# Finite Speed for Symmetrizable Hyperbolic Systems

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**Abstract**

We prove that symmetrizable hyperbolic systems have finite speed of propagation. This is done by constructing the solution by the method of finite differences. The estimate for the speed is not sharp. Proving a precise result is an open problem.

## 1 Introduction

Denote by

$$(t,x) \;=\; (t,x_1,\ldots,x_d) \;=\; y \;=\; (y_0,y_1,\ldots,y_d) \;\in\; \mathbb{R}^{1+d}\,,$$

with dual variable

$$(\tau,\xi) \;=\; (\tau,\xi_1,\ldots,\xi_d) \;=\; \eta \;=\; (\eta_0,\eta_1,\ldots,\eta_d)\,.$$

Consider $N\times N$ systems of linear partial differential operators

$$L(y,\partial_y) \;:=\; \partial_t \;+\; \sum_{j=1}^{d} A_j(y)\,\partial_j + B(y) \tag{1}$$

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on $[0,T] \times \mathbb{R}^d$ which satisfy

$$\forall \alpha, \quad \sup_{y \in [0,T]\times\mathbb{R}^d} \left| \partial_y^\alpha \{A_j(y), B(y)\} \right| \ < \ \infty \,, \tag{2}$$

Finite speed for nonlinear hyperbolic problems is usually proved by constructing a linear equation for the difference of two solutions.

**Definition 1** *A homogeneous polynomial $p(\tau, \eta)$ is hyperbolic with timelike variable $t$ iff $p(1, 0, \dots, 0) \neq 0$ and for all $\xi \in \mathbb{R}^d$, the roots $\tau$ of $p(\tau, \xi) = 0$ are all real.*

**Definition 2** *The system (1) is hyperbolic iff for all $y \in [0,T] \times \mathbb{R}^d$,*

$$p(y, \tau, \xi) \ := \ \det \Big( \tau I + \sum_j A_j(y)\, \xi_j \Big)$$

*is a hyperbolic polynomial in $(\tau, \xi)$ with timelike variable $t$.*

**Definition 3** *The system (1) is symmetric hyperbolic (see [4], [9]) iff the matrices $A_j(y)$ are hermitian symmetric.*

**Definition 4** *The system (1) is strictly hyperbolic iff for each $y \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d \setminus 0$ there are $N$ real distinct roots $\tau$ of $p(y, \tau, \xi)$ with a uniform lower bound on the distance between roots for $(y, \xi) \in [0,T] \times \mathbb{R}^d \times \{|\xi| = 1\}$.*

**Definition 5** *The system (1) is symmetrizable hyperbolic iff there is a smooth invertible $N \times N$ matrix valued function $K(y, \xi) \in C^\infty([0,T] \times \mathbb{R}^d \times \{\mathbb{R}^d \setminus 0\})$, homogeneous of degree 0 in $\xi$ so that so that*

$$K(y, \xi) \ \sum_{j=1}^d \Big( A_j(y)\, \xi_j \Big) \ K(y, \xi)^{-1}$$

*is hermitian symmetric and for all $\alpha$*

$$\partial_{y,\xi}^\alpha K \,, \qquad \text{and} \qquad \partial_{y,\xi}^\alpha (K^{-1}) \tag{3}$$

*are uniformly bounded on $[0,T] \times \mathbb{R}^d \times \{|\xi| = 1\}$.*

Symmetric and strictly hyperbolic systems are both symmetrizable. So are systems so that $\sum A_j \xi_j$ is diagonalisable with real eigenvalues of constant multiplicity. Using the algebra of singular integral operators, a.k.a. pseudodifferential operators, Calderón [2] (resp. Yamaguti [16]), treated the Cauchy problem in the strictly hyperbolic (resp. constant multiplicity) setting. The symmetrizable systems were introduced in the early 1960s. They date at least to [9] where it is shown that the methods of Calderón and Yamaguti yield the following result.

**Theorem 1** *For any* $s \in \mathbb{R}$ *and* $g \in H^s(\mathbb{R}^d)$ *there is a unique solution* $u \in C([0,T]\,;\, H^s(\mathbb{R}^d))$ *to the Cauchy problem*

$$L\,u \;=\; 0 \quad \text{on} \quad [0,T]\times\mathbb{R}^d\,, \qquad \text{and} \qquad u\big|_{t=0} \;=\; g\,. \tag{4}$$

The symmetrizable systems have some annoying instabilities. If one changes variables to $(t',x') := (t+\varepsilon\psi(x),x)$ with $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\varepsilon$ small, it is not known if the new system must be symmetrizable in $t',x'$. Strictly hyperbolic, symmetric hyperbolic, and diagonalisable systems with of constant multiplicities remain so in the new variables. Such perturbations of space like hypersurfaces are at the heart of the usual proofs of finite speed. To my knowledge it was not known that symmetrizable systems have finite speed of propagation. Since finite speed is one of the desirable properties of hyperbolic systems, this was a serious gap.

In this paper we prove that there is finite speed using results that have been known for more than thirty years. We construct a solution with finite speed by the method of finite differences.

**Theorem 2** *If* $u$ *is the solution of the Cauchy problem from Theorem 1 and* $g$ *vanishes on an open set* $\omega$, *then the the solution* $u$ *vanishes on the relatively open subset* $\Omega$ *of* $\{t\geq 0\}$,

$$\Omega \;:=\; \Big\{(t,x)\in[0,T]\times\mathbb{R}^d \;:\; c\,t\,\sqrt{d} \;<\; \min_{\underline{x}\notin\omega}\sum_j |x_j-\underline{x}_j|\Big\}\,,$$

*where*

$$c \;:=\; \sup\Big\{\,\sigma\Big(\sum_{j=1}^{\infty} A_j(y)\,\xi_j\Big) \;:\; (t,x,\xi)\;\in\;[0,T]\times\mathbb{R}^d\times\{|\xi|=1\}\Big\}, \tag{5}$$

*and* $\sigma$ *denotes spectral radius.*

**Remarks.** **1.** The result implies that waves propagate with speeds no greater than $c\sqrt{d}$. When $d > 1$, the factor $\sqrt{d}$ renders this estimate imprecise.
**2.** By a Hölmgren type duality argument it follows that the operator $L$ has local uniqueness in the Cauchy problem at hypersurfaces $t = \phi(x)$ that satisfy $|\nabla_x \phi| < (c\sqrt{d})^{-1}$.
**3.** Precise finite speed as in [8], [11] would follow from local uniqueness at spacelike surfaces. Conversely, if one has such a precise finite speed result, then a Hölmgren argument would prove local uniqueness at spacelike hypersurfaces.
**4.** An equivalent definition of symmetrizability is the existence of a smooth strictly positive hermitian symmetric $R(y, \xi)$ homogeneous of degree 0 in $\xi$ so that

$$R(y,\xi)\Big(\sum_j A_j(y)\,\xi_j\Big)$$

is hermitian symmetric for all $(y, \xi)$. Given such a symmetrizer $R$, the matrix $K := R^{1/2}$ serves for Definition 5. Conversely if one knows a $K$, then $R = K^*\,K$ serves for the equivalent definitio.
**5.** Precise finite speed would follow if one knew that symmetrizable systems could be approximated by symmetrizable systems with analytic or even Gevrey $G^s$ coefficients with $s < N/(N-1)$. In such a case, precise finite speed for the approximate equation follows by combining results of [1] and [8] (see [13] for a new proof of Bronshstein's theorem). The result would follow by a passage to the limit.
**6.** The Kreiss matrix theorem [7] implies that for constant coefficients $A_j$ the condition of symmetrizability by a not necessarily smooth $K$ satisfying the bounds (3) only for $\alpha = 0$ is a necessary and sufficient for $e^{i\sum \xi_j A_j)}$ to be an $L^2(\mathbb{R}^d)$ multiplier, that is

$$\sup_{\xi\in\mathbb{R}^d} \|e^{i\sum \xi_j A_j)}\| \ < \ \infty\,.$$

A convenient reference is [12].
**7.** Friedrichs and Lax [5], [6] introduced a more restrictive class of symmetrizable hyperbolic operators which remain symmetrizable for nearby spacelike hypersurfaces. As a result, finite speed can be proved in that case by the standard methods. The definition given here is simpler and more widely employed.

## 2 Friedrichs' scheme.

Friedrichs' scheme is a finite difference approximation to the operator $L$ which is constructed so that in the symmetric case, the stability is proved by an energy identity analogous to the energy identity for the differential equation (see for example [12]). The scheme replaces the differential operators $\partial_j$ by the symmetric difference operators

$$(\delta_j^h u)(t,x) := \frac{u(t,x+h\mathbf{e}_j) - u(t,x-h\mathbf{e}_j)}{2h}\,,$$

where

$$\mathbf{e}_j := (0,\ldots,0,1,0,\ldots,0),$$

with nonvanishing $j^{\rm th}$ component.

The distinctive feature of the scheme is the treatment of the time derivative. It is replaced by

$$\frac{u(t+k,x) - \big(\sum_{|\alpha|=1} u(t,x+h\alpha)\big)/2d}{k}\,.$$

The origin of this is the forward difference operator

$$\frac{u(t+k,x) - u(t,x)}{k}\,.$$

Then the value $u(t,x)$ is replaced by the average of $u$ at the $2d$ points which differ from $t,x$ by translations of distance $h$ along the coordinate axes. The time interval $k$ is taken to be

$$k \;:=\; \lambda h\,. \tag{6}$$

**Definition 6** *The difference operator $L^h$ is defined by*

$$L^h w := \frac{w(t+k,x) - \frac{\left(\sum_{|\alpha|=1} w(t,x+h\alpha)\right)}{2d}}{k} \;+\; \sum_j A_j(y)\,\delta_j^h w \;+\; B(y)\,w\,. \tag{7}$$

To construct an approximate solution, $u^h(t,x)$ of the initial value problem in Theorem 1 proceed as follows.

For $0 \le t < k$ define $u^h(t,x) = g(x)$. For $k \le t < 2k$ determine $u^h$ by solving

$$L^h u^h \;=\; 0\,. \tag{8}$$

Inductively determine $u^h$ for all $nk \le t < (n+1)k$. so that $u^h$ satisfies (8). This determines $u^h$ on a maximal interval $0 \le t < T(k)$ with $T(k) \in ]T-k, T]$.

Define

$$\Gamma_\lambda \ := \ \Big\{(t,x) \ \ t \ge 0\,, \ \text{and} \ \sum_j |x_j| \ \le \ \frac{t}{\lambda}\Big\}$$

Reasoning inductively on the intervals $[0, nk[$ shows that the value of $u^h$ at $(\underline{t}, \underline{x})$ depends only on the values of $g$ on the intersection of $\underline{x} - \Gamma_\lambda$ with $t = 0$.

The values of the exact solution of (4) at $(\underline{t}, \underline{x})$ could (depending on the system) depend on the values of $g$ at all points of the ball $|x - \underline{x}| \le c\,t$.

The Courrant-Friedrichs-Levy (CFL) condition (see [3], [12]) requires that the domain of dependence of the difference scheme must be at least as large as the domain of dependence for the differential equation. For this to hold one must have

$$\lambda\, c \ \le \ \sqrt{d}\,. \tag{9}$$

In order to have the smallest domain of dependence, we take the extreme value

$$\lambda\, c \ := \ \sqrt{d}\,. \tag{10}$$

If $c = 0$ this choice is not possible. However in that case, there are no $x$ derivatives in the partial differential operaotor and Theorem 2 is elementary. In the sequel we suppose that $c > 0$.

With the choice (10), the value of $u^h(t,x)$ depends only on the values of $g$ on $(t,x) - \Gamma_{\sqrt{d}/c}$. Therefore if $g$ vanishes on $\omega$, then $u^h$ vanishes on the set $\Omega$ in Theorem 2.

# 3 Proof of Theorem 2

We now use the fact (see [14],[16]) that for symmetrizable systems the Friedrichs scheme is stable for all $\lambda$ satisfying the CFL condition (9). This nontrivial result relies on the sharp Gårding inequality of Lax-Nirenberg (see [10], [15]).

The stability means that there is a constant $C(s)$ independent of $h$ and $g$ so that

$$\sup_{0 \le t \le T-k} \ \|u^h(t)\|_{H^s(\mathbb{R}^d)} \ \le \ C(s)\, \|g\|_{H^s(\mathbb{R}^d)}\,.$$

It follows that there is a subsequence $u^{h(n)}$ with $h(n) \to 0$ and a $v$ so that

$$u^{h(n)} \ \rightharpoonup \ v \quad \text{weak} * \ \text{in} \quad L^\infty([0,T] \,:\, H^s(\mathbb{R}^d))\,.$$

To show that $v$ is the solution of (4), introduce the transposed difference operator

$$(L^h)^\dagger w := \frac{w(t-k,x) - \frac{\left(\sum_{|\alpha|=1} w(t,x-h\alpha)\right)}{2d}}{k} \ - \ \sum_j \delta_j^h(A_j(y)^\dagger\, w) \ + \ B(y)^\dagger\, w\,.$$

so that for all $f \in L^\infty([0,T]\,;\, H^(\mathbb{R}^d))$, $w \in C_0^\infty([0,T]\times\mathbb{R}^d)$ and $h$ small,

$$\int_0^T \int L^h(f)\ w\ dx\,dt \ = \ \int_0^T \int f\ (L^h)^* w\ dx\,dt\,.$$

In addition,

$$(L^h)^\dagger w \ = \ L^\dagger w + O(h)\,,$$

where

$$L^\dagger w \ := \ -\partial_t w \ - \ \sum_j \partial_j(A_j^\dagger w) \ + \ B^\dagger w\,,$$

is the transposed differential operator.

We next perform a more careful computation when $w \in C_0^\infty(]-\infty,T[\times\mathbb{R}^d)$ need not vanish near $t=0$. The key is the identity

$$\int_0^T f(t+k)\,w(t)\ dt \ = \ \int_0^T f(t)\,w(t-k)\ dt \ - \ \int_{-k}^0 f(t+k)\,w(t)\,dt\,.$$

Using this yields

$$\int_0^T \int L^h(f)\,w\,dx\,dt \ = \ \int_0^T \int f\ (L^h)^\dagger w\,dx\,dt - \frac{1}{k}\int_{-k}^0 \int f(t{+}k,x)\,w(t,x)\,dx\,dt\,.$$

Use this with $f = u^h$ which satisfies $L^h u^h = 0$ and $u^h(t) = g$ for $t \in [0,k[$ to find,

$$0 \ = \ \int_0^T \int u^h\,(L^h)^\dagger w\ dx\,dt \ - \ \frac{1}{k}\int_{-k}^0 \int g(x)\,w(t,x)\ dx\,dt\,.$$

Pass to the limit $h(n) \to 0$ to find,

$$0 \ = \ \int_0^T \int v\ L^\dagger w\ dx\,dt \ - \ \int g(x)\,w(0,x)\ dx\,.$$

This is the weak form of the initial value problem (4). By uniqueness of solutions it follows that $v = u$.

We have shown that $u^h$ vanishes on the open set $\Omega$ of the Theorem 2. Since $u^{h(n)}$ converges to $u$ it follows that $u$ vanishes on $\Omega$. ∎

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