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GNAVOA, I.
Studies in groups, nonassociative algebras and vertex operator algebras.

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Abstract. This article will be one in a series which take an exploratory look at some VOAs of CFT type, such as the ones of lattice type, their automorphism groups and the automorphism groups of their degree 2 part. In part I, we are concerned mostly with definitions, examples and methods. We will reflect on some general relations between groups, nonassociative algebras and VOAs.

Contents

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1 Introduction

At this time, we are especially interested in questions about VOAs and their automorphism groups[1] mostly along the following lines:

Q1. What groups occur as $Aut(V)$, for a VOA V?

Q2. Are there reasonable methods for determining $Aut(V)$ in cases of interest?

In the seventies decade, the theories of finite simple groups and commutative nonassociative algebras became more closely interconnected. In the mid-eighties, VOA theory became established, and developed with ideas from physics, geometry and Lie theory and the algebraic theories involving the monster simple group.

It seems a good idea to explore interconnections among groups, nonassociative algebras and VOA theory, hence the acronym GNAVOA. Here, we are thinking mainly of finite dimensional commutative nonassociative algebras which occur as some $(V_2, 1^{st})$. It has been known for a long time that the algebra $(V_1, 0^{th})$ is a Lie algebra if $V_n = 0$ for $n < 0$ and $dim(V_0) = 1$. We shall say little about this well-studied role of Lie algebras, and concentrate on degree 2. We have a little to say about higher k.

A few examples will indicate some relevant ideas from group theory.

Example 1.1 The algebra A is the direct sum of n copies of $\mathbb{C}$. Obviously, $Aut(A) \cong Sym_n$ since there are exactly n primitive idempotents, serving as the units for the n summands.

Definition 1.2 An algebra $A = \bigoplus_{i\in I} A_i$ (I is just an index set, not a grading) is described by a linear map $A \otimes A \to A$, which in a natural way is a linear combination of maps p_{ijk} identified with maps $A_i \otimes A_j \to A_k$. A *deformation of* A *with respect to the direct sum* $A = \bigoplus_{i\in I} A_i$ is an algebra structure on A which is given by a different linear combination of the p_{ijk}. Two deformations with respect to the same direct sum are *equivalent* if each is a deformation of the other. We call such a deformation *invertible, nondegenerate* or *nonsingular*; otherwise, we call it *noninvertible, degenerate* or *singular* A deformation is a *rescaling* if there is a scalar $c \neq 0$ so that every map p_{ijk} is replaced by cp_{ijk}. A subalgebra B of A which is homogeneous with respect to the direct sum has a natural *induced deformation* since for all i, j, k, we have $p_{ijk}(B \otimes B) \leq B$. We may write $(A, \cdot)$ and $(A, *)$ to indicate the original algebra and deformation, respectively.

Remark 1.3 If the group G acts as automorphisms of the algebra A and all the A_i are G-submodules, then the maps p_{ijk} are G-maps and G acts as automorphisms of any deformation. For instance in the case of 1.1, there are deformations which are nonassociative and still carry Sym_n as a group of automorphisms. Possibly, a deformation may have a larger group of automorphisms.

Definition 1.4 Given an algebra $A = \bigoplus_{i\in I} A_i$ as in 1.2, we get an algebra structure on any A_i, called *contraction.* It is given by the map p_{iii}.

[1]We recall a few basic results about a VOA $(V, Y, \mathbf{1}, \omega)$. An *automorphism* is an invertible linear transformation g on V which fixes $\mathbf{1}$ and ω and commutes with the action of Y in the sense that $Y(gv, z) = gY(v, z)g^{-1}$ for all $v \in V$. The subspace of fixed points V^G is a subVOA. If $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is the natural grading on V and $a \in V_i, b \in V_j$ then for every integer k, $a_k b \in V_{i+j-k-1}$, where $Y(a, z) := \sum_{k\in\mathbb{Z}} a_k z^{-k-1}$.

The n^{th} *binary composition on* V is defined by: $a_n b$. By (V_{n+1}, n^{th}), we denote the algebra structure on the homogeneous component V_{n+1} of the VOA $(V, Y, \mathbf{1}, \omega)$ given by the n^{th} binary composition. This is called the $(n + 1)^{th}$ *contraction of* V, in accordance with Definition 1.4.

Example 1.5 Here is an instance of 1.4. Let G be a finite permutation group on the set Ω. Let V be a direct summand of the permutation module $M := \mathbb{C}\Omega$. Let π be the orthogonal projection of M to V and ι the inclusion of V in M. We define a product $*$ on V by $u * v := \pi(\iota(u)\iota(v))$, where the latter product is the natural one on M, as in 1.1. Obviously, $G \leq Aut(V, *)$.

Example 1.6 (a) Let G be a finite group and V a finite dimensional module. Suppose that all of $Hom_G(S^2V, V), Hom_G(S^2V, \mathbb{C}), Hom_G(S^3V, \mathbb{C})$ are 1-dimensional. Then on V, there is an essentially unique nontrivial commutative algebra structure and a G-invariant symmetric bilinear form which is associative: $(xy, z) = (z, yz)$ for all $x, y, z \in V$.

(b) These hypotheses are satisfied if G is a triply transitive group on some set Ω and V is the nontrivial submodule of the permutation module $\mathbb{C}\Omega$. Then $Aut(V) \cong Sym_\Omega$. For an easy proof, see the Appendix of [DG].

The above dimension statements may be verified with the character table and power maps for G, if they are available.

Example 1.7 The monster $\mathbb{M}$ has an irreducible V of dimension 196883 (and no irreducible module of smaller dimension except the trivial module). There is an invariant commutative algebra structure on it which is an instance of both 1.5 and 1.6. This algebra is sometimes denoted $\mathcal{B}_0$ and a 196884-dimensional algebra $\mathcal{B}$, which contains $\mathcal{B}_0$ as a submodule and has a unit, occurs within the moonshine VOA [2] and has $\mathcal{B}_0$ as a contraction.

From the finite group theory viewpoint, these examples have the same general flavor. In the seventies, there was some effort to find a role for the sporadic groups as automorphism groups of a good category of algebras. From the VOA viewpoint, only the last of the above examples has played a role in any really obvious way. In this article, we hope to show that these two philosophies have more common ground.

There is some literature on determining the automorphism group of algebras which arise as in 1.6. See [Smith], [Froh].

We shall take a closer look at how commutative nonassociative algebras come up in the VOA world. Mainly, we are thinking of the cases where $(V_2, 1^{st})$ is commutative. These include classic examples, for instance some Jordan matrix algebras, but also nonfamiliar ones. The algebra $\mathcal{B}$ of 1.7 has no nontrivial low degree identities [GrMont], so one can not hope for a structure theory like those of Lie and Jordan algebras. Probably classic work with identities is not effective in general for the algebras $(V_2, 1^{st})$. As with 1.7, some questions about the algebras may be answered by dealing with the automorphism group. The advantage of this viewpoint is that both the theories of Lie groups and finite simple groups are well-developed.

2 Background

Now, let V be a VOA. We recall the terms *lattice VOA* or LVOA [FLM] and *lattice type VOA* [DG] or LTVOA (a subVOA of a lattice VOA V_L of the form V_L^G, where G is a finite group of automorphisms).

[2] Since the moonshine module turns out to have a VOA structure on it, we prefer the term *moonshine VOA* to moonshine module.

We know $Aut(V)$ for only a limited family of V. The ones we are aware of are the LVOAs [DN], LTVOAs of rank 1 and a few special cases, such as the monster and $O^+(10,2)$. See the survey in [GrRaleigh]. Since that survey, the following basic result has been obtained.

Theorem 2.1 The automorphism group of a finitely generated VOA is an algebraic group.

A proof may be found in the article [DG2], which describes the general structure of automorphism groups and Lie algebra of derivations for the finitely generated case.

A general algebraic group G has the following shape:

AG1. In the algebraic group G, the connected component of the identity G^0 has finite index.

AG2. The unipotent radical R is complemented in G^0 by a reductive group L, so $G^0 = RL$ is a semidirect product. (The group L, is called a Levi factor; it is unique up to conjugacy but is not unique in general.)

As automorphism groups of VOAs, so far we know explicit examples of finite groups, connected positive dimensional reductive groups, and mixtures of these. In [DGR], there is a nontrivial example of a VOA with trivial automorphism group.

For the question of which finite simple groups (or nearly simple ones) occur, we have no global conjecture, but do have the important theme of 3-transposition groups to consider.

Definition 2.2 A set D of *3-transpositions* in a group G is a subset of elements of order 2 so that $x, y \in D$ implies that $|xy| \in \{1, 2, 3\}$. We call G (or, more precisely, the pair (G, D)) a *3-transposition group* if D is a set of 3-transpositions, is a union of conjugacy classes, and generates G.

It is possible that a set of 3-transpositions is contained in a conjugacy class which is not a set of 3-transpositions.

Since two distinct involutions generate a dihedral group, the condition $|xy| = 2$ is equivalent to $xy = yx$. If we define a graph structure on D by joining distinct $x, y \in D$ if and only if $xy \neq yx$, and D_i are the connected components, then $\langle D \rangle$ is a central product of the groups $\langle D_i \rangle$.

We do not know all finite 3-transposition groups, but do know the ones which are essentially simple, due to this basic theorem of Bernd Fischer [Fi].

Theorem 2.3 Let (G, D) be a finite 3-transposition group such that G' is simple. Then (G, D) is one of the following:

Sym_n, the set of transpositions; or $n = 6$ and D is the conjugacy class of fixed point free involutions;

$PSU(n, 2)$, the set of unitary transvections;

$Sp(2m, 2)$, the set of symplectic transvections (except for the case $Sp(4, 2) \cong Sym_6$);

$O^\varepsilon(2m, 2)$, the set of orthogonal transvections;

$O^{\varepsilon,\mu}(n, 3)$, the set of orthogonal reflections $x \mapsto x + (x, y)y$, where $(y, y) = \mu$.

$Fi_{22}, Fi_{23}, Fi_{24}$, one of three sporadic simple groups and in each case D is a uniquely determined conjugacy class of involutions.

(The parameter ε for even dimension is $+$ if the Witt index is maximal, and otherwise is $-$; it is nonexistent for odd dimensional orthogonal groups in characteristic not 2.)

Note that D is not uniquely determined for Sym_6 and that in the even dimensional projective orthogonal groups over the field of three elements we have two different conjugacy classes of reflections (more accurately, of their images in the projective group) which satisfy the 3-transposition condition and may generate different subgroups of the projective orthogonal group.

The special relationship of 3-transpositions with VOA theory occurs via the Virasoro elements, also called conformal elements.

Definition 2.4 A *Virasoro frame* in a VOA $(V, Y, \mathbf{1}, \omega)$ is a collection of Virasoro elements ω_i such that $\omega = \sum_i \omega_i$ and the central charge of each ω_i is $\frac{1}{2}$. We abbreviate this with VF. A VOA with a VF is called a *framed vertex operator algebra*, FVOA.

The connection was noticed by Miyamoto, whose idea is that to each element of a Virasoro frame is associated an automorphism $t(\omega_i)$ of order 2 (or order 1, in exceptional situations), based on fusion rules involving $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{2}, \frac{1}{16})$, the irreducibles for the Virasoro subVOA generated by the ω_i. In case $L(\frac{1}{2}, \frac{1}{16})$ does not occur in V, $t(\omega_i)$ belongs to a conjugacy class of 3-transpositions in $Aut(V)$. See [Miy], [DGH], [GrRaleigh].

We think of the 3-transposition concept as a link between the worlds of finite simple groups and basic VOA theory, something worth studying.

Let us formally deal with the case of an algebraic group which contains a conjugacy class of 3-transpositions.

Lemma 2.5 If G is an algebraic group containing a conjugacy class of 3-transpositions, D, then D generates a finite normal subgroup of G.

Proof. Let H be the Zariski closure of the group generated by D and suppose that H is positive dimensional. Then H^0 has a positive dimensional Lie algebra. Take $x \in D$. If x acts nontrivially on the Lie algebra, it has an eigenvector for eigenvalue -1 and so in H^0, x inverts a nontrivial 1-parameter subgroup under conjugation. This violates the 3-transposition condition. Therefore D centralizes H^0, whence H^0 is abelian and so is a torus. An old theorem of Schur says that if the center of a group has finite index, its commutator subgroup is finite. Therefore, H' is finite. Also, H/H' is an abelian group generated by involutions, so has exponent 2 (but is possibly infinite). Clearly then H has finite exponent, whereas H^0 is a positive dimensional torus, a contradiction. □

Definition 2.6 A VOA V has *CFT type* if V_n is 0 for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$ is 1-dimensional.

Definition 2.7 The *OZ property* of a VOA $V = \bigoplus_{n \in \mathbb{Z}} V_n$ means the following set of conditions: $dim(V_n) = 0$ for $n < 0$; $dim(V_0) = 1$; and $dim(V_1) = 0$. (Note that OZ stands for the sequence of dimensions: one, zero). A VOA with the OZ property is called an *OZVOA*, or an *ozzie*, for short.

The OZ property implies the CFT property, but not conversely.

If V has the OZ property, $V_0 = \mathbb{C}\mathbf{1}$ and $(V_2, 1^{st})$ is a commutative nonassociative algebra with an associative, symmetric bilinear form $(x, y) = x_3 y$, $x, y \in V_2$ [FLM].

Definition 2.8 A commutative algebra $(A, *)$ for which there is an OZVOA V such that $(A, *) \cong (V_2, 1^{st})$ is called a *Griess algebra*. We say that such an OZVOA *affords* the algebra $(A, *)$.

The term Griess algebra arose in the VOA literature, due to the role of the 196884-dimensional algebra $\mathcal{B}$ in the construction of the monster and in the theory of $V^\natural$, the moonshine VOA, which has the OZ property. Given a Griess algebra, there seems to be no obvious relation between two VOAs which afford it.

We can create many OZVOAs in the following way.

Definition 2.9 Take a VOA V of CFT type. Let F be a subgroup of $Aut(V)$ which is fixed point free on the degree 1 part. Then the fixed point subVOA V^F is an OZVOA. Call this procedure (of making ozzies from CFTs) *ozzification.*

A given VOA of CFT type may have many ozzifications, depending on choice of F. One can see several rank 1 examples of LTVOA ozzifications in [DG, DGR]. When the lattice is a root lattice, we can use well-developed knowledge of the finite subgroups of Lie groups [GRS][GRQE]. In $E_8(\mathbb{C})$, there are many fixed point free finite subgroups, for example ones isomorphic to $PSL(2,q)$, for at least $q = 5, 9, 16, 31, 32, 41, 49, 61$. A nontoral elementary abelian 2-group of rank 5 in $E_8(\mathbb{C})$ gave the example in [GrCol]. In $E_7(\mathbb{C})$, there is $PSU(3,8)$ and in $E_6(\mathbb{C})$ there is $PSL(2,19)$, for instance. In general, a Lie primitive finite subgroup of a simple Lie group will be fixed point free on the adjoint module (though not conversely). See [GRS], [GRQE] and references therein.

Definition 2.10 Let k be an integer. The *degree-k automorphism group* of a VOA V is $Aut(V,k)$, the restriction of $Aut(V)$ to V_k. It acts as automorphisms of the algebra $(V_k,(k-1)^{th})$, so we have a containment $Aut(V,k) \leq Aut((V_k,(k-1)^{th}))$.

3 Automorphism groups

It would be nice to know that V^G is finitely generated when V is finitely generated and G is a finite subgroup of $Aut(V)$.

In the cases we examine in this article, we think this is so and is probably not difficult to check, but we have not verified this in all cases.

Now we begin a search for VOAs whose automorphism groups are not yet covered by existing theory. We take a clue from [GrCol]; see 3.2.

PS. (Program of Study.) We take a VOA V, a finite subgroup F of $Aut(V)$, then study V^F and $Aut(V^F)$. We have a map $\varphi : N_{Aut(V)}(F) \to Aut(V^F)$.

PO. (Possible Outcomes.)

PO1. $Im(\varphi) = Aut(V^F)$.

PO2. $Im(\varphi) \underset{\neq}{\triangleleft} Aut(V^F)$.

PO3. $Im(\varphi) \ntriangleleft Aut(V^F)$.

In each case PO2, PO3, we can further subdivide into cases of where the index is small or large, or in the case of a containment of algebraic groups, we can consider the difference or ratio of dimensions.

Example 3.1 We look at analogues of (PO1,2,3) for containments of Lie algebras. We get corresponding statements about containment of lattice type VOAs whose automorphism group is the same as the underlying Lie algebra [DN].

Consider the containment of Lie algebras $D_8 \subset E_8$, realized by taking fixed points of an involution t from class $2B$. Let $F := \langle t \rangle$, a group of order 2. The subalgebra has a graph automorphism but it is not realized by an element of $N(F) \cong HSpin(16,\mathbb{C})$, a connected group. So we are in (PO2).

The containment $D_4 \subset D_5$ gives an example of (PO3).

The containment $D_4 \subset E_6$ gives an example of (PO1).

Strategy. We input good choices V, F like
(S1.) $V = V_L$, a LVOA based on the lattice L;
(S2.) F not in the natural torus based on L;
(S3.) some nonabelian F, with $N(F)/F$ not small.

The reason for lattice type in (S1) is that their automorphism groups are familiar. If the restriction in (S2) does not hold, $V^F = V_M$ for a finite index sublattice M of L, where M is the common kernel of all the maps in $Hom(L, \mathbb{C}^\times)$ which correspond to elements of F; this is uninteresting for us since $Aut(V_M)$ is nothing new. The reason for nonsmallness in (S3) is to give some visible structure to $Aut(V^F)$. For an idea of difficulty in "small" cases, see [DG] and [DGR].

Example 3.2 [GrCol]. If we take $L = \sqrt{2}L_{E_8}$ and let F be the group of order 2 generated by a lift of -1 [DGH], then $N(F)/F \cong 2^8{:}O^+(8,2)$. Since $Aut(V^F) \cong O^+(10,2)$, we have a case of (PO3), of index $17.31 = 527$. There is an embedding of L as a sublattice of $M \cong L_{E_8}$. Since $Aut(V_M) \cong E_8(\mathbb{C})$, there is $E \cong 2^5$ (unique up to conjugacy) in $Aut(V_M)$ which is $2B$-pure of rank 5. It is nontoral, and any subgroup E_0 of index 2 in E is toral. It happens that $V_M^E \cong V_L^F$. In more detail, we may take the embedding of L in M and choice of E and E_0 to satisfy $V_M^{E_0} = V_L$ and $V_M^E = V_L^F$. So, we have an action of $N_{Aut(V_M)}(E) \cong 2^{5+10}GL(5,2)$ with kernel containing E on the VOA V_L^F. It turns out that $Aut(V_L^F) \cong O^+(10,2)$ is generated by its two subgroups produced as above, of respective shapes $2^8{:}O^+(8,2)$ and $2^{10}{:}GL(5,2)$. The determination of the automorphism group depends critically on the presence of a class of 3-transpositions and the Fischer classification. Note that $dim_q(V_L^F) = 1 + 156q^2 + \cdots$.

Example 3.3 Take L to be the lattice $\sqrt{2}L_{D_4} \cong 2L_{D_4}^*$. The VOA V_L^+, with q-dimension $1 + 22q^2 + \cdots$, is probably isomorphic to the code VOA V_{MM} studied by Matsuo and Matsuo [MM], who determined that $Aut(V_{MM}) \cong 2^6{:}[GL(3,2) \times GL(2,2)]$. Let us assume that there is such an isomorphism. From the ozzification viewpoint, we simply observe that we have a containment of lattices $L \leq M \cong L_{D_4}$ and we can recognize $V_L \leq V_M$ and $V_L^+ = V_M^E$, where E is a nontoral subgroup of $D_4(\mathbb{C}){:}Sym_3$ of order 8 whose normalizer has shape $2^{3+6}[GL(3,2) \times GL(2,2)]$.

Example 3.4 Now take the Borovik group, B, which contains $B_5 \times B_6$ with index 4, where $B_5 \cong Alt_5$ and $B_6 \cong Alt_6$ [Bor][CGB][FG]. An inner product calculation shows that $V_{L_{E_8}}^{B_5}$ has dimension 64 and $V_{L_{E_8}}^{B_6}$ has dimension 10. The respective commutative subalgebras have not been identified yet. By dimensions only, $Mat_8(\mathbb{C})^+$ and $SymMat_4(\mathbb{C})^+$ are reasonable guesses.

Remark 3.5 More examples for $L = L_{E_8}$, F a maximal nontoral elementary abelian p-group, are being investigated. Here, $Aut(V^F)$ in most cases seems to be only $N(F)/F$.

4 27-dimensional algebras.

We shift emphasis from the prime 2 to the prime 3 and find an interesting situation along the lines of the previous section. In particular, we create a 27-dimensional nonassociative commutative algebra, denoted $\mathcal{A}$ (it is defined in 4.3). We prove some basic properties of this algebra and settle its automorphism group. For some arguments, we assume the classification of finite simple groups.

At first, we thought that $\mathcal{A}$ could be the exceptional Jordan algebra (for example, each has a group of automorphisms isomorphic to $ASL(3,3)$ (see Notation 4.2) and a system of subalgebras isomorphic to $Mat_3(\mathbb{C})^+$). This algebra turns out not to be Jordan, and is possibly a previously unknown algebra. After our results were obtained, Gerald Höhn called our attention to [CH] which defines a 27-dimensional algebra in an E_6-related situation similar to ours. It seems possible that these two algebras are isomorphic (but look ahead to the Suspect List after 4.6). For completeness, we note the article [GuHy] on a related topic, that of realizing an affinization of the exceptional Jordan algebra.

Let $\omega := e^{2\pi i/3}$, a primitive cube root of unity. [3] For an even integral lattice L, we use the common notation $L_n := \{x \in L \mid (x,x) = 2n\}$, the set of lattice vectors of type n. In V_L, $\mathbb{C}L_n$ denotes the span of all e^λ, for $\lambda \in L_n$. See [GrCol] for work with similar notations.

Notation 4.1 Let $L = L_{E_6}$, $V = V_L$. Then $Aut(V) \cong E_6(\mathbb{C})$:2 and it has a nontoral subgroup $E \cong 3^3$ so that $J := N(E) \cong 3^{3+3}{:}GL(3,3)$. See [GrElAb] (for the adjoint form, $E_6(\mathbb{C})$). Also, $J \cap Aut(V)^0$ has shape $3^{3+3}{:}SL(3,3)$. It is obtained from the normalizer of a maximal torus $\mathbb{T}$ as follows. There is an element $\theta \in N(\mathbb{T})$ of order 3 so that the corresponding element of the Weyl group has minimal polynomial x^2+x+1. Then $E_1 := C_{\mathbb{T}}(\theta) \cong 3^3$ and we may arrange for $E = \langle E_0, \theta\rangle$ where $E_0 := E \cap \mathbb{T}$ is a subgroup of order 3^2 in $C_{\mathbb{T}}(\theta)$. See [GrElAb] and Section 8.2.

Notation 4.2 For an integer n and field K, $AGL(n,K)$ denotes the *affine general linear group*, the group of affine transformations on K^n. It is the semidirect product of the group of translations by the complement $GL(n,K)$. If $SL(n,K)$ is used instead, the semidirect product is called $ASL(n,K)$, the *affine special linear group.*

We take V^E and note that we have an action of $N(E)/E \cong AGL(3,3)$ on this. The action is faithful on ${V_2}^E$ since $N(E) \cap \mathbb{T}$ acts nontrivially (because it contains $C_{\mathbb{T}}(\theta) \cong 3^3$) and $N(E)/E$ has a unique minimal normal subgroup. Also, faithfulness follows from the Galois theory for VOAs, Section 8.5.

It is easy to study the homogeneous pieces of V:

$V_0 = \mathbb{C}\mathbf{1}$;

$V_1 = H_1 + \mathbb{C}L_1$;

$V_2 = S^2H_1 + H_2 + H_1\mathbb{C}L_1 + \mathbb{C}L_2$.

On $H := \mathbb{C} \otimes L$, θ has spectrum ω with multiplicity 3 and $\bar{\omega}$ with multiplicity 3. Write $H_{k,\alpha}$ for the α-eigenspace of θ in H_1.

Since E_0 is in the natural torus associated to L, V^{E_0} is the natural subVOA V_M, where M is the index 9 sublattice of L which is the common kernel of all the elements of $Hom(L, \mathbb{C}^\times)$ which correspond to the elements of $E_0 \le \mathbb{T}$. The lattice M satisfies $M_1 = \emptyset$ and $|M_2| = 54$. See Section 8.2.

Now, $V^E = V_M^{\langle\theta\rangle}$. Since θ is fixed point free on the lattice and has the above spectrum, we have ${[V_M]_1}^{\langle\theta\rangle} = 0$ and ${[V_M]_2}^{\langle\theta\rangle} = H_{1,\omega}H_{1,\bar{\omega}} \oplus \bigoplus_{\mathcal{O}} \mathbb{C}s(\mathcal{O})$, where $\mathcal{O}$ runs over the 18 orbits of $\langle\theta\rangle$ on M_2 and $s(\mathcal{O})$ is an orbit sum, that is $s(\mathcal{O})$ is a sum of three elements in $\hat{M}$ in an orbit of $\langle\theta\rangle$, one from each 1-space $\mathbb{C}e^x$, where x

[3]Unfortunately, ω is a common abbreviation for both a cube root of unity and for the principal Virasoro element of a VOA; the latter meaning will not be used in this section.

runs over the three vectors of an orbit of $\langle\theta\rangle$ on M_2. See Section 8.2 (but note the different meanings of M and J).

Notation 4.3 Write $\mathcal{A} := ({V_2}^E, 1^{st})$. We write 1 for the unit (in fact, 1 is $\frac{1}{2}$ times the principal Virasoro element). Write ϕ for the representation of J on $\mathcal{A}$.

A basis for $\mathcal{A}$ is the union of these two sets (1) the set of all $u \otimes v$, where u runs over a basis of $H_{1,\omega}$ and v runs over a basis of $H_{1,\bar{\omega}}$; and (2) the set of all above $s(\mathcal{O})$.

Lemma 4.4 As a module for $O_3(J)/E$, $\mathcal{A}$ is the regular representation and if X, Y are two eigenspaces, then $XY \neq 0$, whence XY is an eigenspace.

Proof. Since J/E acts faithfully and on $O_3(J)/E$ the action is transitive on non-identity elements, $\mathcal{A}$ contains a copy of the regular representation with the principal representation removed. Since the identity element of the algebra is fixed, the principal representation of $O_3(J)/E$ is present, and we have the first statement. For the second, just observe that X and Y are centralized by an order 3 subgroup $O_3(J)/E$, whence they lie in a subalgebra conjugate under J to $H_{1,\omega}H_{1,\bar{\omega}} \cong Mat_3(\mathbb{C})^+$, where $XY \neq 0$ follows from inspection. Obviously, XY is an eigenspace. □

Lemma 4.5 The fixed point space of E_1 on V_2^E is just $H_{1,\omega}H_{1,\bar{\omega}}$.

Proof. *First proof.* Since V_2^E is isomorphic to the regular representation of $O_3(J)^\phi$ (see 4.4, it follows that the fixed points of any subgroup of order 3, such as $E_1^\phi \cong 3$, has dimension $27/3 = 9$. It follows that the 9-dimensional subspace $H_{1,\omega}H_{1,\bar{\omega}}$ is the full set of fixed points for E_1 on V_2^E.

Second proof. Since the action of E_1 fixes the polynomial factor of V_L pointwise, it suffices to show that each $s(\mathcal{O})$ is an eigenvector for a nontrivial linear character of E_1. This is proved by inner product calculations in the E_6-lattice (see Appendix 8.2 for notations; we may take E_0 to be the exponentials of the lattice spanned by $\sum_j \alpha_j$, $3\alpha_i, 3\beta_i, \alpha_i - \beta_i$, for $i = 1, 2, 3$, and E_1 as the exponentials of the span of E_0 and $\frac{1}{3}\sum_i \alpha_i - \beta_i$; check that inner products of any norm 4 vector and the second lattice contains $\pm 1 \notin 3\mathbb{Z}$). □

Lemma 4.6 The image of $O_3(J)/E$ in $Aut(\mathcal{A})$ is selfcentralizing.

Proof. Suppose that g centralizes $O_3(J)/E$. Then g acts as scalar c_λ on the eigenspace U_λ for the character λ. Since the product of U_λ and U_μ is $U_{\lambda\mu}$ (see 4.4) and since g is an automorphism, the function $\lambda \mapsto c_\lambda$ is a group homomorphism, so gives a character of the group of characters of $O_3(J)/E$. This means that g acts like an element of $O_3(J)/E$. □

We start our analysis of its automorphism group by making some observations. On V_2 and ${V_2}^E$, we have nondegenerate symmetric bilinear forms which are invariant by their automorphism groups. Therefore, the annihilator of 1 is a submodule for the automorphism group which complements $\mathbb{C}1$.

Consider the following commutative 27 dimensional algebras, which, because of the action of $ASL(3,3)$, seem at first to be reasonable candidates for $\mathcal{A}$:

Suspect list of 27 dimensional algebras.

(a) The exceptional Jordan algebra, automorphism group $F_4(\mathbb{C})$;

(b) the associative algebra $\bigoplus_1^{27} \mathbb{C}$, automorphism group Sym_{27}; or deformations of these (see 1.2).

(c) an algebra with automorphism group containing $G := O^{+,\eta}(6,3)$, for $\eta = \pm$; there is an irreducible X of degree 26 for the simple group $G' \cong \Omega^+(6,3) \cong PSL(4,3)$ and its character χ satisfies $(S^2\chi, \chi) = 1 = (S^3\chi, 1) = (S^2\chi, 1)$; see 1.6. Therefore there is an essentially unique nontrivial commutative algebra structure on X and it has an associative symmetric bilinear form. There are several G'-invariant commutative algebra with unit structures on the direct sum of X and a trivial module which contracts to X.

(d) There are 27 dimensional algebras whose automorphism group is just $ASL(3,3)$ or $AGL(3,3)$.

Next, we show that $\mathcal{A}$ is not Jordan, as can be verified by attempting to verify the linearized form of the Jordan identity $u^2(uv) = u(u^2v)$.

Lemma 4.7 Assume that 2 is not a zero divisor. The linearized Jordan identiy is valid in a commutative nonassociative algebra if and only if

$$(yz.x)w - yz.xw + (wy.x)z - wy.xz + (wz.x)y - wz.xy = 0.$$

Proof. [Jac]. □

Lemma 4.8 $\mathcal{A}$ is not a Jordan algebra.

Proof. It suffices to produce a substitution for which the linearized Jordan identity is not valid.

Now let $\mathcal{O}, O', O''$ be three orbits of $\langle\theta\rangle$ on type 2 elements of $\hat{M}$ so that there exist $\lambda \in \mathcal{O}$, $\lambda' \in \mathcal{O}'$, $\lambda'' \in \mathcal{O}''$ so that $\lambda, \lambda', -\lambda''$ form a triangle of type 222 in $\hat{M}$. See Section 8.2. For an orbit $\mathcal{O}$, let $s(\mathcal{O})$ be the orbit sum $\sum_{\lambda\in\mathcal{O}} e^\lambda$. Let pq, rs stand for $p\otimes q + q\otimes p$, $r\otimes s + s\otimes r$. Set $a := pq, b := rs, c := s(\mathcal{O}), d := s(\mathcal{O}')$. See 8.1 for how to multiply these elements in $\mathcal{A}$.

All six terms of 4.7 are multiples of $s(\mathcal{O}'')$. When $\alpha, \beta, \alpha\beta \in \hat{M}$ have type 2, $(e^\alpha)_1(e^\beta) = e^{\alpha+\beta}$. When we take $\lambda, \lambda', \lambda''$ as above, we get $s(\mathcal{O})_1 s(\mathcal{O}') = s(\mathcal{O}'')$. [4] Therefore the coefficients at $s(\mathcal{O}'')$ for each of the six terms of 4.7 are, respectively, η times

$+(p,\lambda'')(q,\lambda'')(r,\lambda'')(s,\lambda'')$
$-[(p,s)(r,\lambda'')(q,\lambda'') + (q,r)(p,\lambda'')(s,\lambda'')]$
$+(p,\lambda)(q,\lambda)(r,\lambda)(s,\lambda)$
$-(p,\lambda)(q,\lambda)(r,\lambda')(s,\lambda')$
$+(p,\lambda')(q,\lambda')(r,\lambda')(s,\lambda')$
$-(p,\lambda')(q,\lambda')(r,\lambda)(s,\lambda)$.

If the Jordan identity holds, the sum of these six terms is 0.

Let $\kappa : \hat{M} \to M$ be the covering map. It is straighforward to check that the projections of $span(\kappa(\{\lambda, \lambda', \lambda''\}))$ to each summand in $H_1 = H_{1,\omega} \oplus H_{1,\bar{\omega}}$ are two dimensional (Hint: show nonsingularity of the Gram matrix with respect to the set of four vectors obtained by projecting each of $\kappa(\lambda), \kappa(\lambda')$ to these two subspaces). For $\{x, y\} = \{\omega, \bar{\omega}\}$, let A_x be the image of $span(\kappa(\{\lambda, \lambda', \lambda''\}))$ in $H_{1,x}$ and let B_y be the annihilator in $H_{1,x}$ of A_y. Then B_x, B_y are in duality as are A_x, A_y and $H_{1,x} = A_x \oplus B_x$.

Now take $p, r \in H_{1,\omega}$ and $q, s \in H_{1,\bar{\omega}}$ so that $(p, s) = 1$ and $(s, A_\omega) = 0$; for instance, we may take nonzero $p \in B_\omega$ and $s \in B_{\bar{\omega}}$. Then all terms are zero except possibly $-(p,s)(r,\lambda'')(q,\lambda'')$, and it can be arranged nonzero by choosing r, q to satisfy $(r,\lambda'') = (q,\lambda'') = 1$. □

[4] Alternatively, we may use triangles in M and the epsilon-function.

Since $\mathcal{A}$ is not Jordan, it follows that (a) is out, as is the first candidate in (b).

We shall get more information about $\mathcal{A}$ relative to the other candidates on the list after we determine $Aut(\mathcal{A})$ which is an algebraic group (since the automorphism group of a finite dimensional algebra is an algebraic group).

Lemma 4.9 Let $\mathbb{F}$ be a field of characteristic not 3 and $R := \mathbb{F}[x]/(x^3-1)$, $R_0 := (x-1)/(x^3-1)$. Consider deformations with respect to $R = R_1 \oplus R_0$, where $R_1 = \mathbb{F}1$. Use basis $y_i := x^i + (x^3-1)$, $i = 0, 1, 2$.

(i) An invertible deformation $(R, *)$ is given by nonzero scalars a, b, c, d so that $y_0 * y_0 = ay_0$, $y_0 * y_i = by_i$ for $i = 1, 2$, and $y_i * y_i = cy_j$ and $y_i * y_j = dy_0$ for $\{i, j\} = \{1, 2\}$,

(ii) An invertible deformation is Jordan (hence associative) if only if $a = b$ and $c^2 = db$. Furthermore, the basis $y_i' := c^{-1}y_i$ for $i \neq 0$, $y_0' := a^{-1}y_0$ and the correspondence $y_k \mapsto y_k'$ give an isomorphism of the original algebra R with the deformation.

Proof. (i) is clear. For (ii), observe that, assuming Jordan, we have power associativity and we get associativity since y_i generates R. Using associativity, work with $y_0 * y_0 * y_i$ gives $a = b$; work with $y_i * y_i * y_i * y_i$ gives $c^3 = cdb$ or $c^2 = db$; work with $y_i * y_j * y_j * y_i$ gives $d^2a = c^2d$, or $da = c^2$. On the other hand, if these conditions hold, define a new basis by $y_i' := c^{-1}y_i$ for $i \neq 0$, $y_0' := a^{-1}y_0$. These elements satisfy $y_p' * y_q' = y_{p+q}'$ for all p, q (indices are integers mod 3). It is clear that the deformation is isomorphic to R. □

Notation 4.10 Let $\mathbb{F}$ be a field of characteristic not 3. Let $\mathcal{J}$ be a 27-dimensional (exceptional) Jordan algebra constructed by a triple basis $\mathcal{B}$ as in [GJ]. The definition of $\mathcal{B}$ was based on a Moufang loop, constructed from a factor set on $\mathbb{F}_3^3$. We have $R < S < \mathcal{J}$, subalgebras of respective dimensions 3, 9, 27 spanned by basis elements corresponding to a flag of subspaces of $\mathbb{F}_3^3$ of respective dimensions 1, 2, 3.

Lemma 4.11 Use notation 4.10. Consider the deformations of R and S induced by an invertible deformation with respect to $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_0$, where $\mathcal{J}_1 = \mathbb{F}1$ and the latter summand is spanned by all basis elements corresponding to nonzero vectors of $\mathbb{F}_3^3$. Then, equivalent are

(i) the deformation of $\mathcal{J}$ is Jordan and is isomorphic to $\mathcal{J}$ by rescaling basis elements;

(ii) the induced deformation of S is Jordan and is isomorphic to S by rescaling basis elements;

(iii) the induced deformation on R is Jordan and is isomorphic to R by rescaling basis elements;

Proof. It suffices to prove (iii) implies (i). Recall the basis of e_λ and factor set construction of [GJ] using the Moufang loop $\mathbb{O}_{81}$.

For any nontrivial character ν, we have a subalgebra $R_\nu := span\{e_1, e_\nu, e_{\nu^{-1}}\}$ where we may apply 4.9 and get a system of nonzero scalars a, b, c, d. These scalars are actually independent of ν since our deformation of $\mathcal{J}$ depends on multiples of the maps p_{ijk}, which restrict to nonzero maps on R (except for $p_{001} = 0$) giving the induced deformation.

As in 4.9 , we may rescale basis elements within each R_ν to satisfy the original multiplication rules for the e_λ by using the formulas in 4.9. The rescaling is independent of ν and has the property that each of $R, S, \mathcal{J}$ is isomorphic to its respective deformation by the linear isomorphism represented by the rescaling.

Therefore, this deformation of $\mathcal{J}$ is a Jordan algebra, isomorphic to $\mathcal{J}$. $\square$

Proposition 4.12 $Aut(\mathcal{A})$ is finite.

Proof. Let $K := Aut(\mathcal{A})^0$. We shall prove that $K = 1$.

Suppose that $Aut(\mathcal{A})$ is positive dimensional. Since J' acts irreducibly on the 26-dimensional space $1^\perp$ and since a proper subgroup of J' of index dividing 26 has index 26 or 13, either K acts irreducibly, or acts with 13 irreducibles of degree 2 or 26 irreducibles of degree 1. In the latter case, K is abelian, hence is a torus. In the middle case, K' is a direct product of 13 copies of $SL(2,\mathbb{C})$. In either of these cases, there is a torus of rank 13 in K.

By the Borel-Serre theorem [BoSe], $O_3(J)$ normalizes a maximal torus $\mathbb{T}$ of K. It centralizes a nontrivial subgroup of $\mathbb{T}_{(3)} := \{x \in \mathbb{T} | x^3 = 1\}$. Since $O_3(J)^\phi$ is self-centralizing 4.6, $O_3(J)^\phi \cap K \geq O_3(J)^\phi \cap \mathbb{T} \neq 1$. Since $O_3(J)^\phi$ is an irreducible module for J, it follows that $O_3(J)^\phi \leq K$. Similarly, we argue that $Z(K) \cap O_3(J)^\phi$ is $O_3(J)^\phi$ or 1. If the former, we have a contradiction to 4.6. Now, 4.6 implies that each component is actually simple and there is just one component, i.e., K is simple.

Since $O_3(J)^\phi \leq K$, the action of J implies that K acts irreducibly. Also, 4.6 and $O_3(J)^\phi$ is not contained in a maximal torus.

The classification of maximal nontoral elementary abelian p-groups [GrElAb] implies that $K = K' \cong F_4(\mathbb{C})$.

There is just one dimension of invariants giving commutative algebra structures on the 26-dimensional irreducible for $F_4(\mathbb{C})$. This follows from the tensor decomposition $S^2 26 = 1 + 26 + 324, \wedge^2 26 = 52 + 273$. From 4.11 and noting that $H_{1,\omega} H_{1,\bar{\omega}} \cong Mat_3(\mathbb{C})^+$, we see that $\mathcal{A}$ is a Jordan algebra, a contradiction to 4.8. $\square$

Notation 4.13 We now have finiteness of $G := Aut(\mathcal{A})$. From 4.6, $O_3(J)^\phi \ntrianglelefteq G$ if $J^\phi < G$. Let S be a minimal normal subgroup of G.

We want to prove that $G = J^\phi$, so let us assume $J^\phi < G$, equivalently that $O_3(J^\phi) \ntrianglelefteq G$.

Lemma 4.14 Assume that $J^\phi < G$. Then S is a finite simple group containing $(J^\phi)'$.

Proof. Assume that $O_3(J)^\phi \cap S = 1$. Then S is a group of order prime to 3. If S were an elementary abelian p-group, then $p \neq 3$ and S has rank $26k$, for some integer $k \geq 1$. Since S acts faithfully on $1^\perp$, $k = 1$, and S acts on $1^\perp$ as a full diagonal group with entries p^{th} roots of unity. Let v_i be a set of eigenvectors. Then $v_i * v_j = 0$ for all i, j, a contradiction to $\mathcal{A}^2 \neq 0$. Therefore, assuming the classification of finite simple groups, we deduce that S is a direct product of isomorphic copies of $Sz(q)$, for $q \geq 8$ an odd power of 2. Since S acts faithfully and J acts irreducibly in degree 26, $q = 8$. The character degrees and Clifford theory give a contradiction.

We conclude that $O_3(J)^\phi \cap S \neq 1$, whence $O_3(J)^\phi \leq S$. From 4.6, $C_G(S) \leq O_3(J)^\phi \leq S$ and $Z(S) \leq O_3(J)^\phi$. Since $O_3(J)^\phi \ntrianglelefteq G$, S is nonabelian, hence a direct product of simple groups. By 4.6, S is simple. The classification of finite simple groups implies that $Out(S)$ is solvable, whence $(J^\phi)' \leq S$. $\square$

The classification of finite simple groups will now be used. The requirements that a simple group contain $(J^\phi)'$, $O_3(J)^\phi = C(O_3(J)^\phi)$ and have an irreducible of degree 26 are met by exactly one group in the Atlas[5]: $PSL(4,3)$. This possibility leads to a contradiction, as we show next. To complete the contradiction in the case $J < G$, we shall later check the full list of finite simple groups.

Proposition 4.15 Then S is not isomorphic to $PSL(4,3)$.

Proof. We know that $G' \cong PSL(4,3)$, so G/G' corresponds to a subgroup of $Out(PSL(4,3)) \cong 2 \times 2$. Since $J \leq G$, this subgroup must contain at least the group of order 2 corresponding to $PGL(4,3)$. However, there is no extension of either degree 26 irreducible of $PSL(4,3)$ to this group, a contradiction. □

Proposition 4.16 Assume that $J^\phi < G$. If S is a finite simple group, then $S \cong PSL(4,3)$.

Proof. We assume the classification of finite simple groups.

Suppose S has Lie type in characteristic 3. Then the Borel-Tits theorem [BT], says that S has a parabolic subgroup P so that $O_3(J)^\phi \leq O_3(P)$ and $(J')^\phi \leq P$. It follows that $P = N_S(O_3(J)^\phi) = (J')^\phi$ or J^ϕ, whence $S \cong A_3(3)$ which has been eliminated by 4.15.

Suppose that S has Lie type in characteristic $p \neq 3$. Since S contains $(J')^\phi$, if S is a classical group, the dimension of its defining representation is at least 26, whence S has rank at least 12. Since S contains a copy of $PSL(12,q)$ or $SL(12,q)$ for some $q = p^f$, $f \geq 1$, we deduce that S contains an elementary abelian p-group of rank 36. This is impossible since S embeds in $GL(26,\mathbb{C})$.

Now suppose that S has Lie type in characteristic $p \neq 3$ but S does not have classical type. Suppose that it contains a copy of $G_2(q)'$, for $q = p^f$. The parabolic of shape $q^{1+6}GL(2,q)$ has the property that a faithful representation has degree at least $q^3(q-1)$. So, $q^3(q-1) \leq 26$ implies that $q = 2$. Therefore, S is defined over the field of 2 elements only and is bigger than $G_2(2)$, so contains $F_4(2)$, a group listed in the Atlas, hence already eliminated. Finally, if S contains no $G_2(q)$, it must be one of $Sz(q)$, ${}^2F_4(q)'$ or ${}^2G_2(q)$. The latter is out since its Sylow 2-groups are abelian and those of $(J^\phi)'$ are not. If S contains some $Sz(q)$ (which is in ${}^2F_4(q)$), a Borel subgroup has a faithful representation of degree 27, whence $27 \geq q(q-1)$, whence $q < 6$; since q is an odd power of 2, $q = 2$. The remaining case is $S \cong {}^2F_4(2)'$. This group has Sylow 3-subgroup of order 27 so can not contain $(J^\phi)'$, a contradiction.

Suppose that S is sporadic. Then S is listed in the Atlas, so has already been eliminated.

Finally, suppose that $S \cong Alt_m$. Since $S \geq (J')^\phi$, $m \geq 27$, so $m = 27$. Since the action of J in a faithful degree 27 permutation representation contains an odd permutation, $K \cong Sym_{27}$. Thus, $\mathcal{A}$ is either the standard permutation module for K or that module tensored with the sign representation.

Since S is triply transitive, the S-invariant algebra structure contracts to an essentially unique algebra on the degree 26 irreducible 1.6 whose full automorphism group is Sym_{27}. Also, $\mathcal{A}$ as a module is the permutation module.

Let t be a automorphism acting as a transposition. Then the fixed point subalgebra $\mathcal{A}^{\langle t \rangle}$ has dimension 26. From this we shall derive a contradiction.

[5]Some errors have been noted in [Atlas], notably in lists of maximal subgroups; see [ABC]. However, our argument depends mainly on group orders, which one expects to be reliably reported.

If F is any subgroup of order 3 in $O_3(J)^\phi$, $\mathcal{A}^F \cong Mat_3(\mathbb{C})^+$ (reason: all are conjugate by the action of J, and we have one represented by a subgroup of the natural torus acting on $[V_M]_2{}^{\langle\theta\rangle} = H_{1,\omega}H_{1,\bar\omega} \oplus \bigoplus_{\mathcal{O}} \mathbb{C}s(\mathcal{O})$, which fixes exactly $H_{1,\omega}H_{1,\bar\omega} \cong Mat_3(\mathbb{C})^+$ (see 8.4).

Also, $\mathcal{A}$ is generated by all such $\mathcal{A}^F$, for $F \leq O_3(J)^\phi$, $|F| = 3$. It follows that there exists such an F so that t does not centralize $\mathcal{A}^F$. The previous paragraph then implies that $\mathcal{A}^F \cap \mathcal{A}^{\langle t\rangle}$ has codimension 1 in $\mathcal{A}^F$. From [Rac], every proper subalgebra has codimension at least 2, a contradiction. □

We have proved:

Proposition 4.17 The automorphism group of the commutative and non-Jordan (therefore nonassociative) algebra $\mathcal{A}$ is isomorphic to $AGL(3,3)$.

Remark 4.18 If ${V_L}^E$ is generated by its degree 2 part, $Aut({V_L}^E) \cong AGL(3,3)$.

5 Associative subalgebras of a commutative nonassociative algebra

One can write down a set of 48 mutually annihilating idempotents in $\mathcal{B}$, using the Leech lattice. Meyer and Neutsch [MN] made an interesting analysis of maximal associative subalgebras of $\mathcal{B}$ and conjectured that any associative subalgebra of $\mathcal{B}$ has dimension at most 48 and is is a direct sum of at most 48 copies of the field.

This conjecture was proven by Miyamoto who showed that such a system of idempotents is connected with a Virasoro frame in the moonshine VOA [Miy]. This is a striking application of infinite dimensional Lie theory to a problem in finite dimensional algebras.

6 A nonassociative commutative algebra in the degree 3 part of a VOA

So far, the VOA-derived finite dimensional algebras (V_{k+1}, k^{th}) of special interest are for $k = 1$ and $k = 2$. For other values of k, it is not clear what to expect. For one thing, the natural candidate for a unit (a scalar multiple of the principal Virasoro element) is missing in degrees other than 2. However, we can report something for $k = 3$, whic occurred in [DG]. One of the problems in [DG] was to restrict possible automorphisms of a VOA of the form $V = V_L^G$. Let W be the space of highest weight vectors for the principal Virasoro subalgebra. The 2^{nd} product on V_3 restricts to a commutative algebra structure on the subspace $W_3 := W \cap V_3$. In this situation, $dim(W_3) = 2$ and it comes with an action of Sym_3 which preserves the product. This can be recognized as an instance of 1.6(b), whence Sym_3 is the full group of algebra automorphisms. This proved what we needed, that the given automorphisms are all the automorphisms. It was nice to see the elementary idea 1.5 explored in the mid-seventies reappear and be useful in the context of VOAs.

7 Towers of VOAs and fluctuating finiteness.

Example 7.1 We give an example of $V > V' > V'' > V''' > V''''$, all VOAs with vacuum and Virasoro elements, so that $Aut(V_2)$, $Aut(V_2')$, $Aut(V_2'')$, $Aut(V_2''')$ are respectively infinite, infinite, finite, infinite, finite.

We take $V = V_L$, $V' = V_L^{E_0}$ and $V'' = V_L^E$ as in Section 4. Next, we take $E < P \leq O_3(N(E))$, so that $|P : E| = 3$. The analysis in Section 1.3 shows that $V''' := V^P$ has $(V_2''', 1^{st}) \cong Mat_3(\mathbb{C})^+$, whose automorphism group contains $PGL(3,\mathbb{C}){:}2$. For V'''', we take the subVOA generated by the principal Virasoro element.

Example 7.2 Let V be the moonshine VOA. In $Aut(V) \cong \mathbb{M}$, there is an involution z whose centralizer $C(z)$ has shape $O_2(C) \cong 2^{1+24}, C/O_2(C) \cong Co_1$. Let $P := O_2(C)$. Then $(V^P)_2$ has dimension 300, and under the 1^{st} product is the Jordan algebra of degree 24 matrices, whose automorphism group is $PO(24, \mathbb{C})$. This has the subVOA generated by the Virasoro element. So, here is a length 3 chain of VOAs whose automorphism groups are finite, infinite, finite (in fact, the identity).

Remark 7.3 Whenever we have VOAs $W > W'$ such that $Aut(W)$ is smaller than $Aut(W')$, there exist automorphisms of W' which do not extend to W.

In general, we have a homomorphism of groups $Aut(V) \to Aut((V_k, (k-1)^{th})$, which is neither a monomorphism nor an epimorphism in general; the image is $Aut(V, k)$ 2.10. These examples raise a general question about how much fluctuation is possible for the groups $Aut((V_k, (k-1)^{th})$ and $Aut(V, k)$ as we pass to subVOAs.

8 Appendices.

8.1 The first product on V_2.

Notation 8.1 $pq = p \otimes q + q \otimes p$ for $p, q \in H_1$ (this is twice the projection of $p \otimes q$ to the symmetric tensors. We use the standard double cover of an even integral lattice.

Let $V = V_L$ be a LVOA. Write $\times$ for the 1^{st} product on V_2. This product on $(V_2, 1^{st})$ satisfies the following rules (among others):

*Prod*1. $$x^2 \times y^2 = 4(x, y)xy, \qquad pq \times y^2 = 2(p, y)qy + 2(q, y)py,$$
$$pq \times rs = (p, r)qs + (p, s)qr + (q, r)ps + (q, s)pr;$$
$$pq \times rs = (p, s)qr + (q, r)ps \qquad \text{when } p, r \in H_\omega \text{ and } q, s \in H_{\bar{\omega}} \;;$$

*Prod*2. $$x^2 \times e^\lambda = (x, \lambda)^2 e^\lambda, \qquad xy \times e^\lambda = (x, \lambda)(y, \lambda)e^\lambda$$

*Prod*3. $$e^\alpha \times e^\beta = e^{\alpha\beta}, \text{ when } \alpha, \beta, \alpha\beta \in \hat{L}_2$$

8.2 The lattice L_{E_6} and the automorphism θ.

Notation 8.2 For an even lattice, Q, write $Q_n := \{ x \in Q \mid (x, x) = 2n \}$, the *vectors of type* n in Q. If $\hat{Q}$ is the usual double cover of Q, we define the set of *type* n *elements of* $\hat{Q}$ as $\hat{Q}_n$, the preimage in $\hat{Q}$ of Q_n.

Lemma 8.3 Let L be the E_6-lattice and Y a root lattice of type A_2, with set of fundamental roots α, β.

1. There is exactly one subgroup of index three in Y which does not contain roots. It is $3Y + \mathbb{Z}(\alpha - \beta)$ and is the radical modulo 3 of Y, i.e., $\{x \in Y \mid (x, Y) \leq 3\mathbb{Z}\} = 3Y^* \cap Y = 3Y^*$.
2. We consider N, an orthogonal direct sum of three type A_2 lattices, called N_i, with respective fundamental roots α_i, β_i, for $i = 1, 2, 3$. A sublattice J of index 3 in N which does not contain roots is one of the following: $J = 3N^* + \mathbb{Z}u + \mathbb{Z}v$, where the ordered pair (u, v) is $(\alpha_1 \pm \alpha_2, -\alpha_1 \mp \alpha_3)$, $(\alpha_1 \pm \alpha_2, -\alpha_1 \pm \alpha_3)$, $(\alpha_1 \mp \alpha_2, -\alpha_1 \mp \alpha_3)$ or $(\alpha_1 \mp \alpha_2, -\alpha_1 \pm \alpha_3)$. Define the vector $w \in J$ by $u + v + w = 0$.

3. The set J_2 has cardinality 54 and is partitioned into the six 9-sets $x + 3N^* \cap J_2$, for $x \in \{\pm u, \pm v, \pm w\}$.
4. This set of four sublattices forms an orbit under the Weyl group of N, isomorphic to $Sym_3 \times Sym_3 \times Sym_3$.
5. A rootless index 3 sublattice of N is isometric to $\sqrt{3}L^*$.
6. An element of order 3 in the Weyl group of L which has 0 fixed points on L and stabilizes N stabilizes and acts nontrivially on each of the three orthogonally indecomposable summands N_i of N. The 54 lattice vectors of type 2 form 18 orbits of length 3.

Proof. The first statement is an exercise, and it is easy since $Y/3Y$ has exactly four subgroups of index 3.

For the second, let P be such a sublattice. We claim that P contains $3N_i^*$, for all i. Suppose this is false for index i. Then $P \cap N_i$ is proper in N_i, so has index exactly 3. Examination of the two cosets of $3N_i^*$ in N_i shows that P contains a root, contradiction. Also, rootlessness of P implies that $N_i \cap P = 3N_i^*$, for all i. Now it is clear that for distinct indices i, j, one of $\alpha_i \pm \alpha_j$ lies in P. Such elements and $3N^*$ generate P.

The third statement follows easily from properties of the A_2-lattices N_i.

The fourth statement is easy since a reflection at α_i takes α_i to its negative and acts trivially on the other N_j.

For the fifth statement take $J := 3N^* + \mathbb{Z}(\alpha_1 - \alpha_2) + \mathbb{Z}(\alpha_2 - \alpha_3)$, determinant 3^5. Its dual is $N^* + \mathbb{Z}\frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)$. Note that $N^* \cong \frac{1}{\sqrt{3}}N$ (because $L_{A_2}^* \cong \frac{1}{\sqrt{3}}L_{A_2}$) and $(N^*, \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)) \leq \frac{1}{3}\mathbb{Z}$ and $(\frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3), \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)) = \frac{2}{3}$. Clearly then this is a root lattice rescaled by $\frac{1}{3}$ and by indecomposability and determinant, the classification of root systems implies that this lattice is the E_6-lattice.

The sixth statement follows since an isometry of order 3 which cycles the three summands has fixed point sublattice of rank 2. Therefore, such an element of order 3 in the Weyl group fixes each summand. □

8.3 An additional product. Here, $\mathcal{O}$, $\mathcal{O}'$ denote orbits of $\langle\theta\rangle$ on $\hat{J}_2$. Here, L, M, J are as in 8.3 and θ is a fixed point free element of order 3 from the Weyl group of L, lifted to an automorphism of $\hat{L}$. Let $s(\mathcal{O}) := \sum_{\alpha \in \mathcal{O}} e^\alpha \in V_L$.

*Prod*4.

$$s(\mathcal{O}) \times s(\mathcal{O}') = 0, s((\lambda\lambda')^{\langle\theta\rangle}), \frac{1}{2}[\lambda(-1)^2 + \mu(-1)^2 + \nu(-1)^2]$$

as $\bar{\mathcal{O}} = \bar{\mathcal{O}}'$, $\lambda\lambda' \in \hat{J}_2$, $\lambda\lambda' = 1$, respectively.

8.4 A subalgebra of S^2H. Let H be a finite dimensional vector space with a nondegenerate symmetric bilinear form, (,). The degree 2 tensors $H \otimes H$ has a product based on $w \otimes x \cdot y \otimes z = (x, y) w \otimes z$ which makes an algebra isomorphic to square matrices of degree $dim(H)$.

The degree 2 symmetric tensors S^2H correspond to the Jordan algebra of symmetric matrices under the product $A, B \mapsto A \circ B := \frac{1}{2}(AB + BA)$. The orthogonal group on H acts as automorphisms of this Jordan algebra.

We need to identify a subalgebra of S^2H.

Lemma 8.4 Let g be an orthogonal transformation on H. Let $\alpha \neq \pm 1$ be an eigenvalue. Take bases v_i of the α-eigenspace and w_j of the α^{-1} eigenspace so that

$(v_i, w_j) = \delta_{ij}$ for all $i, j = 1, \dots, d$ where d is the common multiplicity of α and α^{-1}. Let The subalgebra of S^2H spanned by the linearly independent set $v_i w_j$, for $i, j = 1, \dots, d$, is isomorphic to $Mat_d(\mathbb{C})^+$.

Proof. Let E_{ij} be the usual matrix units, with zeroes everywhere except for a 1 in the i, j position, for $i, j = 1, \dots, d$.

The hypothesis on α imply that the α-eigenspace and α^{-1}-eigenspace are totally singular, whence $u_i \otimes w_j \cdot u_k \otimes w_l = \delta_{jk} u_i \otimes w_l$.

It is straightforward to verify that the linear isomorphism $v_i w_j \mapsto 2E_{ij}$ gives an isomorphism of Jordan algebras. □

8.5 Galois Theory for VOAs. Combined work of Akihide, Daisuke, Dong, Mason and Miyamoto [DMQ], [DMGT], [ADM] shows that if V is a simple VOA and G is a finite group of automorphisms, then the lattice of subgroups of G is in bijection with the VOAs between V^G and V. This is a clear analogy with the Galois theory for field extensions.

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