse never sees the caustic. The $V^{\rm foc}$ term represents an outgoing spherical pulse created when the incoming pulse crosses the focus.

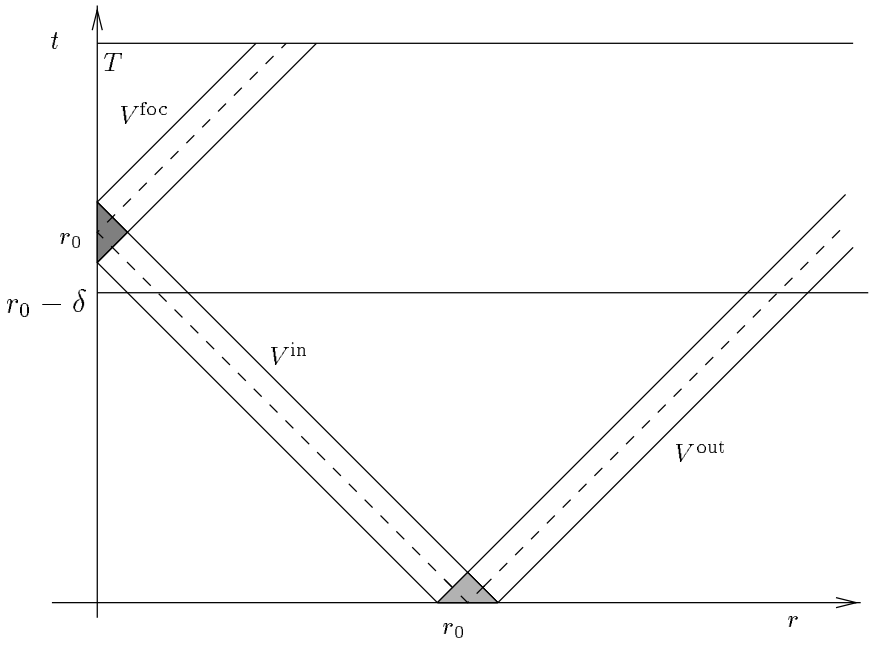

FIGURE 2. Geometry of rays.

The profiles defined for $r>0$ are determined as solutions of the initial boundary value problems,

$$\begin{cases} (\partial_t - \partial_r)V^{\rm in}(t,r,z_1) = r^{1-p}g(V^{\rm in})(t,r,z_1), \\ V^{\rm in}\big|_{t=0} = P_0(r,z_1), \end{cases} \tag{1.7}$$

$$\begin{cases} (\partial_t + \partial_r)V^{\rm out}(t,r,z_3) = r^{1-p}g(V^{\rm out})(t,r,z_3), \\ V^{\rm out}\big|_{t=0} = P_0(r,z_3), \\ V^{\rm out}\big|_{r=0} = 0\,, \end{cases} \tag{1.8}$$

$$\begin{cases} (\partial_t + \partial_r) V^{\rm foc} = r^{1-p} g(V^{\rm foc})(t, r, z_2), \\ (V^{\rm in} + V^{\rm foc})(t, 0, z) = 0\,, \\ V^{\rm foc}(t, 0, z)\big|_{t=0} = 0\,. \end{cases} \tag{1.9}$$

These in turn are solved by integrating nonlinear ordinary differential equations along the rays of geometric optics.

The second equation in (1.9) shows that on traversing the focus the amplitude of the profile is multiplied by $-1 = e^{i2\pi/2}$. This is the classical Maslov index for a focal point of multiplicity equal to 2 (see e.g. [5]).

Theorem 1. *Define the approximate solution $((v^\varepsilon_-)_{\rm app}, (v^\varepsilon_+)_{\rm app})$ by (1.6), (1.7), (1.8) and (1.9). As in* [2]*, nonlinear geometric optics is valid before the focus in the sense that for any $\delta > 0$,*

$$\|v^\varepsilon_\pm - (v^\varepsilon_\pm)_{\rm app}\|_{L^\infty([0, r_0 - \delta]\times[0,\infty[)} \;=\; O(\varepsilon)\,,$$

while $v^\varepsilon = O(1)$. For larger times nonlinear geometric optics is again valid but the error can be larger, precisely

$$\|v^\varepsilon_\pm - (v^\varepsilon_\pm)_{\rm app}\|_{L^\infty([0,T]\times[0,\infty[)} \;=\; O(\varepsilon^{2-p})\,.$$

This result asserts that for ε small the pulse is described by nonlinear geometric optics before and after the focal point. The only influence of the focal point on the asymptotic description is the sign change which also occurs for the linear problem. Away from the focus the nonlinearity plays a crucial role as it appears in the transport equations of geometric optics, but for the crossing of the focus the effects are as in the linear case.

The next result refines this in two ways. In §3 we will show that the approximate solution defined above does not give good pointwise estimates for $u^\varepsilon_t = (v^\varepsilon_+ + v^\varepsilon_-)/2r$. Fix $1 \geq \alpha$. Consider new approximations $\mathbf{v}^\varepsilon := (\mathbf{v}^\varepsilon_-, \mathbf{v}^\varepsilon_+)$ in $\{r \leq \varepsilon^\alpha\}$ which are exact solutions of the linear wave equation in characteristic form. These solutions are chosen to match with the geometric optics solution along $r = \varepsilon^\alpha$ (see §5). They are defined by

$$\operatorname{supp} \mathbf{v}^\varepsilon_- \;\subset\; \{t + r > r_0 - z_0\varepsilon\} \qquad \operatorname{supp} \mathbf{v}^\varepsilon_+ \;\subset\; \{t - r > r_0 - z_0\varepsilon\} \qquad ,$$

and in these domains

$$\begin{aligned} \mathbf{v}^\varepsilon_-(t, r) \;&=\; V^{\rm in}\Big(r_0, \varepsilon^\alpha, \frac{t + r - r_0}{\varepsilon}\Big)\,, \\ \mathbf{v}^\varepsilon_+(t, r) &= -V^{\rm in}\Big(r_0, \varepsilon^\alpha, \frac{t - r - r_0}{\varepsilon}\Big)\,. \end{aligned} \tag{1.10}$$

Theorem 2. *For fixed $2 - p \leq \alpha \leq 1$, the family $\mathbf{v}^\varepsilon$ of solutions of the linear wave equation in $\{r \leq \varepsilon^\alpha\}$ is a better approximation in the sense that*

$$\|v^\varepsilon - \mathbf{v}^\varepsilon\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{2-p})\,,$$

$$\|(v^\varepsilon_+ + v^\varepsilon_-) - (\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O\big(r\,\varepsilon^{\alpha(1-p)}\big) \;=\; o(r/\varepsilon)\,,$$

while $v^\varepsilon_+ + v^\varepsilon_- = O(r/\varepsilon)$.

The first estimate shows that $\mathbf{v}^\varepsilon$ is a good an approximation to v^ε. The last relation shows that $(\mathbf{v}^\varepsilon_- + \mathbf{v}^\varepsilon_+)/2r$ is a good pointwise approximation for u^ε_t. The factor of r on the right of the second error estimate is crucial. In contrast, $((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app})/2r$ is **not** a good approximation to u^ε_t. In fact we show in §3 that it is more singular than u^ε_t.

In the last section of the paper we apply Theorem 2 to prove that a more natural looking inner approximation near $\{r = 0\}$ is in fact less accurate.

2. Existence of the profiles

Since the profiles are defined by ordinary differential equations along the rays, the existence results for equations (1.7), (1.8) and (1.9) follow from the theory of ordinary differential equations. The main point is to show that the time interval of existence includes $[0, r_0]$, so that the first profile exists past the focusing time (Figure 2). This is achieved in at least the following two cases;

- when the equation is dissipative, that is, $a \geq 0$, or
- when the initial data P_0 is sufficiently small.

Proposition 2.1. *Assume that $a \geq 0$, or that P_0 is sufficiently small. Then there exists a time $T > r_0$ such that Equations (1.7), (1.8) and (1.9) have unique solutions $V^{\rm in}$, $V^{\rm out}$, $V^{\rm foc} \in (C \cap L^\infty)([0,T] \times \overline{\mathbb{R}_+} \times \mathbb{R})$. Moreover, these profiles are compactly supported in their last variable,*

$$\operatorname{supp} V^{\rm in}(t,r,.),\ \operatorname{supp} V^{\rm out}(t,r,.),\ \operatorname{supp} V^{\rm foc}(t,r,.) \ \subset\ [-z_0, z_0]\,, \tag{2.1}$$

and their derivatives with respect to both t and z are bounded, that is

$$\partial_{t,z}\Big\{V^{\rm in}, V^{\rm out}, V^{\rm foc}\Big\} \ \in\ (C \cap L^\infty)([0,T] \times \overline{\mathbb{R}_+} \times \mathbb{R})\,.$$

The boundedness of the derivatives follows upon differentiating the equation and noting that these derivatives satisfy an equation with the same form as the profile itself. The same is **not** true of the derivatives with respect to r which are singular, $\sim r^{1-p}$, near the focal point.

3. Proof of Theorem 1

Define the first remainder

$$w^\varepsilon := (w^\varepsilon_-, w^\varepsilon_+)\,, \qquad \text{with} \qquad w^\varepsilon_\pm := v^\varepsilon_\pm - (v^\varepsilon_\pm)_{\rm app}\,.$$

The nonlinear wave equation in characteristic form (1.4) together with the equation (1.7) for the incoming profile implies that $(v^\varepsilon)_{\rm app}$ satisfies

$$(\partial_t - \partial_r)(v^\varepsilon_-)_{\rm app} \ =\ r^{1-p}\, g\big((v^\varepsilon_-)_{\rm app}\big)\,.$$

Similarly $(v^\varepsilon_+)_{\rm app}$ satisfies

$$(\partial_t + \partial_r)(v^\varepsilon_+)_{\rm app} \ =\ r^{1-p}\, g\big((v^\varepsilon_+)_{\rm app}\big)\,,$$

as soon as ε is so small that the waves with profiles $V^{\rm foc}_+$ and $V^{\rm out}_+$ do not overlap.

Subtracting these equations from the equations satisfied by $v^\varepsilon_\pm$ yields

$$(\partial_t \pm \partial_r) w^\varepsilon_\pm \ =\ r^{1-p}\big(g(v^\varepsilon_- + v^\varepsilon_+) - g((v^\varepsilon_\pm)_{\rm app})\big)\,. \tag{3.1}$$

In this expression we make two replacements. First,

$$g(v^\varepsilon_- + v^\varepsilon_+) = g\big((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app}\big) + \Big(g(v^\varepsilon_- + v^\varepsilon_+) - g\big((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app}\big)\Big)\,.$$

Using Taylor's Theorem, the second term is written as

$$g(v^\varepsilon_- + v^\varepsilon_+) - g\big((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app}\big) = (w^\varepsilon_- + w^\varepsilon_+) h((v^\varepsilon)_{\rm app}, w^\varepsilon) := (w^\varepsilon_- + w^\varepsilon_+)\, f^\varepsilon(t,r)\,.$$

Since $g \in C^1$ and the approximate solutions $(v^\varepsilon)_{\rm app}$ are uniformly bounded, it follows that f^ε is uniformly bounded on any set on which the family v^ε is uniformly bounded. Summarizing, we have the following error equation

$$(\partial_t \pm \partial_r) w^\varepsilon_\pm \ =\ r^{1-p} f^\varepsilon\ w^\varepsilon_\pm + S^\varepsilon_\pm(t,r)\,, \tag{3.2}$$

$$S^\varepsilon_\pm(t,r) \ :=\ r^{1-p}\,\big(g\big((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app}\big) \ -\ g\big((v^\varepsilon_\pm)_{\rm app}\big)\big)\,. \tag{3.3}$$

We use an inequality of Haar type for the differential operator

$$L \;:=\; \frac{\partial}{\partial t} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial r}\,.$$

Definition 3.1. *The characteristics for L are the lines traveling with speed ± 1. In $[0,T]\times\overline{\mathbb{R}}_+$, the* **incoming characteristics** *are speed minus one characteristics starting at $t=0$, and the* **outgoing characteristics** *are speed one characteristics starting at $t=0$ or at $r=0$. The set of incoming characteristics is denoted $\Gamma_-(T)$, and the set of outgoing characteristics is denoted $\Gamma_+(T)$.*

Proposition 3.1. *Let $w=w(t,r)=:(w_-,w_+)$. Assume $w\in (C\cap L^\infty)([0,T]\times\mathbb{R}_+)$ is a solution of*

$$Lw = r^{1-p} f w + \begin{pmatrix} S_- \\ S_+ \end{pmatrix}, \qquad (w_- + w_+)\big|_{r=0} = 0\,,$$

where f is a matrix with coefficients in $(C\cap L^\infty)([0,T]\times\mathbb{R}_+)$. Then there exists C depending on T, p and f such that for all $0\le t\le T$,

$$\big\|w(t)\big\|_{L^\infty([0,\infty[)} \le C\Big(\big\|w(0)\big\|_{L^\infty([0,\infty[)} + \sup_{\gamma\in\Gamma_-(T)}\int_\gamma |S_-| + \sup_{\gamma\in\Gamma_+(T)}\int_\gamma |S_+|\Big).$$

Proof. The first step is the elementary fact that if

$$Lw \;=\; G \quad\text{in}\quad r\ge 0\,, \qquad (w_-+w_+)\big|_{r=0} \;=\; 0\,,$$

then

$$\|w_-(t)\|_{L^\infty([0,\infty[)} \le \|w_-(0)\|_{L^\infty([0,\infty[)} + \sup_{\gamma\in\Gamma_-(T)}\int_\gamma |G_-|, \quad\text{and}$$

$$\|w_+(t)\|_{L^\infty([0,\infty[)} \le \|w_-(\cdot,0)\|_{L^\infty([0,T]\times\{0\})} + \|w_+(0)\|_{L^\infty([0,\infty[)} + \sup_{\gamma\in\Gamma_+(T)}\int_\gamma |G_+|.$$

An application of Gronwall's Lemma completes the proof. □

Recall that $T>r_0$ denotes a time of existence for the profiles from Proposition 2.1. From Proposition 2.1, there exists $C_0>0$ independent of ε such that

$$\big\|(v^\varepsilon)_{\rm app}\big\|_{L^\infty([0,T]\times[0,\infty[)} \;\le\; C_0. \tag{3.4}$$

A straightforward local existence argument shows that either $v^\varepsilon_\pm$ exists throughout $[0,T]\times[0,\infty[$, or, the maximal local solution belongs to $C([0,T_\varepsilon[\times\mathbb{R}_+)$ with $0<T_\varepsilon<T$ and

$$\liminf_{t\to T_\varepsilon} \|v^\varepsilon(t)\|_{L^\infty([0,\infty[)} \;=\; \infty\,.$$

In the latter case there is a first time, t_ε such that

$$\|w^\varepsilon(t_\varepsilon)\|_{L^\infty([0,\infty[)} \;=\; 2C_0\,.$$

We prove that there is a constant $C>0$, so that for $\varepsilon<1$ and $t\le t_\varepsilon$,

$$\|w^\varepsilon\|_{L^\infty([0,t^\varepsilon]\times[0,\infty[)} \;\le\; C\,\varepsilon^{2-p}\,. \tag{3.5}$$

This implies the $O(\varepsilon^{2-p})$ error estimate of Theorem 1. To see this choose $\varepsilon_1<1$ so that $C\,\varepsilon_1^{2-p}<2C_0$. Then if $t^\varepsilon<T$ we have the contradiction

$$2C_0 = \|w^\varepsilon(t^\varepsilon)\|_{L^\infty([0,\infty[)} \le C\,\varepsilon^{2-p} < 2C_0\,.$$

It follows that $t^\varepsilon=T$. In that case (3.5) is the desired result.

We next prove the estimate (3.5). On $[0,t^\varepsilon]$, w^ε is pointwise bounded by $2C_0$. Since we know that the family $(v^\varepsilon)_{\rm app}$ is uniformly bounded it follows that v^ε is

uniformly bounded. As remarked above, this implies that f^ε is uniformly bounded so Proposition 3.1, implies that

$$\|w^\varepsilon(t)\|_{L^\infty([0,\infty[)} \le C\left(\varepsilon\|P_1\|_{L^\infty_r} + \max_{\gamma\in\Gamma_-(t)}\int_\gamma |S^\varepsilon_-| + \max_{\gamma\in\Gamma_+(t)}\int_\gamma |S^\varepsilon_+|\right).$$

A key step is to estimate the integrals on the right hand side.

The arguments for the $\pm$ cases are entirely analogous so we treat only the minus case.

The first key observation is that

$$g\big((v^\varepsilon_-)_{\rm app} + (v^\varepsilon_+)_{\rm app}\big) \;\approx\; g\big((v^\varepsilon_-)_{\rm app}\big) \;+\; g\big((v^\varepsilon_+)_{\rm app}\big).$$

In fact the two sides are equal except on the small set where **both** $(v^\varepsilon_+)_{\rm app}$ and $(v^\varepsilon_-)_{\rm app}$ are nonzero.

The two functions are nonzero at most on the union of two isoceles right triangles with side $O(\varepsilon)$. An important one is at the focal point. Less important is the triangle with hypotenuse on $t=0$ and centered at $r=r_0$. Denote the union of the two triangles by $E(\varepsilon)$. Then

$$S^\varepsilon_-(t,r) = r^{1-p}\left(g\big((v^\varepsilon_+)_{\rm app}\big) + h^\varepsilon(t,r)\,\chi_{E(\varepsilon)}\right),$$

with uniformly bounded h^ε.

The term in parentheses is uniformly bounded and is supported in the two rightward moving strips from Figure 2. Thus the support of S^ε_- intersects each γ in at most two intervals with total length $O(\varepsilon)$. Thus

$$\int_\gamma |S^\varepsilon_-| \;\le\; C\int_0^{c\varepsilon} r^{1-p}\,dr = O(\varepsilon^{2-p}).$$

Moreover, if one is interested in $t \le r_0-\delta$, the upper bound is sharpened to

$$\int_\gamma |S^\varepsilon_-| \;\le\; C\int_\delta^{\delta+c\varepsilon} r^{1-p}\,dr = O(\varepsilon).$$

The first of these estimates completes the proof of (3.5) and therefore the second part of Theorem 1.

For $t \le r_0-\delta$ the second estimate shows that the "so long as" argument can be sharpened by replacing ε^{2-p} in (3.5) by ε and thereby proving the first assertion of Theorem 1. □

Remarks. 1. The estimate for $\int_\gamma |S_-|$ is sharp. In fact, considering the incoming characteristic, γ, which arrives at $t=r_0, r=0$ it is easy to see that

$$\int_\gamma S_- \;=\; cg\big(-V^{\rm in}(t_0,0,0)\big)\,\varepsilon^{2-p} + o(\varepsilon^{2-p})$$

which in general is no smaller than $O(\varepsilon^{2-p})$. It is not hard to show that this implies that $w^\varepsilon_-(t_0,0)$ is in general no smaller than $O(\varepsilon^{2-p})$ proving that (3.5) is also sharp.
2. For later use we show that the natural estimate

$$\big\|\partial_t w^\varepsilon\big\|_{L^\infty([0,T]\times[0,\infty[)} \;=\; O(\varepsilon^{(2-p)-1}),$$

is valid. The derivative with respect to r is less well behaved near $\{r=0\}$. To derive the estimate for the time derivative differentiate (3.1) with respect to time to find

$$\Big(\partial_t \pm \partial_r\Big)\partial_t w^\varepsilon \;=\; r^{1-p}\,L^\infty\,\partial_t w^\varepsilon + r^{1-p}\,L^\infty\,\frac{1}{\varepsilon}.$$

Proposition 5.1 implies the desired estimate.

4. Weakness of the approximation near the caustic

At the focal point $t = r_0, r = 0$, the approximation is imprecise. For the exact solution, the relation

$$\partial_t u^\varepsilon(t,r) = \frac{v_-^\varepsilon + v_+^\varepsilon}{2r}, \tag{4.1}$$

and the fact that v_-^ε and v_+^ε are bounded yield

$$\partial_t u^\varepsilon(t,r) = O\left(\frac{1}{r}\right). \tag{4.2}$$

The fact that $(v_-^\varepsilon + v_+^\varepsilon)\big|_{r=0} = ru_t\big|_{r=0} = 0$ suggests that there is a better estimate for $\partial_t u^\varepsilon$. In fact, the family $\partial_t u^\varepsilon$ satisfies the same pointwise bounds as focusing pulse solutions of the linear wave equation. In particular, for fixed ε, u_t^ε is bounded on $[0,T]\times\mathbb{R}^3$.

Proposition 4.1. *For any time $T > 0$, there is a constant C so that for all $\varepsilon \in]0,1]$,*

$$|\partial_t u^\varepsilon(t,r)| \ \le \frac{C}{r+\varepsilon} \quad \text{on} \quad [0,T]\times[0,\infty]. \tag{4.3}$$

Proof. Differentiating (1.4) with respect to time and letting $w := \big(\partial_t v_-^\varepsilon, \partial_t v_+^\varepsilon\big)$ one obtains an equation as in Proposition 3.1 with $S_\pm = 0$. It follows that on compact time intervals

$$\big\|\partial_t v_-^\varepsilon, \partial_t v_+^\varepsilon\big\|_{L^\infty([0,T\times[0,\infty[)} \ \le \ C\,\big\|\partial_t v_-^\varepsilon(0,.), \partial_t v_+^\varepsilon(0,.)\big\|_{L^\infty([0,\infty[)} \ \le \ \frac{C}{\varepsilon}.$$

From the definition of $v_\pm^\varepsilon$ one has that

$$(\partial_t - \partial_r)v_-^\varepsilon = (\partial_t + \partial_r)v_+^\varepsilon,$$

so

$$\partial_r(v_-^\varepsilon + v_+^\varepsilon) \ = \ \partial_t(v_-^\varepsilon - v_+^\varepsilon) \ = \ O(1/\varepsilon).$$

Since $v_-^\varepsilon + v_+^\varepsilon$ vanishes when $r = 0$ one finds that

$$v_-^\varepsilon + v_+^\varepsilon \ = \ O(r/\varepsilon).$$

This, compact support, and the crude estimate (4.2) implies (4.3). □

The same estimate does **not** hold for $r^{-1}\big((v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app}\big)$ which is the natural approximation for u_t^ε. To see this begin by remarking that the transport equations satisfied by the the first profiles imply that

$$\begin{aligned}\partial_r\big((v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app}\big) \ &= \ \partial_t\big((v_-^\varepsilon)_{\rm app} - (v_+^\varepsilon)_{\rm app}\big)\\ &+ \ r^{1-p}\left(g(V^{\rm in}(t,r,z_1)) - g(V^{\rm foc}(t,r,z_2))\right)\big|_{z_1=\frac{t+r-r_0}{\varepsilon}, z_2=\frac{t-r-r_0}{\varepsilon}}.\end{aligned}$$

Since the derivatives $\partial_{t,z}V^{\rm in,foc}$ are bounded, it follows that

$$\partial_t\big((v_-^\varepsilon)_{\rm app} - (v_+^\varepsilon)_{\rm app}\big) \ = \ O(1/\varepsilon).$$

On the other hand the second summand is in general of order exactly r^{1-p} near the focal point since using the boundary condition shows that at $r = 0$ where $z_1 = z_2$,

$$\begin{aligned}\big(g(V_{0-}^{\rm in}(t,0,z_1)) - g(V^{\rm foc}(t,0,z_2)\big) &= g(V^{\rm in})(t,0,z) - g(-V^{\rm in})(t,0,z)\\ &= 2g(V^{\rm in})(t,0,z) \neq 0.\end{aligned} \tag{4.4}$$

Therefore, for ε fixed, $\partial_r\big((v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app}\big) \sim r^{1-p}$, as $r \to 0$ so integrating from $r = 0$ implies that

$$(v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app} \ = \ O(r^{2-p}) >> O(r).$$

Note that it follows that $\big((v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app})\big)/r$ is **not** a good approximation of $\partial_t u^\varepsilon$ when $r^{1-p} \gg \varepsilon^{-1}$, that is, for $r \ll \varepsilon^{\frac{1}{p-1}}$.

On the other hand, from Theorem 1 and (4.3), $\big((v_-^\varepsilon)_{\rm app} + (v_+^\varepsilon)_{\rm app})\big)/r$ is a good approximation of $\partial_t u^\varepsilon$ when $r \gg \varepsilon^{3-p}$. Notice that since $1 < p < 2$, one has $(3-p) < (p-1)^{-1}$ (see Figure 3).

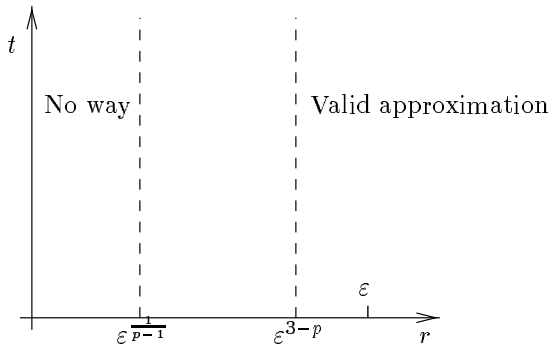

FIGURE 3. Is $(\partial_t u^\varepsilon)_{\rm app}$ a good approximation to u_t^ε?

These observations pose the problem of finding a more accurate description near $r = 0$, that would give at least a good pointwise approximation for $\partial_t u^\varepsilon$. That is accomplished in the next section.

5. Asymptotics near the caustic

We have proved that nonlinear geometric optics is valid before and after the focusing. Though the pulses propagate nonlinearly away from the focus, the crossing of the focus is described with only the phase shift that would be present in the linear case. As in [4], this is called the regime of nonlinear propagation with a linear caustic crossing. The cumulative nonlinear effects from a neighborhood of $r = 0$ are negligible. In this section we will prove a stronger result of this sort. Near $r = 0$, there are solutions of the homogeneous linear wave equation which furnish good approximations to the family u^ε. The strategy is to compare the exact system in characteristic form,

$$(\partial_t \pm \partial_r) v_\pm^\varepsilon \;=\; r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\,,$$

with the free wave equation, $\Box\,\mathbf{u}^\varepsilon = 0$ also written in characteristic form,

$$(\partial_t \pm \partial_r)\mathbf{v}_\pm^\varepsilon \;=\; 0\,, \qquad \mathbf{v}_\pm^\varepsilon \;:=\; (\partial_t \mp \partial_r)\, r\mathbf{u}^\varepsilon\,. \tag{5.1}$$

We know that the family $v_\pm^\varepsilon$ is uniformly bounded so the right hand side of the transport equations for $v_\pm^\varepsilon$ are uniformly integrable along characteristics.

For fixed $\alpha \le 1$, $\mathbf{v}_\pm^\varepsilon$, defined in $r \le \varepsilon^\alpha$, is matched with the approximate solution given by nonlinear geometric optics along $r = \varepsilon^\alpha$. The analysis shows that the resulting matched asymptotics are accurate for all r.

Therefore, the equation (5.1) is supplemented by the boundary conditions

$$\mathbf{v}_+^\varepsilon(t,0) + \mathbf{v}_-^\varepsilon(t,0) \;=\; 0\,, \qquad \mathbf{v}_-^\varepsilon(t,\varepsilon^\alpha) = V^{\rm in}\Big(r_0, \varepsilon^\alpha, \frac{t+\varepsilon^\alpha - r_0}{\varepsilon}\Big)\,, \tag{5.2}$$

and the homogeneous initial conditions

$$\mathbf{v}_\pm^\varepsilon(0,r) \;=\; 0\,. \tag{5.3}$$

There are two important remarks to make about this definition. The first concerns the choice $t = r_0$ in the boundary values at $r = \varepsilon^\alpha$. A more natural choice

might be to take the actual geometric optics approximation $V^{\rm in}\big(t,\varepsilon^\alpha,\frac{t+\varepsilon^\alpha-r_0}{\varepsilon}\big)$. The difference is that in (5.2) the value of t is taken equal to r_0. However, all these incoming waves are supported on an ε^α neighborhood of $t=r_0$ and $V^{\rm in}_t$ is bounded so the difference between the two choices is small (see equation (5.5)).

Secondly, with the choice in (5.2) the solution in $r\le\varepsilon^\alpha$ is given by the simple explicit formula (1.10). To see this, note that the values of $\mathbf{v}^\varepsilon_-$ are determined from their values on $\{r=\varepsilon^\alpha\}\cup\{t=0\}$ since $\mathbf{v}^\varepsilon_-$ is constant on incoming characteristics. Thus,

$$\mathbf{v}^\varepsilon_-(t,r)\;=\;\mathbf{v}^\varepsilon_-(t+r-\varepsilon^\alpha,\varepsilon^\alpha)\;=\;V^{\rm in}\Big(r_0,\varepsilon^\alpha,\frac{t+r-r_0}{\varepsilon}\Big)\,. \tag{5.4}$$

Similarly $\mathbf{v}_+$ is constant on outgoing characteristics so its values determined from $\mathbf{v}_+$ on $\{r=0\}\cup\{t=0\}$. Using the boundary conditions shows that the values are determined from the values of $\mathbf{v}_-$ on $\{r=0\}\cup\{t=0\}$. This yields,

$$\mathbf{v}^\varepsilon_+(t,r)=\mathbf{v}^\varepsilon_+(t-r,0)=-\mathbf{v}^\varepsilon_-(t-r,0)=-V^{\rm in}\Big(r_0,\varepsilon^\alpha,\frac{t-r-r_0}{\varepsilon}\Big)\,,$$

recovering the formula (1.10).

The solution $\mathbf{v}^\varepsilon$ is supported in an $O(\varepsilon)$ neighborhood of the pair of characteristics through the focal point $t=r_0, r=0$.

Proof of Theorem 2. A key element is a Haar inequality like that used in §3.

Definition 5.1. *With L as in §3, the characteristics for L in $[0,T]\times[0,R]$ are the lines traveling with speed ±1. The* **incoming characteristics** *are speed minus one characteristics starting and the* **outgoing characteristics** *are speed one characteristics. The set of incoming characteristics is denoted Γ_-, and the set of outgoing characteristics is denoted Γ_+.*

The next Haar inequality follows immediately upon integrating along characteristics following the same geometry as in the derivation of (1.10).

Proposition 5.1. *Let $w=w(t,r):=(w_-,w_+)$. Assume $w\in(C\cap L^\infty)([0,T]\times[0,R])$ vanishes at $\{t=0\}$ and is a solution of*

$$Lw=f:=(f_1,f_2)\,,\qquad (w_-+w_+)(t,0)=0\,,\qquad w_-(t,R)=\sigma(t)\,,$$

where $f\in C([0,T]\times[0,R])$. Then

$$\big\|w\big\|_{L^\infty([0,T]\times[0,R])}\le\big\|\sigma\big\|_{L^\infty([0,R])}+\sup_{\gamma\in\Gamma_-}\int_\gamma|f_1|+\sup_{\gamma\in\Gamma_+}\int_\gamma|f_2|.$$

The inequality of this Proposition is applied to $w^\varepsilon:=v^\varepsilon-\mathbf{v}^\varepsilon$ which satisfies the hypothesis with

$$R:=\varepsilon^\alpha\,,\quad \sigma:=v^\varepsilon_-(t,\varepsilon^\alpha)-V^{\rm in}\Big(r_0,\varepsilon^\alpha,\frac{t+\varepsilon^\alpha}{\varepsilon}\Big)\,,\quad f:=\frac{g(v^\varepsilon_++v^\varepsilon_-)}{r^{p-1}}\,.$$

To estimate γ write it as a sum

$$\sigma(t)=\Big(v^\varepsilon_-(t,\varepsilon^\alpha)-(v^\varepsilon_-)_{\rm app}(t,\varepsilon^\alpha)\Big)+\Big((v^\varepsilon_-)_{\rm app}(t,\varepsilon^\alpha)-\mathbf{v}^\varepsilon(t,\varepsilon^\alpha)\Big)\,.$$

The first summand is $O(\varepsilon^{2-p})$ and the second summand is equal to

$$V^{\rm in}\Big(t,\varepsilon^\alpha,\frac{t+r-r_0}{\varepsilon}\Big)-V^{\rm in}\Big(r_0,\varepsilon^\alpha,\frac{t+r-r_0}{\varepsilon}\Big)\,. \tag{5.5}$$

In the support of σ, $t+r-r_0=O(\varepsilon)$ and $r=\varepsilon^\alpha$ so $t-r_0=O(\varepsilon^\alpha)$. In addition $\partial_tV^{\rm in}$ is bounded so the difference (5.5) is $O(\varepsilon^\alpha)$. Since $\alpha\ge2-p$ this is $O(\varepsilon^{2-p})$.

Proposition 5.1 then implies that

$$\|v^\varepsilon-\mathbf{v}^\varepsilon\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])}=O(\varepsilon^{2-p})\,.$$

So far this is no better than the error estimate for $(v^\varepsilon_\pm)_{\rm app}$. The improvement involves the approximation of u^ε_t by $(\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-)/r$. From the formula (1.10) and the boundedness of $\partial_z V$ it follows that $\partial_{t,r}\mathbf{v}^\varepsilon = O(1/\varepsilon)$. Integrating dr from $r=0$ then shows that

$$\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_- \;=\; O(r/\varepsilon)\,,$$

which is the size of $v^\varepsilon_+ + v^\varepsilon_-$ proved in Proposition 4.1.

To estimate the error in the approximation $\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-$ differentiate the error equation with respect to t to obtain

$$L\big(\partial_t w^\varepsilon\big) = \partial_t\Big(g(v^\varepsilon_+ + v^\varepsilon_-)/r^{p-1}\Big)\,.$$

The source terms on the right hand side and on the boundary $\{r=\varepsilon^\alpha\}$ are larger by a factor $1/\varepsilon$ than those for w^ε. One obtains the error estimate

$$\|\partial_t\big(v^\varepsilon - \mathbf{v}^\varepsilon\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{1-p})\,.$$

The error equation then yields

$$\|\partial_r\big(v^\varepsilon - \mathbf{v}^\varepsilon\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{1-p}) + O(r^{1-p})\,.$$

Integrating from $r=0$ then implies that

$$\|\big(v^\varepsilon_+ + v^\varepsilon_-\big) - \big(\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(r\varepsilon^{1-p}) + O(r^{2-p})\,.$$

Since $r\le\varepsilon^\alpha$ the last term is $O(r*r^{1-p}) = O(r\,\varepsilon^{\alpha(1-p)})$, which is larger than the first term. Thus,

$$\|\big(v^\varepsilon_+ + v^\varepsilon_-\big) - \big(\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(r\,\varepsilon^{\alpha(1-p)})\,.$$

This is $o(r/\varepsilon)$ since $\alpha < 1/(p-1)$. This completes the proof of Theorem 2. □

6. A natural inner expansion is less accurate

The preceding section shows that a good approximation for the solution in $\{r\le\varepsilon^\alpha\}$ can be constructed using the values of the incoming profile $V^{\rm in}(t,r,z)$ on $r=\varepsilon^\alpha$. Since this is an inner expansion one might expect that one could or even should use the values of $V^{\rm in}$ at the origin, $r=0$. This yields new inner approximations

$$\mathbf{v}^\varepsilon_{\rm new} \;:=\; \big(\mathbf{v}^\varepsilon_{{\rm new},-}\,,\,\mathbf{v}^\varepsilon_{{\rm new},+}\big)\,,$$

where

$$\begin{aligned}\mathbf{v}^\varepsilon_{{\rm new},-}(t,r) \;&:=\; V^{\rm in}\Big(r_0,0,\frac{t+r-r_0}{\varepsilon}\Big)\,,\\ \mathbf{v}^\varepsilon_{{\rm new},+}(t,r) \;&:=\; -V^{\rm in}\Big(r_0,0,\frac{t-r-r_0}{\varepsilon}\Big)\,.\end{aligned}\tag{6.1}$$

The next result shows that the relative error of this approximation tends to zero as $\varepsilon\to0$, but, at a slower rate than the error for $\mathbf{v}^\varepsilon$

Theorem 3. *The new approximation is good but not as good as* $\mathbf{v}^\varepsilon$*, precisely*

$$\|v^\varepsilon - \mathbf{v}^\varepsilon_{\rm new}\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{\alpha(2-p)})\,,\tag{6.2}$$

and

$$\|\big(v^\varepsilon_+ + v^\varepsilon_-\big) - \big(\mathbf{v}^\varepsilon_{{\rm new},+} + \mathbf{v}^\varepsilon_{{\rm new},-}\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O\Big(\frac{r}{\varepsilon}\,\varepsilon^{\alpha(2-p)}\Big) \;=\; o\Big(\frac{r}{\varepsilon}\Big)\,.\tag{6.3}$$

Proof. Since $\partial_r V^{\rm in} = O(r^{1-p})$ integrating from $r=0$ shows that

$$\mathbf{v}^\varepsilon(t,r) - \mathbf{v}^\varepsilon_{\rm new}(t,r) \;=\; O(r^{2-p}) \;=\; O(\varepsilon^{\alpha(2-p)})\,.$$

This together with the second estimate of Theorem 1 proves (6.2).

To prove the second estimate of the Theorem, use the error equation,

$$L\big(v^\varepsilon - \mathbf{v}^\varepsilon_{\rm new}\big) \;=\; r^{1-p}\,\big(g(v^\varepsilon_+ + v^\varepsilon_-) - g(\mathbf{v}^\varepsilon_{\rm new})\big)\,, \qquad \text{in} \quad 0<r<\varepsilon^\alpha\,,$$

which follows from the profile equation satisfied by $V^{\rm in}$.

Differentiating with respect to time and using the fact that $\partial_t V^{\rm in}$ is bounded shows that

$$L\Big(\partial_t\big(v^\varepsilon - \mathbf{v}^\varepsilon_{\rm new}\big)\Big) \;=\; O\Big(\frac{r^{1-p}}{\varepsilon}\Big)\,.$$

The estimate in the second remark of §3, shows that

$$\|\partial_t\big(v^\varepsilon - \mathbf{v}^\varepsilon\big)\|_{L^\infty([0,T]\times\{r=\varepsilon^\alpha\})} \;=\; O(\varepsilon^{\alpha(2-p)-1})\,.$$

Proposition (5.1) then implies that

$$\|\partial_t\big(v^\varepsilon - \mathbf{v}^\varepsilon\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{\alpha(2-p)-1})\,.$$

The error equation then yields

$$\|\partial_r\big(v^\varepsilon - \mathbf{v}^\varepsilon\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(\varepsilon^{\alpha(2-p)-1}) + O(r^{1-p})\,.$$

Integrating from $r=0$ implies

$$\|\big(v^\varepsilon_+ + v^\varepsilon_-\big) - \big(\mathbf{v}^\varepsilon_+ + \mathbf{v}^\varepsilon_-\big)\|_{L^\infty([0,T]\times[0,\varepsilon^\alpha])} \;=\; O(r\,\varepsilon^{\alpha(2-p)-1}) + O(r^{2-p})\,.$$

Since $r\le\varepsilon^\alpha$ the last term is $O(r * r^{1-p}) = O(r\,\varepsilon^{\alpha(1-p)})$. which is smaller than the first since $\alpha\le 1$. The proof of the Theorem is complete. □

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