Nonassociativity in VOA theory and finite group theory

Mile High Conference on Nonassociative Mathematics,
25 June, 2009

Robert L. Griess, Jr.
Department of Mathematics,
University of Michigan
Ann Arbor, MI 48109

Abstract

We give some examples of nonassociative algebras which occur in VOA theory and finite group theory. A good theory of these algebras could be useful.

1 Introduction

This talk is motivated by concerns from finite group theory.

The first is that we do not have good axiom systems for all of the finite simple groups.

The second is that we do not really understand how the sporadic simple groups fit into mathematics.

A satisfactory answer to the first concern could help us with the second. A possible answer to the first concern could be a good theory of some relevant nonassociative algebras.

The finite simple groups
The alternating groups, $Alt_n, n \geq 5$
Finite groups of Lie type, $A_n(q), B_n(q), \ldots, E_8(q), {}^2A_n(q), \ldots {}^2F_4(q)$ (q is a prime power) (for example, $A_n(q)$ is $PSL(n+1,q)$, determinant 1 matrices mod scalars over $\mathbb{F}_q$, the finite field of q elements; ${}^2A_n(q)$ is $PSU(n+1,q)$, $B_n(q)$ is $PSO(2n+1,q)$, etc.)
The 26 sporadic groups: M_{11} (the smallest, order $7920 = 2^4 3^2 5 \cdot 11$) ... $F_1 = \mathbb{M}$ (the largest, order $2^{46}3^{20}5^9 7^6 11^2 13^3 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \times 10^{53}$)

2 Theory for groups of Lie type

Let me concentrate on the groups of Lie type. These are, roughly, analogues over finite fields of the simple real Lie groups.

We start with Lie algebras. These are given by a few simple axioms. One can quickly derive consequences to make a structure theory. In a one term course, one can classify all the simple finite dimensional Lie algebras over the complex numbers.

For groups of Lie type, we have an axiom system derived from Lie theory:

Definition 2.1. A *(B, N) pair* is a pair of subgroups B and N of a group G such that the following axioms hold:

G is generated by B and N.

The intersection $H := B \cap N$ is a normal subgroup of N.

The group $W := N/H$ is generated by a nonempty set of elements S of order 2 such that

if sH is one of the generators of W and n is any element of N, then $sBn \subseteq BsnB \cup BnB$;

if $sH \in S$ then sH contains no element which normalizes B

Example 2.2. $G = GL(m, K)$, $B =$ invertible upper triangular matrices, $N=$ all invertible monomial matrices (diagonal times permutation matrix), $H=$ diagonal matrices in G, $W \cong Sym_m$.

These axioms lead to uniform proofs for structure theory, representation theory, conjugacy, etc. They predict and explain a lot. For example, there are uniform arguments for many aspects of GL_n, orthogonal, symplectic groups, G_2, etc.

3 Theory for sporadic groups

We would like theory with similar qualities for sporadic groups. None is known at this time.

There are reasons to favor the world of commutative associative algebras as a clue to finding a theory. One can see that the following is generally true. Let G be any finite group and Ω a G-set. Form the permutation module $\mathbb{C}\Omega$. This is a direct sum of fields, indexed by Ω, and G permutes the indecomposalbe summands and acts as automorphisms.

Take any submodule A of this and let π be the orthogonal projection $\mathbb{C}\Omega \to A$. Then the product $*$ on A, defined by $x * y := \pi(x \cdot y)$, is a G-invariant commutative algebra structure on A, though it is often not associative. It has an associative bilinear form $(x, y * z) = (x * y, z)$.

There is literature on many cases of this where A is a sporadic group (or central extension of such). In some cases, there are results on $Aut(A, *)$.

Example 3.1. If G is the Monster simple group, order about 10^{54}, and Ω is the conjugacy class $2A$, then $|\Omega|$ is about 10^{20}. There is a direct summand A of the permutation module so that $dim(A) = 196884$ and that A is an algebra with 1 so that A_0, the annihilator in A of 1, is an irreducible module and has dimension 196883.

Along comes the theory of VOAs in the mid 1980s.

An abbreviated definition of VOA.

$(V, Y, \mathbf{1}, \omega)$ is a VOA means

$V = \bigoplus_{n \in \mathbb{Z}} V_n$, each V_n finite dimensional

$Y : V \to End(V)[[z, z^{-1}]]$ (Y is the "vertex operator"), so for $a \in V$, $Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, for $a_n \in End(V)$.

satisfying MANY axioms including a kind of Jacobi identity (power series in several variables; $\mathbf{1}$ is a vacuum element, ω is a Virasoro element)

Consequences.

(1) For each k, we have a product on V, $a, b \mapsto a_k(b)$, where $Y(a, z) = \sum_i a_i z^{-i-1}$. It takes $V_i \times V_j \to V_{i+j-k-1}$. If $i = j = k+1$, then (V_{k+1}, k^{th}) is a finite dimensional algebra.

(2) If V is *CFT type* ($V_n = 0$ for $n \leq -1$, $V_0 = \mathbb{C}\mathbf{1}$), then $(V_1, 0^{th})$ is a Lie algebra.

(3) If V is an *OZVOA* (=CFT type and $V_1 = 0$), then $(V_2, 1^{st})$ is *commutative.*

(Here we get associative algebras, classical Jordan algebras, $\mathcal{B}$ and many others).

(4) If G is any group of automorphism of a VOA, V, the set of fixed points V^G is also a VOA. (By definition, an automorphism of a VOA preserves $\mathbf{1}$ and ω and so preserves each V_n. Therefore, $V^G = \bigoplus_n V_n^G$.)

4 Lattice type VOAs and algebra for the degree 2 term

Suppose that L is a lattice, i.e. a free abelian group in Euclidean space so that $(x, y) \in \mathbb{Z}$ for all $x, y \in L$.

There is a standard way to make a VOA from L. As a linear space it looks like $V = \mathbb{S} \otimes \mathbb{C}[L]$, where $\mathbb{C}[L]$ is the group algebra of L (basis e^{α},
$a \in L$) and where $\mathbb{S}$ is the symmetric algebra on the vector space $H \otimes t^{-1} \oplus H \otimes t^{-2} \oplus H \otimes t^{-3} \oplus \cdots$.

Grading on V based on $deg(e^{\alpha}) = \frac{1}{2}(\alpha, \alpha)$, $deg(H \otimes t^{-m}) = m$.

In particular, if we take an isometry of the lattice L, it can be lifted to an automorphism of V_L. The -1 isometry lifts, and the set of fixed points of the lift is denoted V_L^+.

A context for many commutative algebras associated to finite simple groups is in VOAs, as the degree 2 piece.

Note. Formulas for multiplying the standard basis elements of $(V_L)_2$ are given in several places, including [1] and [2]. Examples: $S^2(H \otimes t^{-1})$ is isomorphic to the algebra of symmetric matrices with product $A \circ B = \frac{1}{2}(AB + BA)$; $(e^{\alpha} + e^{-\alpha}) * (e^{\beta} + e^{-\beta}) = \pm(e^{\alpha+\beta} + e^{-\alpha-\beta})$ when α, β are norm 4 vectors such that $(\alpha, \beta) = -2$.

Some care is needed for the "sign function" (though it is identically 1 for the 156-dimensional example).

5 The 196884-dimensional algebra $\mathcal{B}$

The dimension 196884 dimensional algebra for the Monster, usually denoted $\mathcal{B}$, occurs as the degree 2 piece of $V^\natural$ the Moonshine VOA of Frenkel-Lepowsky-Meurman. This VOA has graded dimension $1+196884q^2+21493760q^3+864299970q^4+20245856256q^5+\cdots$, which is essentially the elliptic modular function + constant with degrees shifted.

Theorem 5.1. *There is no nonzero homogeneous polynomial identity for $\mathcal{B}$ of degree less than or equal to 5.*

Question. A nontrivial homogeneous polynomial identity exists, but can it be of practical use?

There is a lot of work on subalgebras of $\mathcal{B}$ generated by idempotents, by Conway, Matsuo, Meyer-Neutsch, Norton, ...

Theorem 5.2. *(Miyamoto) A maximal set of pairwise orthogonal idempotents in $\mathcal{B}$ has cardinality 48. Therefore, 48 is the maximum dimension of an associative semisimple subalgebra.*

The proof uses VOA theory. A finite dimensional statement uses infinite dimensional techniques!

6 A 156-dimensional algebra

This uses an 8-dimensional lattice $M \cong \sqrt{2}E_8$. Define H to be the ambient complex vector space $\mathbb{C} \otimes M$.

We take the lattice VOA V_M and its subVOA V_M^+. This is lattice type and the degree 1 term is 0. This has degree 2-part which looks, as a linear space, like $S^2(H) \oplus \bigoplus \mathbb{C}(e^\alpha + e^{-\alpha})$, sum over pairs $\alpha, -\alpha$ of norm 4 vectors in M (there are 120 such pairs). The dimension of H is 8 and the dimension of $S^2(H)$ is $\binom{9}{2} = 36$. So $(V_M^+)_2$ has dimension 120+36=156. Its automorphism group is $O^+(10, 2)$.

This algebra is a subalgebra of $\mathcal{B}$.

7 A 27-dimensional algebra

Here is an example coming from fixed points of a group of odd order.

Take the root lattice $L = E_6$ and form V_L. There is a group of automorphism, E of order 3^3 so that $(V_L^E)_1 = 0$ and $(V_L^E)_2$ has dimension 27 and is commutative. It has $3^3{:}SL(3,3)$ in its automorphism group. It fixes 1 and has an irreducible 26-dimensional complement.

There is a long list of finite groups with an irreducible degree 26 representation and which contains $3^3{:}SL(3,3)$ as a subgroup. Some of these groups leave invariant a commutative algebra structure. One is the finite group $PGL(4,3)$ and another is the Lie group $F_4(\mathbb{C})$.

The algebra $(V_L^E)_2$ does not satisfy the Jordan identity, so is not the exceptional 27-dimensional Jordan algebra. Finally, it turns out to have finite automorphism group $3^3{:}GL(3,3)$.

8 Possible directions

(1) Study some VOAs V with small dimensional degree 2 term to determine identities, connections between the algebra product on V_2 and automorphisms (e.g., idempotents and involutions).

(2) If the algebra $\mathcal{B}$ has a uniqueness result (aside the requirement that it supports the monster as automorphism group), it would have important applications to VOA theory. One could approach this by proving uniqueness results for some of its subalgebras. We need a characterization of $\mathcal{B}$ in some way as an algebra.

(3) Can one take a commutative algebra, A, and create a VOA V so that $(V_2, 1st) \cong A$? This is open, and hard.

(4) Is the study of identities on a finite dimensional commutative algebra the right way to go? Is there a good alternative?

9 References

There will be references and other material on my web page, in the research section on nonassociative algebras and loops.

References

[1] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, 1988.

[2] Robert L. Griess, Jr. A vertex operator algebra related to E_8 with automorphism group $O^+(10,2)$, article in The Monster and Lie Algebras, ed. J. Ferrar and K.Harada, deGruyter, Berlin, 1998.