Group Velocity at Smooth Points of Hyperbolic Characteristic Varieties

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Dedication. To my friend Jean-Michel Bony with best wishes and appreciation for what he has taught me of mathematics other things.

Suppose that $P(D)$ is a homogeneous hyperbolic polynomial of degree $m \geq 1$ with time-like covector θ. Here $D = \partial/i\partial y$ with $y \in \mathbb{R}^n$. The symbol $P(\eta)$ is a homogeneous polynomial on $(\mathbb{R}^n)^*$. Hyperbolicity with respect to $\theta \in (\mathbb{R}^n)^*$ means that for any $\eta \in (\mathbb{R}^n)^*$ the equation

$$P(\eta + s\theta) = 0 \tag{1}$$

has only real roots s. In particular, $P(\theta) \neq 0$.

The characteristic variety

$$\operatorname{Char} P \ := \ \big\{\eta \in \mathbb{R}^n \setminus 0 \ : \ P(\eta) = 0\big\}$$

is a conic real algebraic variety in $(\mathbb{R}^n)^*$. Since the equation (1) has m complex roots (counting multiplicity), and they all are real, it follows that every real line $\eta + s\theta$ intersects the variety in at least one point and no more than m points which shows that the variety has codimension 1 in $(\mathbb{R}^n)^*$. The fundamental stratification of real algebraic geometry (see [BR]) asserts that except for a set of codimension at least 2, the variety consists of smooth, points, that is points where locally the variety is equal to the zero set of a real analytic function with nonvanishing gradient.

Definitions. *If $\underline{\eta} \neq 0$ is a point of the characteristic variety then $Q_{\underline{\eta}}(\eta)$ is the homogeneous polynomial of degree $k \geq 1$ which is the leading term in the expansion of $P(\underline{\eta} + \eta)$ about $\underline{\eta}$,*

$$P(\underline{\eta} + \eta) = Q_{\underline{\eta}}(\eta) + \text{higher order terms in } \eta\,, \qquad Q_{\underline{\eta}} \neq 0\,.$$

At a smooth point $\underline{\eta}$, the annihilator of the tangent space $T_{\underline{\eta}}(\operatorname{Char} P)$ is a one dimensional linear subspace $L_{\underline{\eta}} \in \big(T_{\underline{\eta}}(\operatorname{Char} P)\big)^* = \mathbb{R}^n$. The lines in $\mathbb{R}^n$ parallel to $L_{\underline{\eta}}$ are those moving with the **group velocity** *(see [AR]).*

This velocity describes the propagation of wave packets, pulses, and singularities associated with the frequencies $(\mathbb{R} \setminus 0)\,\underline{\eta}$.

For variable coefficient operators, the above computations are performed in the tangent and cotangent spaces at a fixed point and P is the principal symbol at that point. They are pertinent for example for symmetric hyperbolic systems and points of the characteristic variety which are microlocally of constant multiplicity.

If $\underline{\eta} \in \operatorname{Char} P$ is a smooth point of multiplicity one, that is $P(\underline{\eta}) = 0$ and $dP(\underline{\eta}) \neq 0$, then $dP(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$ and one has a simple way of recovering the velocity from the symbol.

In an analogous way, at a smooth point one can write the variety locally as $q = 0$ with $dq \neq 0$, then $dq(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$. However, in real algebraic geometry it is not in general easy to find a function q starting from the defining function P when the roots have multiplicity greater than one. The following two results provide a straigtforward algorithm to compute the group velocity for our hyperbolic operators.

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Theorem. *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ is as above, then there is a linear form $\ell(\eta)$ so that the tangent plane at $\underline{\eta}$ to the characteristic variety of P is equal to $\{\ell(\eta - \underline{\eta}) = 0\}$, and, $Q_{\underline{\eta}}(\eta) = \ell(\eta)^k$.*

Corollary. *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ and $L_{\underline{\eta}}$ are as above, then for all η which are not in the characteristic variety of $Q_{\underline{\eta}}$ (e.g. $\eta = \theta$), $dQ_{\underline{\eta}}(\eta)$ is a basis for $L_{\underline{\eta}}$.*

These results both rely on the fundamental theorems concerning Local Hyperbolicity (see [G]). That theory is closely related to the ideas of microhyperbolicity introduced by Bony and Shapira in [BS] (see [H, §8.7]).

The proof of the Theorem begins with the fact from [G] that $Q_{\underline{\eta}}(\eta)$ is hyperbolic with time-like covector θ. Then for every real η the equation

$$Q_{\underline{\eta}}(\eta + s\theta) = 0 \tag{2}$$

has only real roots s.

Lemma 1. *For every real η the equation (2) has exactly one root s.*

Proof. Since k is the degree of $Q_{\underline{\eta}}$, one has as $\epsilon \to 0$,

$$\epsilon^{-k} P(\underline{\eta} + \epsilon(\eta + s\theta)) \;=\; Q_{\underline{\eta}}(\eta + s\theta) + O(\epsilon)\,. \tag{3}$$

If (2) had two roots s_1 and s_2, then Rouché's theorem would imply that the characteristic variety of P would have points near $\underline{\eta} + \epsilon(\eta + s_j\theta)$ as $\epsilon \to 0$ violating the smooth variety hypothesis. ∎

The next Lemma is then applied to $R = Q_{\underline{\eta}}$.

Lemma 2. *If $R(\eta)$ is a homogeneous polynomial hyperbolic with respect to the time-like covector θ and for all real η the equation $R(\eta + s\theta) = 0$ has exactly one real root s, then there is a linear form $\ell(\eta)$ such that*

$$R(\eta) = \ell(\eta)^{\deg R}\,.$$

Proof. Introduce coordinates $(\tau, \xi_1, ..., \xi_{n-1})$ in $(\mathbb{R}^n)^*$ so that $\theta = (1, 0, \ldots, 0)$. Then

$$R(\tau, \xi) = R(1, 0, \ldots, 0)\Big(\tau^k + a_1(\xi)\tau^{k-1} + \cdots + a_{k-1}(\xi)\tau + a_k(\xi)\Big)$$

with $a_j(\xi)$ homogeneous of degree j and $k = \deg R \geq 1$.

By hypothesis, for each real ξ the equation $R(\tau, \xi) = 0$ has a unique root $\tau = r(\xi)$. Therefore

$$R(\tau, \xi) = R(1, 0, \ldots, 0)\big(\tau - r(\xi)\big)^k\,.$$

Equating coefficients of τ^{k-1} shows that

$$-k\, r(\xi) \;=\; a_1(\xi)\,,$$

so $r(\xi)$ is a homogeneous polynomial of degree 1. The Lemma follows with $\ell(\tau, \xi) = c(\tau - r(\xi))$ provided that c is chosen to satisfy $c^k \;=\; R(1, 0, \ldots, 0)$. ∎

The constant c and the functional ℓ are uniquely determined up to a factor which is a k^{th} root of unity.

Proof of Theorem. Combining the above lemmas implies that $Q_{\underline{\eta}}(\eta) = \ell(\eta)^k$. It remains to show that the tangent plane to the characteristic variety of P is given by the equation $\ell(\eta - \underline{\eta}) = 0$.

Introduce local coordinates (τ, ξ) as in the proof of Lemma 2. Since $\theta = (1, 0, \ldots, 0)$ is noncharacteristic for P, the variety of P is given by the roots τ of $P(\tau, \xi) = 0$ with ξ ranging over $\mathbb{R}^n \setminus 0$.

The points near $\underline{\eta} = (\underline{\tau}, \underline{\xi})$ are then given by the roots τ of

$$P(\underline{\tau} + \epsilon\tau, \underline{\xi} + \epsilon\xi) \;=\; 0\,, \tag{4}$$

with $|\xi| \leq 1$. Equation (2) takes the form

$$\epsilon^{-k}\, P(\underline{\tau} + \epsilon\tau, \underline{\xi} + \epsilon\xi) \;=\; Q_{\underline{\eta}}(\tau, \xi) + O(\epsilon)\,. \tag{5}$$

Since $Q_{\underline{\eta}} = \ell^k$, the equation $Q_{\underline{\eta}}(\tau, \xi) = 0$ is equivalent to the equation $\ell(\tau, \xi) = 0$. Since $\ell(\theta)^k = Q_{\underline{\eta}}(\theta) \neq 0$, it follows that the solutions of $\ell(\tau, \xi) = 0$ are given by $\tau = \underline{x}.\xi$ for an appropriate $\underline{x}$.

Rouché's Theorem applied to (5) shows that for $|\xi| < 1$ the roots of (4) are given by

$$\tau = \underline{x}.\xi + O(\epsilon)\,.$$

The corresponding points $\eta = (\underline{\tau} + \epsilon\tau, \underline{\xi} + \epsilon\xi)$ of the characteristic variety of P differ from $\underline{\eta}$ by $O(\epsilon)$ and satisfy

$$\ell(\eta - \underline{\eta}) = O(\epsilon^2)\,.$$

This shows that the equation of the tangent plane is $\ell(\eta - \underline{\eta}) = 0$. ∎

Proof of Corollary. Since $Q_{\underline{\eta}} = \ell^k$ one has

$$dQ_{\underline{\eta}}(\eta) \;=\; k\,\ell(\eta)^{k-1}\, d\ell(\eta)\,.$$

Since ℓ is a linear form on $(\mathbb{R}^n)^*$, $d\ell(\eta)$ is a vector which does not depend the point η where the derivative is evaluated. The Theorem implies that $d\ell$ is a basis for $L_{\underline{\eta}}$. Therefore, $dQ_{\underline{\eta}}(\eta)$ is a basis whenever it is nonvanishing. This holds exactly for η which satisfy $\ell(\eta) \neq 0$ which is exactly those η which are not in the characteristic variety of Q_η. ∎

References

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