Precise Finite Speed and Uniqueness in the Cauchy Problem for Symmetrizable Hyperbolic Systems

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Key words. Finite speed, domain of influence, symmetrizable hyperbolic system, uniqueness in the Cauchy problem.

Mathematics subject classification numbers. Primary 35L45. Secondary 35S10.

Abstract

Precise finite speed, in the sense of that the domain of influence is a subset of the union of influence curves through the support of the initial data is proved for hyperbolic systems symmetrized by pseudodifferential operators in the spatial variables. From this, uniqueness in the Cauchy problem at spacelike hypersurfaces is derived by a Hölmgren style duality argument. Sharp finite speed is derived from an estimate for propagation in each direction. Propagation in a fixed direction is proved by regularizing the problem in the orthogonal directions. Uniform estimates for the regularized equations is proved using pseudodiffential techniques of Beals-Fefferman type.

§1. Introduction.

In $\mathbb{R}^{1+d}_{t,x}$, consider the system of partial differential equations,

$$0 \;=\; \partial_t u \;+\; \sum_{j=1}^{d} A_j(t,x)\partial_j u \;+\; B(t,x)u \;:=\; L\,u\,, \qquad \partial_j \;:=\; \frac{\partial}{\partial x_j}\,. \tag{1.1}$$

Here

$$u(t,x) \;=\; (u_1(t,x),\ldots,u_N(t,x)), \qquad (t,x) \;\in\; \mathbb{R}^d,$$

is complex N-vector valued and the coefficients A_j, B are smooth $N \times N$ matrix valued functions so that for all α,

$$\partial^{\alpha}_{t,x}A_j\,,\, \partial^{\alpha}_{t,x}B \;\in\; L^{\infty}(\mathbb{R}^{1+d})\,.$$

Introduce the symbol,

$$a(t,x,\xi) \;:=\; \sum_{j=1}^{d} A_j(t,x)\, i\xi_j\,, \tag{1.2}$$

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* Research partially supported by the National Science Foundation under grant NSF DMS 0405899

and *characteristic polynomial*

$$p(t,x,\tau,\xi) \ := \ \det\big(i\,\tau I \ + \ a(t,x,\xi)\big)\,.$$

Definitions. *The system is* **weakly hyperbolic** *when for all* $(t,x,\xi)\in\mathbb{R}^{1+2d}$*, the roots* τ *of* $p(t,x,\tau,\xi)$ *are real. The system is* **symmetric hyperbolic** *when* $i\,a$ *is hermitian symmetric for all* $(t,x,\xi)\in\mathbb{R}^{1+2d}$*. The system is* **strictly hyperbolic** *when for all real* $(t,x,\xi)\in\mathbb{R}^{1+2d}$ *with* $\xi\neq 0$*, the equation* $p=0$ *has has* N *distinct real roots* τ*. The system has* **constant multiplicity** *when* $i\,a(t,x,\xi)$ *is diagonalisable for all real* $(t,x,\xi)\in\mathbb{R}^{1+2d}$ *with real eigenvalues whose multiplicity is independent of* t,x *and* $\xi\neq 0$*. The system is* **symmetrizable***, when there is a smooth hermitian* $N\times N$ *matrix valued function* $r(t,x,\xi)$ *defined on* $\mathbb{R}^{1+d}_{t,x}\times\mathbb{R}^d_\xi$ *homogeneous of degree zero in* ξ *for* $|\xi|\geq 1$ *so that*

i. $\exists C>0, \quad \forall t,x,\xi, \qquad r \ \geq \ CI,$

ii. $\forall\,\alpha,\beta, \qquad \langle\xi\rangle^{|\beta|}\partial^\alpha_{t,x,}\partial^\beta_\xi r \ \in \ L^\infty(\mathbb{R}^{1+d}\times\mathbb{R}^d)\,.$

iii. *The matrix* $r(t,x,\xi)a(t,x,\xi)$ *is hermitian antisymmetric for all* t,x,ξ *with* $|\xi|\geq 1$*.*

For weakly hyperbolic systems, the Cauchy problem is well set in suitable Gevrey spaces provided the coefficients have sufficient Gevrey regularity [B] [ST]. They are usually not well posed in C^∞. The other four classes define Cauchy problems so that for arbitrary s and initial value $f\in H^s(\mathbb{R}^d)$ there is a unique solution $u\in\cap_{j\geq 0}\,C^j(\mathbb{R}\,;\,H^{s-j}(\mathbb{R}^d))$ with $u|_{t=0}=f$. All four are special cases of the fourth.

Convention. *When not otherwise specified,* $(\cdot,\cdot)$ *and* $\|\cdot\|$ *denote the scalar product and norm in* $L^2(\mathbb{R}^d_x)$*.*

The crux in showing that the Cauchy problem is well set for the last four classes above is to prove that there is a function $c(t)$ independent of the intitial data so that for solutions $u\in C(\mathbb{R}\,;\,H^1(\mathbb{R}^d))$ of (1.1), $\|u(t)\|\leq c(t)\,\|u(0)\|$.

The strategy for proving such estimates is to find a smooth family of strictly positive bounded family of self adjoint operators on $L^2(\mathbb{R}^d))$ so that

$$CI\geq R(t)\geq cI>0, \qquad \text{and,}$$

$$\sup_{t,x,\xi\in\mathbb{R}^{1+2d}}\Big\|R\big(\sum_{j=1}^d A_j(t,x)\partial_j u\big) \ + \ \Big(R\big(\sum_{j=1}^d A_j(t,x)\partial_j u\big)\Big)^*\Big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \ := \ K<\infty.$$

In such case one finds that for solutions of (1.1),

$$\partial_t(R(t),u,u)\leq C_1(R(t)u,u),$$

so

$$(R(t)u(t),u(t)) \ \leq \ e^{C_1|t|}\ (R(0)u(0),u(0)).$$

In the symmetric hyperbolic case, one may take $R=I$. The case where $r=r(t,x)$ is multiplication by a matrix valued function was introduced by Friedrichs [F]. When r is not the identity this class is sometimes called symmetrizable. Our principal interest is the case where r depends on ξ. Precise finite speed when the symmetrizer is independent of ξ has several known proofs ([JMR], [R1]). In the constant multiplicity case, including the strictly hyperbolic case, order the distinct real eigenvalues of $i\,a$,

$$\lambda_1(t,x,\xi) \;<\; \lambda_2(t,x,\xi) \;<\; \cdots \;<\; \lambda_n(t,x,\xi)\,.$$

Since the multiplicities do not change, the eigenvalues do not cross, and such an ordering of the smooth eigenvalues is possible. Define $\pi_\mu(t,x,\xi)$ smooth and homogeneous of degree zero to be the spectral projection along the range of $a(t,x,\xi)-\lambda_\mu(t,x,\xi)\,I$ onto its kernel. Then the $R(t)$ can be taken to be classical pseudodifferential operator with principal symbol is equal to

$$\sum_\mu \pi_\mu(t,x,\xi)^*\,\pi_\mu(t,x,\xi)\,.$$

This construction is due to Calderón [Ca] in the strictly hyperbolic case and to Yamaguti [Y] in the constant multiplicity case. The symmetric, strictly, and constant multiplicity systems are all symmetrizable. The earliest appearance of the symmetrizable class that I know of is in [La]. The definition is in my opinion the most natural one leading to pseudodifferential estimates without loss of derivatives.

Remarks. 1. Condition **ii.** implies that r belongs to the classical symbol class $S^0(\mathbb{R}^{1+d}\times\mathbb{R}^d)$. *

2. The Kreiss Matrix Theorem [K] shows that constant coefficient initial value problems which generate a C_0 semigroup on $L^2(\mathbb{R}^d))$ are characterized by the existence of such an $r(\xi)$ without the smoothness. That is, **ii** holds only for $\alpha=\beta=0$.

Though the initial value problem is solvable, many natural questions are not easily settled for symmetrizable systems. For example to merit the name hyperbolic one would like to know that there is finite speed of propagation. The difficulty for systems symmetrized by pseudodifferential operators in x is that the definition is rigidly anchored in the choice of the time variable. For example, if one has such a system and one perturbs the time variable,

$$\tilde t = t + \sum \alpha_j x_j, \quad |\alpha|_{\mathbb{R}^d} < \epsilon, \qquad \tilde x = x,$$

it is not clear whether their is a symmetrizer in the new variables. The results of the present paper show that the initial value problem with initial data given at $\tilde t = 0$ is well posed.

Finite speed of propagation for systems symmetrized by pseudodifferential operators has only recently been established [R2] by constructing solutions as the limit of approximate solutions satisfying finite difference equations. The stabililty of those schemes is proved by nontrivial pseudodifferential techniques [LN], [YN], [V1], [V2]. Variant definitions for

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* A smooth symbol $c(t,x,\xi)$ belongs to $S^m(\mathbb{R}^{1+d}\times\mathbb{R}^d)$ when for all α,β, $\langle\xi\rangle^{\beta-m}\partial^\alpha_{t,x}\partial^\beta_\xi c \in L^\infty(\mathbb{R}^{1+d}\times\mathbb{R}^d)$.

symetrizability are proposed in [FL1], [FL2] which are stable under perturbations of the timelike variable. They have not assumed the classic status of the energy estimate for symmetrizable systems.

In this paper we settle two open problems concerning symmetrizable hyperbolic initial value problems, *uniqueness in the Cauchy problem at spacelike hypersurfaces*, and, *precise finite speed of propagation.*

Definitions. *Suppose that L is weakly hyperbolic. The* **characteristic variety** *of L at t, x is the set of $(\tau, \xi) \in \mathbb{R}^{1+d} \setminus 0$ such that $p(t, x, \tau, \xi) = 0$. The* **forward timelike cone** *of L at (t, x), denoted $\mathcal{T}^+_{t,x}$ is the set of τ, ξ belonging to the connected component of $dt = (1, 0, \ldots, 0)$ in the complement of the characteristic variety at t, x. A smooth embedded hypersurface, M, is* **spacelike** *at m when half of its conormal line at m lies in $\mathcal{T}^+_m$.*

For each (t, x), $\mathcal{T}^+_{t,x}$ is a nonempty open convex cone in (τ, ξ) space. For this and other properties of hyperbolic polynomials see [Co], [Gå], [Ho1], [La].

Theorem 1.1. *If L is symmetrizable hyperbolic and $M \subset \mathbb{R}^{1+d}$ is a smooth embedded hypersurface which is spacelike at $m \in \mathbb{R}^{1+d}$ and $u \in \mathcal{D}'(\mathbb{R}^{1+d})$ satisfies $L u = 0$ on a neighborhood of m and vanishes on one side of M near m, then $u = 0$ on a neighborhood of m in $\mathbb{R}^{1+d}$.*

For each of the more restricted classes, symmetric, strictly, and constant multiplicity, the systems are also of that type for any time variable $\tilde{t} = \tilde{t}(t, x)$ so that $d\tilde{t}$ and dt belong to the same connected component of the noncharacteristic points. Thus for these more restricted systems, one easily proves Theorem 1.1.

By precise finite speed we mean the bound on the support of solutions of $L u = 0$ in Theorem 1.2 which we describe now. For $\xi \in \mathbb{R}^d \setminus 0$, define,

$$\tau_{\max}(t, x, \xi) \ := \ \max \ \big\{ \tau \in \mathbb{R} \ : \ p(t, x, \tau, \xi) = 0 \big\} .$$

As a function of ξ, $\tau_{\max}(t, x, \xi)$ is positively homogeneous of degree one, continuous, and, convex. The set $\mathcal{T}^+_{t,x}$ has equation,

$$\mathcal{T}^+_{t,x} \ = \ \big\{ (\tau, \xi) \ : \ \tau > \tau_{\max}(t, x, \xi) \big\} .$$

Definitions. *The* **forward propagation cone** *for L at (t, x) is the closed convex dual cone,*

$$\Gamma^+_{t,x} \ := \ \Big\{ (T, X) \ : \ \forall \, \tau, \xi \in \mathcal{T}^+_{t,x}, \quad T\tau + X\xi \geq 0 \Big\} . \tag{1.3}$$

An **influence curve** *for L is a lischitzean curve $\gamma(t) = (t, x(t))$ defined for t in a nontrivial closed interval and so that the tangent vector to γ lies in $\Gamma^+_{\gamma(t)}$ for Lebesgue almost all t.*

Since $dt = (1, 0, \ldots, 0) \in \mathcal{T}^+_{t,x}$ it follows from the definition that $T > 0$ in $\Gamma^+_{t,x}$. Thus, the ≥ 0 in (1.3) is hardest to satisfy when τ is small. The infinum of the values τ is

$-\tau_{\max}(t,x,-\xi)$. Therefore, the propagation cone has equation

$$\Gamma^+_{t,x} \;=\; \Big\{(T,X)\;:\;T\ge 0 \quad\text{and}\quad \forall\,\xi,\;\; -T\,\tau_{\max}(t,x,-\xi)+X.\xi\ge 0\Big\}\,.$$

By homogeniety, it suffices to consider $|\xi|=1$. Following Leray [Le] (see also [JMR]), define emissions as follows.

Definition. *If $K\subset\mathbb{R}^{1+d}$ is a closed set, the* **forward emission of** *K denoted $\mathcal{E}^+(K)$ is the union of forward influence curves $\gamma:[\underline{t},\infty[\to\mathbb{R}^{1+d}$ with $\gamma(\underline{t})\in K$. The backward emission is denoted $\mathcal{E}^-$.*

The emissions are closed subsets of $\mathbb{R}^{1+d}$.

Theorem 1.2. *If L is symmetrizable hyperbolic, $s\in\mathbb{R}$, $u\in C(\mathbb{R}:H^s(\mathbb{R}^d))$ satisfies $Lu=0$ in the sense of distributions, then,* $\operatorname{supp}u\,\cap\,\{t\ge 0\}\subset\mathcal{E}^+(\operatorname{supp}u(0))$.

Remark. Duhamel's representation shows that one has the same conclusion provided that $Lu\in L^1_{loc}([0,\infty[\,;H^s(\mathbb{R}^d))$ has support in $\mathcal{E}^+(\operatorname{supp}u(0))$.

Uniqueness in the Cauchy problem at spacelike hypersurfaces implies precise finite speed (see [JMR], [R1]). Thus for symmetric, strictly and constant multiplicity systems precise finite speed is known. For symetrizable systems, we reverse the logic proving Theorem 1.2 and then deriving Theorem 1.1 by a duality argument of Hölmgren type.

Our strategy for proving precise finite speed is to estimate the propagation in a single direction, say x_1, by regularizing the equation in the directions $x_2,\dots,x_d$. For the regularized equation, the propagation in x_1 is elementary. The original equation is recovered by removing the regularization. The crux is to prove uniform energy estimates for the regularized problems. This stability step uses pseudodifferential techniques beyond the classical calculus.

A main step is Proposition 2.1 concerning propagation in x_1. The proof of that result occupies the next three sections. Theorem 1.2 is derived from Proposition 2.1 in §5 by a global geometric argument. The geometric arguments in [Le], [R1], and this one form a sequence reducing the input required to deduce precise finite speed. The present result derives sharp finite speed whenever the conclusion of Proposition 1.2 is available. Theorem 1.1 is derived from Theorem 1.2 in §6.

Acknowledgment. When I was struggling to prove the central estimate, Proposition 4.2, I was aided by conversations with David Dos Santos, Patrick Gérard, and Guy Métivier. I thank them warmly.

§2. Propagation in x_1.

Proposition 2.1. *Suppose that L is symmetrizable, and, for all t,x the spectrum of $A_1(t,x)$ belongs to the interval $[\lambda_{\min},\lambda_{\max}]$. If $s\in\mathbb{R}$, and $u\in C\big(\mathbb{R}\,;H^s(\mathbb{R}^d)\big)$ satisfies $Lu=0$ and*

$$\operatorname{supp}u(0)\;\subset\;\{-\infty\le a\le x_1\le b\le\infty\}\,,$$

then for $t \geq 0$,

$$\operatorname{supp} u \ \subset \Big\{ a + \lambda_{\min} t \ \leq \ x_1 \ \leq \ b + \lambda_{\max} t \Big\} . \tag{2.1}$$

Remarks. 1. For $t \leq 0$ the support is in $a + \lambda_{\max} t \leq x_1 \leq b + \lambda_{\min} t$. This follows from the $t \geq 0$ result upon reversing the time.

2. The derivation, in §5, of Theorem 1.2 from Proposition 2.1 is essentially geometric.

To prove Proposition 2.1, the derivatives with respect to $x_2, \ldots, x_d$ are regularized leaving a differential operator in t, x_1 whose propagation in x_1 is easily analysed. The proposition follows upon passing to the limit. The difficulty is to prove *uniform bounds* for solutions of the regularized operators. The regularized equations are symmetrized by a *nonclassical* pseudodifferential operator.

Use the notations

$$\xi = (\xi_1, \xi'), \qquad \xi' \ := \ (\xi_2, \ldots, \xi_d), \qquad \big\langle \epsilon \xi' \big\rangle \ := \ \big(1 + \epsilon^2 |\xi'|^2 \big)^{1/2} .$$

Introduce the tangentially regularized symbols,

$$a_\epsilon(t, x, \xi) \ := \ a\Big(t, x, \xi_1, \frac{\xi'}{\langle \epsilon \xi' \rangle} \Big) \ = \ A_1 \, i\xi_1 + \sum_{j=2}^{d} A_j \, \frac{i\xi_j}{\langle \epsilon \xi' \rangle} .$$

For $\epsilon = 0$ this is equal to $a(t, x, \xi)$. It defines a smooth family of symbols for the compact parameter set $0 \leq \epsilon \leq 1$.

We use the Weyl calculus of pseudodifferential operators as presented in [H, v.IIII, chap. 18]. The metrics and weights depend on the parameter $\epsilon \in [0, 1]$ and we verify that the continuity and temperance hypotheses are satisfied with constants uniform in ϵ. The uniformity allows us to conclude that when the calculus is used to derive estimates, they are uniform in ϵ.

A superscript w denotes the Weyl quantization. The Weyl operator, $a^w(t, x, D)$, with symbol $a(t, x, \xi)$ is equal to

$$a^w(t, x, D) \ = \ \sum_j \Big(A_j(x) \partial_j - \frac{1}{2} (\partial_j A)(x) \Big) .$$

Therefore,

$$L\, u \ = \ \partial_t u \ + \ a^w(t, x, D) u \ + \ \tilde{B}(t, x) u \, , \qquad \tilde{B} \ := \ B + \frac{1}{2} \sum_{j=1}^{d} \frac{\partial A_j}{\partial x_j} . \tag{2.2}$$

Approximate solutions are defined as solutions of the regularized system

$$\partial_t u^\epsilon \ + \ a^w_\epsilon(t, x, D) u^\epsilon \ + \ \tilde{B}(t, x) u^\epsilon \ = \ 0 \, , \qquad u^\epsilon|_{t=0} = u|_{t=0} .$$

The key step is to derive estimates for u^ϵ independent of ϵ. From such estimates one immediately concludes that $u^\epsilon \to u$. The bound on the support of u follows when one proves that the u^ϵ are supported in the set in the right hand side of (2.1). The technical difficulty is that a_ϵ is not a classical symbol of order 1. The derivatives $\partial_\xi^\beta a$ do not decay like $\langle \xi \rangle^{1-|\beta|}$. The symbols behave more and more like this classical behavior as $\epsilon \to 0$. Our strategy is to use the Weyl calculus with weights and metrics depending on $\epsilon \in [0,1]$. The hypotheses of the Weyl calculus are satisfied *uniformly in* ϵ . The uniformity is verified in the next section. It is applied in §4 to prove uniform bounds.

§3. The parameters of the Weyl calculus.

Suppressing the (t,x) dependence,

$$\frac{\partial a_\epsilon}{\partial \xi_1} \;=\; \frac{\partial a}{\partial \xi_1}\Big(\xi_1, \frac{\xi'}{\langle \epsilon \xi' \rangle}\Big), \qquad \frac{\partial a_\epsilon}{\partial \xi'} \;=\; \frac{\partial a}{\partial \xi'}\Big(\xi_1, \frac{\xi'}{\langle \epsilon \xi' \rangle}\Big)\, \frac{\partial}{\partial \xi'} \frac{\xi'}{\langle \epsilon \xi' \rangle}\,.$$

The last factor is uniformly bounded,

$$\sup_{\xi, \epsilon} \left| \frac{\partial}{\partial \xi'} \frac{\xi'}{\langle \epsilon \xi' \rangle} \right| \;<\; \infty\,.$$

As $\partial_\xi a = O(1/|\xi|)$ for $|\xi| \geq 1$, one has

$$\big|\partial_\xi a_\epsilon\big| \;\leq\; \frac{C}{\big\langle \xi_1\,,\, \xi'/\langle \epsilon \xi' \rangle \big\rangle}\,.$$

Continuing in this way shows that with constants independent of $\epsilon \in [0,1]$,

$$\sup_{t,x} \big| \partial_{t,x}^\beta \partial_\xi^\alpha a_\epsilon(x,\xi) \big| \;\leq\; \frac{C(\alpha,\beta)}{\big\langle \xi_1\,,\, \xi'/\langle \epsilon \xi' \rangle \big\rangle^{|\alpha|-1}} \;<\; \infty\,.$$

For the Weyl calculus, introduce the familly of metrics g_ϵ and weights m_ϵ,

$$g_{\epsilon,(x,\xi)} \;:=\; dx^2 \;+\; \frac{d\xi^2}{\langle \xi_1, \xi'/\langle \epsilon \xi' \rangle \rangle^2}, \qquad m_\epsilon \;:=\; \big\langle \xi_1\,,\, \xi'/\langle \epsilon \xi' \rangle \big\rangle\,.$$

The formula for $g_{\epsilon,(x,\xi)}$ means that,

$$g_{\epsilon,(x,\xi)}(y,\eta) \;=\; |y|^2 \;+\; \frac{|\eta|^2}{\big\langle \xi_1\,,\, \xi'/\langle \epsilon \xi' \rangle \big\rangle^2}\,.$$

An orthonormal set of vector fields at x, ξ is,

$$\frac{\partial}{\partial y_j}, \qquad \big\langle \xi_1\,,\, \xi'/\langle \epsilon \xi' \rangle \big\rangle\, \frac{\partial}{\partial \eta_j}\,.$$

Recall ([H, Def. 4.9.4.3]) that $b(x,\xi) \in S(m_\epsilon, g_\epsilon)$ when

$$\forall k \in \mathbb{N}, \qquad \sup_{x,\xi} \frac{|b|_k^{g_\epsilon}(x,\xi)}{m_\epsilon(x,\xi)} < \infty ,$$

where $|b|_k^{g_\epsilon}(x,\xi)$ denotes the norm of the k-multilinear k^{th} derivative of $b(x,\xi)$ with respect to x,ξ. The important thing is that the the norm of the derivatives at x,ξ are taken with respect to the metric $g_{\epsilon,(x,\xi)}$.

Thus, $a_\epsilon \in S(m_\epsilon, g_\epsilon)$, with estimates uniform in t,ϵ. The same is true of $\partial_t^j a_\epsilon$ for each j.

Definition. *A family of symbols $b_\epsilon(t) \in S(m_\epsilon, g_\epsilon)$ is said to be bounded if and only if*

$$\forall j \in \mathbb{N},\ k \in \mathbb{N}, \qquad \sup_{\epsilon\in[0,1],t,x,\xi} \frac{|\partial_t^j b_\epsilon(t)|_k^{g_\epsilon}(x,\xi)}{m_\epsilon(x,\xi)} < \infty .$$

Examples. i. The family of symbols a_ϵ is bounded in $S(m_\epsilon\,,\,g_\epsilon)$. **ii.** The family of symmetrizer symbols

$$r_\epsilon\Big(t, x, \xi_1, \frac{\xi'}{\langle \xi_1, \epsilon\xi'\rangle}\Big)$$

is bounded in $S(1, g_\epsilon)$. (The weight 1 corresponds to L^2 bounded operators.)

For $\epsilon = 0$, the metric and and weight reduce to,

$$g_{0,(x,\xi)} \;=\; dx^2 \;+\; \frac{d\xi^2}{\langle\xi\rangle^2}, \qquad m_0(x,\xi) \;=\; \langle\xi\rangle ,$$

and the symbol class $S(m_0, g_0)$ is the classical space $S^1(\mathbb{R}^{1+d} \times \mathbb{R}^d)$.

The weight m_ϵ is increasing in ϵ. For ϵ fixed the weights are bounded as $\xi' \to \infty$ with ξ_1 bounded. It's maximum values is $O(1/\epsilon)$. As ϵ decreases, the symbols a_ϵ increase. becomes closer and closer to first order in ξ'.

The metric, $dx^2 + d\xi^2/m_\epsilon^2$, is decreasing in ϵ. This encodes the fact that the derivatives of a_ϵ have improving decay properties as ϵ decreases. As ϵ decreases, the derivatives are better controled. The stability result in the next section is proved relying on a precise harmony between these two countervailing effects.

The first step is to compute the metric g_ϵ^σ which is derived from g_ϵ. Recall one of the characterizations. Fix x,ξ. Denote by W' the space of y,η and W its dual with coordinates $\hat y, \hat\eta$ and duality,

$$\Big\langle (y,\eta)\,,\,(\hat y,\hat\eta)\Big\rangle \;=\; y\hat y + \eta\hat\eta \;=\; \sum_j \Big(y_j\hat y_j + \eta_j\hat\eta_j\Big) .$$

The symplectic form

$$\sigma\big((y,\eta)\,,\,(z,\zeta)\big) \;:=\; z\eta \;-\; y\zeta ,$$

is a real nondegenerate quadratic form on $W' \oplus W'$. It induces the isomorphism

$$W \ni (\hat{z}, \hat{\zeta}) \quad \mapsto \quad A(\hat{z}, \hat{\zeta}) = (\hat{\zeta}\,,\, -\hat{z}) \in W'\,,$$

so that

$$\sigma(A(\hat{y}, \hat{\eta}), (x, \xi)) = \Big\langle (y, \eta)\ ,\ (x, \xi) \Big\rangle \ =\ y\hat{y}\ +\ \eta\hat{\eta}\,.$$

A metric γ on W is given by transporting g by the map A,

$$\gamma(\hat{y}, \hat{\eta})\ :=\ g_{\epsilon,(x,\xi)}\big(A(\hat{y}, \hat{\eta})\big)\ =\ g_{\epsilon,(x,\xi)}\big(\hat{\eta}, -\hat{y}\big)\ =\ |\hat{\eta}|^2\ +\ \frac{|\hat{y}|^2}{\langle \hat{\xi}_1\,,\, \hat{\xi}'/(\epsilon\hat{\xi}')\rangle^2}\,.$$

By duality, this induces a form on W' ([H, eqn. 4.9.4.11]),

$$g^{\sigma}_{\epsilon,(x,\xi)}(y, \eta)\ :=\ \max_{\gamma(\hat{y},\hat{\eta}))=1} \Big\langle (y, \eta)\ ,\ (\hat{y}, \hat{\eta}) \Big\rangle^2\,.$$

Equivalently,

$$g^{\sigma}_{\epsilon,(x,\xi)}(y, \eta)\ :=\ \max\Big\{ \big(y.\hat{y} + \eta.\hat{\eta}\big)^2\ :\ |\hat{\eta}|^2\ +\ \frac{|\hat{y}|^2}{\big\langle \xi_1\,,\, \xi'/\langle\epsilon\xi'\rangle\big\rangle^2}\ =\ 1 \Big\}\,.$$

This minimization problem yields

$$g^{\sigma}_{\epsilon,(x,\xi)}(y, \eta)\ =\ \big\langle \xi_1\,,\, \xi'/\langle\epsilon\xi'\rangle\big\rangle^2 |y|^2\ +\ |\eta|^2\ =\ \big\langle \xi_1\,,\, \xi'/\langle\epsilon\xi'\rangle\big\rangle^2\ g_{\epsilon,(x,\xi)}(y, \eta)\,. \tag{3.1}$$

From (3.1) one finds the formula for the regulariizing weigth h_ϵ,

$$h_\epsilon(x, \xi)\ :=\ \Bigg(\sup_{(y,\eta)\neq 0} \frac{g_{\epsilon(x,\xi)}(y, \eta)}{g^{\sigma}_{\epsilon,(x,\xi)}(y, \eta)} \Bigg)^{1/2}\ =\ \big\langle \xi_1\,,\, \xi'/\langle\epsilon\xi'\rangle\big\rangle^{-1}\ =\ \frac{1}{m_\epsilon(x, \xi)}\,. \tag{3.2}$$

With these formulas in hand, we verify the hypotheses of the Weyl calculus.

Lemma 3.1. i. *The metrics g_ϵ are uniformly of slow variation ([H, Definition 4.9.4.1]),*

$$\exists N > 0, c > 0,\quad \forall \epsilon < 1,\quad g_{\epsilon,(x,\xi)}(y, \eta)\ \leq\ \frac{1}{N^2}\quad \Longrightarrow\quad g_{\epsilon,(x+y,\xi+\eta)}(z, \zeta)\ \leq\ C\, g_{\epsilon,(x,\xi)}(z, \zeta).$$

ii. *The weights m_ϵ is g_ϵ continuous uniformly in ϵ ([H, Definition 4.9.4.3]),*

$$\exists\, 0 < c < C,\ \forall \epsilon < 1,\ g_{\epsilon,(x,\xi)}(y, \eta) \leq c\ \Longrightarrow\ m_\epsilon(x, \xi)/C\ \leq\ m_\epsilon(x{+}y, \xi{+}\eta)\ \leq\ C m_\epsilon(x, \xi)\,. \tag{3.3}$$

Therefore, $m_\epsilon^{1/2}$ is also uniformly g_ϵ continuous.

iii. *The metric g_ϵ is σ-temperate, uniformly in ϵ. In fact,*

$$g_{\epsilon,(y,\eta)}(z,\zeta) \ \le \ g_{\epsilon,(x,\xi)}(z,\zeta)\Big(1+g^\sigma_{\epsilon,(y,\eta)}(x-y,\xi-\eta)\Big) \tag{3.4}$$

verifying [H, (4.9.5.11′)] with $C=N=1$.
iv *The weight m_ϵ is σ, g_ϵ temperate, uniformly in ϵ. In fact,*

$$m_{\epsilon,(y,\eta)} \ \le \ m_{\epsilon,(x,\xi)}\Big(1+g^\sigma_{\epsilon,(y,\eta)}(x-y,\xi-\eta)\Big)$$

verifying [H, (4.9.5.12)] with $C=N=1$. Therefore $m_\epsilon^{1/2}$ is uniformly σ, g_ϵ temperate.

Proof. i. First note that

$$g_{\epsilon,(x,\xi)}(y,\eta) \ \le \ \frac{1}{N^2} \quad\Longrightarrow\quad \frac{|\eta|^2}{1+\xi_1^2+\langle \xi'/\epsilon\xi'\rangle^2} \le \frac{1}{N^2} \quad\Longrightarrow\quad |\eta| \ \le \ \frac{1}{N}\Big|\xi_1\,,\,\xi'/\langle\epsilon\xi'\rangle\Big|\,. \tag{3.5}$$

Slow variation is implied by the existence of a $N, c>0$ so that for all such t,x,ξ,η

$$\Big|\xi_1+\eta_1\,,\,\frac{(\xi'+\eta')}{\langle\epsilon(\xi'+\eta')\rangle}\Big| \ \ge \ c\Big|\xi_1\,,\,\frac{\xi'}{\langle\epsilon\xi'\rangle}\Big|\,. \tag{3.6}$$

Take $N=100$. Split the proof of (3.6) into two cases depending on whether $|\xi_1|\le|\xi|/10$ or not.
When $\xi_1>|\xi|/10$, use (3.5) to show that

$$\eta_1 < \frac{|\xi|}{N} < \frac{10|\xi_1|}{N} = \frac{|\xi_1|}{10}\,. \tag{3.7}$$

Then

$$|\xi_1+\eta_1| \ge \frac{9|\xi_1|}{10} \ge \frac{9|\xi|}{100} \ge \frac{9}{100}\Big|\xi_1\,,\,\frac{\xi'}{\langle\epsilon\xi'\rangle}\Big|\,,$$

implying (3.6).
On the other hand, when $|\xi_1|\le|\xi|/10$, one has

$$|\xi'| \ge \frac{9}{10}|\xi| \ge 9|\xi_1|\,.$$

Use (3.5) to find,

$$|\eta'| \le \frac{1}{N}\Big(\Big(\frac{|\xi'|}{9}\Big)^2+\Big(\frac{|\xi'|}{\langle\epsilon\xi'\rangle}\Big)^2\Big)^{1/2} \le \frac{2|\xi'|}{N} = \frac{|\xi'|}{500}\,.$$

Then,

$$\frac{501}{500}|\xi'| \ge |\xi'+\eta'| \ge \frac{499}{500}|\xi'|$$

and (3.6) follows.

ii. It suffices to treat m_ϵ. For that, it suffices to verify the second, inequality in (3.3), since one can then write $(x,\xi) = (x+y, \xi+\eta) + (-y, -\eta)$ to derive the first.

The proof of slow variation showed that there is a $c > 0$ so that $g_{\epsilon,(x,\xi)} \leq 1/(4.100)^2$ implies (3.6). This implies the desired second inequality in (3.3).

iii. Compute

$$g_{\epsilon,(x,\xi)}(z,\zeta) \;=\; |z|^2 + \frac{|\zeta|^2}{\big\langle \xi_1\,,\, \xi'/\langle \epsilon\eta'\rangle \big\rangle^2},$$

and,

$$g_{\epsilon,(y,\eta)}(z,\zeta) \;=\; |z|^2 + \frac{|\zeta|^2}{\big\langle \eta_1\,,\, \eta'/\langle \epsilon\eta'\rangle \big\rangle^2} \;=\; |z|^2 + \frac{|\zeta|^2}{\big\langle \xi_1\,,\, \xi'/\langle \epsilon\xi'\rangle \big\rangle^2}\;\frac{\big\langle \xi_1\,,\, \xi'/\langle \epsilon\xi'\rangle \big\rangle^2}{\big\langle \eta_1\,,\, \eta'/\langle \epsilon\eta'\rangle \big\rangle^2}$$

Peetre's inequality is the first of the sequence,

$$\frac{\big\langle \xi_1\,,\, \xi'/\langle \epsilon\xi'\rangle \big\rangle^2}{\big\langle \eta_1\,,\, \eta'/\langle \epsilon\eta'\rangle \big\rangle^2} \leq 1 + \Big|\Big(\xi_1 - \eta_1\,,\, \frac{\xi'}{\langle \epsilon\xi'\rangle} - \frac{\eta'}{\langle \epsilon\eta'\rangle}\Big)\Big|^2 \leq 1 + |\xi-\eta|^2 \leq \big(1 + g^\sigma_{\epsilon,(y,\eta)})(x-y, \xi-\eta)\big). \tag{3.8}$$

Combining the last three lines yields (3.4).

iv. Estimate (3.8) establishes the desired estimate for m_ϵ. For $m_\epsilon^{1/2}$ it follows that

$$m^{1/2}_{\epsilon,(y,\eta)} \;\leq\; m^{1/2}_{\epsilon,(x,\xi)}\Big(1 + g^\sigma_{\epsilon,(y,\eta)}(x-y, \xi-\eta)\Big)^{1/2}$$

verifying [H, (4.9.5.12)] with $C = 1$ and $N = 1/2$. ▮

§4. Estimate for the propagation in x_1.

We use three facts about the Weyl calculus when the weights and metrics satisfy the uniform continuity and temperance estimates as in the Lemma 3.1.

W.1. *The adjoint with respect to the* $L^2(\mathbb{R}^d)$ *scalar product of* $b^w(x,D)$ *is equal the Weyl operator with symbol* $b^*(x,\xi)$.

W.2. *If* b_ϵ *and* c_ϵ *are bounded in* $S(m_{\epsilon,1}, g_\epsilon)$ *and* $S(m_{\epsilon,2}, g_\epsilon)$ *respectively then the product* $b^w_\epsilon(x,D)c^w_\epsilon(x,D)$ *is a Weyl operator with symbol bounded in* $S(m_{\epsilon,1}m_{\epsilon,2}\,, g_\epsilon)$. *And, the operators*

$$c^w_\epsilon(x,D)b^w_\epsilon(x,D) \;-\; (c_\epsilon b_\epsilon)^w(x,D)$$

have Weyl symbols bounded in $S(h_\epsilon m_{\epsilon,1}m_{\epsilon,2}\,, g_\epsilon)$.

W.3. *If* $b_\epsilon(x,\xi)$ *is bounded in* $S(1, g_\epsilon)$, *then* $b^w_\epsilon(x,D)$ *is bounded in* $\mathcal{L}(L^2(\mathbb{R}^d))$.

Proofs. The first is immediate from the definition ([H2, pg. 151].

The second is the leading symbol part of [H, Theorem 18.5.4].

The third is a consequence of the proof of [H2, Theorem 18.6.3]. The reader can verify that the bound proved in Theorem 18.6.3 depends only on the constants in the slow variation, continuity and termperance estimates together with the $S(1, g_\epsilon)$ bounds on the symbol. ▮

Example. Consider **W.2** when one of the operators has symbol in $S(m_\epsilon, g_\epsilon)$ and the other in $S(1, g_\epsilon)$. The first operator is not as singular as an operator of order 1 since m_ϵ is smaller than $\langle \xi \rangle$ most particularly when $\xi' \to \infty$. The error term when one uses the product of the leading symbols belongs to $S(h_\epsilon m_\epsilon, g_\epsilon) = S(1, g_\epsilon)$. The gain $h_\epsilon = 1/m_\epsilon$ is less than one order but is exactly what is needed so that the error has weight 1 and will therefore be a bounded operator.

The next result is elementary. For ϵ fixed it considers a_ϵ^w as a bounded perturbation of $A_1 \partial_1$. The bounded terms grow like $1/\epsilon$ which leads to very crude estimates as $\epsilon \to 0$.

Lemma 4.1. i. *For any* $f \in H^\infty(\mathbb{R}^d) := \cap_s H^s(\mathbb{R}^d)$, *there is a unique solution*

$$u^\epsilon \in \cap_s \ C^s(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$$

to the initial value problem

$$\partial_t u^\epsilon + a_\epsilon^w(t,x,D) u^\epsilon + \tilde{B}(t,x) u^\epsilon \ = \ 0\,,$$

$$u^\epsilon \big|_{t=0} \ = \ f\,,$$

with $\tilde{B}$ *defined in (2.2). There is a constant* c *independent of* t, ϵ, f, *so that*

$$\sup_{t \in \mathbb{R}} \ e^{-c|t|/\epsilon} \, \|u^\epsilon(t)\|_{L^2(\mathbb{R}^d)} \ \le \ c\, \|f\|_{L^2(\mathbb{R}^d)}\,. \tag{4.1}$$

ii. *If the spectrum of* A_1 *belongs to* $[\lambda_{\min}, \lambda_{\max}]$ *for all* (t,x) *and* $\operatorname{supp} f \ \subset \{a \le x_1 \le b\}$ *then,*

$$\operatorname{supp} u^\epsilon \ \cap \ \{t \ge 0\} \ \ \subset \ \Big\{ a + \lambda_{\min}\, t \ \le \ x_1 \ \le \ b + \lambda_{\max}\, t \Big\}\,. \tag{4.2}$$

Proof. i. Write,

$$\Big(A_\epsilon^w + \tilde{B} \Big) \ - \ A_1(t,x)\partial_x \ = \ c_\epsilon^w(t,x,D'),$$

with

$$c_\epsilon(t,x,\xi) \ := \ \sum_{j=2}^d A_j(t,x)\, \frac{i\xi_j}{\langle \epsilon \xi' \rangle}.$$

This symbol defines a classical pseudodifferrential operator in x', D' depending smoothly on t, x_1. In fact

$$\epsilon\, c_\epsilon(t,x,\xi) \ \text{ is bounded in } \ S^1(\mathbb{R}^{d+1}_{t,x} \times \mathbb{R}^{d-1}_{\xi'}) \qquad \text{(hence in } S(1, g_\epsilon))\,.$$

Thus,

$$\forall s, \quad \exists c_s, \quad \forall t, \qquad \|c^w_\epsilon(t,x,D')\|_{\mathcal{L}(H^s(\mathbb{R}^d_x))} \;\le\; \frac{c(s)}{\epsilon}\,. \tag{4.3}$$

In addition, c^w_ϵ is local in x_1. That is, for any t,

$$\omega \subset \mathbb{R} \ \text{ and } \ \operatorname{supp} k \;\subset\; \omega \times \mathbb{R}^{d-1}_{x'} \quad\Longrightarrow\quad \operatorname{supp}\, c^w_\epsilon(t,x,D')k \;\subset\; \omega \times \mathbb{R}^{d-1}_{x'}\,.$$

Write the differential equation defining u^ϵ as

$$\mathbf{r}\partial_t u^\epsilon + \mathbf{r}A_1(t,x)\partial_1 u^\epsilon + \mathbf{r}(t,x)\, c^w_\epsilon(t,x,D)u^\epsilon \;=\; 0, \quad \mathbf{r}(t,x) \;:=\; r\big(t,x,(1,0,\dots,0)\big), \tag{4.4}$$

where $r(t,x,\xi)$ is the symmetrizer. Since $\mathbf{r}$ is strictly positive and $\mathbf{r}A_1$ is hermitian symmetric, the operator

$$G := \mathbf{r}\partial_t + \mathbf{r}A_1\partial_1,$$

is symmetric hyperbolic. Introduce the growth matrix

$$Z(t,x) \;:=\; \partial_t\mathbf{r} + \partial_1(\mathbf{r}A_1).$$

The differential energy identity for G is,

$$\partial_t\langle \mathbf{r}v,v\rangle \;+\; \partial_1\langle \mathbf{r}A_1\, v,v\rangle \;=\; 2\operatorname{Re}\langle Gv,v\rangle \;+\; \langle Zv,v\rangle,$$

The Cauchy problem for (4.1) is solved by Picard iteration. The first approximatiion u^ϵ_1 is the solution of

$$\Big(\mathbf{r}\partial_t + \mathbf{r}A_1\partial_1\Big)u^\epsilon_1 \;=\; 0, \qquad u^\epsilon_1\big|_{t=0} \;=\; f\,.$$

This first approximation is independent of ϵ. For $\nu > 1$, approximations are defined by

$$\Big(\mathbf{r}\partial_t + \mathbf{r}A_1\partial_1\Big)u^\epsilon_\nu \;=\; -\mathbf{r}(t,x)\, c^w_\epsilon(t,x,D')\, u^\epsilon_{\nu-1}, \qquad u^\epsilon_\nu\big|_{t=0} \;=\; f\,,$$

Integrating the differential energy law for $v = u^\epsilon_1$ over $\mathbb{R}^d$ yields

$$\partial_t(\mathbf{r}u^\epsilon_1, u^\epsilon_1) \;=\; (Z(t,x)u^\epsilon_1, u^\epsilon_1) \;\le\; c\,(\mathbf{r}u^\epsilon_1\,,\, u^\epsilon_1).$$

In particular, there is a constant c independent of $\epsilon, t \ge 0$ so that

$$\|u^\epsilon_1\| \;\le\; c\,e^{ct}\,\|f\|\,.$$

Define $u^\epsilon_0 := 0$. For $\nu \ge 2$, use the differential energy identity for $v := u^\epsilon_\nu - u^\epsilon_{\nu-1}$. Estimate

$$\big\|(G+Z)\big(u^\epsilon_\nu - u^\epsilon_{\nu-1}\big)\big\| \;=\; \big\|\big(-\mathbf{r}\,c^w_\epsilon + Z\big)(u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2}\big)\big\| \;\le\; \frac{c}{\epsilon}\,\|u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2}\|\,.$$

Integrating the differential energy identity over $\mathbb{R}^d_x$ yields

$$\partial_t\big(\mathbf{r}(u^\epsilon_\nu - u^\epsilon_{\nu-1})\,,\,u^\epsilon_\nu - u^\epsilon_{\nu-1}\big) \;\le\; \frac{c}{\epsilon}\,\big(\mathbf{r}(u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2})\,,\,u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2}\big)\,.$$

Integrate to find

$$\|(u^\epsilon_\nu - u^\epsilon_{\nu-1})(t)\| \;\le\; \frac{c}{\epsilon}\int_0^t \|(u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2})(s)\|\;ds \;:=\; \frac{c}{\epsilon}\,I(\|(u^\epsilon_{\nu-1} - u^\epsilon_{\nu-2})(s)\|\,,$$

involving the integration operator I,

$$(I\,g)(t) \;:=\; \int_0^t g(s)\;ds\,.$$

Iterating yields

$$\big\|(u^\epsilon_\nu - u^\epsilon_{\nu-1})(t)\big\| \;\le\; \Big(\frac{c}{\epsilon}\Big)^{\nu-1}\; I^{\nu-1}(\|u^\epsilon_1(s)\|)\,.$$

Use the estimate

$$I^\nu(g)(t) \;\le\; \frac{1}{(\nu-1)!}\;\max_{0\le s\le t}|g(s)|\,,$$

to find,

$$\big\|(u^\epsilon_\nu - u^\epsilon_{\nu-1})(t)\big\| \;\le\; \Big(\frac{c}{\epsilon}\Big)^{\nu-1}\;\frac{1}{(\nu-2)!}\;c\,e^{ct}\,\|f\|\,.$$

This implies that as $\nu \to \infty$, u^ϵ_v converges to a solution $u \in L^\infty_{\rm loc}(\mathbb{R}\,;\,L^2(\mathbb{R}^d))$ satisfying the estimate in **i.**

Estimates for the derivatives of the u^ϵ_ν can be proved by differentiating the equation u^ϵ_ν and reasoning as above. The details are left to the reader.

Uniqueness is proved by using the energy identity for $v = u^\epsilon - w^\epsilon$, the difference of two solutions. Reasoning as above yields

$$\partial_t(\mathbf{r}v, v) \;\le\; \frac{c}{\epsilon}\,\big(\mathbf{r}v\,,\,v\big)\,.$$

As the initial value of v vanishes, it follows that $v = 0$.

ii. We prove that for $t \ge 0$, u^ϵ vanishes for $x \le a + \lambda_{\rm min}t$. The proof that u vanishes for $x_1 \ge b + \lambda_{\rm max}t$ is analogous.

Consider the domain $\Omega := \{0 \le t \le \underline{t},\ \ x_1 \le a + \lambda_{\rm min}t\}$. Denote by $\Gamma := \{0 \le t \le \underline{t},\ x_1 = a + \lambda_{\rm min}t\}$ the lateral boundary. Integrate the diffential energy identity applied to $v = u^\epsilon$ over Ω to find,

$$\int_\Omega \partial_t\langle \mathbf{r}v, v\rangle \;+\; \partial_1\langle \mathbf{r}A_1\,v, v\rangle \;-\; 2\,\mathrm{Re}\,\langle Gv, v\rangle \;-\; \langle Zv, v\rangle\;dt\,dx \;=\; 0. \tag{4.5}$$

Let

$$E(t) \ := \ \Big(\int_{x_1 \le a + \lambda_{\min} t} \langle \mathbf{r} u^\epsilon \,,\, u^\epsilon \rangle \ dx \Big)^{1/2} .$$

Integrate by parts to show that

$$\int_\Omega \partial_t \langle \mathbf{r} v, v\rangle \ + \ \partial_1 \langle \mathbf{r} A_1 \, v, v\rangle \ dt\, dx \ = \ E(\underline{t})^2 \ - \ E(0)^2 \ + \int_\Gamma \big\langle (n_0 \mathbf{r} + n_1 \mathbf{r} A_1) u^\epsilon \,,\, u^\epsilon \big\rangle \ d\sigma, \tag{4.6}$$

where $(n_0, n_1, 0, \cdots, 0)$ is the unit outward normal at Γ and $d\sigma$ is the element of d-dimensional area on Γ. Since u^ϵ vanishes for $x_1 \le a$, it follows that $E(0) = 0$.

Since

$$(G + Z) u^\epsilon \ = \ \mathbf{r}(t, x)\, c^\epsilon(t, x, D') u^\epsilon,$$

one has

$$\Big| \int_\Omega 2 \,\mathrm{Re}\, \langle G v, v\rangle \ + \ \langle Z v, v\rangle dx\, dt \Big| \ \le \ \frac{c}{\epsilon} \int_0^t E(s)^2 \ ds.$$

Combining with (4.5) and (4.6) yields,

$$E(\underline{t})^2 \ \le \ - \int_\Gamma \big\langle (n_0 \mathbf{r} \ + \ n_1 \mathbf{r} A_1) u^\epsilon \,,\, u^\epsilon \big\rangle \ d\sigma \ + \ \frac{c}{\epsilon} \int_0^{\underline{t}} E(s)^2 \ ds \,. \tag{4.7}$$

The unit outward normal to Γ is a positive multiple of $(-\lambda_{\min}, 1, 0, \dots, 0)$. Thus $n_0 \mathbf{r} + n_1 \mathbf{r} A_1$ is a hermitian matrix which is a positive multiple of

$$-\lambda_{\min} \mathbf{r} + \mathbf{r} A_1 \ = \ \mathbf{r}^{1/2} \Big(- \lambda_{\min} \, I \ + \ \mathbf{r}^{1/2} \, A_1 \, \mathbf{r}^{-1/2} \Big) \mathbf{r}^{-1/2} \,.$$

The matrix $\mathbf{r}^{1/2} \, A_1 \, \mathbf{r}^{-1/2}$ has the same eigenvalues as A_1, thus real and $\ge \lambda_{\min}$. Therefore the matrix in parenthesis has real nonnegative eigenvalues. Thus the eigenvalues of $-\lambda_{\min} \mathbf{r} + \mathbf{r} A_1$ are nonnegative, so this matrix is nonnegative hermitian. Therefore

$$\int_\Gamma \big\langle (n_0 \mathbf{r} \ + \ n_1 \mathbf{r} A_1) u^\epsilon \,,\, u^\epsilon \big\rangle \ d\sigma \ \ge \ 0.$$

Use this in (4.7) to find,

$$E(\underline{t})^2 \ \le \ \frac{c}{\epsilon} \int_0^{\underline{t}} E(s)^2 \ ds \,.$$

Gronwall's inequality implies that $E(t)$ is identically zero completing the proof of **ii.** ▮

The main analytical result proves estimates for u^ϵ that are *independent of* ϵ. It is here that the Weyl calculus is used.

Proposition 4.2. *For each* $s \in \mathbb{R}$, *There is a constant* $c = c(s)$ *independent of* ϵ, t, f *so that*

$$\sup_{t \in \mathbb{R}} \ e^{-c|t|} \, \| u^\epsilon(t) \|_{H^s(\mathbb{R}^d)} \ \le \ c \, \| f \|_{H^s(\mathbb{R}^d)} \,.$$

Proof. The case $s = 0$ implies the general case by a straight forward commutation argument estimating $(1 + |D|^2)^{s/2}u$. We prove the estimate for $s = 0$. Since the $r_\epsilon(t, x, \xi)$ are hermitian, the operators $r_\epsilon^w(t, x, D)$ are self adjoint by **W.1**. We first show that there are strictly positive constants C_j independent of ϵ so that,

$$R^\epsilon(t) \ := \ r_\epsilon^w(t,x,D) + C_1 \big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{-1} \ \geq \ C_2\, I\,. \tag{4.8}$$

To prove (4.8), choose $C_2 > 0$ so that

$$\forall \ \ t, x, \xi, \qquad r(t,x,\xi) \ \geq \ 2\, C_2\, I\,.$$

Choose smooth $s(t, x, \xi)$ equal to the positive hermitian square root of $r(t, x, \xi) - C_2 I$ Then, s belongs to the classical symbol class $S^1(\mathbb{R}^{1+d} \times \mathbb{R}^d)$. Let

$$s_\epsilon(t,x,\xi) \ := \ s\big(t,x,\xi_1\,,\, \xi'/\langle \epsilon \xi'\rangle\big)\,.$$

Then s_ϵ and r_ϵ are bounded families of symbols in $S(1, g_\epsilon)$.
Since $s_\epsilon(t, x, \xi)^2 = r_\epsilon(t, x, \xi) - C_2 I$, it follows that

$$r_\epsilon^w(t,x,D) \ = \ C_2 I \ + \ (s_\epsilon^w(t,x,D))^2 \ + \ \rho_\epsilon^w(t,x,D) \tag{4.9}$$

W.2 shows that the hermitian symbols $\rho_\epsilon(t, x, \xi)$ are bounded in $S(h_\epsilon\,,\, g_\epsilon) = S(1/m_\epsilon\,,\, g_\epsilon)$.
The family of operators

$$\big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{1/2} \rho_\epsilon^w(t,x,D) \big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{1/2}$$

is the product of operators with symbols in $S(m_\epsilon^{1/2}, g_\epsilon)$, $S(m_\epsilon^{-1}, g_\epsilon)$, and, $S(m_\epsilon^{1/2}, g_\epsilon)$ respectively. **W.2** shows that its symbols are bounded in $S(1, g_\epsilon)$.
By **W.3**, there is a constant C_1 so that for all ϵ, f

$$\Big(\big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{1/2} \rho_\epsilon^w(t,x,D) \big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{1/2} f\,,\, f\Big) \ \leq \ C_1 \Big(f\,,\, f\Big)\,.$$

The substitution $g = \langle D'/\epsilon D'\rangle\big\rangle^{1/2} f$ shows that this is equivalent to,

$$\begin{aligned}\Big(\rho_\epsilon^w(t,x,D) g\,,\, g\Big) \ &\leq \ C_1 \Big(\big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{-1/2} f\,,\, \big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{-1/2} f\Big) \\ &= \ C_1 \Big(\big\langle D_1\,,\, \langle D'/\epsilon D'\rangle\big\rangle^{-1} f\,,\, f\Big)\,.\end{aligned}$$

This together with (4.9) proves the desired positivity (4.8).
Continuing with the proof of the Proposition, compute,

$$\partial_t (R^\epsilon u^\epsilon(t), u^\epsilon(t)) \ = \ (R^\epsilon u^\epsilon_t, u^\epsilon) + (R^\epsilon u^\epsilon, u^\epsilon_t) + (R^\epsilon_t u^\epsilon, u^\epsilon)\,.$$

The operators $R^\epsilon_t = (r_t)^w_\epsilon(t,x,D)$ have symbols bounded in $S(1,g_\epsilon)$. By **W.3**, the last term is bounded by $C\,\|u^\epsilon(t)\|^2_{L^2(\mathbb{R}^d)}$. Such terms with constant independent of ϵ are negligible in the computation that follows.

The sum of the other two terms is equal to

$$\Big(\big[R^\epsilon a^w_\epsilon(t,x,iD) - \big(R^\epsilon a^w_\epsilon(t,x,iD)\big)^*\big]u\,,\,u\Big)\,.$$

The operator R^ϵ is a sum of two terms and we treat them in turn.

The first summand is $r^w_\epsilon(t,x,D)$. Since $r_\epsilon(t,x,\xi)\in S(1,g_\epsilon)$, $a_\epsilon(t,x,\xi)\in S(m_\epsilon,g_\epsilon)$, and $r_\epsilon(t,x,\xi)a_\epsilon(t,x,\xi)$ is hermitian, **W.1** together with **W.2** imply that the family of symbols of

$$r^w_\epsilon(t,x,D)\,a^w_\epsilon(t,x,D)\;-\;(r^w_\epsilon(t,x,D)\,a^w_\epsilon(t,x,D))^*\quad\text{is bounded in}\quad S(h_\epsilon m_\epsilon\,,\,g_\epsilon)\,.$$

As $h_\epsilon = 1/m_\epsilon$, this shows that the symbols are bounded in $S(1,g_\epsilon)$.

The second summand is $C_1\langle D_1,\langle D'/\epsilon D'\rangle\rangle^{-1}$. Since

$$\big\langle \xi_1,\xi'/\langle\epsilon\xi'\rangle\big\rangle^{-1}\;\;\text{is bounded in}\;\;S(1/m_\epsilon\,,\,g_\epsilon),$$

W.2 implies that the two families of operators,

$$C_1\big\langle D_1\,,\,\langle D'/\epsilon D'\rangle\big\rangle^{-1}\;a^w_\epsilon(t,x,D)\qquad\text{and,}\qquad\Big(C_1\big\langle D_1\,,\,\langle D'/\epsilon D'\rangle\big\rangle^{-1}a^w_\epsilon(t,x,D)\Big)^*$$

have symbols bounded in $S(1,g_\epsilon)$.

Combining shows that the symbols of the family of operators

$$R^\epsilon a^w_\epsilon(t,x,iD)\;-\;\big(R^\epsilon a^w_\epsilon(t,x,iD)\big)^*$$

are bounded in $S(1,g_\epsilon)$. By **W.3**, there is a constant C independent of ϵ so that

$$\Big\|\Big(R^\epsilon a^w_\epsilon(t,x,iD) - \big(R^\epsilon a^w_\epsilon(t,x,iD)\big)^*\Big)u(t)\Big\|\;\le\;C\,\|u(t)\|\,.$$

Therefore,

$$\partial_t(R^\epsilon u^\epsilon(t),u^\epsilon(t))\;\le\;C\,\|u^\epsilon(t)\|^2,$$

with C independent of $\epsilon\in[0,1]$.

From the uniform positivity of the R^ϵ it follows that

$$\partial_t(R^\epsilon u^\epsilon(t),u^\epsilon(t))\;\le\;C_1\,(R^\epsilon u^\epsilon(t),u^\epsilon(t))\,.$$

So,

$$(R^\epsilon(t)u^\epsilon(t),u^\epsilon(t))\;\le\;e^{C_1|t|}(R^\epsilon(0)u^\epsilon(0),u^\epsilon(0)).$$

Then, with constants independent of ϵ

$$\|u^\epsilon(t)\|^2 \;\le\; \frac{1}{C_2}\,(R^\epsilon u^\epsilon(t), u^\epsilon(t)) \;\le\; \frac{1}{C_2}\,e^{C_1|t|}(R^\epsilon(0)u^\epsilon(0), u^\epsilon(0)) \;\le\; C_3 e^{C_1|t|}\,\|u(0)\|^2\,.$$

This completes the proof. ∎

Corollary 4.3. *For any $T > 0$, u^ϵ converges weak star in $L^\infty([-T,T]\,;\,L^2(\mathbb{R}^d))$ to the solution u of equation (2.2) with initial value f.*

Proof. We have just proved that the family u^ϵ is bounded in $L^\infty([-T,T]\,;\,L^2(\mathbb{R}^d))$. Suppose that $w \in L^\infty([-T,T]\,;\,L^2(\mathbb{R}^d))$ is the weak star limit of a subsequence u^{ϵ_k} with $\epsilon_k \to 0$ as $k \to \infty$. To prove the proposition it suffices to show that w solves the initial value problem which uniquely determines u. That is, it suffices to establish that

$$\partial_t w \;+\; \sum_{j=1}^{d} A_j(t,x)\partial_j w \;+\; B(t,x)w \;=\; 0\,,$$

in the sense of distributions, and that the initial value of w at $t = 0$ is f.

We split the verification into $t \ge 0$ and $t \le 0$ and present the details of the first. For the $t \ge 0$ half, it suffices to show that for all $\phi \in C_0^\infty(\mathbb{R}^{1+d})$,

$$0 \;=\; \int_0^\infty \int_{\mathbb{R}^d} \big\langle w\,,\, -\partial_t\phi + a_0^w(t,x,D)^*\phi + \tilde{B}^*\phi \big\rangle\; dt\,dx \;+\; \int_{\mathbb{R}^d} \big\langle f\,,\, \phi(0,x)\big\rangle\; dx\,. \tag{4.10}$$

Begin with the weak form of the equation for u^ϵ,

$$0 \;=\; \int_0^\infty \int_{\mathbb{R}^d} \big\langle u^\epsilon\,,\, -\partial_t\phi + a_\epsilon^w(t,x,D)^*\phi + \tilde{B}^*\phi \big\rangle\; dt\,dx \;+\; \int_{\mathbb{R}^d} \big\langle f\,,\, \phi(0,x)\big\rangle\; dx\,. \tag{4.11}$$

Choose $T > 0$ so that ϕ is supported in $|t| < T$. Then,

$$u^{\epsilon_k} \quad\to\quad w \quad \text{weakly in} \quad L^2([0,T]\times\mathbb{R}^d).$$

Compute

$$a_\epsilon^w(t,x,D)^* \;=\; \Big(A_1\partial_1 - \frac{\partial_1 A_1}{2}\Big)^* + c_\epsilon^w(t,x,D)^* \;=\; -A_1^*\frac{\partial}{\partial x_1} + \frac{\partial_1 A_1^*}{2} + (c_\epsilon^*)^w(t,x,D).$$

The first term from $(a_\epsilon^w)^*\phi$,

$$\Big(-A_1^*\frac{\partial}{\partial x_1} + \frac{\partial_1 A_1^*}{2}\Big)\phi\,, \quad \text{is independent of } \epsilon.$$

The classical calculus of pseudodifferential operators implies that for all s,

$$(c_\epsilon^*)^w(t,x,D')\phi \quad\to\quad (c_0^*)^w(t,x,D')\phi \quad \text{in} \quad H^s([0,T]\times\mathbb{R}^d), \quad \text{as } \epsilon \to 0.$$

Therefore as $k \to \infty$,

$$-\partial_t \phi + a^w_{\epsilon_k}(t,x,D)^*\phi + \tilde{B}^*\phi \ \to\ -\partial_t\phi + a^w_0(t,x,D)^*\phi + \tilde{B}^*\phi \quad \text{in} \quad H^s([0,T]\times\mathbb{R}^d).$$

Passing to the limit $k \to \infty$ in (4.11) yields (4.10), completing the proof. ∎

Proof of Proposition 2.1. Combining the Corollary 4.3 with Lemma 4.1.**ii** proves Propostion 2.1. ∎

§5. End of proof of Theorem 1.2.

Begin with two corollaries of Proposition 2.1.

Corollary 5.1. *Suppose that L is symmetrizable hyperbolic, ξ is a unit covector, and, for all (t,x) the eigenvalues of $\sum_j A_j(t,x)\,\xi_j$ are $\le \lambda_{\max}$. Then if $s\in\mathbb{R}$, $u\in C(\mathbb{R}:H^s(\mathbb{R}^d))$ satisfies $Lu=0$, and, $\operatorname{supp} u(0)\subset\{x.\xi\le b\}$, then for $t\ge 0$,*

$$\Big(\operatorname{supp} u\ \cap\ \{t\ge 0\}\Big)\ \subset\ \Big\{x.\xi\ \le\ b\ +\ \lambda_{\max}\,t\Big\}\,. \tag{5.1}$$

Proof. An approximation argument reduces to the case $u(0)\in\cap_s H^s(\mathbb{R}^d)$.

An orthogonal tranformation taking ξ to $(1,0,\dots,0)$ reduces this to Proposition 2.1 in the case $a=-\infty$. ∎

Corollary 5.2. *Suppose that L is symmetrizable hyperbolic and that Γ is a proper convex cone in $\{t\ge 0\}$ so that for all t,x, $\Gamma^+_{t,x}\subset\Gamma$. Then if $s\in\mathbb{R}$, and $u\in C(\mathbb{R}:H^s(\mathbb{R}^d))$ satisfies $L\,u=0$, then,*

$$\Big(\operatorname{supp} u\ \cap\ \{t\ge 0\}\Big)\ \subset\ \cup_{x\in\operatorname{supp} u(0)}\Big(x\ +\ \Gamma\Big)\,. \tag{5.2}$$

Proof. Writing the initial datum as a sum of distributions with support in small balls, it suffices to prove the Corollary for data with support in a small ball.

By translation invariance it suffices to prove the Corollary for data with support in balls with center at the origin. Suppose that $\operatorname{supp} u(0)\subset\{|x|\le r\}$.

Denote by $\mathcal{T}_\Gamma$ the dual cone to Γ. Since Γ is a proper future cone it follows that $dt=(1,0,\dots,0)$ belongs to the interior of $\mathcal{T}_\Gamma$.

Since Γ contains $\Gamma^+_{t,x}$ for all t,x it follows that $\mathcal{T}\subset\mathcal{T}^+_{t,x}$ for all t,x. Therefore $\mathcal{T}$ is given by an equation

$$\tau\ge\tau_\Gamma(\xi)\ \ge\ \sup_{t,x}\ \tau_{\max}(\xi)\,. \tag{5.3}$$

The eigenvalues of $\sum_j A_j(t,x)\xi_j$ are the negatives of the roots τ of $p(t,x,\tau,\xi)=0$. Therefore,

$$\forall\ (t,x),\qquad \operatorname{spec}\Big(\sum_j A_j(t,x)\xi_j\Big)\ \subset\ \big\{\lambda\le\tau_{\max}(t,x,-\xi)\big\}\ \subset\ \big\{\lambda\le\tau_\Gamma(-\xi)\big\}\,. \tag{5.4}$$

Corollary 5.1 implies that the support of u is contained in $\{x.\xi \ \le\ r + \tau_\Gamma(-\xi)t\}$. This is equivalent to $t\tau_\Gamma(\xi) + x.\xi \ge 0$. By (5.3), this implies that $t\tau + x.\xi \ge 0$ for all $(\tau,\xi) \in \mathcal{T}$ with $|\xi| = 1$. By positive homogeneity it extends to all $(\tau,\xi) \in \mathcal{T}$. Therefore,

$$\operatorname{supp} u \ \subset\ \Big\{(t,x)\ :\ \forall(\tau,\xi) \in \mathcal{T}_\Gamma,\quad t\tau + x.\xi\ \le\ r\Big\} \tag{5.5}$$

From the duality between Γ and $\mathcal{T}$, (5.5) is equivalent to

$$\operatorname{supp} u\ \subset\ \{|x| \le r\}\ +\ \Gamma\,.$$

This proves the desired result when the support of $u(0)$ is contained in a ball centered at the origin. ▮

To prove Theorem 1.2, use fattened cones as in [R1] to generate wiggle room. We fatten Γ by shrinking $\mathcal{T}$.

Definition. *For $\epsilon > 0$, define the shrunken time like cone,*

$$\mathcal{T}^{+,\epsilon}_{t,x}\ :=\ \Big\{(\tau,\xi) \in \mathbb{R}^{1+d}\ :\ \tau\ >\ \tau_{\max}(t,x,\xi)\ +\ \epsilon|\xi|\Big\}\,.$$

Define the fattened propagation cone, $\Gamma^{+,\epsilon}_{t,x}$, to be the closed dual cone. Denote by $\mathcal{E}^{\pm,\epsilon}$ the emissions defined with the $\Gamma^{\pm,\epsilon}_{t,x}$.

The fattened cones, $\Gamma^{+,\epsilon}_{t,x}$ are strictly convex, increasing in ϵ and contain $\Gamma^{+,\epsilon/2}_{t,x} \setminus 0$ in their interior. In addition, $\cap_{0<\epsilon<1}\Gamma^{+,\epsilon}_{t,x} = \Gamma^{+}_{t,x}$. It follows that in the limit $\epsilon \to 0$ the emissions $\mathcal{E}^{+,\epsilon}$ decrease to $\mathcal{E}^{+}$.

The proof of Theorem 1.2 proceeds by a sequence of Lemmas.

Lemma 5.3. *For any $\epsilon > 0$ there is a $\delta_1 > 0$ so that*

$$\max\big\{|t-\tilde t|\,,\,|x-\tilde x|\big\}\ \le\ 2\delta_1 \quad\Longrightarrow\quad \Gamma^{+,\epsilon}_{t,x}\ \subset\ \Gamma^{+,2\epsilon}_{\tilde t,\tilde x}\,. \tag{5.6}$$

Proof. Since

$$\tau I + \sum_j A_j\xi_j\ =\ r^{-1}\Big(\tau r + r\sum_j A_j\xi_j\Big)\ =\ r^{-1}\ r^{1/2}\Big(\tau I + r^{-1/2}(r\sum_j A_j\xi_j)r^{-1/2}\Big)r^{1/2},$$

the roots τ of $p(t,x,\tau,\xi) = 0$ are, for real ξ, the negatives of the eigenvalues of the hermitian matrix $r^{-1/2}(r\sum_j A_j\xi_j)r^{-1/2}$. As the matrix is uniformly lipshitzean, it follows that $\tau_{\max}$ is uniformly lipschitzean on $\mathbb{R}^{1+2d}$. Lemma 5.3 follows. ▮

Lemma 5.4. *Suppose that L is symmetrizable. For each $0 < \epsilon < 1$ there is a $\delta > 0$ so that if $u \in C(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$ satisfies $Lu = 0$ and $\underline{t} \in \mathbb{R}$, then*

$$\operatorname{supp} u\ \cap\ \{\underline{t} \le t \le \underline{t} + \delta\}\ \subset\ \mathcal{E}^{+,\epsilon}(\operatorname{supp} u(\underline{t}))\,.$$

Proof. Given $\epsilon > 0$ choose $\delta_1 > 0$ so that (5.6) holds.

Decomposing u with a partition of unity it is sufficient to prove the assertion for u with $\operatorname{supp} u(\underline{t}) \subset \{|x - \underline{x}| \leq \delta_1/2\}$. Translating coordinates we may suppose that $\underline{t} = 0$ and $\underline{x} = 0$.

Let $\Gamma := \Gamma^{+,2\epsilon}_{0,0}$. Then

$$\max\{|t|, |x|\} \leq 2\delta_1 \quad \Longrightarrow \quad \Gamma^+_{t,x} \subset \Gamma\,.$$

Use a localization idea dating at least to Leray [Le],. Define a new symmetrizable operator $\tilde{L}$ which agrees with L when $\max\{|t|, |x|\} \leq \delta_1$ and for which the propagation cones are all contained in Γ. Choose a smooth

$$\Phi \;:\; \mathbb{R}^{1+d} \quad \rightarrow \quad \{|t, x| < 2\delta_1\} \;\subset\; \mathbb{R}^{1+d}$$

so that

$$\max\{|t|, |x|\} \leq \delta_1 \;\Longrightarrow\; \Phi(t, x) \;=\; (t, x) \qquad \max\{|t|, |x|\} \geq 2\,\delta_1 \;\Longrightarrow\; \Phi(t, x) \;=\; (0, 0)\;.$$

Define a modified system of partial differential operators by

$$\tilde{L} \;:=\; \partial_t + a^w(\Phi(t, x), \partial) + \tilde{B}(\Phi(t, x))).$$

Then, $\tilde{L}$ is symmetrized by $r(\Phi(t, x), D)$. In addition, $\tilde{L}$ is equal to L when $\max\{|t|, |x|\} \leq \delta_1$.

Since the range of Φ is contained in the set of points where $\Gamma^+_{t,x} \subset \Gamma$, it follows that for all t, x the forward propagation cones of $\tilde{L}$ are contained in Γ. Define $\tilde{u}$ to be the solution of

$$\tilde{L}\tilde{u} = 0, \qquad \tilde{u}(0) = u(0)\,.$$

Corollary 5.2 implies that for $t \geq 0$,

$$\operatorname{supp} \tilde{u} \;\subset\; \operatorname{supp} u(0) \;+\; \Gamma \;\subset\; \{|x| \leq \delta_1/2\} + \Gamma\,.$$

Use the elementary bound

$$|\tau_{\max}| \;\leq\; \sup_{\mathbb{R}^{1+d} \times \{|\xi|=1\}} \Big\| \sum_j A_j(t, x)\xi_j \Big\| \;:=\; K,$$

together with $\epsilon < 1$ to conclude that

$$\Gamma \;\subset\; \{|x| \leq (1 + K)t\}\,.$$

Choose $\delta \leq \delta_1$ so that

$$(1 + K)\delta + \delta_1/2 < \delta_1. \tag{5.7}$$

Then, for $0 \le t \le \delta$, the support $\tilde{u}$ is contained in $|x| \le \delta_1$ where $L = \tilde{L}$. Thus $\tilde{u}$ solves the same Cauchy problem as u so by uniqueness, $\tilde{u} = u$. Thus,

$$\operatorname{supp} u \ \cap \ \{0 \le t \le \delta\} \ \subset \ \operatorname{supp} u(0) + \Gamma\,.$$

Since on the support of u, $\Gamma^{+,\epsilon}_{t,x} \subset \Gamma$, this implies that

$$\operatorname{supp} u \ \cap \ \{0 \le t \le \delta\} \ \subset \ \mathcal{E}^{+,\epsilon}(\operatorname{supp} u(0))\,.$$

This proves the desired propagation result for data of small support and therefore completes the proof of Lemma 5.4. ▮

Proof of Theorem 1.2. For ϵ fixed, one can iterate Lemma 5.4 to conclude that on $t \ge 0$

$$\operatorname{supp} u \ \subset \ \mathcal{E}^{+,\epsilon}(\operatorname{supp} u(0))\,.$$

Since the cones $\Gamma^{+,\epsilon}$ are convex and decrease to Γ^+, passing to the limit $\epsilon \to 0$, yields

$$\operatorname{supp} u \ \subset \ \mathcal{E}^{+}(\operatorname{supp} u(0))\,,$$

completing the proof. ▮

§6. Proof of Theorem 1.1.

The proof is of Hölmgren type, requiring the solution of initial value problems for the transposed operator,

$$L^\dagger \ := \ -\partial_t \ - \ \sum_j A_j(t,x)^\dagger \partial_j \ + \ \Big(B^\dagger - \sum_j \partial_j A_j^\dagger\Big)\,.$$

This is possible because the transposed system is symmetrizable.

Proposition 6.1. i. *If L is symmetrized by $r(t,x,\xi)$ then the transposed operator is symmetrized by $(r(t,x,\xi)^{-1})^\dagger$.* **ii.** *The timelike and propagation cones for $L^\dagger$ are identical to those of L.*

Proof. By hypothesis

$$s(t,x,\xi) \ := \ r(t,x,\xi) \sum_j A_j \xi_j$$

is hermitian symmetric for $|\xi| \ge 1$. Therefore,

$$r^{-1}\, s\, r^{-1} \ = \ \Big(\sum_j A_j \xi_j\Big)\, r^{-1}$$

is hermitian symmetric for $|\xi| \ge 1$. Take transpose to find that

$$(r^{-1})^\dagger \Big(\sum_j A_j^\dagger \xi_j\Big)$$

is hermitian symmetric for $|\xi| \geq 1$. This proves **i**.
The characteristic polynomial for $L^\dagger$ is

$$0 = \det\Big(-\tau I - \sum_j A_j^\dagger \xi_j \Big) = (-1)^N \det\Big(\tau I + \sum_j A_j^\dagger \xi_j \Big) = (-1)^N \det\Big(\tau I + \sum_j A_j \xi_j \Big)^\dagger$$
$$= (-1)^N \det\Big(\tau I + \sum_j A_j \xi_j \Big) = (-1)^N p(t, x, \tau, \xi).$$

The roots $\tau(\xi)$ are the same for L and $L^\dagger$. Therefore the timelike cones and propagation cones are identical. ▮

If M is an embedded hypersurface in $\mathbb{R}^{1+d}_{t,x}$ which is spacelike at m and $\psi(t,x)$ vanishes on M with $d\psi(m) \neq 0$ then on a small neighborhood of m, ψ is a defining function of M. The covector $d\psi(m)$ is conormal to M so replacing ψ by $-\psi$ if necessary, $d\psi \in \mathcal{T}^+_m$ expresses the fact that M is spacelike. Then small balls $B_r(m)$ have a future half defined by $\psi > 0$. Decreasing r if necessary one has $d\psi(t,x) \in \mathcal{T}^+_{t,x}$ for all (t,x) with $|(t,x) - m| \leq r$. In particular, M is spacelike at all $\tilde m \in M$ with $|m - \tilde m| \leq r$.

Lemma 6.2. *With the choices of the preceding paragraph, and $0 < \rho < r$ consider the closed balls*
$$\overline{B}_\rho(m) \ := \ \Big\{ (t,x) \ : \ |(t,x) - m| \leq \rho \Big\}.$$
There is a ρ so that,

$$\mathcal{E}^-(\overline{B}_\rho(m)) \ \cap \ \Big\{ |(t,x) - m| = r\,, \ \psi(t,x) \geq 0 \Big\} \quad = \quad \phi\,.$$

Proof. If not there would be a sequence of points (t_n, x_n) converging to m and influence curves
$$\gamma_n(t) = (t, x_n(t)) \quad \text{with} \quad -\infty < t \leq \underline{t}$$
and $\tilde t_n < \underline{t}$ satisfying,

$$\big|\gamma(\tilde t_n\,,\, x_n(\tilde t_n) - m\big| = r\,, \qquad \text{and,} \qquad \psi(\tilde t_n\,,\, x_n(\tilde t_n)) \geq 0\,.$$

Writing $m = (\underline{t}, \underline{m})$, the first conditions implies that $\tilde t_n \in [\underline{t} - r, \underline{t}]$.
For an influence curve, $\gamma' = (1, x') \in \Gamma^+_{\gamma(t)}$. This implies the uniform bound,

$$\|x'(t)\| \ \leq \ \sup_{t,x,|\xi|=1} \ \Big\| \sum_j A_j(t,x)\xi_j \Big\|, \qquad \text{Lebesgue a.e. } t\,.$$

Ascoli's Theorem implies that there is a subsequence, still denoted γ_n which converges uniformly to an influence curve, $\gamma : [\underline{t} - r\,,\, \underline{t}] \to \mathbb{R}^{1+d}$, $(t_n, x_n) \to m$, and, $\tilde t_n \to \tilde t \in [\underline{t} - r, \underline{t}]$. Passing to the limit one finds,

$$\gamma(\underline{t}) = m, \quad \big|\gamma(\tilde t) - m\big| = r, \quad \text{and} \quad \psi(\gamma(\tilde t)) = 0\,.$$

Since γ is an influence curve, and $d\psi$ is timelike it follows that

$$\frac{d}{dt}\,\psi(\gamma(t)) \;>\; 0 \quad \text{for Lebesgue almost all } \; t \in [\underline{t} - r, \underline{t}]\,.$$

Therefore

$$0 = \psi(m) = \psi(\gamma(\underline{t})) \;>\; \psi(\gamma(\tilde{t}) = 0.$$

This contradiction proves the Lemma. ▮

Proof of Theorem 1.1. We show that u vanish in $B_\rho(m)$ with ρ from Lemma 6.2. Since $L^\dagger$ is symmetrizable hyperbolic from Proposition 6.2, for any $\phi \in C_0^\infty(B_\rho(m))$ we can define $v \in \cap_{s\in\mathbb{N}} C^s(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$ to be the solution of of the Cauchy problem

$$L^\dagger v \;=\; \phi, \qquad v\big|_{t=\underline{t}+\rho} \;=\; 0.$$

The precise finite speed result together with Duhamel's formula implies that,

$$\operatorname{supp} v \;\subset\; \mathcal{E}^-(\operatorname{supp}\phi) \;\subset\; \mathcal{E}^-(B_\rho(m))\,. \tag{6.1}$$

We next complete the proof in the case that $u \in C^1(B_r(m))$. Since u is supported in the future half of B_ρ, the Lemma implies that at each point of $\partial B_r(m)$ one of u or v vanishes. Therefore, integration by parts shows that

$$\int_{B_r(m)} \big\langle Lu\,,\, v\big\rangle\; dx \;=\; \int_{B_r(m)} \big\langle u\,,\, L^\dagger v\big\rangle\; dx\,.$$

From the differential equations satisfied by u and v, one finds

$$0 \;=\; \int_{B_r(m)} \big\langle u\,,\, \phi\big\rangle\; dx\,.$$

Since $\phi \in C_0^\infty(B_\rho(m))$ is arbitrary, this is equivalent to $u = 0$ on $B_\rho(m)$ and Theorem 1.1 is proved for $u \in C^1(B_r(m))$.

In the case of distribution solutions, $u \in \mathcal{D}'(B_r(m))$ reason as follows. Lemma 6.2 implies that there is a compact set $K \subset B_r(m)$ so that for all $\phi \in C_0^\infty(B_\rho(m))$,

$$\operatorname{supp} v \;\cap\; \operatorname{supp} u \;\subset\; K\,.$$

Choose a test function $\chi \in C_0^\infty(B_r(m))$ so that $\chi = 1$ on a neighborhood of K. Since $Lu = 0$ in $B_r(m)$, one has

$$\big\langle u\,,\, L^\dagger(\chi v)\big\rangle \;=\; 0\,. \tag{6.2}$$

Expand

$$L^\dagger(\chi v) \;=\; \chi L^\dagger v \;+\; \big(\partial_t\chi + \sum_j A_j\partial_j\chi\big)v \;=\; \chi\,\phi \;+\; \big(\partial_t\chi + \sum_j A_j\partial_j\chi\big)v\,. \tag{6.3}$$

The support of the second term is disjoint from K, it follows that for all $(t,x) \in B_r(m)$ at least one of the factors

$$u\,, \qquad \partial_t\chi + \sum_j A_j\partial_j\chi, \qquad \text{or} \qquad v,$$

vanishes on a neighborhood of (t,x). Therefore,

$$\big\langle u\,,\,\big(\partial_t\chi + \sum_j A_j\partial_j\chi\big)v\big\rangle \;=\; 0\,. \tag{6.4}$$

Since $\chi = 1$ on

$$\operatorname{supp} v \cap \operatorname{supp} u \;\supset\; \operatorname{supp} \phi \cap \operatorname{supp} u\,,$$

it follows that

$$\big\langle u\,,\,\chi\phi\big\rangle \;=\; \big\langle u\,,\,\phi\big\rangle\,. \tag{6.5}$$

Combining equations (6.2) to (6.5) yields

$$\big\langle u\,,\,\phi\big\rangle \;=\; 0\,.$$

Therefore $u = 0$ on $B_\rho(m)$. ∎

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