A new existence proof of the Monster by VOA theory

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Abstract

We use uniqueness of a VOA (vertex operator algebra) extension of $(V_{EE_8}^+)^3$ to a Moonshine type VOA to give a new existence proof of a finite simple group of Monster type. The proof is relatively direct. Our methods depend on VOA representation theory and are free of many special calculations which traditionally occur in theory of the Monster.

1

Contents

1 Introduction

We define a finite group G to be of *Monster type* if it has an involution z whose centralizer $C_G(z)$ has the form $2^{1+24}Co_1$, is 2-constrained (i.e., satisfies $\langle z \rangle = C_G(O_2(C_G(z)))$) and z is conjugate to an element in $C_G(z) \setminus \{z\}$. A short argument proves that such a G must be simple (e.g., see [20, 49]). The present article gives a new and relatively direct existence proof of a group of Monster type. In fact, a group of Monster type is unique up to isomorphism [21], so the group constructed in this article can be called "the" Monster, the group constructed in [20]. To avoid specialized finite group theory in this article, we work with a group of Monster type and refer to [21] for uniqueness.

    Our basic strategy is described briefly in the next paragraph. It was inspired by the article of Miyamoto [38], which showed how to make effective use of *simple current modules and extensions*. Later in this introduction, we sketch these important concepts. In a sense, our existence proof is quite short. The hard group theory and case-by-case analysis of earlier proofs have essentially been eliminated. We use the abbreviation VOA for *vertex operator algebra* [15]. Most of this article is dedicated to explaining how existing VOA theory applies.

In [44], Shimakura takes $(V_{EE_8}^+)^3$ and builds a candidate V for the Moonshine VOA (a short account is in Section 2.1 of the present article). His construction furnishes a large subgroup of $Aut(V)$. From this subgroup, we take a certain involution and analyze V^+, V^-, its fixed point VOA and its negated space on V, respectively. We can recognize V^+ as a Leech lattice type VOA. The group $Aut(V^+)$ and its extension (by projective representations) to irreducibles of the fixed point VOA are understood. One of these irreducibles is V^-. We thereby get a new subgroup of $Aut(V)$ which has the shape $2^{1+24}Co_1$ and is moreover isomorphic to the centralizer of a 2-central involution in the Monster. These two subgroups of $Aut(V)$ generate the larger group $Aut(V)$, which we then prove is a finite group of Monster type.

We refer to the overVOA of $(V_{EE_8}^+)^3$ constructed in (2.20) as a VOA of *Moonshine type*, meaning a holomorphic VOA $V = \oplus_{n=0}^\infty V_n$ of central charge 24, so that V_0 is 1-dimensional, $V_1 = 0$ and the Monster acts as a group of automorphisms with faithful action on V_2. We mention that such a VOA is isomorphic to the standard Moonshine VOA constructed in [15], by [8, 29]. For the purpose of this article, it is not necessary to quote such characterizations.

The theory of simple current modules originated in the papers [17] and [41]. In [9, 32], certain simple current modules of a VOA are constructed using weight one semi-primary elements and extensions of a VOA by its simple current modules are also studied. The notion of simple current extension turns out to be a very powerful tool for constructing new VOAs from a known one [6, 27, 30, 32, 34, 38]. Let V be a VOA and let $\mathcal{M} = \{M^i \mid i \in I\}$ be a finite set of irreducible modules of V with integral weights. If $V \in \mathcal{M}$ and $\oplus_{M \in \mathcal{M}} M$ is closed under the fusion rules, it is possible that $\oplus_{M \in \mathcal{M}} M$ carries the structure of a VOA for which V is a subVOA. In general, it is extremely difficult to determine if $\oplus_{M \in \mathcal{M}} M$ has a VOA structure. There may be no such VOA structures, or there could be many. When the simple current property holds (2.4), there is a VOA structure on $\oplus_{M \in \mathcal{M}} M$ extending the given action of V and the VOA structure is unique if the underlining field is algebraically closed [12, Proposition 5.3] (see also [13, 34]). This "rigidity" of simple current extensions is useful in structure analysis and leads to certain transitivity results, which reduce the need for calculations.

As described in [20], existence of the Monster implies existence of several other sporadic groups which had originally been constructed with special methods, including computer work. We hope that the present article may suggest useful viewpoints for other sporadic groups.

1.1 About existence proofs of the Monster

The first existence proof of the Monster was made in 1980, and published in [20]; see also [19]. A group $C \cong 2^{1+24}Co_1$ and a representation of degree 196883 was described. The hard part was to choose a C-invariant algebra structure, give an automorphism σ of it which did not come from C, then identify the group $\langle C, \sigma \rangle$ by proving finiteness and proving that C is an involution centralizer in it.

During the decade that followed [20], there were analyses, improvements and alternate viewpoints by Tits [46, 47, 48, 49] and Conway [1]. In the mid-80s, the theory of vertex algebras was developed. The Frenkel-Lepowsky-Meurman text [15] established the important construction of a Moonshine VOA and became a basic reference for VOA theory. The construction of the Monster done in [15] followed the lines of [20], but in a broader VOA setting. The articles [2, 3] constructed a VOA and gave a physics field theory interpretation to aspects of [15, 20].

In 2004, Miyamoto [38], made significant use of simple current extensions to give a new construction of a Moonshine VOA and of the Monster acting as automorphisms. An existence proof of the Monster was recently announced in [26], which uses theories of finite geometries and group amalgams.

Uniqueness was first proved in [21]. A different uniqueness proof is indicated in [26].

1.2 Acknowledgments

1.3 Table of Notations

Notation	Explanation	Examples in text
$A.B, A{:}B,$ $A{\cdot}B$	group extension of normal subgroup A by quotient B, split extension, nonsplit extension, respectively	(2.25) (3.8),
BW_{16}	the Barnes-Wall lattice of rank 16	Lemma 2.28
C	the centralizer of an involution in $Aut(V)$	(3.9)
$cvcc\frac{1}{2}$	simple conformal vector of central charge $\frac{1}{2}$	Sec. 3.1
Co_1	the first Conway group $O(\Lambda)/\{\pm 1\}$	Page 2
EE_8	lattice isometric to $\sqrt{2}$ times the famous E_8 lattice	Page 2 Sec 2.1
$\mathbb{F}\{L\}$	a twisted group algebra of a lattice L over a field $\mathbb{F}$	Coro. 3.7
$g \circ M$	g-conjugate module of a V-module M	Notation 2.5
H_Λ	the subgroup of $Aut(V_\Lambda^+)$ generated by Miyamoto involutions of AA_1-type	Notation 3.12
$\tilde{H}_\Lambda$	the subgroup of $Aut(V_\Lambda^+)$ generated by Miyamoto involutions of AA_1-type	Notation 3.12
Λ	Leech lattice, the unique even unimodular lattice of rank 24 with no roots	Sec. 3 Coro. 3.7
$M(1)$	the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha \otimes t^n 1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n > 0$ and $K = 1$, where $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} L$ and $\hat{\mathfrak{h}} = \oplus_{n \in \mathbb{Z}}(\mathfrak{h} \otimes t^n) \oplus \mathbb{C}K$	Coro. 3.7
$M \times_V N$	the fusion product of V-modules M, N	Notation 2.1
$O_p(G)$	maximal normal p-subgroup of G	Page 2
$O_{p'}(G)$	maximal normal subgroup of G of order prime to p	Page 2
$R(U)$	the set of all inequivalent irreducible modules of U	Sec. 2.1
$Stab_G(X)$	subgroup of the group G which stabilizes the set X	(4.10)
$t(e)$	Miyamoto involution associated to a $cvcc\frac{1}{2}$ e	Notation 3.15
U	the VOA $V_{EE_8}^+$	(2.15)
2^{1+2n}	extra-special 2-group of order 2^{1+2n}	Page 2
$2^{1+24}Co_1$	an extension of Co_1 by 2^{1+24}	Page 2, Sec. 3
V	a VOA which is a simple current extension of U	(2.21)
V_L	lattice VOA for positive definite even lattice L	Lem. 3.6, Coro. 3.7
$V_{L,\mathbb{R}}$	lattice VOA over $\mathbb{R}$ for positive definite even lattice L	Lem. 3.6, Coro. 3.7
$\tilde{V}_{L,\mathbb{R}}$	a real form of the lattice VOA V_L whose invariant form is positive definite	Prop. 4.17
V_L^+	the fixed point subVOA of V_L by a lift of the (-1)-isometry of L	Sec. 2.1 Coro. 3.7

2 Simple current extensions

In this section, we shall recall the notion of simple current extensions and their basic properties [9, 44].

Let V be a VOA and let M_1, M_2, M_3 be V-modules. Let $I_V\binom{M_3}{M_1\ M_2}$ be the space of all V-intertwining operators of type $\binom{M_3}{M_1\ M_2}$. We refers to [14, Chapter 5] for the definition of intertwining operators.

The *fusion coefficient (or fusion rule)* is defined to be the integer

$$N_{M_1\ M_2}^{M_3} = dim\left(I_V\binom{M_3}{M_1\qquad M_2}\right).$$

Notation 2.1. *Suppose V is rational. Then V has only finitely many inequivalent irreducible modules and all V-modules are direct sums of finitely many irreducibles. Moreover, all fusion coefficients are finite [16]. We define a formal product of two isomorphism types of irreducible modules M_1 and M_2:*

$$M_1 \times_V M_2 = \sum_{M_3} N_{M_1\ M_2}^{M_3} M_3,$$

where M_3 runs through the finitely many isomorphism types of irreducible modules of V. This product is often called the fusion product.

Definition 2.2. *[9] Let V be a rational VOA. A V-module M is called a simple current module if the fusion product $M \times_V N$ is again irreducible for any irreducible V-module N, i.e.,*

$$\sum_{W\ irred} N_{M\ N}^W = 1.$$

By definition, it is clear that $M \cong M \times_V V$ is irreducible if M is a simple current module.

Definition 2.3. *A full subVOA is a subVOA which contains the principal Virasoro element of the larger VOA.*

Now let V^0 be a simple rational C_2-cofinite VOA of CFT type and let D be a finite abelian group. Let $\{V^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible

V^0-modules indexed by D. Assume that the weights of $V^\alpha, \alpha \in D$, are integral and

$$V^\alpha \times_{V^0} V^\beta = V^{\alpha+\beta}$$

for all $\alpha, \beta \in D$.

Definition 2.4 ([9]). *A VOA V is called a (D-graded) simple current extension of V^0 if*
1. *V^0 is a full subVOA of V ;*
2. *$V = \oplus_{\alpha \in D} V^\alpha$;*
3. *All V^α, $\alpha \in D$, are simple current V^0-modules.*

Next we shall recall the notion of g-conjugate modules [11, 44].

Definition 2.5. *Let (M, Y_M) be a V-module and $g \in Aut(V)$ an automorphism. The g-conjugate module of M is defined to be the V-module $(g \circ M, Y_{g \circ M})$, where $g \circ M = M$ as a vector space and $Y_{g \circ M}(v, z) = Y_M(g^{-1}v, z)$ for all $v \in V$.*

Remark 2.6. *By definition, there exists a linear isomorphism $\nu : g \circ M \to M$ such that $\nu Y_{g \circ M}(v, z) = Y_M(g^{-1}v, z)\nu$ for any $v \in V$. In fact, one can assume $\nu = id_M$ by identifying $g \circ M$ with M as vector spaces.*

The following lemma can be shown easily by definition.

Lemma 2.7 (cf. [44]). *Let V be a VOA and $g \in Aut(V)$.*

1. *If W is an irreducible V-module, then $g \circ W$ is also irreducible.*

2. *If $M \cong W$ as a V-module, then $g \circ M \cong g \circ W$.*

3. *Let M_1, M_2, M_3 be V-modules. For each $x \in \{1, 2, 3\}$, let $\nu_x : g \circ M_x \to M_x$ be a linear isomorphism such that $\nu_x Y_{g \circ M_x}(v, z) = Y_{M_x}(g^{-1}v, z)\nu_x$ for any $v \in V$. Then for any nonzero intertwining operator $\mathcal{Y}$ of type $\binom{M_3}{M_1\ M_2}$, the map $\nu_3 \mathcal{Y}(\nu_1 u, z)\nu_2$ is a nonzero intertwining operator of type $\binom{g \circ M_3}{g \circ M_1\ g \circ M_2}$. In particular, the fusion rules are preserved by g-conjugation.*

The following theorem follows easily by the fusion rules $V^\alpha \times V^\beta = V^{\alpha+\beta}$.

Theorem 2.8. *Let V^0 be a rational C_2-cofinite VOA of CFT type and let $V = \oplus_{\alpha \in D} V^\alpha$ be a (D-graded) simple current extension of V^0. Let D^* be the group of all irreducible characters of D. Then for any $\chi \in D^*$, the linear map*

$$\tau_\chi(v) = \chi(\alpha)v \qquad \text{for any } v \in V^\alpha, \alpha \in D,$$

defines an automorphism of V. In particular, $\{\tau_\chi \mid \chi \in D^*\} \cong D^*$ is an abelian subgroup of $Aut(V)$.

Notation 2.9. *By abuse of notation, we often denote the group $\{\tau_\chi \mid \chi \in D^*\} < Aut(V)$ defined in (2.8) by D^*.*

Notation 2.10. *Let W be an irreducible V-module. We shall use $[W]$ to denote the isomorphism class containing W.*

The next theorem gives a criterion for lifting an automorphism of V^0 to V and can be proved using the general arguments for simple current extensions [40, 42].

Theorem 2.11 (cf. [42])**.** *Let V^0 be a rational C_2-cofinite VOA of CFT type and $V = \oplus_{\alpha \in D} V^\alpha$ a (D-graded) simple current extension of V^0. Let $g \in Aut(V^0)$. Then there exists an automorphism $\tilde{g} \in Aut(V)$ such that $\tilde{g}|_{V^0} = g$ if and only if $\{[g \circ V^\alpha] \mid \alpha \in D\} = \{[V^\alpha] \mid \alpha \in D\}$.*

Notation 2.12. *Let $\mathcal{X}$ be a set of isomorphism types of irreducible modules of a VOA V. We denote*

$$g \circ \mathcal{X} = \{[g \circ W] \mid [W] \in \mathcal{X}\}$$

for any $g \in Aut(V)$.

The next theorem follows easily from Theorem 2.11 and a proof can be found in [44].

Theorem 2.13 (Corollary 2.2 of [43])**.** *Let $V = \oplus_{\alpha \in D} V^\alpha$ a (D-graded) simple current extension of V^0. Denote*

$$N_D = \left\{ g \in Aut(V^0) \mid \{[g \circ V^\alpha] \mid \alpha \in D\} = \{[V^\alpha] \mid \alpha \in D\} \right\}.$$

Then there exists an exact sequence

$$1 \longrightarrow D^* \longrightarrow N_{Aut(V^0)}(D^*) \xrightarrow{\eta} N_D \longrightarrow 1,$$

where η is the restriction map to V^0 and D^* is identified with the group $\{\tau_\chi \mid \chi \in D^*\}$.

8

2.1 Simple current extension of $(V_{EE_8}^+)^3$

In this section, we shall recall a construction of the Moonshine VOA by Shimakura [44]. First, we sketch an outline of the Shimakura construction.

It is known [42] (see also [29]) that all irreducible modules of $V_{EE_8}^+$ are simple current modules and the fusion group $R(V_{EE_8}^+)$ of $V_{EE_8}^+$ has a quadratic spaces over $\mathbb{Z}_2$. Moreover, the automorphism group $Aut(V_{EE_8}^+) \cong O^+(10, 2)$ acts faithfully on $R(V_{EE_8}^+)$. By the general theory of simple current extensions, one knows that holomorphic (simple current) extensions of $(V_{EE_8}^+)^3$ correspond to maximal totally singular subspaces of $R(V_{EE_8}^+)^3$ up to the action of $Aut((V_{EE_8}^+)^3) \cong O^+(10, 2) \wr Sym_3$. The main idea of Shimakura is to construct a maximal totally singular subspace of $R(V_{EE_8}^+)^3$ with large minimal weight and compute certain automorphism subgroups using the standard theory of quadratic spaces.

Next we shall review some basic properties of the lattice type VOA $V_{EE_8}^+$ [23, 42].

Notation 2.14. *Let $R(U)$ be the set of all inequivalent irreducible modules of a VOA U. If $U = V_{EE_8}^+$, then $R(V_{EE_8}^+)$ forms a 10-dimensional quadratic space over $\mathbb{Z}_2$ with respect to the fusion rules and the quadratic form*

$$q([M]) = \begin{cases} 0 & \text{if the weights of } M \text{ are in } \mathbb{Z}, \\ 1 & \text{if the weights of } M \text{ are in } \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{1}$$

We shall denote the corresponding bilinear form by $\langle\, ,\, \rangle$.

Recall that $Aut(V_{EE_8}^+) \cong O^+(10, 2)$ and $Aut(V_{EE_8}^+)$ acts as on $R(V_{EE_8}^+)$ as a group of isometries [23, 42].

Notation 2.15. *From now on, we use U to denote the VOA $V_{EE_8}^+$. and we use U^n to denote the tensor product of n copies of U.*

The proof of the following proposition can be found in [42].

Proposition 2.16. *The group $Aut(U) \cong O^+(10, 2)$ acts transitively on non-zero singular elements and non-singular elements of $R(U)$, respectively.*

1. If $[W]$ is a non-zero singular element in $R(U)$, then the minimal weight of the irreducible module W is 1 and $dim(W_1) = 8$.

2. If $[W]$ is a non-singular element, then the minimal weight of W is $1/2$ and $dim(W_{\frac{1}{2}}) = 1$.

Notation 2.17. *Since $R(U^3) \cong R(U)^3$, we shall view $R(U^3)$ as a direct sun of quadratic spaces. See (2.14) for notations. The quadratic form and the associated bilinear form are given by*

$$q(a, b, c) = q(a) + q(b) + q(c) \quad and$$
$$\langle (a, b, c), (a', b', c') \rangle = \langle a, a' \rangle + \langle b, b' \rangle + \langle c, c' \rangle$$

for $(a, b, c) \in R(U)^3$.

Following the analysis of [44], let Φ and Ψ be maximal totally singular subspaces of $R(U)$ such that $\Phi \cap \Psi = 0$. Then the space

$$\mathcal{S} := span_{\mathbb{Z}_2}\{(a, a, 0), (0, a, a), (b, b, b) \mid a \in \Phi, b \in \Psi\} \tag{2}$$

is a maximal totally singular subspace of $R(U)^3 \cong R(U^3)$.

Definition 2.18. *Let W be an irreducible module. We define the minimal weight of $[W]$ to be the minimal weight of W.*

Lemma 2.19. *Let $[W] \in \mathcal{S}$. Then the minimal weight of $[W]$ is ≥ 2. If the minimal weight of $[W]$ is 2, then up to a permutation of the 3 coordinates, $[W]$ has the form:*

1. $(a, a, 0)$, where $a \in \Phi \setminus \{0\}$ or

2. $(a + b + c, a + c, b + c)$, where $a, b \in \Phi \setminus \{0\}$, $c \in \Psi$, $a + c, b + c$ are non-singular and $a + b + c$ is non-zero singular.

Proof. First we note that a general element in $\mathcal{S}$ has the form

$$(a + b + c, a + c, b + c),$$

where $a, b \in \Phi$ and $c \in \Psi$.

Since $\Phi \cap \Psi = 0$, $a + c = 0$ implies that $a = c = 0$ if $a \in \Phi$, $c \in \Psi$. Thus, up to permutations of the 3 coordinates, a nonzero element in $\mathcal{S}$ has the form

1. $(a, a, 0)$, where $a \in \Phi \setminus \{0\}$ or

2. $(a + b + c, a + c, b + c)$, where $a, b \in \Phi \setminus \{0\}$ and $c \in \Psi$.

In Case 1, the minimal weight is 2. In Case 2, the minimal weight is $> \frac{3}{2}$ and thus ≥ 2.

If $(a+b+c, a+c, b+c)$, a, b, c nonzero, has the minimal weight 2, then two of the $a + b + c, a + c, b + c$ have minimal weights $1/2$ and the remaining one has minimal weight 1. Therefore, two are non-singular and one is a nonzero singular. $\square$

By Lemma 2.19, we have the following theorem.

Theorem 2.20 (Theorem 4.10 of [44]). *Let $V := \mathcal{V}(\mathcal{S}) = \oplus_{[W] \in \mathcal{S}} W$. Then V is a holomorphic framed VOA of central charge 24 and $V_1 = 0$.*

Notation 2.21. *For the rest of this article, V shall denote the VOA $V := \mathcal{V}(\mathcal{S}) = \oplus_{[W] \in \mathcal{S}} W$ of (2.20).*

Remark 2.22. *It was also shown in [44] that the singular space $\mathcal{S}$ defined in (2) is the unique (up to $Aut(U^3)$) maximal totally singular subspace of $R(U^3)$ such that $\mathcal{V}(\mathcal{S}) = \oplus_{[W] \in \mathcal{S}} W$ has trivial weight one subspace.*

Theorem 2.23 ([44]). *$Aut(V)$ acts transitively on the set of all subVOAs of V which are isomorphic to U^3.*

Now let H be the stabilizer of Φ in $Aut(U)$ and K the stabilizer of Ψ in H. Then $H = 2^{10} : L_5(2)$ and $K = L_5(2)$ (cf. [44, Proposition 2.5]). Set $H_{(i,j)} = \{(g_1, g_2, g_3) \mid g_i \in H, g_i = g_j \text{ and } g_k = 1, \text{ for } k \neq i, j\}$, where $i \neq j$ and $i, j \in \{1, 2, 3\}$ and $K_{(1,2,3)} = \{(g, g, g) \mid g \in K\}$.

Lemma 2.24 (Proposition 2.5 of [44]). *Let S be defined as in (2). Then the stabilizer $N_{\mathcal{S}}$ of $\mathcal{S}$ in $Aut(U^3)$ is generated by $O_2(H_{(1,2)}), O_2(H_{(1,3)})$, $K_{(1,2,3)}$ and Sym_3 and it has the shape $2^{20} : (L_2(5) \times Sym_3)$.*

As a consequence, we have

Proposition 2.25 (Corollary 4.18 of [44]). *Let $\mathcal{S}$ be defined as in (2). We identify $\mathcal{S}^*$ with the subgroup $\{\tau_\chi \mid \chi \in \mathcal{S}^*\}$ of $Aut(V)$ which was defined in (2.8). Then*

$$N_{Aut(V)}(\mathcal{S}^*) = Stab_{Aut(V)}(U \otimes U \otimes U) \cong 2^{15}(2^{20} : (L_5(2) \times Sym_3)).$$

Set $\mathcal{S}_\Phi = span\{(a, a, 0), (0, a, a) \mid a \in \Phi\}$. Then

$$V_\Phi = \mathcal{V}(\mathcal{S}_\Phi) = \oplus_{[W] \in \mathcal{S}_\Phi} W$$

is a subVOA of V.

For any coset $x + \mathcal{S}_\Phi \in \mathcal{S}/\mathcal{S}_\Phi$, the subspace $\mathcal{V}(x + \mathcal{S}_\Phi) = \oplus_{[W] \in x + \mathcal{S}_\Phi} W$ is an irreducible module of V_Φ. In fact, $\mathcal{V}(x + \mathcal{S}_\Phi)$ is a simple current module of V_Φ.

Proposition 2.26. *Let $V = \oplus_{x + \mathcal{S}_\Phi \in \mathcal{S}/\mathcal{S}_\Phi} \mathcal{V}(x + \mathcal{S}_\Phi)$ be a simple current extension of V_Φ. Then $N_{Aut(V)}((\mathcal{S}/\mathcal{S}_\Phi)^*)$ stabilizes the subVOA $U^{\otimes 3}$ and*

$$N_{Aut(V)}((\mathcal{S}/\mathcal{S}_\Phi)^*) = Stab_{Aut(V)}(U \otimes U \otimes U) \cong 2^5(2^{10}(2^{20} : (L_5(2) \times Sym_3))).$$

Notation 2.27. *Now let* $\mathcal{S}_1 = \{(0, a, a) \mid a \in \Phi\}$. *Then* $\mathcal{S}_1$ *is a totally singular subspace of* $\mathcal{S}$, *where* $\mathcal{S}$ *is defined as in (2.17). Let* $V' = \mathcal{V}(\mathcal{S}_1) = \oplus_{[W] \in \mathcal{S}_1} W < V$.

Lemma 2.28. *Let* $\mathcal{S}_1, V'$ *be as in (2.27). We have* $V' \cong V^+_{EE_8} \otimes V^+_{BW_{16}}$, *where* BW_{16} *denotes the Barnes-Wall lattice of rank* 16.

Proof. Let $E \cong E_8$ be an overlattice of EE_8. Then the lattice

$$B = span\{EE_8 \perp EE_8\} \cup \{(\alpha, \alpha) \mid \alpha \in E\} < E \perp E$$

has the discriminant group 2^8 and is isomorphic to BW_{16}.

Since there is only one orbit of maximal totally singular spaces of $R(U)$ under $Aut(U)$, we may assume

$$\Phi = span\{V^+_{\alpha + EE_8} \mid \alpha + EE_8 \in E/EE_8\} \cup \{V^-_{EE_8}\}.$$

Then we have $V' = \mathcal{V}(\mathcal{S}_1) \cong V^+_{EE_8} \otimes V^+_B \cong V^+_{EE_8} \otimes V^+_{BW_{16}}$ as desired. $\square$

Recall from [42] that $Aut(V^+_{BW_{16}}) \cong 2^{16}.\Omega^+(10, 2)$.

Remark 2.29. *Let* $\mathcal{S}_1, V'$ *be as in (2.27). Let* $\bar{\mathcal{S}} = \mathcal{S}/\mathcal{S}_1$. *Then for any coset* $a + \mathcal{S}_1 \in \bar{\mathcal{S}}$, *the subspace* $\mathcal{V}(a + \mathcal{S}_1) = \oplus_{[W] \in a + \mathcal{S}_1} W$ *is an irreducible* V'-*module. Moreover, the dual group* $\bar{\mathcal{S}}^*$ *acts on* V.

Proposition 2.30. *Let* $\mathcal{S}_1, V'$ *be as in (2.27). Let* $\bar{\mathcal{S}} = \mathcal{S}/\mathcal{S}_1$. *Then*

$$N_{Aut(V)}(\bar{\mathcal{S}}^*) = Stab_{Aut(V)}(V^+_{EE_8} \otimes V^+_{BW_{16}}) \cong 2^{10}(2^{16}\Omega^+(10, 2)).$$

Proof. Let $\tilde{\mathcal{S}} = \{\mathcal{V}(a + \mathcal{S}_1) \mid a + \mathcal{S}_1 \in \bar{\mathcal{S}}\} \subset R(V') = R(V^+_{EE_8}) \oplus R(V^+_{BW_{16}})$. Then $|\tilde{\mathcal{S}}| = 2^{10}$.

Recall from [42] that both $R(V^+_{EE_8})$ and $R(V^+_{BW_{16}})$ are 10-dimensional quadratic spaces of maximal Witt index. Moreover, $Aut(V^+_L)/O_2(Aut(V^+_L))$ acts faithfully on $R(V^+_L)$ for $L = EE_8$ or BW_{16} [42, Theorem 3.20].

Let $\rho_1 : \tilde{\mathcal{S}} \to R(V^+_{BW_{16}})$ and $\rho_2 : \tilde{\mathcal{S}} \to R(V^+_{EE_8})$ be natural projections. Then both ρ_1 and ρ_2 are bijections and they determine an isomorphism

$$\phi = \rho_2(\rho_1)^{-1} : R(V^+_{BW_{16}}) \to R(V^+_{EE_8}).$$

For each $g \in Aut(V^+_{BW_{16}}) \cong 2^{16}.\Omega^+(10, 2)$, define $^\phi \bar{g} = \phi \bar{g} \phi^{-1} \in \Omega^+(10, 2)$, where $^-: 2^{16}.\Omega^+(10, 2) \to \Omega^+(10, 2)$ is the natural map. Then the stabilizer $N_{\tilde{\mathcal{S}}}$ of $\tilde{\mathcal{S}}$ is given by $\{(^\phi\bar{g}, g) \mid g \in 2^{16}.\Omega^+(10, 2)\} \cong 2^{16}.\Omega^+(10, 2)$. The theorem now follows from Theorem 2.13. $\square$

3 Centralizer of an involution

In this section, we shall show that the automorphism group of $V = \mathcal{V}(\mathcal{S})$ (2.21) has an involution z such that $V^z \cong V_\Lambda^+$ and $C_{Aut(V)}(z) \cong 2^{1+24}Co_1$, where Λ denotes the Leech lattice.

Notation 3.1. *Let Φ, Ψ and $\mathcal{S}$ be defined as in Notation (2.17). Let $x \in \Phi$ be a non-zero element. We denote*

$$\mathcal{S}^0 = \{(a, b, c) \in \mathcal{S} \mid \langle(a, b, c), (x, 0, 0)\rangle = 0\},$$
$$\mathcal{S}^1 = \{(a, b, c) \in \mathcal{S} \mid \langle(a, b, c), (x, 0, 0)\rangle = 1\}.$$

Definition 3.2. *Let $V = \mathcal{V}(\mathcal{S}) = \oplus_{[W]\in\mathcal{S}}W$ be the VOA defined in Theorem 2.20. Define a linear map $z : V \to V$ by*

$$z = \begin{cases} 1 & \text{on } W \text{ for } [W] \in \mathcal{S}^0, \\ -1 & \text{on } W \text{ for } [W] \in \mathcal{S}^1. \end{cases} \tag{3}$$

Then z is automorphism of order 2 of V.

Lemma 3.3. *Let $\mathcal{B}$ be the weight 2 subspace of V. Then $dim(\mathcal{B}) = 196884$.*

Proof. By Lemma 2.19, an irreducible module in $\mathcal{S}$ has minimal weight 2 if and only if it has the form
  1. $(a, a, 0)$, where $a \in \Phi \setminus \{0\}$ or
  2. $(a + b + c, a + c, b + c)$, where $a, b \in \Phi \setminus \{0\}$, $c \in \Psi \setminus \{0\}$, $a + c, b + c$ are non-singular and $a + b + c$ is non-zero singular.

Note that there are $2^5 - 1$ non-zero vectors in Ψ and $a + c$ is singular if and only if $\langle a, c \rangle = 1$. Thus, there are $(2^5 - 1) \times 2^4 \times 2^4 \times 3$ such vectors in case 2 while there are $(2^5 - 1) \times 3$ vectors in case 1. By Proposition 2.16, we have

$$dim(\mathcal{B}) = 156 \times 3 + (2^5 - 1) \times 3 \times 8^2 + (2^5 - 1) \times 2^4 \times 2^4 \times 3 \times 8 = 196884.$$

Note that $dim\, U_2 = 156$ and $dim(U^3)_2 = 156 \times 3$. $\square$

Lemma 3.4. *The trace of z on $\mathcal{B}$ is 276.*

Proof. Since $x \in \Phi \setminus \{0\}$, $(x, 0, 0)$ is orthogonal to all elements of the shape $(a, a, 0)$, $(0, a, a)$, $(a, a, 0)$ in $\mathcal{S}$.

Moreover, $(a + b + c, a + c, b + c)$ is orthogonal to $(x, 0, 0)$ if and only if $\langle x, c \rangle = 0$. Therefore, there are $(2^4 - 1) \times 2^4 \times 2^4 \times 3$ vectors of the form $(a + b + c, a + c, b + c)$ in $\mathcal{S}^0$ and $2^4 \times 2^4 \times 2^4 \times 3$ vectors in $\mathcal{S}^1$ that have minimal weights 2.

Thus, the trace of z on $\mathcal{B}$ is

$$(156 \times 3 + (2^5 - 1) \times 3 \times 8^2 + (2^4 - 1) \times 2^4 \times 2^4 \times 3 \times 8) - (2^4 \times 2^4 \times 2^4 \times 3 \times 8) = 276$$

as desired. $\square$

Notation 3.5. *Let $x \in \Phi \setminus \{0\}$ be as in Notation (3.1). Let A be an irreducible U-module such that $[A] = x \in R(U)$ and $M = A \otimes U \otimes U$. Then $[M] = (x, 0, 0) \in R(U)^3$. Let V and z be as in (3.2). Denote*

$$\tilde{V} = V^z \oplus (V^z \times_{U^3} M).$$

Then $\tilde{V}$ is also a holomorphic framed VOA of central charge 24 [30].
    Note also that

$$V^z \times_{U^3} M = \Big(\bigoplus_{[W] \in \mathcal{S}^0} W \Big) \times_{U^3} M = \bigoplus_{[W] \in \mathcal{S}^0} (W \times_{U^3} M) = \bigoplus_{[W] \in (x,0,0) + \mathcal{S}^0} W.$$

Thus, we also have $\tilde{V} = \oplus_{[W] \in \tilde{\mathcal{S}}} W$, where $\tilde{\mathcal{S}} = \mathcal{S}^0 \cup ((x, 0, 0) + \mathcal{S}^0)$.

The following is easy to prove.

Lemma 3.6. *Let $\tilde{V}$ be defined as in Notation (3.5).*
    1. $dim(\tilde{V}_1) = 24$.
    2. $\tilde{V}$ contains a subVOA isomorphic to $(V_{EE_8})^3$.

Proof. Let $x \in \Phi \setminus \{0\}$ be as in Notation (3.1). Since $V_1 = 0$, we have $(V^z)_1 = 0$ and thus $\tilde{V}_1 < V^z \times_{U^3} M = \bigoplus_{[W] \in (x,0,0) + \mathcal{S}^0} W$.

By the definition of $\mathcal{S}$ (2.17), we know that $(0, 0, 0), (x, x, 0), (x, 0, x) \in \mathcal{S}^0$. Thus, we have $(x, 0, 0), (0, x, 0)$ and $(0, 0, x)$ in $(x, 0, 0) + \mathcal{S}^0$ and they have minimal weight 1.

Now let $(a + c, b + c, a + b + c), a, b \in \Phi, c \in \Psi$, be an element of $\mathcal{S}^0$. If $s = (x, 0, 0) + (a + c, b + c, a + b + c) = (x + a + c, b + c, a + b + c)$ has minimal weight 1, then at least one of the coordinates must be zero; otherwise, the minimal weight $\geq \frac{3}{2}$. Since $\Phi \cap \Psi = 0$, we have $c = 0$ and $s = (x + a, b, a + b)$ for some $a, b \in \Phi$. Since $x + a, a$ and $a + b$ are singular and s has minimal weight

14

1, only one coordinate is non-zero. Hence, $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$ are the only elements in $(x, 0, 0) + \mathcal{S}^0$ which have minimal weight 1.

Moreover by (2.16),

$$dim(A \otimes U \otimes U)_1 = dim(U \otimes A \otimes U)_1 = dim(U \otimes U \otimes A)_1 = 8.$$

Hence, we have $dim\tilde{V}_1 = 8 + 8 + 8 = 24$.

Since $Aut(V_{EE_8}^+)$ acts transitively on non-zero singular vectors (2.16)(see also [44]), we have $V_{EE_8}^+ \oplus A \cong V_{EE_8}^+ \oplus V_{EE_8}^- = V_{EE_8}$.

Now let $\tilde{S}' = span\{(x, 0, 0), (0, x, 0), (0, 0, x)\} < \tilde{S}$. Then $\oplus_{[W] \in \tilde{S}'} W$ is isomorphic to $(V_{EE_8})^3$ and thus $\tilde{V}$ contains $(V_{EE_8})^3$ as a subVOA. $\square$

As a direct consequence, we have

Corollary 3.7. $\tilde{V} \cong V_\Lambda$ and $V^z \cong V_\Lambda^+$ (see (3.2) and (3.5)).

Proof. Since $V_{EE_8^3} \cong (V_{EE_8})^3$ is a full subVOA of $\tilde{V}$, $\tilde{V}$ is a direct sum of irreducible $V_{EE_8^3}$-modules. Thus by [4], there exists an even lattice $EE_8^3 < L < (EE_8^3)^*$ such that

$$\tilde{V} = \bigoplus_{\alpha + EE_8^3 \in L/(EE_8^3)} V_{\alpha + EE_8^3} = M(1) \otimes \mathbb{C}\{L\}.$$

Hence $\tilde{V}$ is isomorphic to the lattice VOA V_L [32]. Note that $V_{\alpha + EE_8^3}$, $\alpha \in L/(EE_8^3)$, are irreducible $V_{EE_8^3}$-modules and $\tilde{V}$ is a direct sum of simple current modules of $V_{EE_8^3}$ [4]. Therefore, $\tilde{V} \cong V_L$ also follows from the uniqueness of simple current extensions [12].

Recall that
$$dim(V_L)_1 = rank\, L + |L(2)|,$$

where $L(2) = \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$ [15].

Thus $L(2) = \emptyset$ as $rank\, L = 24$ and $dim(\tilde{V}_1) = 24$. Moreover, L is unimodular since $\tilde{V}$ is holomorphic [4]. Since the Leech lattice is the only even unimodular lattice with no roots, $L \cong \Lambda$ and $\tilde{V} \cong V_\Lambda$.

Recall that $\tilde{V} = V^z \oplus (V^z \times_{U^3} M)$ (3.5). We define an automorphism g on $\tilde{V}$ by

$$g = \begin{cases} 1 & \text{on } V^z, \\ -1 & \text{on } V^z \times_{U^3} M. \end{cases}$$

15

Since $V_1 = 0$, we have $V_1^z = 0$ and $\tilde{V}_1 \subset V^z \times_{U^3} M$. Therefore, g acts as -1 on $\tilde{V}_1$ and thus it is conjugate to the automorphism θ, a lift of the (-1)-isometry of Λ by [6]. Hence, we have

$$V^z = \tilde{V}^g \cong V_\Lambda^+$$

as desired. $\square$

Let V^- be the (-1)-eigenspace of z in V. Then $V^- = \oplus_{[W] \in \mathcal{S}^1} W$ is a direct sum of simple current modules of U^3. Since z acts trivially on U^3, $U^3 \subset V^z$. Hence, by [35, Theorem 5,4], for any irreducible V^z-module X, we have

$$\sum_W N_{V^- \; X}^W \leq 1$$

where W runs through all isomorphism types of irreducible V^z-modules. Moreover, $V^- \times_{V^z} X \neq 0$ since V^z is rational. Thus V^- is also a simple current module of V^z.

Thus, by Theorem 2.13, we have

Theorem 3.8. *Let z be defined as in (3.2). Then we have an exact sequence*

$$1 \to \langle z \rangle \to C_{Aut(V)}(z) \xrightarrow{\eta} Aut(V^z) \to 1$$

and $C_{Aut(V)}(z)$ has the shape $2 \cdot 2^{24} Co_1$.

Proof. Let N be an irreducible V^z-module. Since V is a framed VOA, by Theorem 1 of [30], the V^z-module $V \times_{V^z} N$ has a structure of an irreducible V-module or an irreducible z-twisted V-module.

Thus, every irreducible module of V^z can be embedded into V or the unique irreducible z-twisted module of V. Note that V is holomorphic. Therefore, it has only one irreducible module, namely V and a unique z-twisted module, up to isomorphisms [10].

Note that $V = V^z \oplus V^-$ and $V \times_{U^3} M = (V^z \times_{U^3} M) \oplus (V^- \times_{U^3} M)$ is the unique z-twisted module [30], where M is defined as in (3.5). Thus, V^z has exactly 4 inequivalent irreducible modules, namely, V^z, V^-, $V^z \times_{U^3} M$ and $V^- \times_{U^3} M$. Since $V_1 = 0$, it is clear that the minimal weight of V^z and V^- are 0 and 2, respectively. On the other hand, the minimal weight of $V^z \times_{U^3} M$ is 1 and the weights of $V^- \times_{U^3} M$ are in $\frac{1}{2} + \mathbb{Z}$. Thus, $g \circ V^- \cong V^-$ for all $g \in Aut(V^z)$ since $g \circ V^-$ and V^- have the same characters. The theorem now follows from Theorem 2.13 and the fact that $Aut(V_\Lambda^+) \cong 2^{24} Co_1$ [42]. $\square$

16

Notation 3.9. *The group C is the group which is in the middle of the short exact sequence of (3.8). It has the shape $2^{1+24}Co_1$. (There are two 2-constrained groups of the shape $2^{1+24}Co_1$ [18].)*

Remark 3.10. *Recall from [5] that V_Λ^+ has exactly 4 inequivalent irreducible modules, namely, V_Λ^+, V_Λ^-, $V_\Lambda^{T,+}$ and $V_\Lambda^{T,-}$, and their minimal weights are 0, 1, 2 and $\frac{3}{2}$, respectively. Since $V^z \cong V_\Lambda^+$ and $V_1 = 0$, it is easy to see that $V = V^z \oplus V^- \cong V_\Lambda^+ \oplus V_\Lambda^{T,+}$ as a V_Λ^+-module.*

We can furthermore say that V is isomorphic to the Frenkel-Lepowsky-Meurman Moonshine VOA $V^\natural$, by the uniqueness of simple current extensions [12, 25] or by use of [29] or [8] since V is framed. If we wish to avoid quoting these results, then after we prove (4.24), we can claim the more modest result that V has Moonshine type.

3.1 Conformal vectors of central charge $\frac{1}{2}$

Notation 3.11. *We shall use $cvcc\frac{1}{2}$ to mean simple conformal vectors of central charge $\frac{1}{2}$.*

Since $V^z \cong V_\Lambda^+$, all $cvcc\frac{1}{2}$ in the VOA V^z are classified in [28]. There are exactly two classes of $cvcc\frac{1}{2}$, AA_1-type and EE_8-type, up to the conjugacy of $Aut(V_\Lambda^+)$. Since Λ is generated by norm 4 vectors, it can be shown by [28, Proposition 3.2] that Miyamoto involutions associated to AA_1-type $cvcc\frac{1}{2}$ generate a normal subgroup $H_\Lambda \cong Hom(\Lambda/2\Lambda, \mathbb{Z}_2) \cong 2^{24}$ in V_Λ^+.

Notation 3.12. *Let H_Λ be the normal subgroup generated by AA_1-type Miyamoto involution in V_Λ^+.*

Next we shall show that $\eta^{-1}(H_\Lambda)$ is an extra-special group of order 2^{25}.

Definition 3.13. *For any $\alpha \in L(4)$, define*

$$\omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot \mathbf{1} \pm \frac{1}{4}(e^\alpha + e^{-\alpha}).$$

Then $\omega^\pm(\alpha)$ are $cvcc\frac{1}{2}$ and we call them $cvcc\frac{1}{2}$ of AA_1-type.

Lemma 3.14. *Let $\omega^{\epsilon_1}(\alpha)$ and $\omega^{\epsilon_2}(\beta)$ be AA_1-type $cvcc\frac{1}{2}$ in V_Λ^+, where $\alpha, \beta \in \Lambda(4)$. Then,*

$$\langle \omega^{\epsilon_1}(\alpha), \omega^{\epsilon_2}(\beta) \rangle = \begin{cases} \frac{1}{2^7}(\langle \alpha, \beta \rangle)^2 & \text{if } \alpha \neq \pm\beta, \\ \frac{1}{4}\delta_{\epsilon_1,\epsilon_2} & \text{if } \alpha = \pm\beta. \end{cases}$$

17

Proof. The lemma follows easily from the formulas [15, Chapter 8] that

$$\langle \alpha(-1)^2 \cdot \mathbf{1}, \beta(-1)^2 \cdot \mathbf{1} \rangle = 2(\langle \alpha, \beta \rangle)^2,$$

$$\langle e^\alpha + e^{-\alpha}, e^\beta + e^{-\beta} \rangle = \begin{cases} 0 & \text{if } \alpha \neq \pm\beta, \\ 2 & \text{if } \alpha = \pm\beta \end{cases}$$

and the definition of $\omega^\pm(\alpha)$. $\square$

Notation 3.15. *For a cvcc$\frac{1}{2}$, e, we denote by $t(e)$ the associated Miyamoto involution [33].*

By Lemma 3.14 and Sakuma's Theorem [39], we know that $t(\omega^{\epsilon_1}(\alpha))$ commutes with $t(\omega^{\epsilon_2}(\beta))$ unless $\langle \alpha, \beta \rangle = \pm 1$.

Lemma 3.16. *Let $\omega^{\epsilon_1}(\alpha)$ and $\omega^{\epsilon_2}(\beta)$ be AA_1-type cvcc$\frac{1}{2}$ in V_Λ^+, where $\alpha, \beta \in \Lambda(4)$. Then, as automorphisms of V,*

$$\left(\tau_{\omega^{\epsilon_1}(\alpha)} \tau_{\omega^{\epsilon_2}(\beta)} \right)^2 = \begin{cases} 1 & \text{if } \langle \alpha, \beta \rangle \text{ is even,} \\ z & \text{if } \langle \alpha, \beta \rangle = \pm 1. \end{cases}$$

Proof. Since $t(\omega^{\epsilon_1}(\alpha))t(\omega^{\epsilon_2}(\beta))t(\omega^{\epsilon_1}(\alpha)) = t\big(t(\omega^{\epsilon_1}(\alpha))\omega^{\epsilon_2}(\beta)\big)$, we have

$$t(\omega^{\epsilon_1}(\alpha))t(\omega^{\epsilon_2}(\beta))t(\omega^{\epsilon_1}(\alpha)) = \begin{cases} t(\omega^{-\epsilon_2}(\beta)) & \text{if } \langle \alpha, \beta \rangle = \pm 1 \\ t(\omega^{\epsilon_2}(\beta)) & \text{if } \langle \alpha, \beta \rangle \text{ is even.} \end{cases}$$

Thus, we have $(t(\omega^{\epsilon_1}(\alpha))t(\omega^{\epsilon_2}(\beta)))^2 = t(\omega^{-\epsilon_2}(\beta))t(\omega^{\epsilon_2}(\beta)) = z$ by [28, Lemma 5.14] if $\langle \alpha, \beta \rangle = \pm 1$ and $(t(\omega^{\epsilon_1}(\alpha))t(\omega^{\epsilon_2}(\beta)))^2 = 1$ if $\langle \alpha, \beta \rangle$ is even. $\square$

As a corollary, we have

Theorem 3.17. *The Miyamoto involutions $\{t(\omega^\pm(\alpha)) \mid \alpha \in \Lambda(4)\}$ generate a subgroup isomorphic to 2^{1+24} in $Aut(V)$ and thus $C_{Aut(V)}(z) \cong 2^{1+24}.Co_1$.*

Proof. First we note that Λ is generated by norm 4 vectors. Moreover, $\Lambda/2\Lambda$ forms a non-degenerate quadratic space over $\mathbb{Z}_2$ with the quadratic form $q(\alpha + 2\Lambda) = \frac{1}{2}\langle \alpha, \alpha \rangle \mod 2$.

Let T be the subgroup of $Aut(V)$ generates by AA_1-type Miyamoto involutions. Then the restriction map η induces an group homomorphism $\eta : T \to H_\Lambda$ (3.12) and we have an exact sequence

$$1 \to \langle z \rangle \to T \xrightarrow{\eta} H_\Lambda \cong Hom(\Lambda/2\Lambda, \mathbb{Z}_2) \to 1.$$

Moreover, by Lemma 3.16, we have

$$[t(\omega^{\epsilon_1}(\alpha)), t(\omega^{\epsilon_2}(\beta))] = z^{\langle\alpha,\beta\rangle}$$

and thus $T = \eta^{-1}(H_\Lambda) \cong 2^{1+24}$. Note that $z = t(\omega^+(\alpha))t(\omega^-(\alpha)) \in T$. $\square$

Remark 3.18. *Let $\omega^\epsilon(\alpha)$ be an AA_1-type cvcc$\frac{1}{2}$ in $V_\Lambda^+ < V_\Lambda$. Then we may also consider $t(\omega^\epsilon(\alpha))$ as an automorphism of V_Λ. In this case, $t(\omega^\epsilon(\alpha))$ acts trivially on the Heisenberg part $M(1)$ and thus acts trivially on $(V_\Lambda)_1$ [28].*

Recall (3.12), (3.17). Now let $\tilde{H}_\Lambda < Aut(V_\Lambda)$ be the group generated by $\{t(\omega^\pm(\alpha)) \mid \alpha \in \Lambda(4)\}$. Then $\tilde{H}_\Lambda \cong Hom(\Lambda, \mathbb{Z}_2) \cong 2^{24}$ and the restriction map

$$g \in \tilde{H}_\Lambda \to g|_{V_\Lambda^+} \in H_\Lambda < Aut(V_\Lambda^+)$$

is an isomorphism from $\tilde{H}_\Lambda$ to H_Λ [28]. By the discussion above, $\tilde{H}_\Lambda$ acts trivially on $(V_\Lambda)_1$.

4   Analysis of the finite group $Aut(V)$

In the first subsection, we prove that $Aut(V)$ is finite. This involves a discussion of framed VOAs over both the real and complex field. Then, in the second subsection, we use finite group theory to complete our analysis of $Aut(V)$.

4.1   Framed VOA over $\mathbb{R}$

First, we recall some facts about framed VOA over $\mathbb{R}$ from [37, 38].

Notation 4.1. *Let $Vir_\mathbb{R} = \oplus_{n\in\mathbb{Z}}\mathbb{R}L_n \oplus \mathbb{R}\mathbf{c}$ be the Virasoro algebra over $\mathbb{R}$. For $c, h \in \mathbb{R}$, let $L(c, h)_\mathbb{R}$ be the irreducible highest weight module of $Vir_\mathbb{R}$ of highest weight h and central charge c over $\mathbb{R}$.*

The following results can be found in [38].

Proposition 4.2 (Corollary 2.2, 2.3 and Theorem 2.4 of [38])**.** *$L(\frac{1}{2}, 0)_\mathbb{R}$ is a rational VOA, that is, all $L(\frac{1}{2}, 0)_\mathbb{R}$-modules are completely reducible. Moreover, $L(\frac{1}{2}, 0)_\mathbb{R}$ has only 3 inequivalent irreducible modules, namely, $L(\frac{1}{2}, 0)_\mathbb{R}$, $L(\frac{1}{2}, \frac{1}{2})_\mathbb{R}$ and $L(\frac{1}{2}, \frac{1}{16})_\mathbb{R}$, and*

$$L(\frac{1}{2}, h) \cong \mathbb{C} \otimes L(\frac{1}{2}, h)_\mathbb{R}$$

for all $h = 0, 1/2$ or $1/16$.

19

Lemma 4.3 (Lemma 2.5 of [38])**.** *Let W be a VOA over $\mathbb{R}$ and let M^1, M^2, M^3 be W-modules. Then*

$$dim\left(I_W\begin{pmatrix}M^3\\M^1\quad M^2\end{pmatrix}\right) \leq dim\left(I_{\mathbb{C}\otimes W}\begin{pmatrix}\mathbb{C}\otimes M^3\\\mathbb{C}\otimes M^1\quad \mathbb{C}\otimes M^2\end{pmatrix}\right)$$

.

Proposition 4.4 (cf. (2.5) and (2.6) of [38])**.** *For $h_1, h_2, h_3 \in \{0, 1/2, 1/16\}$, we have*

$$dim\left(I_{L(\frac{1}{2},0)_{\mathbb{R}}}\begin{pmatrix}L(\frac{1}{2},h_3)_{\mathbb{R}}\\L(\frac{1}{2},h_1)_{\mathbb{R}}\quad L(\frac{1}{2},h_2)_{\mathbb{R}}\end{pmatrix}\right) = dim\left(I_{L(\frac{1}{2},0)}\begin{pmatrix}L(\frac{1}{2},h_3)\\L(\frac{1}{2},h_1)\quad L(\frac{1}{2},h_2)\end{pmatrix}\right).$$

In particular, the fusion rules for $L(\frac{1}{2},0)_{\mathbb{R}}$ over $\mathbb{R}$ are exactly the same as the fusion rules for $L(\frac{1}{2},0)$ over $\mathbb{C}$.

Definition 4.5. *A simple VOA W over $\mathbb{R}$ is framed if it contains a full subVOA T isomorphic to $L(\frac{1}{2},0)_{\mathbb{R}}^n$.*

Notation 4.6. *For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$ and an even binary linear code E, we define*

$$M_{\mathbb{R}}^{\alpha} = \otimes_{i=1}^n L(\frac{1}{2}, \frac{\alpha_i}{2})_{\mathbb{R}} \quad and \quad M_{E,\mathbb{R}} = \oplus_{\alpha \in E} M_{\mathbb{R}}^{\alpha}.$$

Unlike the complex case, a simple current extension over $\mathbb{R}$ is no longer unique and there is more than one VOA structure on $M_{E,\mathbb{R}}$ (see for example [38]). Nevertheless, the following holds.

Proposition 4.7 (Proposition 3.5 of [38])**.** *Let E be an even linear binary code. Then $M_{E,\mathbb{R}}$ has a unique VOA structure over $\mathbb{R}$ such that the invariant form on $M_{E,\mathbb{R}}$ is positive definite.*

Proposition 4.8. *Let W be a framed VOA over $\mathbb{R}$ such that its invariant form is positive definite. Then there exists two binary codes E and D such that $D < E^{\perp}$ and*

$$W = \oplus_{\beta \in D} W^{\beta},$$

where $W^0 \cong M_{E,\mathbb{R}}$ and for each $\beta \in D$, W^{β} is an irreducible M_E-module with the 1/16-word β. The 1/16-word for an irreducible $M_{E,\mathbb{R}}$-module is defined as in the complex case [36].

Corollary 4.9. *Let W be a framed VOA over $\mathbb{R}$ such that its invariant form is positive definite. Then W is finitely generated as a VOA.*

Proof. By Proposition, 4.8, W contains a subVOA $W^0 \cong M_{E,\mathbb{R}}$ and W is a direct sum of finitely many irreducible W^0-modules.

It is clear that $M^0 = \otimes_{i=1}^n L(\frac{1}{2}, 0)_{\mathbb{R}}$ is generated by n $cvcc\frac{1}{2}$. Since $W^0 \cong M_{E,\mathbb{R}}$ is direct sum of finite many irreducible M^0-modules, W^0 is finitely generated. Moreover, W is a direct sum of finitely many W^0-irreducible modules. Hence W is finitely generated. $\square$

Notation 4.10. *Let W be a framed VOA with a positive definite invariant form over $\mathbb{R}$ and $T \cong L(\frac{1}{2}, 0)_{\mathbb{R}}^n$ a Virasoro frame. Denote*

$$Stab_{Aut(W)}(T) = \{g \in Aut(W) \mid g(T) = T\},$$
$$Stab_{Aut(W)}^{pt}(T) = \{g \in Aut(W) \mid g(v) = v \text{ for all } v \in T\}.$$

Since $L(\frac{1}{2}, 0)$ and $L(\frac{1}{2}, 0)_{\mathbb{R}}$ have the same fusion rules, the following can be proved by the same argument as in the complex case.

Proposition 4.11 ([6, 30]). *Let W be a framed VOA with a positive definite invariant form over $\mathbb{R}$. Then*

1. $Stab_{Aut(W)}^{pt}(T)$ is a finite 2-group.

2. $Stab_{Aut(W)}^{pt}(T)$ is normal in $Stab_{Aut(W)}(T)$ and $Stab_{Aut(W)}(T)/Stab_{Aut(W)}^{pt}(T)$ is isomorphic to a subgroup of $Aut(D)$.

In particular, $Stab_{Aut(W)}(T)$ is a finite group.

Lemma 4.12 (Theorem 5.1 of [36]). *Let W be a CFT type VOA over $\mathbb{R}$. Suppose $W_1 = 0$ and the invariant form on W is positive definite. Then for any pair of distinct $cvcc\frac{1}{2}$ e and f in W, we have*

$$0 \leq \langle e, f \rangle \leq \frac{1}{12}.$$

In particular, W has only finitely many $cvcc\frac{1}{2}$.

Proposition 4.13 (cf. [38]). *Let W be a framed VOA over $\mathbb{R}$. Suppose the invariant form on W is positive definite and $W_1 = 0$. Then $Aut(W)$ is a finite group.*

Proof. By Lemma 4.12, W has only finitely many $cvcc\frac{1}{2}$ and thus W has only finitely many Virasoro frames. By Proposition 4.11 (see also [6]), the stabilizer of a Virasoro frame is a finite group. Hence $Aut(W)$ is finite. $\square$

Proposition 4.14 (cf. [38])**.** *Let W be a framed VOA over $\mathbb{R}$. Suppose the invariant form on W is positive definite and $W_1 = 0$. Then $Aut(\mathbb{C} \otimes W)$ is a finite group.*

Proof. In this proof, $\otimes$ means $\otimes_{\mathbb{R}}$. There is a semilinear automorphism, denoted γ, on $\mathbb{C} \otimes_{\mathbb{R}} W$ which fixes $\mathbb{R} \otimes W$ and is -1 on $\mathbb{R}\sqrt{-1} \otimes W$.

By (4.9) (see also [6]), W is a finitely generated VOA. From [7], $Aut(\mathbb{C} \otimes W)$ is a finite dimensional algebraic group and on it, γ has finite fixed points (4.13). Its corresponding action on $Der(\mathbb{C} \otimes W)$, the complex Lie algebra of derivations, is therefore fixed point free and so acts as -1 on this complex Lie algebra. In fact, we note that the (-1)-eigenspace of γ on $End(\mathbb{C} \otimes W)$ may be identified with the real subspace $\mathbb{R}\sqrt{-1} \otimes End(W)$ of $\mathbb{C} \otimes End(W) \cong End(\mathbb{C} \otimes W)$. Since this real subspace contains no nontrivial complex subspaces, we conclude that $Der(\mathbb{C} \otimes W) = 0$. It follows that the connected component of the identity $Aut(\mathbb{C} \otimes W)^0$ is 0-dimensional, whence $Aut(\mathbb{C} \otimes W)$ is finite.

Definition 4.15. *A real form of a complex VOA V is a real subspace W which is closed under the VOA operations and such that a real basis for W is a complex basis for V. Given a real form W of V, a real form of a V-module M is a real subspace N which is closed under action by W and such that a real basis for N is a complex basis for M. We say that N is a real form of M with respect to the real form W of V.*

Next we shall show that the VOA V constructed in (2.20) has a real form with a positive definite invariant form.

First we recall that the lattice VOA constructed in [15] can be defined over $\mathbb{R}$. Let $V_{L,\mathbb{R}} = S(\hat{H}_{\mathbb{R}}^-) \otimes \mathbb{R}\{L\}$ be the real lattice VOA associated to an even positive definite lattice, where $H = \mathbb{R} \otimes_{\mathbb{Z}} L$, $\hat{H}^- = \oplus_{n \in \mathbb{Z}^+} H \otimes \mathbb{R}t^{-n}$. As usual, we use $x(-n)$ to denote $x \otimes t^{-n}$ for $x \in H$ and $n \in \mathbb{Z}^+$.

Notation 4.16. *Let $\theta : V_{L,\mathbb{R}} \to V_{L,\mathbb{R}}$ be defined by*

$$\theta(x(-n_1) \cdots x(-n_k) \otimes e^{\alpha}) = (-1)^k x(-n_1) \cdots x(-n_k) \otimes e^{-\alpha}.$$

Then θ is an automorphism of $V_{L,\mathbb{R}}$, which is a lift of the (-1)-isometry of L [15]. We shall denote the (± 1)-eigenspaces of θ on $V_{L,\mathbb{R}}$ by $V_{L,\mathbb{R}}^{\pm}$.

The following result is well-known [15, 38].

Proposition 4.17 (cf. Proposition 2.7 of [38]). *Let L be an even positive definite lattice. Then the real subspace $\tilde{V}_{L,\mathbb{R}} = V^+_{L,\mathbb{R}} \oplus \sqrt{-1}V^-_{L,\mathbb{R}}$ is a real form of V_L. Furthermore, the invariant form on $\tilde{V}_{L,\mathbb{R}}$ is positive definite.*

Notation 4.18. *Let $U_{\mathbb{R}} = V^+_{EE_8,\mathbb{R}}$ be a real form of U. Since all $cvcc^{\frac{1}{2}}$ in U are contained in $V^+_{EE_8,\mathbb{R}}$ [13, 23], $U_{\mathbb{R}}$ is a real framed VOA. In fact, $U_{\mathbb{R}} \cong M_{RM(2,4),\mathbb{R}}$ (see (4.6)) since $V^+_{EE_8} \cong M_{RM(2,4)}$, where $RM(2,4)$ is the 2nd order Reed-Muller code of degree 4.*

Lemma 4.19 (Lemma 3.10 of [38]). *Let E be an even binary code. Let X be an irreducible $M_{E,\mathbb{R}}$-module. Then $\mathbb{C} \otimes X$ is an irreducible M_E-module.*

Lemma 4.20. *Let M be an irreducible module of U over $\mathbb{C}$. Suppose $[M] \in R(U)$ is a non-zero singular element. Then M has a positive definite real form.*

Proof. Since $Aut(V^+_{EE_8})$ acts transitively on non-zero singular vectors (2.16) (see also [44]), there is $g \in Aut(U)$ such that $M \cong g \circ V^-_{EE_8}$. Recall from [23] that $Aut(U)$ is generated by σ-involutions associated to $cvcc^{\frac{1}{2}}$ and all $cvcc^{\frac{1}{2}} \in U$ are contained in $U_{\mathbb{R}}$. Therefore, g keeps $U_{\mathbb{R}}$ invariant and define an automorphism on $U_{\mathbb{R}}$, also.

By Proposition 4.17, the invariant form on $\tilde{V}_{EE_8,\mathbb{R}} = U_{\mathbb{R}} \oplus (\mathbb{R}\sqrt{-1} \otimes V^-_{EE_8,\mathbb{R}})$ is a positive definite. Set $W = \mathbb{R}\sqrt{-1} \otimes V^-_{EE_8,\mathbb{R}}$. Then $U_{\mathbb{R}} \oplus (g \circ W) \cong \tilde{V}_{EE_8,\mathbb{R}}$ also has a positive definite invariant form. Moreover, $\mathbb{C} \otimes (g \circ W) \cong g \circ V^-_{EE_8,\mathbb{R}} \cong M$. Thus, $g \circ W$ is a positive definite real form of M. $\square$

Notation 4.21. *Let Φ, Ψ and $\mathcal{S}$ be as in (2.17). Then $\mathcal{V}(\Phi) = \oplus_{[M]\in\Phi}M \cong V_{E_8}$. By Proposition 4.17, $\mathcal{V}(\Phi)$ has a positive definite real form $W \cong \tilde{V}_{E_8,\mathbb{R}}$. Then W is a direct sum of irreducible $U_{\mathbb{R}}$-modules. Let X be an irreducible $U_{\mathbb{R}}$-submodule of W. Then, $\mathbb{C} \otimes X$ is an irreducible U-module by Lemma 4.19. Since $\mathbb{C} \otimes W = \mathcal{V}(\Phi)$, $[\mathbb{C} \otimes X] = a$ for some $a \in \Phi$. Hence, for each $a \in \Phi$, there exists a real $U_{\mathbb{R}}$-module M^a such that $a = [\mathbb{C} \otimes M^a]$ and $\oplus_{a\in\Phi}M^a \cong \tilde{V}_{E_8,\mathbb{R}}$. Similarly, for each $b \in \Psi$, there exists a real submodule N^b such that $b = [\mathbb{C} \otimes W^b]$ and $\oplus_{a\in\Psi}N^b \cong \tilde{V}_{E_8,\mathbb{R}}$.*

Recall that a general element in $\mathcal{S}$ has the form $(a+b+c, a+c, b+c)$ for some $a,b \in \Phi$ and $c \in \Psi$ (cf. (2.17) and (2.19)). For any $s = (a+b+c, a+c, b+c) \in \mathcal{S}$, we define

$$W^s := (M^{a+b} \times_{U_{\mathbb{R}}} N^c) \otimes (M^a \times_{U_{\mathbb{R}}} N^c) \otimes (M^b \times_{U_{\mathbb{R}}} N^c)$$

as a $(U_{\mathbb{R}})^3$-module. Note that $W^0 \cong (U_{\mathbb{R}})^3$.

23

Theorem 4.22. *Let V be the framed VOA constructed in (2.20). Then V has a real form W such that the invariant form on W is positive definite and W is framed. Thus, $Aut(V)$ is finite by (4.14).*

Proof. Let $W^s, s \in \mathcal{S}$, be defined as in (4.21). We shall show that $W = \oplus_{s \in \mathcal{S}} W^s$ has a real VOA structure such that the invariant form on W is positive definite.

By Lemma (4.3), we know that all $W^s, s \in \mathcal{S}$, are simple current modules of $(U_\mathbb{R})^3$. Thus, by [38, Theorem 5.25], it suffices to show $\oplus_{s \in \mathcal{T}} W^s$ has a real VOA structure with a positive definite invariant form for any 2-dimensional subspace $\mathcal{T}$ of $\mathcal{S}$.

Let $(a + b + c, a + c, b + c)$ and $(a' + b' + c', a' + c', b' + c')$ be a basis of $\mathcal{T}$, where $a, b, a', b' \in \Phi$ and $c, c' \in \Psi$. Take $0 \neq x \in \Phi$ such that x is orthogonal to c, c'. Then as in (3.2), we define z' on W by

$$z' = \begin{cases} 1 & \text{on } W^s \text{ if } \langle s, (x, 0, 0) \rangle = 0, \\ -1 & \text{on } W^s \text{ if } \langle s, (x, 0, 0) \rangle = 1. \end{cases}$$

Then by the same argument as in (3.7), one can show that the fixed point subspace W^z can be embedded into $V_{\Lambda, \mathbb{R}}^+$, which has a real VOA structure with a positive definite invariant form.

Since $\mathcal{T}$ is orthogonal to $(x, 0, 0)$, we have $\oplus_{s \in \mathcal{T}} W^s < W^z$ and $\oplus_{s \in \mathcal{T}} W^s$ has a real VOA structure with a positive definite invariant form. $\square$

4.2 $Aut(V)$ is of Monster type

Proposition 4.23. *Let H be a finite group containing an involution z so that $C_H(z) \cong C$, the group of (3.9). Then*
(i) $H = O_{2'}(H)C$; or
(ii) H is a simple group of order $2^{46}3^{20}5^9 7^6 11^2 13^3 17 \cdot 19 \cdot 23 \cdot 39 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.

Proof. [49]. A similar conclusion was obtained in [45] under the additional assumption that z is conjugate in H to an element of $O_2(C) \setminus \{z\}$. That fusion assumption was verified in the situation of [20]. For completeness, we give a verification of this fusion result for $H = Aut(V)$ in an appendix (A.1). $\square$

Theorem 4.24. *(i) $Aut(V)$ is a finite simple group;*
(ii) $|Aut(V)| = 2^{46}3^{20}5^9 7^6 11^2 13^3 17 \cdot 19 \cdot 23 \cdot 39 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.

Proof. By (4.22), $Aut(V)$ is a finite group. By (3.8), $C_{Aut(V)}(z) = C$. By (2.25), C is a proper subgroup of $Aut(V)$.

To prove (i), we quote (4.23) or [49]. Observe that the structure of the group in (2.25) shows that 31 divides the order of $Aut(V)/O_{2'}(Aut(V))$, whence the alternative (i) of (4.23) does not apply here.

For (ii), use (4.23) or [21]. $\square$

Remark 4.25. *(i) So far, determinations of the group order still depend on [45] or [21].*

(ii) Our VOA construction of the Monster has an easy proof of finiteness (4.12), whereas proof of finiteness in [20] was more troublesome. A short proof of finiteness, using the theory of algebraic groups, was given in [49].

Corollary 4.26. *The VOA V (2.21), constructed by Shimakura [44], is of Moonshine type (see the Introduction for the definition).*

A Appendix: A fusion result

The following is relevant to the alternate argument for (4.23) and in fact proves more about fusion. It could be of some independent interest.

Lemma A.1. *The involution z is conjugate in $Aut(V)$ to elements of $O_2(C) \setminus \langle z \rangle$ and to elements of $C \setminus O_2(C)$.*

Proof. We see this by examination of the group in (2.25).

Let $x \in \Phi$, $\mathcal{S}^0$, $\mathcal{S}^1$ and $z \in Aut(V)$ be defined as in (3.1) and (3.2). Without loss, we may assume $x = [V_{EE_8}^-]$.

Recall the bilinear form on $R(U)$ from [42] that

$$
\begin{aligned}
\langle [V_{\frac{\alpha}{2}+EE_8}^{\pm}], [V_{\frac{\beta}{2}+EE_8}^{\pm}] \rangle &= \langle \alpha, \frac{\beta}{2} \rangle \mod 2, \\
\langle [V_{EE_8}^-], [(V_{EE_8}^{T_\chi})^{\pm}] \rangle &= 1,
\end{aligned}
\tag{4}
$$

where $\alpha, \beta \in EE_8$ and $V_{EE_8}^{T_\chi}$ is an irreducible twisted module V_{EE_8} for some character χ of EE_8^*/EE_8.

Let $p_1 : R(U)^3 \to R(U)$ be the natural projection to the first component. Then for any $s \in \mathcal{S}^0$, $\langle p_1(s), x \rangle = 0$ and by (4), we have $p_1(s) = [V_{\frac{\beta}{2}+EE_8}^{\epsilon}]$ for some $\beta \in EE_8$.

Let $x' \in \Phi$ with $x' \neq x$. Then as in (3.2), we can define an automorphism z' by

$$z' = \begin{cases} 1 & \text{on } W \text{ if } [W] \in \mathcal{S}, \langle [W], (x', 0, 0) \rangle = 0, \\ -1 & \text{on } W \text{ if } [W] \in \mathcal{S}, \langle [W], (x', 0, 0) \rangle = 1. \end{cases}$$

Again we may assume $x' = [V^\epsilon_{\frac{\alpha}{2}+EE_8}]$ for some $\alpha \in EE_8$ and $\epsilon = +$ or $-$. Then z' acts on $V^z = \oplus_{[W]\in\mathcal{S}^0} W$ and by (4)

$$z'|_{V^z} = (-1)^{\langle \alpha, \frac{\beta}{2} \rangle} \quad \text{on } W \text{ with } p_1([W]) = [V^\pm_{\frac{\beta}{2}+EE_8}].$$

Thus, $z'|_{V^z} \in H_\Lambda$ and $z' \in O_2(C)$, where H_Λ is defined as in (3.12).

By the same argument as in (3.6) and (3.7), we also have $V^{z'} \cong V^+_\Lambda$. Thus, by the uniqueness of simple current extensions, there exists an automorphism g that maps V^z to $V^{z'}$ and hence $z' = gzg^{-1}$.

Next we shall show that z is conjugate to an element in $C \setminus O_2(C)$.

Let $y \in \Psi$ such that $\langle x, y \rangle = 1$. Define z_y by

$$z_y = \begin{cases} 1 & \text{on } W \text{ if } [W] \in \mathcal{S}, \langle [W], (y, 0, 0) \rangle = 0, \\ -1 & \text{on } W \text{ if } [W] \in \mathcal{S}, \langle [W], (y, 0, 0) \rangle = 1. \end{cases}$$

Then we again have $V^{z_y} \cong V^+_\Lambda$ and z_y is conjugate to z in $Aut(V)$.

Note that z_y also acts on $V = V^z \oplus (V^z \times_{U^3} M) \cong V_\Lambda$ (see Notation (3.5)). Since $\langle x, y \rangle = 1$, z_y acts as -1 on M and thus acts non-trivially on $\hat{V}_1$. By (3.18), $z_y \notin \tilde{H}_\Lambda$ and thus $z_y|_{V^z} \notin H_\Lambda$.

Therefore, $z_y \in C \setminus O_2(C)$ as desired. $\square$

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