Hyperbolic Domains of Determinacy and Hamilton-Jacobi Equations

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Abstract. If $L(t,x,\partial_t,\partial_x)$ is a linear hyperbolic system of partial differential operators for which local uniqueness in the Cauchy problem at spacelike hypersurfaces is known, we find nearly optimal domains of determinacy of open sets $\Omega_0 \subset \{t=0\}$. The frozen constant coefficient operators $L(\underline{t},\underline{x},\partial_t,\partial_x)$ determine local convex propagation cones, $\Gamma^+(\underline{t},\underline{x})$. Influence curves are curves whose tangent always lies in these cones. We prove that the set of points Ω which cannot be reached by influence curves beginning in the exterior of Ω_0 is a domain of determinacy in the sense that solutions of $L\,u=0$ whose Cauchy data vanish in Ω_0 must vanish in Ω. We prove that Ω is swept out by continuous space like deformations of Ω_0 and is also the set described by maximal solutions of a natural Hamilton-Jacobi equation (HJE). The HJE provides a method for computing approximate domains and is also the bridge from the raylike description using influence curves to that depending on spacelike deformations. The deformations are obtained from level surfaces of mollified solutions of HJEs.

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§0. Introduction.

The question addressed in this paper is to describe as accurately as possible the property of finite speed of propagation for solutions of general hyperbolic systems. This is a problem of propagation of zeros. Typically the speed depends on position in space time and also on the direction of propagation.

The analysis addresses $m^{\rm th}$ order $N\times N$ system of partial differential operators with complex matrix valued coefficients

$$L(t,x,\partial_t,\partial_x) \ =\ \sum_{|\beta|\le m} A_\beta(t,x)\,\partial^\beta_{t,x} \ =\ \partial_t^m \ +\ \text{lower order in } t\,. \tag{0.1}$$

Hypothesis 0.1. *The coefficients satisfy for all $T>0$,*

$$A_\beta\in L^\infty([-1,T]\times\mathbb{R}^d)\,,\qquad \text{and for }\ |\beta|=m,\ \ \nabla_{t,x}A_\beta\in L^\infty([-1,T]\times\mathbb{R}^d)\,.$$

The characteristic polynomial is

$$P(t,x,\tau,\xi)\ :=\ \det\Big(\sum_{|\beta|=m} A_\beta(t,x)\,(\tau,\xi)^\beta\Big)\,,\qquad \deg P = mN\,. \tag{0.2}$$

The operator is **hyperbolic** *with timelike variable t in the sense that the equation*

$$P(t,x,\tau,\xi)=0$$

has only real roots τ for real $\xi\in\mathbb{R}^d$.

Define

$$\tau_{\max}(t,x,\xi)\ :=\ \max\big\{\tau\ :\ P(t,x,\tau,\xi)=0\big\}, \tag{0.3}$$

and the associated convex cone of time like codirections (see e.g. [Gå, Co, H03])

$$\mathcal{T}(t,x)\ :=\ \big\{(\tau,\xi)\ :\ \tau>\tau_{\max}(t,x,\xi)\big\}\,. \tag{0.4}$$

$\mathcal{T}(t,x)$ a subset of the cotangent space at (t,x) and yields, by duality, the forward propagation cone in the tangent space at (t,x)

$$\Gamma^+(t,x)\ :=\ \big\{(T,X)\ :\ \forall(\tau,\xi)\in\mathcal{T}(t,x)\,,\ \ (T,X).(\tau,\xi)\ \ge\ 0\big\}\,. \tag{0.5}$$

The set $\Gamma^+(t,x)$ depends only on the principal symbol of L. Both $\mathcal{T}$ and Γ^+ are convex. The former is open and the latter has compact intersection with the planes $T=\text{const}>0$.

Examples. 1.,2. For the scalar real constant coefficient operators $\partial_t+\mathbf{v}.\partial_x$ and $\partial_t^2-c^2\Delta_x:=\Box$ with $c>0$, the forward cones Γ^+ are

$$\big\{(T,X)\ :\ T\ge0\,,\ X=\mathbf{v}\,T\big\}\qquad\text{and}\qquad \big\{(T,X)\ :\ |X|\ \le\ cT\big\}$$

respectively. **3.** For the diagonal real systems in dimension $d=1$

$$\partial_t\ +\ \text{diag}\Big(c_1(t,x)\,,\,c_2(t,x)\,,\,\dots\,,\,c_N(t,x)\Big)\partial_x\ +\ \text{order zero}\,, \tag{0.6}$$

the forward propagation cone is

$$\Gamma^+(t,x) \;=\; \big\{(T,X)\;:\;T\ge 0\,,\;\big(\min_j\{c_j(t,x)\}\big)\,T\;\le X\;\le\;\big(\max_j\{c_j(t,x)\}\big)\,T\big\}\,. \tag{0.7}$$

Definitions. *An embedded hypersurface $\Sigma\subset\mathbb{R}^{1+d}$ is* **space like** *when its conormal vectors belong to $\mathcal{T}(t,x)\cup-\mathcal{T}(t,x)$ for every $(t,x)\in\Sigma$. A relatively open set $\Omega\subset[0,\infty[\times\mathbb{R}^d$ is called a* **domain of determinacy** *of the relatively open subset $\Omega_0\subset\{t=0\}$ when every $H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ solution of $Lu=0$ whose Cauchy data vanish in Ω_0 must vanish in Ω. A closed subset $S\subset[0,\infty[\times\mathbb{R}^d$ is called* **a domain of influence** *of the closed set $S_0\subset\{t=0\}$ if every $H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ solution of $Lu=0$ whose Cauchy data is supported in S_0 is supported in S. An* **influence curve** *is a lipschitzian curve $x(t):[a,b]\to\mathbb{R}^d$ so that the tangent vector to $\big(s,x(s)\big)$ belongs to $\Gamma^+\big(s,x(s)\big)$ for Lebesgue almost all s.*

The definitions imply that a set Ω is a domain of determinacy of Ω_0 if and only if $S:=\big([0,\infty[\times\mathbb{R}^d\big)\setminus\Omega$ is a domain of influence of $S_0:=\mathbb{R}^d\setminus\Omega_0$. The problems of finding large domains of determinacy and small domains of influence are therefore equivalent and amount to accurately describing the speed of propagation for solutions of $Lu=0$.

The intersection of a family of domains of influence of a fixed set S_0 is a domain of influence. Thus there is a smallest such domain called the **the exact domain of influence** and sometimes just **the domain of influence**. For example, the exact domain of influence of the origin for the operator $\Box+m^2$ is the solid cone $|x|^2\le c^2t^2$ when $m\neq 0$ or if $m=0$ and $d\neq 3,5,7,\ldots$. For $m=0$ and odd $d\ge 3$, the exact domain of influence is just the boundary of the cone, $|x|^2=c^2t^2$. The case of $d=3,5,\ldots$ and $m\approx 0$ shows that the exact domain of influence depends sensitively on the operator and not only on the principal part even in the constant coefficient case. On the other hand, in the constant coefficient case the convex hull of the domain of influence of the origin is always equal to Γ^+. We prove that the bound of the domain of influence given by Γ^+ extends naturally to a domain of influence in the variable coefficient case.

The union of a family of domains of determination of a fixed set is also a domain of determination. The largest domain of determination is called the **exact domain of determination**.

The most natural description of domains of influence and determinacy for hyperbolic problems use influence curves ([Co, §VI.7], [Le, §VI.4], [La1, Thm 2.2]). The natural theorem is that if (t,x) is not connected by an influence curve to the set S_0 in $\{t=0\}$, then the values of solutions of $Lu=0$ at (t,x) are not influenced by the Cauchy data in S_0. This geometric description of the domain of influence does not immediately suggest a method of proof.

There is a second approach to the problem, the method of spacelike deformations, which has the opposite character of leading directly to a proof.

Hypothesis 0.2 *The operator L has the property of local uniqueness in the Cauchy problem at space like hypersurfaces, that is, for every embedded space like hypersurface $\Sigma\subset]-1,\infty[\times\mathbb{R}^d$ and point $p\in\Sigma$, if $u\in H^{m-1}_{\rm loc}(]-1,\infty[\times\mathbb{R}^d)$ satisfies $Lu=0$ on a neighborhood of p in $\mathbb{R}^{1+d}$ and the Cauchy data of u vanish on a neighborhood of p in Σ, then u vanishes on a neighborhood of p in $\mathbb{R}^{1+d}$.*

Examples. 1. Constant coefficient systems and systems with analytic coefficients using Hölmgren's Theorem. **2.** Symmetric and symmetrizeable (see §3) hyperbolic systems of first order. **3.** Strictly hyperbolic systems. **4.** See Remark 3 before Proposition 3.1. **5.** It is not known whether microlocally symmetrizable systems (see §3) have local uniqueness at space like hypersurfaces.

The second approach to domains of determination is that domains swept out by spacelike surfaces with their feet in Ω_0 describe domains of determination. The two simple examples of $\partial_t^2 - \partial_x^2$ and $\partial_t + \partial_x$ both in dimension $d = 1$ with an initial set $\Omega_0 =]-1, 1[$ give the essential idea of the method. The sharp domain of determinacies are the triangle $\{|x| < 1 - t\}$ and the strip $\{-1 < x - t < 1\}$ respectively. These domains are swept out by space like deformations sketched in Figure 0.1

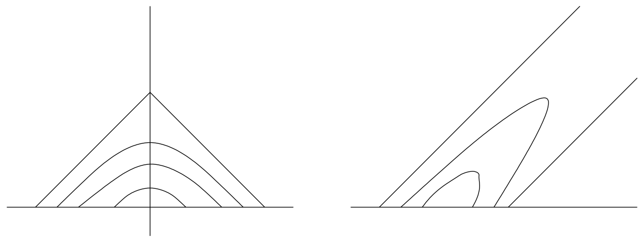


Figure 0.1

In the next result the deformation is by the level sets $\{F = c\}$ with c increasing from 0 to 1. The proof is recalled in the Appendix.

John's Global Hölmgren Theorem [Jo]. *Suppose that L satisfies Hypothesis 0.1-0.2, that $\mathcal{O}$ is a relatively open subset of $\{t \geq 0\}$ and $F \in C^1(\mathcal{O})$ has the following properties.*

i. $F > 0$ *on* $\mathcal{O} \cap \{t > 0\}$.

ii. $F^{-1}([0, 1]) \subset\subset \mathcal{O}$.

iii. *For all* $(t, x) \in F^{-1}\big([0, 1]\big)$, $dF(t, x) \in \mathcal{T}(t, x)$.

Then, if $u \in H^{m-1}_{\rm loc}(\{t \geq 0\})$ *satisfies* $Lu = 0$ *on* $\mathcal{O}$, *and, the Cauchy data of* u *vanish on a neighborhood of* $F^{-1}([0, 1]) \cap \{t = 0\}$, *then* $u = 0$ *on* $F^{-1}([0, 1])$.

Neither the influence curve nor the spatial deformation approach provides a method to compute accurate approximations to the domains. The computation of first arrival times in problems of geology and elsewhere amounts to the same fundamental question and in those communities, computational strategies have been proposed based on using (maximal) solutions of Hamilton-Jacobi equations (see e.g [SF], [RMO], [FJ], and references therein). Computational expertise from shock capturing methods can be used. To our knowledge the relation of this Hamilton-Jacobi approach to the other two has not been investigated except in simple cases. Our main result is that all three descriptions yield the same sets. One can profit from the numerical advantages of Hamilton-Jacobi, the geometry of influence curves, or the analytic advantages of space like deformations, secure in the knowledge that they agree. We use the solution of the Hamilton-Jacobi equation to construct space like deformations. The Hamilton-Jacobi approach is the bridge between the influence curves and the space like deformations.

Several classical results motivate and are special cases of the general result.

Haar ([Ha]) proved that in the one dimensional diagonal case (0.6), the domain of influence of an interval $[a, b]$ is contained in the set $\big\{(t, x) \in [0, \infty[\times\mathbb{R} \ : \ x_1(t) \leq x \leq x_2(t)\big\}$ where

$$\frac{dx_1(t)}{dt} = \min_j\{c_j(t, x)\}, \quad x_1(0) = a\,, \qquad \frac{dx_2(t)}{dt} = \max_j\{c_j(t, x)\}, \quad x_2(0) = b\,.$$

This is exactly the set swept out by all influence curves starting in $[a, b]$. The proof uses the method of characteristics to derive Haar's inequality for solutions of $Lu = 0$,

$$\max_{x_1(t)\le x\le x_2(t)} |u(t,x)| \;\le\; e^{C(T)t} \max_{x_1(0)\le x\le x_2(0)} |u(0,x)|\,, \qquad 0\le t\le T\,.$$

If the principal part of L is equal to D'Alembert's wave operator $\square$, the natural domain of determinacy of an open set Ω_0 is

$$\Omega \;=\; \Big\{(t,x)\;:\; t\le \frac{\mathrm{dist}(x,\mathbb{R}^d\setminus\Omega_0)}{c} := \zeta(x)\Big\}\,. \tag{0.8}$$

The function $\zeta(x)$ is the largest uniformly lipschitz continuous solution of the Hamilton-Jacobi boundary value problem

$$|\nabla\zeta| = 1\,, \qquad \zeta = 0 \;\text{ on }\; \mathbb{R}^d\setminus\Omega_0\,. \tag{0.9}$$

In the constant coefficient hyperbolic case, the convex hull of the support of the forward fundamental solution E defined by

$$L(\partial)\,E = \delta\,, \qquad \mathrm{supp}\,E\subset\{t\ge 0\}\,.$$

is equal to Γ^+ (see [Gå, Ho3]). The fact that it is contained in Γ^+ implies that in the constant coefficient hyperbolic case the set swept out by all influence curves starting in S_0 is a domain of influence. The set Ω defined by $(\underline{t},\underline{x})\in\Omega$ if and only if no influence curve $x:[0,t]\to\mathbb{R}^d$ satisfies both

$$x(0)\in S_0\,, \qquad \text{and} \qquad x(\underline{t}) = \underline{x}\,,$$

is a domain of determinacy of Ω_0 since Ω is the complement of S.

To prove a variable coefficient analogue, we impose a lipschitz regularity assumption on the propagation cones. Hypothesis 0.1 implies that $\tau_{\max}$ is uniforly bounded on $[-1,T]\times\mathbb{R}^d\times\{|\xi|=1\}$. This implies that the propagation sets $\Gamma_1^+(t,x) := \Gamma^+\cap\{T=1\}$ are bounded independent of (t,x) with $0\le t\le \underline{t}$. We need lipschitz regularity.

Hypothesis 0.3. *For each $T>0$, the function $\tau_{\max}(t,x,\xi)$ is uniformly lipschitzian on $[-1,T]\times\mathbb{R}^d\times\{|\xi|=1\}$.*

Examples. 1. If L is symmetric or symmetrizeable Hypothesis 0.1 implies Hypothesis 0.3. **2.** The theorem of Bronshtein [B,W] guarantees that for systems of order m with coefficients which are C^{mN}, $\tau_{\max}$ is uniformly lipschitzian on compact subsets of $[-1,\infty[\times\mathbb{R}^d\times\{|\xi|=1\}$. We could have worked with this weaker condition. The stronger assumption is made for simplicity.

The natural candidate $\Omega\subset[0,\infty[\times\mathbb{R}^d$ for a large domain of determinacy of Ω_0 is

$$\Omega \;:=\; \Big\{(\underline{t},\underline{x})\;:\;\text{no influence curve with }\; x(0)\in S_0 \;\text{ can satisfy }\; x(\underline{t})=\underline{x}\Big\}\,. \tag{0.10}$$

Theorem 0.1. *If Hypotheses 0.1 and 0.3 are satisfied, $\psi(x)\in W^{1,\infty}(\mathbb{R}^d)$ vanishes on S_0, and is strictly positive on Ω_0, then the set Ω from (0.10) is exactly the set $\{(t,x)\in[0,\infty[\times\mathbb{R}^d : \Psi(t,x)>0\}$ where $\Psi\in\cap_T W^{1,\infty}([0,T]\times\mathbb{R}^d)$ is the largest uniformly lipschitzian solution of the Hamilton-Jacobi initial value problem*

$$\Psi_t + \tau_{\max}(t,x,-\nabla_x\Psi(t,x)) = 0 \quad \text{a.e. } (t,x)\,, \qquad \Psi(0,x)=\psi(x)\,.$$

Theorem 0.2. *If Hypotheses* 0.1-0.3 *are satisfied, then the natural* Ω *defined in* (0.10) *is a domain of determinacy of* Ω_0.

Theorem 0.3. *If Hypotheses* 0.1-0.3 *are satisfied and* $(\underline{t}, \underline{x})$ *belongs to the natural* Ω *defined in* (0.10), *then there is a deformation by spacelike hypersurfaces as in John's Theorem so that* $(\underline{t}, \underline{x}) \in F^{-1}([0, 1[)$.

Theorem 0.3 together with John's Theorem proves Theorem 0.2.

A proof of Theorem 0.2 in the strictly hyperbolic case is given in [Le]. It uses a result of Marchaud [M] asserting that if Z is a closed set in $\{t \geq 0\}$ with the property that for each point in Z there is a backward semitangent belonging to $-\Gamma^+(t, x)$ then through each point $(\underline{t}, \underline{x}) \in Z$ with $\underline{t} > 0$ there is a backward influence curve belonging to Z and reaching $t = 0$. The outline of proof in [La1] for the symmetric hyperbolic case is not quite complete. It can be completed by appealing to the above result of [M]. Appeal to the long and technical article [M] can be circumvented by proving the result cited above from scratch.

As far as we know, Theorem 0.3 is nowhere suggested in the literature. Theorem 0.2 is proved in only the symmetric and strictly hyperbolic cases relying on Marchaud. And Theorem 0.1 or analogues are known in only very special cases, for example operators with prinicipal part $\partial_{tt} - \sum a_{ij}(t, x)\partial^2_{ij}$ where the natural domains are described in terms of Riemannian distances (see e.g. [Ho1]).

The convexity of $\tau_{\max}$ as a function of ξ is needed at many points in the analysis. If noncharacteritic deformations generated by Hamilton-Jacobi equations were applied to find a globalization of Hölmgren's Theorem in the nonhyperbolic but analytic case, the HJE would involve the convex hull of the noncharacteristic cone $\tau > \tau_{\max}$. It would be interesting to compare such results to those following Theorem 8.6.5 in [Ho2].

We give two proofs of Theorem 0.2, both using the function Ψ from Theorem 0.1. One passes through Theorem 0.3. A second argument is restricted to the symmetric hyperbolic case where a direct proof uses the energy method with weight functions $e^{2\lambda\Psi}$ with $\lambda \to +\infty$. It is our hope that Hamilton-Jacobi equations may serve in other contexts to construct good weights.

The paper is organized as follows. The first section recalls background material on hyperbolic polynomials. The second contains a proof of Theorem 0.1 in the constant coefficient case. The variable coefficient case is presented in §3. The short §4 proves Theorem 0.2 in the symmetrizable hyperbolic case. In §5 the connections between spacelike deformations and Hamilton-Jacobi equations are developed. In particular Theorem 0.3 is proved. The final §6, is devoted to a Hamilton-Jacobi approach like that in (0.9) which is available in the frequently encountered case of systems that have the the property that Γ^+ contains a neighborhood of the vector $(1, 0, \ldots, 0)$. This is the usual situation treated in the numerical analysis literature.

This paper is written with the idea that readers are likely to be knowledgeable about hyperbolic partial differential equations and less so about Hamilton-Jacobi theory. More detail about the latter is presented than would be appropriate for experts.

§1. The constant coefficient case.

Suppose that $L(\partial)$ is a homogeneous constant coefficient system on $\mathbb{R}^{1+d}$ which is hyperbolic with timelike codirection dt. We recall some basic facts and definitions concerning domains of influence and determinacy in this case (see [Gå,Ho3]).

Definition. *The open cone $\mathcal{T} = \mathcal{T}(L, dt)$ is the connected component of dt in the noncharacteristic codirections of L.*

The open cone $\mathcal{T}$ is convex and consists of time like codirections for L. The opposite cone

$$\mathcal{T}(L, -dt) \;=\; -\mathcal{T}(L, dt)$$

is the connected component of $-dt$ in the noncharacteristic codirections.
The example $L = \partial_t^2 - \partial_x^2$ in the case $d = 1$ shows that $\pm\mathcal{T}(L, dt)$ need not exhaust all time like codirections. For this operator, the complement of the characteristic variety has four components and they all consist of time like codirections.
The characteristic polynomial $P(\tau, \xi)$ is defined by

$$P(\tau, \xi) \;:=\; \det\, L(\tau, \xi)\,. \tag{1.1}$$

Define for $\xi \in \mathbb{R}^d \setminus 0$,

$$\tau_{\max}(\xi) \;:=\; \max\, \big\{\tau \in \mathbb{R} \;:\; P(\tau, \xi) = 0\big\}\,. \tag{1.2}$$

Then $\tau_{\max}(\xi)$ is positively homogeneous of degree one, continuous, and convex. The set $\mathcal{T}$ has equation

$$\mathcal{T} \;=\; \big\{(\tau, \xi) \;:\; \tau > \tau_{\max}(\xi)\big\}\,. \tag{1.3}$$

The sharp finite speed result is described in terms of the dual propagation cone.

Definition. *The closed forward propagation cone is defined by*

$$\Gamma^+ \;:=\; \Big\{(T, X) \in \mathbb{R}^{1+d} \;:\; \forall\, (\tau, \xi) \in \mathcal{T},\;\; T\tau + X.\xi \;\geq\; 0\Big\}\,. \tag{1.4}$$

The cone Γ^+ is the set of all points which lie in the future of the origin $(0, 0) \in \mathbb{R}^{1+d}$ with respect to each time like linear form $(\tau, \xi) \in \mathcal{T}$.
By duality one has

$$\mathcal{T} \;=\; \Big\{(\tau, \xi) \;:\; \forall\, (T, X) \in \Gamma^+,\;\; \tau T + \xi.X \;>\; 0\Big\}\,. \tag{1.5}$$

Proposition 1.1. *The propagation cone Γ^+ has equation*

$$\Gamma^+ \;=\; \Big\{(T, X) \;:\; T \geq 0 \;\text{ and }\; \forall \xi,\; T\,\tau_{\max}(\xi) + X.\xi \geq 0\Big\}\,. \tag{1.6}$$

Proof. Taking $(\tau, \xi) = (1, 0)$ in (1.4) shows that $T \geq 0$ in Γ^+.
Writing

$$(T, X).(\tau, \xi) \;=\; T(\tau - \tau_{\max}(\xi)) \;+\; (T\tau_{\max}(\xi) + X.\xi)$$

it follows that

$$\Big\{T \geq 0 \;\text{ and }\; \forall \xi,\; T\tau_{\max}(\xi) + X.\xi \geq 0\Big\} \;\subset\; \Gamma^+\,.$$

Finally, if $T \geq 0$ and there is a ξ so that $X.\xi + T\tau_{\max}(\xi) < 0$ taking $(\tau, \xi) = (\tau_{\max}(\xi) + \epsilon, \xi)$ with ϵ small and positive, shows that $(T, X) \notin \Gamma^+$. ∎

Since $\mathcal{T}$ is convex and contains an open cone about $\mathbb{R}(1, 0, \ldots, 0)$ it follows that

$$\Gamma_1^+ \ := \ \Gamma^+ \cap \{T = 1\}$$

is a compact convex set.
Therefore, in (1.5) it suffices to consider (T, X) with $T = 1$ and $X \in \Gamma_1^+$, so

$$\mathcal{T} \ = \ \Big\{(\tau, \xi) \in \mathbb{R}^{1+d} \ : \ \forall X \in \Gamma_1^+, \ \tau + X.\xi > 0\Big\}.$$

Thus $\mathcal{T}$ has equation $\tau + \min\{X.\xi \ : \ X \in \Gamma_1^+\} > 0$. Comparing with $\tau - \tau_{\max}(\xi) > 0$ yields the duality relations

$$-\tau_{\max}(\xi) \ = \ \min_{X \in \Gamma_1^+} X.\xi \,, \qquad \tau_{\max}(\xi) = \max_{X \in -\Gamma_1^+} -X.\xi \ = \ \max_{v \in \Gamma_1^+} v.\xi \,, \tag{1.7}$$

Examples. 1. *If* $L(D) = \partial_t + \mathbf{v}.\partial_x$ *then* $\Gamma^+ = \{(t, x) \ : \ t \geq 0 \text{ and } x = \mathbf{v}t\}$ *and* $\tau_{\max}(\xi) = -\mathbf{v}.\xi$.
2. *If* $L(D) = \Box = \partial_t^2 - c^2\,\Delta$ *the speed* $c > 0$ *D'Alembertian, then* $\Gamma^+ = \big\{(t, x) \ : \ t \geq 0 \text{ and } c^2t^2 \geq |x|^2\big\}$ *and* $\tau_{\max}(\xi) = c|\xi|$.
3. *If* $L(D) = \partial^2/\partial t^2 - c^2\,\partial^2/\partial x_1^2$, *then* $\Gamma^+ = \big\{(t, x) \ : \ t \geq 0, \ x_2 = x_3 = \ldots = x_d = 0, \text{ and } c^2t^2 \geq x_1^2\big\}$, *and* $\tau_{\max}(\xi) = c|\xi_1|$.
4. *If* $L = L_1L_2$ *then the cone* $\mathcal{T}^+$ *of* L *is the intersection of the cones for* L_1 *and* L_2. *If* $d \geq k \geq 2$ *and*

$$L = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_2^2}\right)\cdots\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_k^2}\right),$$

then Γ_1^+ *is the dimension* k *rectangle*

$$\big\{x \ : \ -1 \leq x_j \leq 1 \text{ for } 1 \leq j \leq k \ \text{ and } \ 0 = x_{k+1} = \cdots = x_d\big\}\,,$$

and $\tau_{\max}(\xi) = \max_{1 \leq j \leq k} |\xi_j|$.

The link between Γ^+ and domains of determination is understood using plane waves and Hölmgren's Theorem. The domains of determination that we construct depend only on the leading order terms of L. Homogeneous m^{th} order hyperbolic equations $Lu = 0$ have plane wave solutions

$$u = f(\tau t + \xi.x).$$

In fact, $L(\partial_t, \partial_x)u = L(\tau, \xi) f^{(m)}(\tau t + \xi.x)$ so $Lu = 0$ whenever (τ, ξ) is characteristic and $f \in C^m(\mathbb{R};\, \mathbb{C}^N)$ takes values in the nullspace of $L(\tau, \xi)$.
Using functions f as in Figure 1.1 whose support is contained in $]-\infty, 0]$ and nearly touches $\{0\}$ shows that the domain of determinacy in $\{t > 0\}$ of the set $\{\xi.x > 0\}$ cannot be larger than the set $\big\{(t, x) \ : \ \tau t + \xi.x > 0\big\}$.

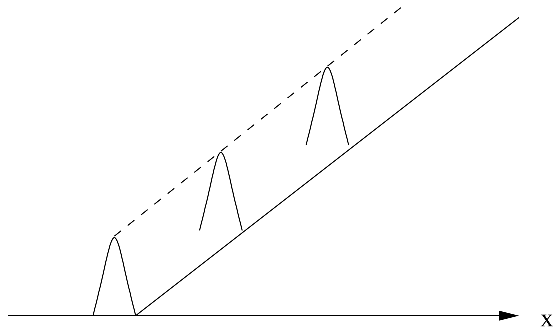
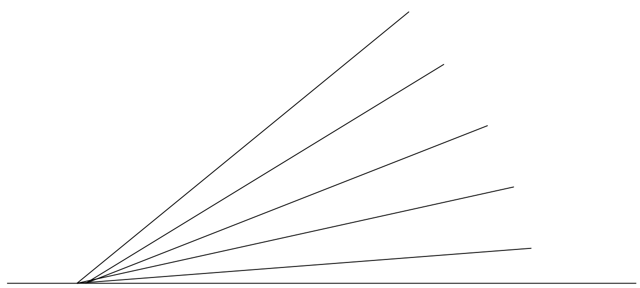


Figure 1.1. Progressing waves limit domain of determinacy

Increasing τ decreases the size of this set so the best bound is

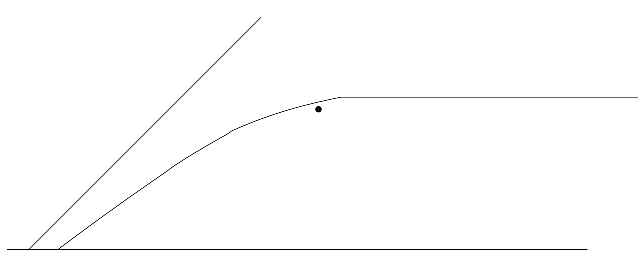
$$\Omega(\xi) \;:=\; \Big\{(t,x)\;:\;t\ge 0,\;\tau_{\max}t+\xi.x>0\Big\}.$$

If $\tau > \tau_{\max}$ then the hyperplane $\tau t + \xi.x = 0$ is noncharacteristic. As $\tau \to \infty$, these hyperplanes approach $\{t = 0\}$, the hyperplanes spanning a wedge.

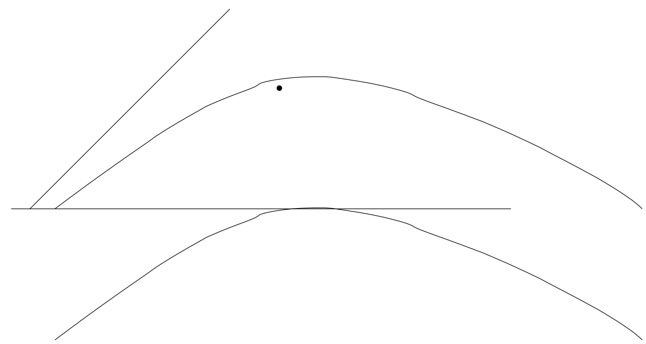


Noncharacteristic hyperplanes sweep out a wedge

This suggests the correct conclusion that one can use John's Theorem to show that $\Omega(\xi)$ is a domain of determination of $\{\xi.x > 0\}$. Changing coordinates we may suppose that $\xi.x = x_1$. For a point below the hyperplane $\{\tau_{\max}t + \xi.x = 0\}$, begin by constructing a spacelike hypersurface $t = g(x_1)$ as in the next figure so that the point lies below the hypersurface. The hypersurface is everywhere less steep than the limiting hyperplane.



Next choose ϵ small and positive so that the surface $t = e^{-\epsilon|x|^2}\,g(x_1)$ is also spacelike and still the desired point lies below. Let $T := \max e^{-\epsilon|x|^2}\,g(x_1)$ so that the translate by T of the hypersurface just touches $t = 0$



Define

$$F := 1 \ - \ \big(t - e^{-\epsilon|x|^2}\, g(x_1)\big)/T\,.$$

Then $F = 1$ on the upper surface, each level surface of F is a translate of the upper one, and $F = 0$ on the lower. John's Theorem implies that u vanishes on $F^{-1}([0,1])$ and therefore at the desired point.

Taking the union over $\xi \neq 0$ and using (1.6) shows that the complement of Γ^+ in $\{t \geq 0\}$ is a domain of determination for the complement of the origin. In particular, the support of the fundamental solution of $L(\partial_t, \partial_x)$ is contained in Γ^+. That this estimate gives exactly the convex hull of the support can be proved using the Paley-Weiner Theorem.

Theorem 1.3. *If E denotes the unique solution of $LE = \delta$ with $\operatorname{supp} E \subset \{t \geq 0\}$ then the exact domain of influence of S_0 is equal to $\cup_{y \in S_0} \big(y + \operatorname{supp} E\big)$. The most useful bound on this set comes from*

$$\text{convex hull}\Big(\operatorname{supp} E\Big) \ = \ \Gamma^+\,,$$

so

$$S \ := \ \cup_{y\in S_0}\ \big(y + \Gamma^+\big)\,, \tag{1.8}$$

is a domain of influence of S_0.

Conjecture. *Suppose that $L_m(\partial_{t,x})$ is a homogeneous hyperbolic $N \times N$ system. Then for arbitrary constant coefficient lower order terms $L_m + M(\partial)$, $\deg(M) \leq m-1$, the operator $L+M$ is weakly hyperbolic in the sense that the Cauchy problem is well set for data in the Gevrey class G^r with $r < mN/(mN-1)$. We conjecture that for most lower order terms one has*

$$\operatorname{supp} E \ = \ \Gamma^+\,.$$

That is, any lacuna which may exist fill in for generic lower order perturbations. Note that in this general setting E will be an ultradistribution (dual of Gevrey). However the conjecture makes sense as well if L_m is strictly hyperbolic or if $L = L_1$ is symmmetric hyperbolic in which case $L_m + M$ is hyperbolic in the standard sense of [Gå] and E is a distribution.

The associated natural description of the domain of determinacy uses the backward cones

$$\Gamma^- \ := \ -\Gamma^+\,.$$

Theorem 1.4. *For constant coefficient L and open $\Omega_0 \subset \mathbb{R}^d$, the exact domain of determinacy is given by*

$$\Omega_{\text{exact}} \ := \ \Big\{(T,X)\,:\, T \geq 0 \ \text{ and } \ \Big(\big((T,X) - \operatorname{supp} E\big) \cap \{t=0\}\Big) \subset \Omega_0\Big\}\,.$$

In particular,

$$\Omega \ := \ \Big\{(T,X)\ :\ T\ge 0 \ \ \text{and}\ \ \Big(\big((T,X)-\Gamma^+\big)\cap\{t=0\}\Big)\subset\Omega_0\Big\}$$

is a domain of determinacy since $\Omega\subset\Omega_{\rm exact}$.

The bound for the domain of influence, S, from Theorem 1.4 is the complement of the domain of determination Ω. These two sets have descriptions in terms of influence curves.

Definition. *A forward influence curve for* L *is a uniformly lipschitzian curve* $x(t)\,:\,I\to\mathbb{R}^{1+d}$ *defined on a compact interval* I *and satisfying*

$$\frac{dx}{dt}\ \in\ \Gamma_1^+\ \ \text{for Lebesgue almost all } t\in I\,. \tag{1.9}$$

With $\gamma\ :=\ (t,x(t))$, *condition* (1.9) *is equivalent to* $d\gamma/dt\in\Gamma^+$. *A backward influence curve satisfies* $dx/dt\in-\Gamma_1^+$.

For a continuous curve $(t(s),x(s))$ a *forward semi-tangent* at $(t(\underline{s}),x(\underline{s}))$ is any limit point as s decreases to $\underline{s}$ of the quotients. $(t(s)-t(\underline{s}),x(s)-x(\underline{s}))/(s-\underline{s})$. In [Le, Def. 90.1], Leray defines influence curve as a continuous curve whose positive semi-tangents belong to Γ^+. To prove that the two definitions are equivalent reason as follows. From our definition one has for all $t>\underline{t}$

$$\frac{x(t)-x(\underline{t})}{t-\underline{t}}\ =\ \frac{1}{t-\underline{t}}\int_{\underline{t}}^{t}x'(t)\ dt\ \in\ \Gamma_1^+$$

since Γ_1^+ is convex. Therefore the semi-tangents to $(t,x(t))$ belong to Γ_1^+.

Conversely, if one has a path satisfying Leray's condition, then $t(s)$ must be a strictly increasing continuous function. Reparameterize with t as new parameter, and one finds that the difference quotients of $x(t)$ lie in Γ_1^+. Since this is compact, it follows that $x(t)$ is lipschitzian with derivative almost everywhere in Γ_1^+.

Theorem 1.5. *The domain of influence* S *in Theorem 1.3 is the union of all forward influence curves beginning in* S_0. *The domain of determinacy* Ω *in Theorem 1.4 is the set of points* $(\underline{t},\underline{x})$ *with the property that no forward influence curve beginning in* S_0 *at* $t=0$ *passes through* $(\underline{t},\underline{x})$.

The input is a disjoint decomposition $\{t=0\}=S_0\cup\Omega_0$ with S_0 closed and Ω_0 relatively open. The output is a disjoint decomposition of $\{t\ge 0\}$ into a relatively closed domain of influence of S and a relatively open domain of determinacy of Ω. The sets S and Ω define domains of influence and determination for all constant coefficient hyperbolic operators with principal symbol L. They have a second desirable property which the exact domain of determination need not have. The algorithm that generates S from S_0 is self reproducing in the sense that if you stop the solution at time $\underline{t}>0$ you know that the Cauchy data at that time are supported in $S\cap\{t=\underline{t}\}$. If you then consider the domain of influence in $t\ge\underline{t}$ generated from that slice by all forward influence curves starting in $S\cap\{t=\underline{t}\}$ that reproduces S in $t\ge\underline{t}$.

Example. *For the* 2×2 *system* $L=\partial_t+\mathrm{diag}(\partial_x,-\partial_x)$ *in dimension* $d=1$ *and with* $S_0=\{0\}$ *the exact domain of influence is the pair of rays* $0\le t=\pm x$. *The exact domain at time* $\underline{t}>0$ *consists of two points* $(\underline{t},\pm\underline{t})$. *The exact domain of influence in* $t\ge\underline{t}$ *of this pair of points is equal to the* *four forward characteristics two each launched from each starting point $(\underline{t}, \pm\underline{t})$. The exact domain is* **not** *self reproducing.*

An algorithm that generates domains of influence must launch the cone $\operatorname{supp} E$ from each initial point. For an algorithm to be self reproducing it must generate a set $\tilde{S}$ with the property that for any $s \in \tilde{S}$, $s + \operatorname{supp} E \subset \tilde{S}$. The next proposition shows that the set S from Theorem 1.5 is the smallest set which satisfies this self reproducing property.

Proposition 1.6. *If $S_0 \subset \{t = 0\}$ and S are as in Theorem 1.5 then $S_0 \subset S$ and for any $s \in S$, $s + \Gamma^+ \subset S$. It is the smallest self reproducing set in the sense that if $\tilde{S} \subset \{t \geq 0\}$ satisfies $S_0 \subset \tilde{S}$ and $s + \operatorname{supp} E \subset \tilde{S}$ for all $s \in \tilde{S}$ then $S \subset \tilde{S}$.*

Proof. A point s belongs to S if and only if there are points $(0, y) \in S_0$ and $v_1 \in \Gamma_1^+$ and $t \geq 0$ so that $s = (t, y + tv)$. The general point of $s + \Gamma^+$ has the form $(t + r, y + tv + rw)$ with $r \geq 0$ and $w \in \Gamma_1^+$. This is exactly

$$\Big(t + r\,,\; y + (t+r)\Big[\frac{t}{t+r}v + \frac{r}{t+r}w\Big]\Big) \;:=\; (\underline{t}, y + \underline{t}z)\,. \tag{1.10}$$

Since Γ_1^+ is convex, $z \in \Gamma_1^+$ so the point in (1.10) belongs to S. This proves the first assertion of the Proposition.

Define a compact set by

$$K \;:=\; \Big\{x \;:\; (1, x) \in \operatorname{supp} E\Big\}\,.$$

The distribution E is positive homogeneous of degree $-d$. Therefore, in $\{t > 0\}$, $\operatorname{supp} E = \{x/t \in K\}$. We must show that for any $(0, y) \in S_0$, $\underline{t} > 0$, and $v \in \Gamma_1^+$ one has $(\underline{t}, y + \underline{t}v) \in \tilde{S}$.

For such a v choose a finite set of points $x_j \in K$ and $0 < t_j$ with $1 \leq j \leq n$,

$$t_1 + t_2 + \ldots + t_n = 1\,, \qquad \text{and} \qquad t_1 x_1 + t_2 x_2 + \ldots + t_n x_n = v\,.$$

The self reproducing property with base point $(0, y)$ shows that $(t, y + tx_1) \in \tilde{S}$ for all $t \geq 0$. In particular $(t_1\underline{t}, y + t_1\underline{t}x_1) \in \tilde{S}$. The self reproducing property with this base point shows that $(t_1\underline{t} + t, y + t_1\underline{t}x_1 + tx_2) \in \tilde{S}$ for all $t \geq 0$. In particular $\big((t_1 + t_2)\underline{t}, y + (t_1x_1 + t_2x_2)\underline{t}\big) \in \tilde{S}$. Continuing one has

$$\Big(\underline{t}, y + \underline{t}v\Big) = \Big((t_1 + t_2 + \ldots + t_n)\underline{t}, y + (t_1x_1 + t_2x_2 + \ldots + t_nx_n)\underline{t}\Big) \in \tilde{S}$$

which is the desired result. ∎

We next look at the regularity of the domains given in Theorems 1.3 to 1.5.

Example. For $L = \partial_t + \mathbf{v}.\partial_x$,

$$\Omega_{\text{exact}} \;=\; \Big\{(t, x) \;:\; t \geq 0 \;\text{ and }\; x - t\mathbf{v} \in \Omega_0\Big\}\,.$$

in this case the open set Ω is generated by translating Ω_0 so is exactly as smooth or rough as is the initial set Ω_0. If Ω_0 is bounded, then the boundary of Ω is not a graph in any linear coordinate system.

In contrast to this last example as soon as Γ^+ has nonempty interior, there is a regularizing effect, the domain of determinacy Ω from Theorem 1.4 is a lipschitz domain even when Ω_0 is not. That assertion follows from the next proposition after a Gallilean transformation.

Proposition 1.7. *Suppose that $\{|x| \le ct\} \subset \Gamma^+,\ c > 0$. Then for any open subset $\Omega_0 \ne \mathbb{R}^d$ and Ω is as in Theorem 1.5, there is a lipschitzian $\zeta : \overline{\Omega_0} \to \mathbb{R}$ so that $|\nabla_x \zeta| \le 1/c$ and*

$$\Omega = \Big\{(t,x)\ :\ x \in \Omega_0\ \text{ and }\ 0 \le t < \zeta(x)\Big\}.$$

Remark. The hypothesis of the proposition is equivalent to $\mathcal{T}(L, dt) \subset \{\tau > c|\xi|\}$ with strictly positive c.

Proof. Since the backward cone Γ^- contains the set $\{t < -c|x|\}$ it follows that if $(\underline{t}, \underline{x}) \in \Omega$ then $(t, \underline{x}) \in \Omega$ for $0 \le t < \underline{t}$. Similarly if $(\underline{t}, \underline{x}) \notin \Omega$ then $(t, \underline{x}) \notin \Omega$ for $t \ge \underline{t}$.
For each $x_0 \in \Omega_0$, $(t, x_0) \in \Omega$ for small nonnegative t. On the other hand, for $ct > \operatorname{dist}\{x_0\,,\ \mathbb{R}^d \setminus \Omega_0\}$, $(t, x_0) \notin \Omega$.
Therefore, Ω is the region under the graph of the function ζ defined by

$$0 < \zeta(x_0)\ :=\ \inf\Big\{t > 0\ :\ (t, x_0) \in S\Big\}\ < \infty\,.$$

The definitions of S implies that if $(T, X) \in S$ then the forward propagation cone $(T, X) + \Gamma^+$ launched from (T, X) also lies in S. Similarly if $(T, X) \in \Omega$ then the backward cone $(T, X) + \Gamma^-$ launched from (T, X) lies in Ω so long as it remains in $\{t \ge 0\}$. Finally if (T, X) with $T > 0$ belongs to the boundary of Ω then the interior of the backward cone $(T, X) + \text{Interior}(\Gamma^-)$ lies in Ω so long as it remains in $\{t \ge 0\}$.
It follows that

$$(\zeta(x_0), x_0) + \Gamma^+ \subset S\,,$$

and that, in $\{t \ge 0\}$:

$$(\zeta(x_0), x_0) - \text{Interior}(\Gamma^+) \subset \Omega\,.$$

Since $\Gamma^+ \supset \{|x| \le ct\}$ these inclusions imply the bound

$$c\,|\zeta(x) - \zeta(x_0)|\ \le\ |x - x_0|\,,$$

and the proof is complete. ∎

§2. Hamilton-Jacobi in the constant coefficient case.

The description in terms of influence curves in Theorem 1.5 yields a second description in terms of solutions of Hamilton-Jacobi partial differential equations. Our treatment of the variational problems is strongly influenced by §1.3 of [Li].

Let $\psi(x)$ be a uniformly lipschitzian function which is strictly positive on the nonempty open set Ω_0 and vanishes on the nonempty complement $S_0 := \mathbb{R}^d \setminus \Omega_0$. For example, $\psi(x) := \operatorname{dist}\{x\,,\ S_0\}$. An example which tends to zero as $|x| \to \infty$ is $\psi(x) := e^{-|x|}\operatorname{dist}\{x\,,\ S_0\}$.

Definitions. *Denote by $\mathcal{X}(T, X)$ the set of forward influence curves $x(t) : [0, T] \to \mathbb{R}^d$ with*

$$x(T) = X\,. \tag{2.1}$$

Define a function $\Psi(T,X)$ *in* $T \ge 0$ *by*

$$\Psi(T,X) \;=\; \inf\, \Big\{ \psi(x(0)) \;:\; x(\cdot) \in \mathcal{X}(T,X) \Big\} \,. \tag{2.2}$$

The infinum in (2.2) is an achieved minimum. To prove this choose a minimizing sequence $x^n(\cdot) \in \mathcal{X}(T,X)$ with $\psi(x^n(0)) \to \Psi(T,X)$. Since $dx^n/dt \in \Gamma_1^+$ is bounded and $x^n(T) = X$ is fixed, Ascoli's Theorem implies that passing to a subsequence if necessary we may assume that $x^n(\cdot)$ converges uniformly to $x(\cdot) \in W^{1,\infty}([0,T])$ and dx^n/dt converges weak star in $L^\infty([0,T])$ to dx/dt. Since Γ_1^+ is convex it follows that $x(\cdot)$ is an admissible influence curve. Passing to the limit shows that $\psi(x(0)) = \Psi(T,X)$, so $x(\cdot)$ is a minimizing influence curve.

An immediate consequence of the Definitions and Theorem 1.6 is the following result.

Corollary 2.1 *The set described in (0.10) is exactly the set* $\{\Psi > 0\}$. *The complementary set* $S := \big([0,\infty[\times\mathbb{R}^d\big) \setminus \Omega$ *is equal to* $\{\Psi = 0\}$.

One interest of the function Ψ is that it is a maximal lipschitzian solution of a Hamilton-Jacobi initial value problem.

Theorem 2.2 *On* $[0,\infty[\times\mathbb{R}^d$ *the function* Ψ *is uniformly lipschitzian and satisfies the Hamilton-Jacobi initial value problem*

$$\partial_t \Psi + \tau_{\max}(-\nabla_x \Psi) \;=\; 0 \;\; a.e. \qquad \Psi(0,x) = \psi(x)\,. \tag{2.3}$$

It is the largest solution in the sense that if $\ell(x) \in W^{1,\infty}_{\mathrm{loc}}\big([0,T]\times\mathbb{R}^d\big)$ *and satisfies*

$$\partial_t \ell + \tau_{\max}(-\nabla_x \ell) \;\le\; 0 \;\; \text{a.e.}\,, \qquad \ell(0,x) \le \psi(x)\,, \tag{2.4}$$

then for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\ell(t,x) \le \Psi(t,x)\,. \tag{2.5}$$

Proof. The first step is to prove that Ψ is lipschitzian. Let $\Lambda_1 \;:=\; \|\nabla_x \psi\|_{L^\infty(\mathbb{R}^d)}$ be the lipschitz constant for ψ and $\Lambda_2 \;:=\; \max\,\big\{|v| \;:\; (1,v) \in \Gamma_1^+\big\}$.

For $(\underline{T},\underline{X})$ with $\underline{T} > 0$, choose an influence curve with $x(\underline{T}) = \underline{X}$ and $\Psi(\underline{T},\underline{X}) = \psi(x(0))$.

Then $x(t) + (X - \underline{X})$ is an influence curve which ends at X so

$$\Psi(\underline{T},X) \;\le\; \psi\Big(x(0) + (X - \underline{x})\Big)\,.$$

Thus,

$$\Psi(\underline{T},X) - \Psi(\underline{T},\underline{X}) \;\le\; \psi\Big(x(0) + (X - \underline{X})\Big) \;-\; \psi(x(0)) \;\le\; \Lambda_1|X - \underline{X}|\,.$$

Reversing the roles of $X, \underline{X}$ yields

$$\Psi(\underline{T},\underline{X}) - \Psi(\underline{T},X) \;\le\; \Lambda_1|X - \underline{X}|\,, \qquad \text{so}$$

$$\Big|\Psi(\underline{T},\underline{X}) - \Psi(\underline{T},X)\Big| \;\le\; \Lambda_1|X - \underline{X}|\,.$$

To estimate increments of Ψ corresponding to changes in T and also to derive the Hamilton-Jacobi equation (2.3) we use the following Lemma whose assertions are sometimes called *Bellman equations.*

Lemma 2.3. Dynamic Programing Principles. I. *If* $\underline{T} > 0$ *and* $x(t)$ *is an influence curve with* $\Psi\big(\underline{T}, x(\underline{T})\big) = \psi(x(0))$ *then for all* $0 \le t \le T$, $\Psi(t, x(t)) = \psi(x(0))$.

II. *For any* $t \in [0, T]$,

$$\Psi(\underline{T}, \underline{X}) = \min_{x(\cdot) \in \mathcal{X}(\underline{T}, \underline{X})} \Psi(t, x(t)) \,. \tag{2.6}$$

Proof of Lemma. There is nothing to prove for $t = 0$ or $t = \underline{T}$. Fix $\underline{t} \in]0, T[$.

Since $x(t)$ is an influence curve which at time $\underline{t}$ reaches $x(\underline{t})$ it follows from the definition of Ψ as a minimum, that $\Psi(\underline{t}, x(\underline{t})) \le \psi(x(0)) = \Psi(\underline{T}, \underline{X})$. On the other hand if $y(t)$ is an influence curve passing through $(\underline{t}, x(\underline{t}))$ then

$$z(t) \ := \ \begin{cases} y(t) & 0 \le t \le \underline{t} \\ x(t) & \underline{t} \le t \le T \end{cases}$$

is an influence curve passing through $(\underline{T}, \underline{X})$. Therefore $\psi(y(0)) \ge \Psi(\underline{T}, \underline{X})$. Taking the minimum over all such y shows that $\Psi(\underline{t}, x(\underline{t})) \ge \Psi(\underline{T}, \underline{X})$ proving **I.**

The proof of **II.** is similar. ∎

Returning to the proof of Theorem 2.2, to estimate $\Psi(T_1, X) - \Psi(T_2, X)$ relabel if necessary so that $T_1 > T_2$. Choose an influence curve $x(t)$ passing through (T_1, X) with $\Psi(T_1, X) = \psi(x(0))$. The first Dynamic Programming Principle implies that $\Psi(T_1, X) = \Psi(T_2, x(T_2))$, so

$$\Psi(T_1, X) - \Psi(T_2, X) = \Psi(T_2, x(T_2)) - \Psi(T_2, x(T_1)) \,.$$

Therefore,

$$\Big|\Psi(T_1, X) - \Psi(T_2, X)\Big| \le \Lambda_1 \Big|x(T_2) - x(T_1)\Big| \le \Lambda_1 \Lambda_2 \Big|T_2 - T_1\Big| \,,$$

so Ψ is lipschitzian.

Rademacher's Theorem (see for example [Mo]) implies that it suffices to prove that (2.3) holds at points $(\underline{T}, \underline{X})$ where Ψ is differentiable. Use the second Dynamic Programing Principle with $t = \underline{T} - \epsilon$ with small positive ϵ. If $x(\cdot) \in \mathcal{X}(\underline{T}, \underline{X})$ then,

$$x(\underline{T} - \epsilon) = \underline{X} - \int_{T-\epsilon}^{\underline{T}} x'(s) \; ds \,,$$

$$\Psi\Big(\underline{T} - \epsilon, x(T - \epsilon)\Big) \ = \ \Psi(\underline{T}, \underline{X}) - \epsilon\Big(\Psi_T(\underline{T}, \underline{X}) + \frac{1}{\epsilon} \int_{T-\epsilon}^{\underline{T}} x'(s) \, ds \,.\, \Psi_X(\underline{T}, \underline{X})\Big) + o(\epsilon) \,.$$

The second Dynamic Programming Principal asserts that for ϵ fixed, the minimum of the left hand side over $\mathcal{X}$ is equal to $\Psi(\underline{T}, \underline{X})$. Taking account of the minus sign this shows that

$$\max_{x(\cdot) \in \mathcal{X}(\underline{T}, \underline{X})} \left\{ \Psi_T(\underline{T}, \underline{X}) + \left(\frac{1}{\epsilon} \int_{T-\epsilon}^{\underline{T}} x'(s) \, ds\right) . \, \Psi_X(\underline{T}, \underline{X}) \right\} = \frac{o(\epsilon)}{\epsilon} = o(1) \,. \tag{2.7}$$

As $x(\cdot)$ runs over $\mathcal{X}(\underline{T},\underline{X})$, x' is an arbitrary L^∞ function with values in Γ_1^+ so

$$\frac{1}{\epsilon}\int_{T-\epsilon}^{T} x'(s)\,ds$$

runs over Γ_1^+ thanks to convexity. Thus the left hand side of (7) is independent of ϵ. Sending $\epsilon \to 0$ yields

$$\max_{v\in\Gamma_1^+} \Big\{ \Psi_T(\underline{T},\underline{X}) + v\,.\,\Psi_X(\underline{T},\underline{X}) \Big\} \;=\; 0\,. \tag{2.8}$$

Equation (1.7) shows that (2.8) is equivalent to the desired equation (2.3).

The proof of the comparison (2.6) is in two steps. In the first step we prove that if $w \in C^1\big([0,T]\times\mathbb{R}^d\big)$, $\delta\in\mathbb{R}$ and w satisfies for $0\le t\le T$,

$$w_t + \tau_{\max}(-\nabla_x w) \le 0\,, \qquad w(0,x)\le \psi(x)+\delta$$

then,

$$w(t,x) \le \Psi(t,x) + \delta \quad \text{on} \quad [0,T]\times\mathbb{R}^d\,. \tag{2.9}$$

To prove the inequality (2.9) at a point $(\underline{t},\underline{x}) \in]0,T]\times\mathbb{R}^d$, choose an influence curve $x(t)$ so that $x(\underline{t}) = \underline{x}$ and $\Psi(\underline{t},\underline{x}) = \psi(x(0))$. The first Dynamic Programming Principle implies that for $t\in[0,\underline{t}]$, $\Psi(t,x(t)) = \psi(x(0))$.

Then $w(t,x(t)) - \Psi(t,x(t)) \in C^1([0,\underline{t}])$ and differentiating yields

$$\frac{d}{dt}\Big(w(t,x(t)) - \Psi(t,x(t))\Big) \;=\; w_t(t,x(t)) + x'(t).\nabla_x w(t,x(t))\,, \quad \text{a.e. } t\in[0,\underline{t}]\,.$$

The differential inequality $w_t + \tau_{\max}(-\nabla_x w)\le 0$ is equivalent to

$$\forall v\in\Gamma_1^+\,,\quad (\partial_t + v.\partial_x)w \le 0\,.$$

Since $x'(t)\in\Gamma_1^+$ it follows that

$$\frac{d}{dt}\Big(w(t,x(t)) - \Psi(t,x(t))\Big) \;\le 0 \quad \text{a.e.} \quad t\in[0,\underline{t}]\,.$$

Integrating from $t=0$ to $t=\underline{t}$ yields

$$w(\underline{t},x(\underline{t})) - \Psi(\underline{t},x(\underline{t})) \le \delta$$

which is the desired estimate (2.9).

To complete the proof of (2.5) we approximate ℓ by $w^\epsilon \in C^\infty([0,T-\epsilon[\times\mathbb{R}^d)$,

$$w^\epsilon(t,x) \;:=\; \int\!\!\int \epsilon^{(-1-d)}\,\rho\big((t,x)-(s,y)\big)\;\; \ell(s,y)\;ds\,dy \;:=\; J_\epsilon(\ell)\,, \tag{2.10}$$

where

$$0\le\rho\in C_0^\infty\big(]-1,0[\times\mathbb{R}^d\big)\,, \qquad \int \rho(t,x)\;dt\,dx = 1\,.$$

Then, w^ϵ converges uniformly to ℓ on compact subsets of $[0,T[\times\mathbb{R}^d$.

The differential inequality in (2.4) is equivalent to

$$\forall v \in \Gamma_1^+, \quad \big(\partial_t + v.\partial_x\big)\,\ell(t,x) \le 0\,.$$

Since the constant coefficient operators $\partial_t + v.\partial_x$ commute with the regularization it follows that $\big(\partial_t + v.\partial_x\big)w^\epsilon \le 0$. Taking the maximum over v yields $w_t^\epsilon + \tau_{\max}(-\nabla_x w^\epsilon) \le 0$. Applying the comparison principle for C^1 comparisons and $\delta = \|\ell(0,x) - w^\epsilon(0,x)\|_{L^\infty(\mathbb{R}^d)}$ yields

$$w^\epsilon(t,x) \le \Psi(t,x) + \|\ell(0,x) - w^\epsilon(0,x)\|_{L^\infty(\mathbb{R}^d)}$$

for all $t < T - \epsilon$. Letting $\epsilon \to 0$ proves (2.5). ∎

Proposition 2.4. Interpretation of the HJ equation *The inequality*

$$\tau + \tau_{\max}(-\xi) \ < \ 0\,, \tag{2.11}$$

is equivalent to $(\tau,\xi) \in -\mathcal{T}$. *In particular at points of differentiability, the equation*

$$w_t + \tau_{\max}(-\nabla_x w) < 0$$

is equivalent to $d_{t,x}w \in -\mathcal{T}(t,x)$.

Proof. Since $P(-\tau,-\xi) = (-1)^m P(\tau,\xi)$, the roots of $P(\tau,-\xi) = 0$ are the negatives of the roots of $P(\tau,\xi) = 0$. Thus with an obvious definition for $\tau_{\min}(\xi)$ one has

$$\tau_{\max}(-\xi) \ = \ -\tau_{\min}(\xi)\,.$$

Thus

$$\tau + \tau_{\max}(-\xi) \ < \ 0 \quad \Longleftrightarrow \quad \tau \ < \ -\big(-\tau_{\min}(\xi)\big)\,.$$

Finally from the definitions it follows that $-\mathcal{T} = \big\{(\tau,\xi) \ : \ \tau < \tau_{\min}(\xi)\big\}$. ∎

§3. Variable coefficient Hamilton-Jacobi.

Suppose that $P(t,x,\tau,\xi)$ is the principal symbol of a variable coefficient hyperbolic system of partial differential equations with time like codirection dt. At each (t,x) one can follow the definitions of the first section to define $\mathcal{T}(t,x) = \mathcal{T}(L(t,x,\cdot,\cdot),dt)$, $\tau_{\max}(t,x,\xi)$, $\Gamma^+(t,x)$, and $\Gamma_1^+(t,x)$. It is crucial for what follows that these objects depend in a lipschitz continuous fashion on (t,x), that is, Hypothesis 0.2 is satisfied. Strictly speaking, $P(t,x,\tau,\xi)$ is a function on the cotangent space so $\mathcal{T}(t,x) \subset T^*_{(t,x)}\mathbb{R}^d$, and $\Gamma^+(t,x)$ and $\Gamma_1^+(t,x)$ are subsets of the tangent space $T_{(t,x)}(\mathbb{R}^d)$.

Definition. *The system* (0.1) *with characteristic polynomial* (0.2) *is said to be hyperbolic with roots of constant multiplicity if there are integers* μ_j, $1 \le j \le M$ *independent of* $t, x, \xi \ne 0$ *and real*

$$\lambda_1(t,x,\xi) < \lambda_2(t,x,\xi) < \cdots < \lambda_M(t,x,\xi)$$

so that

$$P(t,x,\tau,\xi) \ = \ \Pi_{j=1}^M\Big(\tau - \lambda_j(t,x,\xi)\Big)^{\mu_j} \qquad \text{with} \quad \lambda_j \ne \lambda_k \ \text{ when } \ j \ne k\,.$$

The system is strictly hyperbolic when this is true with $\mu_j = 1$ *for all* j.

Definitions. *Consider a first order system*

$$L \ := \ A_0(t,x)\,\partial_t + \sum_{j=1}^{d} A_j(t,x)\partial_j + B(t,x)$$

with coefficients such that $A_\mu\,,\ \nabla_{t,x}A_\mu\,,\ B\ \in\ L^\infty\big([-1,T]\times\mathbb{R}^d\big)$ *for all* $T>0$.

It is **symmetric hyperbolic** *when* $A_\mu = A_\mu^*$, *and for all* $T>0$, *there are constants* $c_1(T)>0$ *such that*

$$A_0(t,x) \ge c_1(T) I > 0, \qquad \forall (t,x)\in[-1,T]\times\mathbb{R}^d\,. \tag{3.1}$$

It is **symmetrizable** *when for every* $T>0$ *there is a square matrix valued* $S\in\cap_T W^{1,\infty}\big([-1,T]\times\mathbb{R}^d\big)$ *so that* $S(t,x)\,L$ *is symmetric hyperbolic.*

It is **microlocally symmetrizable** *if there is a square matrix valued* $S(t,x,\xi)$ *which is positive homogeneous of degree zero in* $\xi\in\mathbb{R}^d\setminus\{0\}$ *so that*

$$S(t,x,\xi)\,\sum \xi_j A_j(t,x) \ = \ \Big(S(t,x,\xi)\,\sum \xi_j A_j(t,x)\Big)^*,$$

and for all $-1\le t\le T$, $x\in\mathbb{R}^d$, $|\alpha|\le 1$, $|\xi|=1$, *and* β,

$$S(t,x,\xi)\,A_0(t,x)\ \ge\ c(T)\,I\ >\ 0\,, \qquad \sup_{0\le t\le T\,,\ |\xi|=1} \Big|\partial^\alpha_{t,x}\partial^\beta_\xi\, S\Big|\ <\ \infty\,.$$

Remarks. 1. Microlocally symmetrizable with S independent of ξ is exactly symmetrizability. **2.** Most of the important examples from mathematical physics are symmetrizable since symmetrizability follows from the existence of a strictly convex entropy (see [FrLa], [Go1,2], [La2]). **3.** If L is first order, has roots of constant multiplicity, and for all $\xi\ne 0$ the matrix $A_0^{-1}\sum_{j\ge 1} A_j\,\xi_j$ is diagonalizable, then L is microlocally symmetrizeable.

Proposition 3.1. *Suppose that the leading order coefficients of* L *belong to* $W^{1.\infty}\big([-1,T]\times\mathbb{R}^d\big)$ *for all* $T>0$. *Then Hypothesis* 0.3 *is satisfied when* L *is a microlocally symmetrizable first order system. Hypothesis* 0.3 *is also satisfied when* L *is hyperbolic with roots of constant multiplicity.*

Proof. In the microlocally symmetrizable case, $\tau_{\max}(t,x,\xi)$ is the minimal eigenvalue of

$$\big(S(\xi)A_0\big)^{-1/2}\ S(\xi)\sum A_j(t,x)\xi_j\ \big(S(\xi)A_0\big)^{-1/2}\,.$$

As this hermitian matrix valued function is bounded and lipschitzian, so is its minimal eigenvalue. In the constant multiplicity case, if one orders the roots λ_j at one point $(\underline{t},\underline{x},\underline{\xi})$ and then extends them by continuity, the order relation is preserved for all (t,x,ξ). It follows that the functions λ_j are lipschitzian in (t,x) and real analytic in ξ. Since $\tau_{\max}=\lambda_M$ this suffices for the verification of Hypothesis 0.3. ∎

An *influence curve* is defined as an $x(t)\in W^{1,\infty}(I)$ where $I\subset\mathbb{R}$ is a compact interval and $x(t)$ satisfies

$$\frac{dx(t)}{dt}\ \in\ \Gamma_1^+(t,x(t))\,, \qquad \text{a.e. } t\in I\,. \tag{3.2}$$

The set $\mathcal{X}(T, X)$ and functions ψ, Ψ are then defined as in §2.

Lemma 3.2. *The Dynamic Programming Principles of Lemma 2.3 are valid in this more general context.*

Proof. Identical to the proof of Lemma 2.3. ∎

The next result describes the main properties of Ψ. In the next two sections we will show how it can be used to establish natural domains of dependence and influence. The next result implies Theorem 0.1 in §0.

Theorem 3.3. *If Hypotheses 0.1 and 0.3 are satisfied and $T > 0$, then Ψ is uniformly lipschitzian on $[0, T] \times \mathbb{R}^d$ and satisfies the Hamilton-Jacobi initial value problem*

$$\partial_t \Psi + \tau_{\max}(t, x, -\nabla_x \Psi) \;=\; 0 \;\; a.e., \qquad \Psi(0, x) = \psi(x). \tag{3.3}$$

It is the largest solution in the sense that if $\ell(x) \in W^{1,\infty}_{\rm loc}\big([0,T] \times \mathbb{R}^d\big)$ and satisfies

$$\partial_t \ell + \tau_{\max}(t, x, -\nabla_x \ell) \;\leq\; 0 \;\; \text{a.e.}, \qquad \ell(0, x) \leq \psi(x), \tag{3.4}$$

then,

$$\ell(t, x) \leq \Psi(t, x) \quad \text{on} \quad [0, T] \times \mathbb{R}^d. \tag{3.5}$$

Proof. Fix $T > 0$. Begin as in Theorem 2.2, except that Λ_2 is a supremum over t, x, v and $t, \underline{T}$ are restricted to be no larger than T. Choose an influence curve with $x(T) = \underline{X}$ and $\Psi(\underline{T}, \underline{X}) = \psi(x(0))$. Then

$$x'(t) \in \Gamma_1^+(t, x(t)) \;\; \text{a.e. } t \in [0, T].$$

For the same a.e. t, define $\tilde{v}(t, x) \in \Gamma_1^+(t, x)$ to be the unique vector so that

$$|\tilde{v}(t, x) - x'(t)| = \text{dist}\Big(x'(t)\,,\, \Gamma_1^+(t, x)\Big). \tag{3.6}$$

Then $\tilde{v}(t, x)$ is uniformly lipschitzian in x thanks to Hypothesis 0.3. For $\overline{X} \in \mathbb{R}^d$, let $\overline{x}(t)$ be the unique influence curve defined by

$$\frac{d\overline{x}(t)}{dt} \;=\; \tilde{v}(t, \overline{x}(t)), \qquad \overline{x}(T) = \overline{X}. \tag{3.7}$$

By definition, $\Psi(\underline{T}, \overline{X}) \leq \psi(\overline{x}(0))$.
Standard continuous dependence results relying on Hypothesis 0.3 imply that

$$|x(0) - \overline{x}(0)| \leq C\, |x(T) - \overline{x}(T)| = C\, |\underline{X} - \overline{X}|.$$

Therefore

$$\Psi(\underline{T}, \underline{X}) \;\leq\; \psi(\overline{x}(0)) \;\leq\; \psi(x(0)) + \Lambda_1\, C\, |\underline{X} - \overline{X}| \;=\; \Psi(\underline{T}, \overline{X}) + \Lambda_1\, C\, |\underline{X} - \overline{X}|.$$

Reversing the roles of $\underline{X}$ and $\overline{X}$, one concludes that Ψ is uniformly lipschitzian in x on $[0,T] \times \mathbb{R}^d$ for any $T > 0$.

The control of increments in t is reduced to increments in X using the Second Dynamic Programing Principle as in the proof of Theorem 2.2.

The proof of the Hamilton-Jacobi Equation (3.3) follows the proof of Theorem 2.2 through equation (2.7).

Lemma 3.4. 1. *For any $v \in \Gamma_1^+(\underline{T}, \underline{X})$ there is an influence curve $x(\cdot) \in \mathcal{X}(\underline{T}, \underline{X})$ so that*

$$\frac{1}{\epsilon} \int_{\underline{T}-\epsilon}^{\underline{T}} x'(s)\, ds \ \to\ v\,, \qquad \text{as } \ \epsilon \to 0\,.$$

2. *For any influence curve $x(\cdot) \in \mathcal{X}(\underline{T}, \underline{X})$*

$$\operatorname{dist}\Big(\frac{1}{\epsilon} \int_{\underline{T}-\epsilon}^{\underline{T}} x'(s)\, ds\,,\, \Gamma_1^+(\underline{T}, \underline{X}) \Big) \ \to\ 0\,, \qquad \text{as } \ \epsilon \to 0\,.$$

Proof. To prove the first assertion for $v \in \Gamma_1^+(\underline{T}, \underline{X})$ define a vector field by

$$\underline{v}(t,x) \in \Gamma_1^+(t,x) \quad \text{and} \quad |\underline{v}(t,x) - v| \ =\ \operatorname{dist}\Big(v, \Gamma_1^+(t,x)\Big)\,. \tag{3.8}$$

Then $\underline{v}$ is lipschitzian thanks to Hypothesis 0.3, and $\underline{v}(\underline{T}, \underline{X}) = v$ so

$$|\underline{v}(t,x) - v| \ \le\ C_1\Big(|t - \underline{T}| + |x - \underline{X}| \Big)\,. \tag{3.9}$$

Introduce the influence curve which is the solution of

$$\frac{dx(t)}{dt} \ =\ \underline{v}(t, x(t))\,, \qquad x(\underline{T}) = \underline{X}\,. \tag{3.10}$$

Then

$$|x'(t) - v| \ \le\ C_2 |t - \underline{T}| \quad \text{a.e.}\,,$$

so

$$\frac{1}{\epsilon} \int_{\underline{T}-\epsilon}^{\underline{T}} x'(s)\ ds - v \ =\ O(\epsilon) \quad \text{as} \quad \epsilon \to 0\,.$$

To prove the second assertion, let $v(t) \in \Gamma_1^+(\underline{T}, \underline{X})$ be the closest element to $x'(t) \in \Gamma_1^+(t, x(t))$. Since x' is bounded and $x(t)$ is lipschitzian Hypothesis 0.3 implies that

$$|v(t) - x'(t)| \le C\, |t - \underline{T}| \quad \text{a.e.}\,.$$

Therefore

$$\frac{1}{\epsilon} \int_{\underline{T}-\epsilon}^{\underline{T}} v(s) - x'(s)\ ds \ =\ O(\epsilon)\,,$$

and

$$\frac{1}{\epsilon}\int_{\underline{T}-\epsilon}^{\underline{T}} v(s)ds \;\in\; \Gamma_1^+(\underline{T},\underline{X})\,,$$

finishes the proof of the lemma. ∎

This lemma together with (2.7) implies (2.8). Equation (2.8) is equivalent to (3.3) thanks to (1.7). The proof that Ψ is the largest solution is in two steps. One first proves that if $w \in C^1([0,T]\times\mathbb{R}^d)$ and satisfies

$$w_t + \tau_{\max}(t,x,-\nabla_x w) \le \delta\,, \qquad w(0,x) \le \psi(x) + \delta\,, \tag{3.11}$$

Then for $t > 0$,

$$w \;\le\; \psi + \delta(1+t) \tag{3.12}$$

The proof is exactly like the proof of (2.9).

The regularized solutions w^ϵ are defined by (2.10). The lipschitz continuity of ℓ implies that

$$\|w^\epsilon(0,x) - \psi(x)\|_{L^\infty(\mathbb{R}^d)} \;=\; O(\epsilon)\,.$$

From formula (1.7) one has that

$$\big(\partial_t + v(t,x).\partial_x\big)\ell \;\le\; 0\,, \tag{3.13}$$

for all bounded vector fields such that $v(t,x) \in \Gamma_1^+(t,x)$ for all (t,x). For $(\underline{T},\underline{X})$ with $\underline{T} \in [0,T]$ and $v \in \Gamma_1^+(\underline{T},\underline{X})$ one uses this for the field $\underline{v}(t,x)$ defined by (3.8). Note that the field $\underline{v} \in W^{1,\infty}\big([0,T]\times\mathbb{R}^d\big)$ and that the lipschitz constant can be chosen to depend on the lipschitz constant for the family $\mathcal{T}(t,x)$ and not on $\underline{T},\underline{X},v$. Denoting by J_ϵ the regularization operator in (2.10) we have

$$0 \;\ge\; J_\epsilon\Big(\big(\partial_t + \underline{v}(t,x).\partial_x\big)\ell\Big) \;=\; \big(\partial_t + \underline{v}(t,x).\partial_x\big)w^\epsilon + [J_\epsilon,\underline{v}]\partial_x\ell\,. \tag{3.14}$$

The commutator $[J_\epsilon,\underline{v}]$ is estimated follows. It has integral kernel

$$K(t,x,s,y) \;=\; \epsilon^{(-1-d)}\;\rho\Big(\frac{(t,x)-(s,y)}{\epsilon}\Big)\;\big(\underline{v}(s,y) - \underline{v}(t,x)\big)\,.$$

The last factor is bounded by $C\,\epsilon$ with the constant independent of $\underline{T},\underline{X},v$ so

$$\int |K(t,x,s,y)|\;ds\,dy \;\le\; C\,\epsilon\,.$$

Since $\partial_x\ell \in L^\infty$, it follows that

$$\Big\|[J_\epsilon,\underline{v}]\partial_x\ell\Big\|_{L^\infty([0,T]\times\mathbb{R}^d)} \;\le\; C\epsilon$$

with constant independent of $\underline{T},\underline{X},v$. Combining yields

$$\big(\partial_t + \underline{v}(t,x).\partial_x\big)w^\epsilon \;\le\; C\,\epsilon\,.$$

This estimate is used at $\underline{T},\underline{X}$ yielding

$$w_t^\epsilon(\underline{T},\underline{X}) + \tau_{\max}(\underline{T},\underline{X},-\nabla_x w^\epsilon(\underline{T},\underline{X})) = \max_{v\in\Gamma_1^+(\underline{T},\underline{X})}\;\big(\partial_t + \underline{v}(\underline{T},\underline{X}).\partial_x\big)w^\epsilon(\underline{T},\underline{X}) \;\le\; C\,\epsilon\,,$$

with a constant independent of $\underline{T}, \underline{X}$.
The smooth comparison result then can be applied to give

$$w^\epsilon \;\le\; \Psi + C'\,\epsilon \quad \text{on} \quad [0, T-\epsilon[\times\mathbb{R}^d\,.$$

Letting $\epsilon \to 0$ proves the final conclusion (3.5). ∎

§4. Sharp domains in the symmetric hyperbolic case; the energy method.

In this section the solution Ψ from §3 is used to give efficient weight functions in the energy method. Recall that the desired natural domain of determination of Ω_0 is the set $\Omega = \{\Psi > 0\}$.

Theorem 4.1 *Suppose that L is a symmetrizable hyperbolic system, $\Omega_0 \subset \mathbb{R}^d$ is a proper open subset, and $u \in L^2_{\rm loc}([0,\infty[\,:L^2(\mathbb{R}^d))$ satisfies*

$$L\,u = 0 \quad \text{in} \;\; \{t>0\} \qquad \text{and} \qquad u(0,x) = 0 \qquad \text{in} \;\; \Omega_0\,.$$

Then u vanishes on Ω.

The result follows from the next general principal applied with Φ equal to the function Ψ from the last section.

Theorem 4.2 *If L is a symmetrizable hypberbolic system, $\Phi \in W^{1,\infty}\big([0,T]\times\mathbb{R}^d\big)$ satisfies*

$$\Phi_t + \tau_{\max}\big(t, x, -\nabla_x\Phi(t,x)\big) \;\le\; 0 \;\; \text{a.e.}\,, \tag{4.1}$$

and $u \in C\big([0,T]\,:\,L^2(\mathbb{R}^d)\big)$ satisfies

$$Lu \;=\; 0\,, \qquad \text{and} \qquad \Phi(0,x) \le 0 \quad \text{for} \;\; x \in \operatorname{supp} u(0,.)\,. \tag{4.2}$$

then

$$\Phi(t,x) \le 0 \quad \text{on} \;\; \operatorname{supp} u\,. \tag{4.3}$$

Proof. The equation $L\,u = 0$ is equivalent to $\tilde L\tilde u = 0$ where

$$\tilde L \;:=\; (SA_0)^{-1/2}\, S(t,x) L\big(t,x,\partial_{t,x}\big)\, (SA_0)^{-1/2}\,, \qquad \tilde u \;:=\; (SA_0)^{1/2} u\,.$$

The operator $\tilde L$ is a symmetric hyperbolic system and the coefficient of ∂_t is equal to I. It's characteristic polynomial is the same as that of L so it has the same $\tau_{\max}$. Thus, swapping $\tilde u$ for u and $\tilde L$ for L we may assume without loss of generality that L is symmetric hyperbolic with $A_0 = I$. Since $Lu = 0$,

$$L(t,x,\partial_{t,x})\Big(e^{\lambda\Phi}\,u\Big) \;=\; \lambda\, e^{\lambda\Phi}\, L_1(t,x,d\Phi)\,u\,, \tag{4.4}$$

with matrix valued

$$L_1(t,x,d\Phi) \;:=\; (\partial_t\Phi)\, I + \sum_{j=1}^d A_j(t,x)\,\partial_j\Phi\,.$$

The Hamilton-Jacobi inequality (4.1) yields

$$L_1(t,x,d\Phi) \;\le\; -\tau_{\max}(-\nabla_x\Phi) + \sum_{j=1}^{d} A_j\,\partial_j\Phi = -\Big(\tau_{\max}(-\nabla_x\Phi) + \sum_{j=1}^{d} A_j\,(-\partial_j\Phi)\Big)\,. \tag{4.5}$$

For this problem the characteristic polynomial is given by

$$P(t,x,\tau,\xi) = \det\big(\tau\, I + \sum A_j\,\xi_j\big)\,.$$

The definition of $\tau_{\max}$ shows that for $\tau > \tau_{\max}(\xi)$ the matrix $\tau\, I + \sum A_j\,\xi_j > 0$. Thus

$$\tau_{\max}(\xi)\, I + \sum A_j\,\xi_j \;\ge\; 0\,, \tag{4.6}$$

Equations (4.5) and (4.6) imply the crucial inequality

$$L_1(t,x,d\Phi(t,x)) \;\le\; 0\,. \tag{4.7}$$

Define the matrix valued function

$$Z(t,x) \;:=\; B + B^* + \sum_{j=1}^{\infty} \frac{\partial A_j}{\partial x_j}\,,$$

and the weighted L^2 energy

$$e_\lambda(t) \;:=\; \int_{\mathbb{R}^d} e^{\lambda\Phi}\; |u(t,x|^2\; dx\,.$$

The standard energy identity, proved with the help of Friedrichs mollifiers, reads

$$e_\lambda(t) - e_\lambda(0) = \int_0^t \int_{\mathbb{R}^d} \big(Z(t,x)\, e^{\lambda\Phi} u\,,\, e^{\lambda\Phi} u\big)\; dt\, dx \;+ \lambda \int_0^t \int_{\mathbb{R}^d} e^{2\lambda\Phi} \big(u, L_1(t,x,d\Psi)u\big)\; dt\, dx.$$

The last term is nonpositive for $\lambda \ge 0.$, Also, for any $T > 0$, $Z \in L^\infty([0,T]\times\mathbb{R}^d)$. Therefore, for $0 \le t \le T$

$$e_\lambda(t) \;\le\; e_\lambda(0) + C(T) \int_0^t e_\lambda(s) ds\,.$$

Gronwall's inequality implies

$$e_\lambda(t) \;\le\; e^{C(T)t}\; e_\lambda(0) \qquad t \in [0,T]\,.$$

Since $u(0,x)$ vanishes on the set where $\Phi > 0$, the right hand side is bounded independent of $\lambda > 0$ and $t \in [0,T]$. Therefore $e_\lambda(t)$ is uniformly bounded for these values. This implies that u vanishes on the set $\{\Phi > 0\}$. ∎

§5. Natural domains whenever there is local uniqueness: spacelike deformations.

We reach the same conclusion as in the preceding section by a different and more general method. The method uses perturbations of the function Ψ and local uniqueness of the Cauchy problem so Hypotheses 0.2 and 0.3 are assumed. The idea is that the level sets $\{\Psi = c > 0\}$ with c decreasing from the maximum value of Ψ, almost give a smooth deformation by spacelike hypersurfaces sweeping out the natural domain of determinacy Ω from (0.10). The lipschitzian weights work in the energy method of §4. But, the level sets of $W^{1,\infty}$ functions are ill behaved so the proof in this section uses regularization of Ψ.

In simple cases, it is clear what regions can be swept out with the constraint of remaining spacelike. In the general case we were surprised and pleased to find that the natural set Ω can be reached by such deformations.

The example of Ω_0 equal to a dumbbell shaped region in Figure 5.2 suggests some of the pitfalls. Take ψ to be equal to the distance from the boundary of Ω. Consider the case of D'Alembert's wave equation in which case $\Psi = \psi(x) - t$.

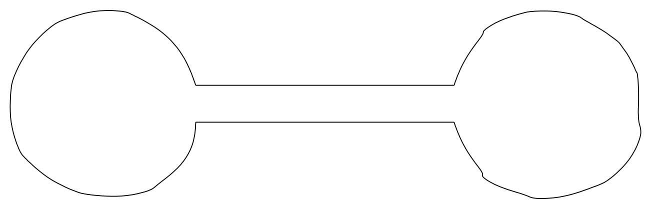


Figure 5.2

The level sets $\{\Psi = c\}$ with c small positive are dumbbell shaped for t small and then for larger t pinch off into two cusped circles. The difficulties are both the cusps and the change of topology. If one puts C^∞ wobbles in the connecting tube of the dumbbell, there can be a countable number of little bubbles pinched off in the tube. We sweep out the set $\{\Psi > \mu\}$ with $\mu > 0$ small. This strict positivity allows us just enough wiggle room to regularize the geometry.

Theorem 0.2 follows from the next general principal applied with Φ equal to the function Ψ from §3 when the initial data ψ are chosen satisfying $\lim_{|x|\to\infty} \psi(x) = 0$. The result is essentially an extension of Theorem 4.2 to the non symmetrizable case.

Theorem 5.1 *If L satisfies Hypotheses 0.1 and 0.2, $\Phi \in W^{1,\infty}\big([0,T]\times\mathbb{R}^d\big)$ for all $T > 0$ and satisfies*

$$\Phi_t + \tau_{\max}\big(t, x, -\nabla_x\Phi(t,x)\big) \ \le\ 0 \ \text{ a.e.}\,, \qquad \lim_{|x|\to\infty} \Phi(0,x) = 0\,, \tag{5.1}$$

and $u \in H^{m-1}_{\text{loc}}\big([0,\infty[\times\mathbb{R}^d\big)$ satisfies

$$Lu \ =\ 0\,, \qquad \text{and} \qquad \Phi(0,x) \le 0 \quad \text{for} \quad x \in \cup_{j=0}^{m-1} \operatorname{supp} \partial_t^j u(0,.)\,. \tag{5.2}$$

then for all $|\alpha| \le m-1$,

$$\Phi(t,x) \le 0 \quad \text{on} \quad \operatorname{supp} \partial^\alpha_{t,x} u\,. \tag{5.3}$$

Theorem 5.1 in turn is proved using the next Lemma which establishes the key link between Hamilton-Jacobi equations and the method of deforming spacelike hypersurfaces.

Lemma 5.2. Spacelike deformations. *Suppose that $\Phi \in \big(C^1\cap W^{1,\infty}\big)\big([0,T]\times\mathbb{R}^d\big)$ for all $T > 0$ satisfies*

$$\Phi_t + \tau_{\max}\big(t, x, -\nabla_x\Phi(t,x)\big) \ <\ 0\,, \qquad \limsup_{|t,x|\to\infty} \ \Phi(t,x) \le 0\,, \tag{5.4}$$

and $u \in H^{m-1}_{\rm loc}\big([0,\infty[\times\mathbb{R}^d\big)$ *satisfies*

$$Lu \ = \ 0\,, \qquad \text{and} \qquad \Phi(0,x)\le 0 \quad \text{for} \quad x\in \cup_{j=0}^{m-1}\,{\rm supp}\,\partial_t^j u(0,.)\,. \tag{5.5}$$

then for all $|\alpha|\le m-1$,

$$\Phi(t,x)\le 0 \quad \text{on} \quad {\rm supp}\ \partial^\alpha_{t,x} u\,, \tag{5.6}$$

Remark. The key additional hypotheses are the continuous differentiability of Φ and the strict Hamilton-Jacobi inequality.

Proof of Lemma 5.2. If $\Phi\le 0$ there is nothing to prove. Since Φ is nonpositive at infinity, if Φ assumes positive values it attains its maximum value. The differential inequality (5.4) shows that Φ has no critical points in $T>0$. Thus the maximum value, $\Phi_{\max}$, is assumed only in $\{t=0\}$.

It suffices to show that u vanishes wherever $\Phi>0$. Therefore it suffices to show that u vanishes on $\Phi^{-1}([\epsilon,\Phi_{\max}]$ for arbitrary $\epsilon>0$. Fix $0<\epsilon<\Phi_{\max}$.

Let

$$\mathcal{O} \ := \ \{t\ge 0\}\,, \qquad F(t,x) \ := \ \frac{\Phi(t,x)-\Phi_{\max}}{\epsilon-\Phi_{\max}}\,. \tag{5.7}$$

Then as Φ decreases from $\Phi_{\max}$ to ϵ, F increases from 0 to 1. It suffices to show that u vanishes on $F^{-1}([0,1])$. It suffices to verify the hypotheses of John's Global Hölmgren Uniqueness Theorem.

The assertion about $\Phi_{\max}$ proves property **i.**

Property **ii.** is a consequence of the fact that $\{F\ge 0\}=\{\Phi\ge\epsilon\}$. The latter is compact thanks to the second part of (5.4).

Property **iii.** follows from the differenital inequality (5.4) and Proposition 2.4.

That the Cauchy data of u vanish on $\{F\ge 0\}\cap\{t=0\}=\{\Phi\ge\epsilon\}\cap\{t=0\}$ follows from the second part of (5.5). That $Lu=0$ is the first part of (5.5) which completes the verification of the hypotheses of John's Theorem. ∎

Proof of Theorem 5.1. Replacing Φ by a larger function satisfying the conditions of the Theorem and with the same initial data, strengthens the conclusion (5.3). Thus it suffices to prove the Theorem for the largest such function Φ. Theorem 3.3 shows that this largest function is given by the formula

$$\Phi_{\rm upper}(t,x) \ := \ \min_{x(\cdot)\in\mathcal{X}(t,x)} \Big\{\Phi(0,x(0))\Big\}\,. \tag{5.8}$$

Then $\Phi_{\rm upper}\in W^{1,\infty}([0,\infty[\times\mathbb{R}^d)$ satisfies

$$\partial_t\Phi_{\rm upper} + \tau_{\max}\big(t,x,-\nabla_x\Phi_{\rm upper}(t,x)\big) \ = \ 0\,, \qquad \Phi_{\rm upper}(0,x)=\Phi(0,x)\,, \tag{5.9}$$

and is the largest such solution. Replace Φ by $\Phi_{\rm upper}$ and drop the subscript.

If (T,X) with $T>0$ satisfies $\Phi(T,X)>0$, we must show that u vanishes on a neighborhood of (T,X). If there are no such points there is nothing to prove. Fix (T,X) with $T>0$ and $\Phi(T,X)>0$.

With $0<\delta$ define

$$\Phi^\delta \ := \ \Phi-\delta\, t\,. \tag{5.10}$$

The Hamilton-Jacobi equation for Φ is equivalent to

$$\Phi^\delta_t + \tau_{\max}(t,x,-\nabla_x\Phi^\delta) \;=\; -\delta\,, \qquad \Phi^\delta(0,x) = \Phi(0,x)\,. \tag{5.11}$$

Fix $0<\delta$ so small that

$$\Phi^\delta(\underline{T},\underline{X}) > 0\,. \tag{5.12}$$

The second assertion in (5.1) together with formulas (5.8) and (5.10) imply that

$$\lim_{|t,x|\to\infty} \;\Phi^\delta(t,x) \;\le\; 0\,. \tag{5.13}$$

Regularize as in (2.10),

$$\Phi^{\epsilon,\delta} \;:=\; J_\epsilon(\Phi^\delta) \;:=\; \int\!\!\int \epsilon^{(-1-d)}\,\rho\Big(\frac{(t,x)-(s,y)}{\epsilon}\Big)\;\; \Phi^\delta(s,y)\;ds\,dy \;\in\; C^\infty([0,\infty[\times\mathbb{R}^d)\,.$$

Equations (5.12) and (5.13) imply that for ϵ small and positive

$$\lim_{|t,x|\to\infty} \;\Phi^{\epsilon,\delta}(t,x) \;\le\; 0 \quad \text{and} \quad \Phi^{\epsilon,\delta}(\underline{T},\underline{X}) > 0\,. \tag{5.14}$$

As in the proof of Theorem 3.3, write the Hamilton-Jacobi equation in the form (3.13). Commute with J_ϵ (using the convexity of $\tau_{\max}$) to get a one sided estimate of the form (3.14). It follows that there is a constant $C(T)$ so that

$$\Big\|\Phi^{\epsilon,\delta}-\Phi^\delta\Big\|_{L^\infty([0,T]\times\mathbb{R}^d)} \;<\; C\,\epsilon\,, \quad \text{and,} \quad \Phi^{\epsilon,\delta}_t + \tau_{\max}(t,x,-\nabla_x\Phi^{\epsilon,\delta}) \;\le\; -\delta + C\,\epsilon\,. \tag{5.15}$$

In addition

$$\sup\Big\{\Phi^{\epsilon,\delta}(0,x) \;:\; x\in\cup_{j\le m-1}\,\mathrm{supp}\,\partial^j u(0,x)\Big\} \;\le C\,\epsilon\,. \tag{5.16}$$

Thus for ϵ small and positive

$$\Phi^{\epsilon,\delta}_t + \tau_{\max}(t,x,-\nabla_x\Phi^{\epsilon,\delta}) < -\delta/2\,, \tag{5.17}$$

and

$$\sup\Big\{\Phi^{\epsilon,\delta}(0,x) \;:\; x\in\cup_{j\le m-1}\,\mathrm{supp}\,\partial^j u(0,x)\Big\} \;<\Phi^{\epsilon,\delta}(\underline{T},\underline{X})/2. \tag{5.18}$$

Theorem 5.1 then follows from Lemma 5.2 applied to the function $\Phi^{\epsilon,\delta}-\Phi^{\epsilon,\delta}(\underline{T},\underline{X})/2$ with ϵ small and positive. ∎

§6. First arrival times when $\{|v|\le c\}\subset\Gamma_1^+$.

In the preceding analysis the object described is the boundary of Ω which is d dimensional. It is disappointing that to do this we solved for a function Ψ of $d+1$ variables which is one variable more than the minimum required. This section provides a description in terms of first arrival times which are functions of d variables. This natural reduced description is available when the propagation cones Γ_1^+ contain a neighborhood of the origin, the same hypothesis which appears in the related Proposision 1.7.

Hypotheses 6.1. *In this section suppose that there is a $c > 0$ so that for all $(t, x) \in [0, \infty[\times\mathbb{R}^d$, $\Gamma_1^+(t,x) \supset \{|v| \le c\}$.*

In this case, curves moving with speed less than or equal to c are influence curves. Suppose that Ω_0 is a bounded nonempty open subset of $\mathbb{R}^d$ with nonempty complement S_0. Define the *first arrival time from* S_0 by

$$\zeta(x) \ := \ \inf\Big\{T \ : \ \text{There is an influence curve } x(\cdot) \text{ with } x(T) = x \text{ and } x(0) \in S_0\Big\}. \qquad (6.1)$$

The hypothesis concerning Γ_1^+ implies that

$$\zeta(x) \ \le \ \frac{\text{dist}(x, S_0)}{c} \ < \ \infty\,.$$

Since the origin belongs to the convex set Γ_1^+ an influence curve which arrives at x at time T can be continued as an influence curve which is stationary at x for times greater than T. It follows that the sets $\{\Psi > 0\}$ and $\{0 \le t < \zeta(x)\}$ are identical. Thus the domain of dependence is as well described by ζ as by Ψ. We turn next to a Hamilton-Jacobi characterization of ζ.

The infinum (6.1) is achieved and the following Dynamic Programming Principles have proofs similar to the proof of Lemma 2.3.

Lemma 6.2. Dynamic Programing Principles. I. *If the infinum (6.1) is achieved on an influence curve $x : [0, T] \to \mathbb{R}^d$, then for $0 \le t \le T$,*

$$\zeta(x(t)) \ = \ t\,. \qquad (6.2)$$

II. *If $\zeta(\underline{x}) = T$, then for $0 \le t \le T$,*

$$\zeta(\underline{x}) = \min_{x(\cdot)\in\mathcal{X}(T,\underline{x})} \Big\{\zeta(x(t))\Big\}\,. \qquad (6.3)$$

Theorem 6.3. *When Hypotheses 0.2 and 6.1 are satisfied, the function ζ is uniformly lipschitzian on $\mathbb{R}^d$ and satisfies*

$$\tau_{\max}\big(\zeta(x), x, -\nabla_x\zeta\big) = 1 \ \text{ a.e. } x \in \Omega_0\,. \qquad \zeta\,\big|_{\partial\Omega_0} = 0\,. \qquad (6.4)$$

It is the largest such solution in the sense that if ℓ is uniformly lipschitzian on Ω_0 and satisfies

$$\tau_{\max}\big(\ell(x), x, -\nabla_x\ell\big) \le 1 \ \text{ a.e. } x \in \Omega_0\,. \qquad \ell\,\big|_{\partial\Omega_0} \le 0\,,$$

then

$$\ell \ \le \ \zeta \qquad \text{on} \quad \overline{\Omega}_0\,. \qquad (6.5)$$

Proof. That ζ is lipschitzian is proved exactly as was Proposition 1.7. To prove (6.4) it suffices to verify the equation at all points where ζ is differentiable. Suppose that $\underline{x}$ is such a point and $\zeta(\underline{x}) = T$.

The second Dynamic Programming Equation with $t = T - \epsilon$ shows that

$$\zeta(\underline{x}) \;=\; \epsilon + \min_{x(\cdot)\in\mathcal{X}(T,\underline{x})} \big\{\zeta(x(T-\epsilon))\big\} = \epsilon + \min_{x(\cdot)\in\mathcal{X}(T,\underline{x})} \zeta\Big(\underline{x} - \int_{T-\epsilon}^{T} x'(s)\,ds\Big).$$

Expand

$$\zeta\Big(\underline{x} - \int_{T-\epsilon}^{T} x'(s)\,ds\Big) = \zeta(\underline{x}) - \epsilon\nabla_x\zeta(\underline{x}).\Big(\frac{1}{\epsilon}\int_{T-\epsilon}^{T} x'(s)\,ds\Big) + o(\epsilon).$$

Combine the last two equations to find

$$1 + \min_{x(\cdot)\in\mathcal{X}(T,\underline{x})} \Big\{ -\nabla_x\zeta(\underline{x}).\frac{1}{\epsilon}\int_{T-\epsilon}^{T} x'(s)\,ds \Big\} = o(1) \qquad \text{as} \quad \epsilon \to 0. \tag{6.6}$$

Lemma 3.4 shows that (6.6) implies

$$1 - \max_{v\in\Gamma_1^+(T,\underline{x})} \Big\{v.\nabla_x\zeta(\underline{x})\Big\} \;=\; 0, \qquad \zeta(\underline{x}) = T.$$

Equation (1.7) implies that this is equivalent to the desired Hamilton-Jacobi equation in (6.4).
The comparison with ℓ is proved in two steps. The first step is a comparison with a C^1 solution. We prove that if $w \in W^{1,\infty}(\Omega_0)\cap C^1(\Omega_0)$ and $\delta \in \mathbb{R}$ satisfy

$$\tau_{\max}(w(x), x, -\nabla_x w) \le 1 + \delta \quad \text{a.e.} \;\; x \in \Omega_0, \qquad \text{and} \quad w|_{\partial\Omega_0} \le \delta,$$

then

$$w \;\le\; \zeta + \delta\Big[\Big(1 + \frac{1}{\Lambda}\Big)e^{\underline{T}\Lambda} + \frac{1}{\Lambda}\Big] \qquad \text{on} \;\; \overline{\Omega_0}, \tag{6.7}$$

where

$$\underline{T} \;:=\; \max\zeta, \qquad \text{and} \qquad \Lambda \;:=\; \Big\|\frac{\partial\tau_{\max}}{\partial t}\Big\|_{L^\infty([0,\underline{T}]\times\mathbb{R}^d)}.$$

To prove (6.7) at $\underline{x} \in \Omega_0$ choose a minimizing influence curve for (6.3), that is $x(T) = \underline{x}$ and $\zeta(\underline{x}) = T$. The first Dynamic Programming Principle implies that $w(x(t)) - \zeta(x(t)) = w(x(t)) - t$. Differentiating yields

$$\big(w(x(t)) - \zeta(x(t))\big)' = \nabla_x w(x(t)).x'(t) - 1. \tag{6.8}$$

Since $x'(t) \in \Gamma_1^+(t, x(t)) = \Gamma_1^+\big(\zeta(x(t)), x(t)\big)$ formula (1.7) shows that

$$\begin{aligned}\nabla_x w(x(t)).x'(t) - 1 &\le \tau_{\max}\big(\zeta(x(t)), x(t), \nabla_x w(x(t))\big) - 1\\ &\le \tau_{\max}\big(w(x(t)), x(t), \nabla_x w(x(t))\big) - 1 + \Lambda|w(x(t)) - \zeta(x(t)|\\ &\le \delta + \Lambda|w(x(t)) - \zeta(x(t)|.\end{aligned}$$

Therefore integrating (6.8) from $t = 0$ to $t = T$ yields

$$\Big|w(x(T)) - \zeta(x(T)\Big| \le \Big(\delta + \frac{\delta}{\Lambda}\Big)\; e^{\Lambda T} + \frac{\delta}{\Lambda}.$$

It follows that

$$w(x(T)) \;\le\; \zeta(x(T)) + \delta\Big[\Big(1 + \frac{1}{\Lambda}\Big)e^{\Lambda T} + \frac{1}{\Lambda}\Big],$$

and the proof of (6.7) is complete.

The second step of the comparison proof is a regularization of ℓ. On Ω_0 let

$$w(x) \ := \ \max\{\ell(x), 0\}\,.$$

be the positive part of ℓ. Extend w to all of $\mathbb{R}^d$ so that w vanishes outside Ω_0. Then since $\ell \le 0$ on the boundary of Ω_0, w is uniformly lipschitzian and

$$\Big\{\nabla_x w(x) = 0 \ \text{ or } \ \nabla_x w(x) = \nabla_x \ell(x)\Big\} \quad \text{for a.e. } \ x \in \mathbb{R}^d\,. \tag{6.9}$$

Regularize to define

$$w^\epsilon(x) \ := \ \int_{\mathbb{R}^d} \epsilon^{-d}\, j\Big(\frac{x-y}{\epsilon}\Big)\ w(y)\ dy\,, \quad 0 \le j \in C_0^\infty(\mathbb{R}^d), \quad \int j(x)dx = 1\,.$$

Then w is smooth and

$$\big\|w^\epsilon\big\|_{W^{1,\infty}(\mathbb{R}^d)} \ \le \ \big\|w\big\|_{W^{1,\infty}(\mathbb{R}^d)}\,.$$

Choose an extension of the function $\tau_{\max}(t,x,\xi)$ to $\mathbb{R}^d \times \mathbb{R}^d$ which is homogeneous and convex in ξ and is uniformly lipschitzian. Then (6.9) implies that

$$\tau_{\max}(w(x), x, -\nabla_x w(x)) \ \le \ 1\,, \quad \text{for a.e. } \ x \in \mathbb{R}^d\,.$$

The same argument as at the end of Theorem 3.3 shows that thanks to Hypothesis 0.3, there is a constant $C > 0$ independent of ϵ so that

$$\tau_{\max}(w^\epsilon(x), x, -\nabla_x w^\epsilon) \ \le \ 1 + C\epsilon \ \text{ on } \ \mathbb{R}^d\,, \qquad w^\epsilon\big|_{\partial\Omega_0} \le C\epsilon\,.$$

The comparison result for C^1 implies that

$$w^\epsilon \ \le \ \zeta + C'\,\epsilon \quad \text{on} \ \ \Omega_0\,.$$

Passing to the limit $\epsilon \to 0$ yields (6.5) and the proof of Theorem 6.3 is complete. ∎

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