

A Priori Estimates for Two-Dimensional Water Waves with Angled Crests

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We consider the two-dimensional water wave problem in the case where the free interface of the fluid meets a vertical wall at a possibly non-right angle and where the free interface can be non- C^1 with angled crests. We assume that the air has density zero, the fluid is inviscid, incompressible, irrotational, and subject to the gravitational force, and the surface tension is zero. In this regime, only a degenerate Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds, with degeneracies at the singularities on the interface and at the point where it meets the wall if the angle is non-right. We construct a low-regularity energy functional and prove an a priori estimate. Our estimate differs from existing work in that it doesn't require a positive lower bound for $-\frac{\partial P}{\partial \mathbf{n}}$.

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1. Introduction

1.1. Water wave problems

A class of water wave problems concerns the dynamics of the free surface separating an incompressible fluid, under the influence of gravity, from a zero-density region (air).

Let $\Omega(t)$ be the fluid region, $\Sigma(t)$ be the free surface between the fluid and the air, and Υ , if it exists, be the fixed rigid boundary of $\Omega(t)$, for time $t \geq 0$; thus $\partial(\Omega(t)) = \Sigma(t) \cup \Upsilon$. We assume that the fluid is inviscid, incompressible and irrotational, and we neglect surface tension. Assume that the fluid density is 1. If the gravity field is $-\mathbf{j}$, the governing equations of motion are

- (1) $\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{j} - \nabla P$ on $\Omega(t)$,
- (2) $\operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 0$ on $\Omega(t)$,
- (3) $P = 0$ on $\Sigma(t)$,
- (4) $(1, \mathbf{v})$ is tangent to the free surface $(t, \Sigma(t))$,
- (5) \mathbf{v} is tangent to the fixed boundary Υ (if it exists),

where \mathbf{v} is the fluid velocity and P is the fluid pressure. There is an important condition for these problems:

$$(6) \quad -\frac{\partial P}{\partial \mathbf{n}} \geq 0$$

pointwise on the interface, where \mathbf{n} is the outward unit normal to the free interface $\Sigma(t)$ [32]; it is well known that when surface tension is neglected and the Taylor sign condition (6) fails, the water wave motion can be subject to the Taylor instability [32, 17, 6].

The study of water waves dates back centuries. Early mathematical works include Stokes [31], Levi-Civita [26] and G.I. Taylor [32]. Nalimov [28], Yosihara [39], and Craig [14] obtained local in time existence and uniqueness of solutions for the 2-D water wave problem for small Sobolev data. In 1997 Wu [33] showed, for the infinite depth two-dimensional water wave problem (1)-(4) with $\Upsilon = \emptyset$, that the strong Taylor stability criterion

$$(7) \quad -\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0,$$

always holds for $C^{1+\epsilon}$ interfaces and that the problem is locally well-posed in Sobolev spaces H^s , $s \geq 4$, for arbitrary data. In [34] Wu proved a similar result in three dimensions. Since then, there have been numerous results on local well-posedness in both two and three dimensions, under the assumption (7), for the water wave equations with nonzero vorticity, with a fixed bottom, and with nonzero surface tension, cf. [10, 20, 5, 25, 27, 13, 30, 41]. Alazard, Burq & Zuily [1, 3] proved local well-posedness of the problem in low regularity Sobolev spaces where the interfaces are only in $C^{3/2}$. Hunter, Ifrim & Tataru [19] obtained a low-regularity result that improves on [1] for 2-D water waves. In addition, in the past several years Wu [35] [36], Germain, Masmoudi & Shatah [18], Ionescu & Pusateri [21] and Alazard & Delort [4] have proved results showing almost global or global well-posedness in two and three dimensions for sufficiently small, smooth and localized initial data. See [9] for other related developments.

All these works either prove or assume the strong Taylor condition (7), and assume either no fixed boundary or else a fixed boundary a positive distance away from the free interface. And the lowest regularity considered are $C^{3/2}$ interfaces.

Consider the water wave equation (1)-(5) in two space dimensions. In the case where the fixed boundary Υ is a vertical wall, by Schwarz reflection, the water wave problem (1)-(5) can be reduced to the one without fixed boundary in the expanded symmetric domain. Assume for example that the fluid is in a region in $\{x \geq 0\}$, bounded by the fixed boundary $\Upsilon : x = 0$ and the free surface. We define, for $\mathbf{v} = (v_1, v_2)$ and $x > 0$, that

$$(8) \quad \mathbf{v}(-x, y, t) = (-v_1(x, y, t), v_2(x, y, t)), \quad P(-x, y, t) = P(x, y, t).$$

Notice that (5) implies that $v_1(\cdot, t) \equiv 0$ on Υ . It is easy to check that equations (1)-(4) continue to hold in the expanded domain. Assume that the interface makes an angle ν with the vertical wall. When $\nu \neq \frac{\pi}{2}$ the extended interface is non- C^1 , with an angled crest in the middle; see Figure 1. In [2] Alazard, Burq & Zuily studied the case where the strong Taylor sign condition (7) holds and the angle $\nu = \frac{\pi}{2}$. We investigate in this paper the question of whether the water wave problem (1)-(5) admits non-right angles ν at the wall, and more generally, whether equations (1)-(5) admit non- C^1 interfaces.¹

¹The reflection/periodization procedure described here dates back to [7]. In [2] it was shown that in order for the strong Taylor sign condition (7) to hold, it is necessary that the angle $\nu = \frac{\pi}{2}$.

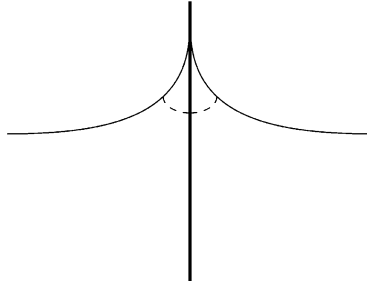


Figure 1: Under a Schwarz reflection, a non-right angle at a vertical wall corresponds to a symmetric angled crest in the middle of surface.

To set up our problem we consider a fixed rigid boundary consisting of two vertical walls at $x = 0, 1$, with water of infinite depth in between the walls. We assume that the fluid region $\Omega_0(t) \subset [0, 1] \times (-\infty, c)$ for some $c < \infty$. Using Schwarz reflection we expand $\Omega_0(t)$ across the y -axis, arriving at a symmetric fluid domain $\Omega(t) \subset [-1, 1] \times (-\infty, c)$. We shall henceforth study the water wave equation (1)-(5) in $\Omega(t)$, with fixed walls Υ at $x = -1, 1$. We denote the angle at $x = 1$ by ν , and refer to the corner of the free surface at the wall $x = \pm 1$ as the *corner*. We assume that

$$(9) \quad \mathbf{v}(x, y, t) \rightarrow 0 \quad \text{as } y \rightarrow -\infty.$$

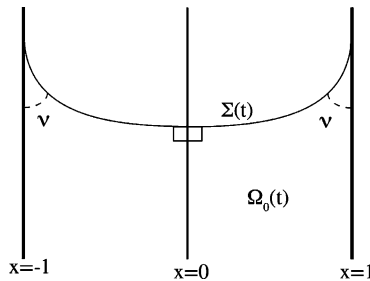


Figure 2: The fluid domain $\Omega_0(t)$ has solid walls at $x = 0, 1$. The figure shows the reflected domain $\Omega(t)$ under a Schwarz reflection across $x = 0$, with a symmetric free surface $\Sigma(t)$. We denote the angle at $x = 1$ by ν .

We note that the extended fluid satisfies the periodic boundary condition

$$(10) \quad \mathbf{v}(-1, y, t) = \mathbf{v}(1, y, t), \quad P(-1, y, t) = P(1, y, t).$$

A serious challenge in allowing non- C^1 interfaces with angled crests and non-right angles at the wall is that in this case, only a degenerate Taylor stability inequality $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds, with a degeneracy $-\frac{\partial P}{\partial \mathbf{n}} = 0$ occurring at the singularity on the interface and at the wall when the interface meets the wall with a non-right angle.² From existing work (cf. [33, 34, 25, 27, 1] etc.) we know the problem of solving the water wave equation (1)-(5) can be reduced to solving a quasilinear equation of the interface $z = z(\alpha, t)$, of type

$$(11) \quad \partial_t^2 \mathbf{u} + a \nabla_{\mathbf{n}} \mathbf{u} = f(\mathbf{u}, \partial_t \mathbf{u}),$$

where $a = -\frac{\partial P}{\partial \mathbf{n}}$, and $\nabla_{\mathbf{n}}$ is the Dirichlet-to-Neumann operator; when the strong Taylor sign condition (7) holds, equation (11) is of the hyperbolic type with the right hand side consisting of lower order terms, and the Cauchy problem can be solved using classical tools. In our case, however, only the degenerate Taylor sign condition (6) holds, with the second term on the left hand side of (11) being degenerate at the singularities; it is not clear if the right hand side of equation (11) is still of the lower order, and what the type of the equation is. This is the main difficulty of the problem.

In this paper, we construct an energy functional and prove an a priori estimate for solutions of the water wave equation. The novelty is that our a priori estimate *does not* require a positive lower bound for $-\frac{\partial P}{\partial \mathbf{n}}$. We follow the general approach of Wu's earlier work [33, 34] and [35], in reducing the water wave problem to an equation on the free surface, and analyzing the free surface equation in the Riemann mapping variable. As an immediate consequence we obtain that, provided that the acceleration is finite, the water wave equation (1)-(5) only allows for angles at the wall $\nu \leq \frac{\pi}{2}$ and, similarly, it only allows for interior angles at angled crests to be no more than π . Using the Riemann mapping we obtain a precise characterization of the singularities of the interface. Two elements played significant roles in the construction of our energy: one is an interface-dependent weight function, which is zero at the singularities; and the other is the convection-driven self-similar solutions constructed in [37], which has an angled crest type singularity. Our energy is finite for all interfaces and velocities in Sobolev spaces $H^s(\mathbb{R})$, $s \geq 3$; it is also finite for interfaces making non-right angles at the wall, and for interfaces with angled crests. Most importantly, it is finite for the self-similar solutions constructed in [37].

The a priori estimate obtained in this paper holds for general periodic water wave equations (1)-(4)-(10). An analogous energy functional can be

²We assume the acceleration is finite.

constructed and the a priori estimate can be proved for the whole line case using a similar approach.

Our energy inequality is a crucial step towards proving local existence, uniqueness and stability in this framework. This will be the content of an upcoming paper.

Remark: This paper was first written and posted on arXiv in June 2014 [24]. Since then, Wu has proved the local existence in the energy class constructed in this paper and posted the work on arXiv in February 2015 [38]. Most recently, in December 2016, Thibault de Poyferré [16] obtained an a priori estimate for the water wave equations (1)-(5) with an emerging bottom that interacts with the free surface. One of the important assumptions in [16] is that the strong Taylor sign condition (7) holds.

1.2. Outline of the paper

In the next subsection, §1.3, we present some of the notations and conventions and introduce the function spaces and norms to be used in the paper. Then, in §2 and §3, we derive the free surface equations of the water wave problem (1)-(5) in Lagrangian and Riemann mapping coordinates, following [33] and [35]. The derivation in §2 and §3 is carried out under the assumption that the interface, velocity and acceleration are smooth. In §4, we define the energy (in §4.1) and state our main result, the a priori inequality (in §4.2) for solutions of the free surface equations. We begin the proof in §4.3, and then in §4.4 we outline the remainder of the proof, which takes up sections §5 through §10.

In §11 we give a characterization of the energy in terms of the velocity and position of the free surface, and we discuss the types of singularities allowed when our energy is finite.

The derivation of the free surface equations and the proof of the main result rely on understanding the boundary behaviors, the holomorphicity, and the means of various quantities; we leave these, as well as some basic identities and inequalities used in the proof of the main result, to appendices §A-§B. The reader may want to read these appendices before certain sections in the main text. We have two additional appendices that might be useful to the reader. In §C, we provide an overview of the notation used in the paper, with cross references to where everything was initially defined. In §D, we list various quantities controlled by the energy, again with cross references.

1.3. Notations, conventions and function spaces

We will define most of our notations throughout the text, as we introduce our various quantities. Here we only list some general conventions and notations.

Since we are in two dimensions, we will often work in complex coordinates $(x, y) = x + iy$. We will use $\Re z := x$ and $\Im z := y$ to represent the real and imaginary parts, respectively, of $z = x + iy$.

Compositions are always in terms of the spatial variable. For example, for $f = f(\alpha, t)$, $g = g(\alpha, t)$, we define $f \circ g = f \circ g(\alpha, t) := f(g(\alpha, t), t)$. An expression $f_x(x, y)$ means $\partial_x f(x, y)$; we occasionally use the notation f' , which is always the spatial derivative in whatever coordinates we are using.

Once we have reduced the water wave equations to an equation on the interface, we will primarily be working with the spatial domain $I := [-1, 1]$. We will often refer to the “boundary”; this refers to what happens at ± 1 . We write $f|_{\partial} := f(1) - f(-1)$. We will use

$$(12) \quad \int_I f := \frac{1}{|I|} \int_I f(x) dx = \frac{1}{2} \int_{-1}^1 f(x) dx$$

for the mean of a function f . Here, and elsewhere for other integrals, when there is no risk of ambiguity, we will often drop the subscript I .

We define

$$(13) \quad [A, B] := AB - BA.$$

We will use the following notation as an abbreviation for a type of higher-order Calderon commutator:

$$(14) \quad [f, g; h](\alpha') := \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta'.$$

We will often deal with 1-d singular integrals of the type $\int k(\alpha', \beta') d\beta'$ where $|k(\alpha', \beta')| = \frac{O(1)}{|\alpha' - \beta'|}$; in this case, the integral $\int k(\alpha', \beta') d\beta'$ is defined to be the principal value of the integral.

We will use C as a placeholder to refer to a universal constant, possibly varying from line to line. We will also often use the notation $f \lesssim g$, which means that there exists some universal constant C such that $f \leq Cg$.

We will at several points have long series of identities or inequalities. When we say “on the RHS” of an equation block with a string of multiple equalities or inequalities, we mean all the terms on the right hand side of the last equality or inequality sign in the string. Similarly, when we say “on

the LHS,” we mean all the terms to the left of the very first equality or inequality sign in the string of equalities and inequalities. We have tried to avoid saying “on the n th line” when any of the mathematical formulas splits into more than one typographic line, but if we have, “line” refers to the mathematical, not typographic, line.

We have tried to give extensive cross references for each time we use a result or estimate. We tend to refer to equation numbers, rather than propositions, since it seems that these will be easier to find as cross references. When we refer to an equation number as part of a proposition, we are of course referring to the whole proposition, including any conditions assumed.

When we are deriving estimates, we sometimes use the cross references within our equations, e.g.:

$$(15) \quad \begin{aligned} f &\leq g \\ &\leq h \end{aligned}$$

and

$$(16) \quad \begin{aligned} h &\lesssim j + f \\ &\lesssim j + (15) \\ &\lesssim j + h. \end{aligned}$$

This means (15) is used to obtain (16). We hope this will help the reader locate the previous estimate or estimates.

In several of our more complicated estimates, we will split terms up $f = I + II$ and then $I = I_1 + I_2$, $I_1 = I_{11} + I_{12}$, etc. Such notation will be *local to each section*. There is an ambiguity between the use of I as a placeholder, its use as the identity operator, and its use as $I := [-1, 1]$. It should be clear from the context which one is being used.

We now introduce the function spaces and norms we will use. We will work with functions $f(\cdot, t)$ defined on $I = [-1, 1]$. Except when necessary to avoid ambiguity, we neglect to write the time variable; when it is not specified, function spaces and norms are in terms of the spatial variable.

We say $f \in C^k(J)$, $J = (-1, 1)$ or $[-1, 1]$, if for every $0 \leq l \leq k$, $\partial_x^l f$ is a continuous function on the interval J . We say $f \in C^k(S^1)$ (i.e., periodic C^k) if for every $0 \leq l \leq k$, $\partial_x^l f \in C^0[-1, 1]$ and $\partial_x^l f(1) = \partial_x^l f(-1)$. ($\partial_x^l f$ at the endpoints 1 or -1 is the derivative from the left or right, respectively.) Note in particular that saying $f \in C^0(S^1)$ implies that $f|_{\partial} = 0$.

For $1 \leq p < \infty$, we define our L^p spaces by the norms

$$(17) \quad \|f\|_{L^p} := \|f\|_{L^p(I)} := \left(\int_I |f|^p \right)^{1/p},$$

and we define L^∞ analogously. We will sometimes deal with weighted L^p spaces. We write

$$(18) \quad \|f\|_{L^p(\omega)} = \|f\|_{L^p(\omega dx)} := \left(\int_I |f(x)|^p \omega(x) dx \right)^{1/p}$$

for weights $\omega \geq 0$. Whenever we write L^p , we will be referring to $L^p(I)$, in the spatial variable. For weighted L^p spaces, we always write $L^p(\omega)$ or $L^p(\omega dx)$, where ω is the weight function.

We now define the periodic Sobolev space $H^k(S^1)$. Let $f \in L^1(I)$, and \tilde{f} be the periodic extension of f to the whole line: $\tilde{f}(x+2) = \tilde{f}(x)$ for all $x \in \mathbb{R}$, and $\tilde{f}(x) := f(x)$ for $x \in I$. We say $f \in H^k(S^1)$ if $\tilde{f} \in H^k((-\infty, \infty))$; and we define

$$(19) \quad \|f\|_{H^k(S^1)} := \left(\sum_{j=0}^k \int_I |\partial_x^j f(x)|^2 dx \right)^{1/2}.$$

By Sobolev embedding, we know $H^{k+1}(S^1) \subset C^k(S^1)$, for $k \geq 0$.

We define the homogeneous half-derivative space $\dot{H}^{1/2}$ by the norm

$$(20) \quad \|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint_{I \times I} \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta' \right)^{1/2}.$$

Through the remainder of the paper, when we say the boundary value of a function G defined on the fluid region $\Omega(t)$ (resp., on $P^- := I \times (-\infty, 0]$), we mean the value of G on the free surface (resp., on $I \times \{0\}$); we do not include the value on vertical boundaries $x = \pm 1$. Except when there's a risk of confusion, we will slightly abuse notation and say that a function f on the free surface (resp., on $I \times \{0\}$) is “holomorphic” (or “antiholomorphic”); what we mean, precisely, is that it is the boundary value of a function that is holomorphic (or antiholomorphic) in the fluid region $\Omega(t)$ (resp., on P^-).

In the next two sections, §2 and §3, we assume the interface, velocity, acceleration and their time derivatives are sufficiently smooth.

2. The free surface equation in the Lagrangian coordinate

Let $z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)$, $\alpha \in I = [-1, 1]$ be a parametrization of the free surface $\Sigma(t)$ in the *Lagrangian* variable α , i.e., $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity and z_{tt} is the acceleration of the particle occupying position $z(\alpha, t)$ at time t . Along the free surface, the Euler equation (1) is $z_{tt} + i = -\nabla P$. By equation (3), we know ∇P is orthogonal to the free surface. Since iz_α is normal to the free surface, we can rewrite our main equation as

$$(21) \quad z_{tt} + i = i\mathbf{a}z_\alpha,$$

where

$$(22) \quad \mathbf{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|} \in \mathbb{R}$$

for $\frac{\partial P}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla P$ the outward-facing normal derivative. The incompressibility and irrotationality condition (2) and the periodicity (10) imply that the conjugate velocity $\bar{\mathbf{v}}$ is periodic holomorphic; therefore \bar{z}_t is the boundary value of a periodic holomorphic function in the fluid region.

2.1. The quasilinear equation

We henceforth focus on the equations on the free surface.³ As in [33] and following works, we differentiate (21) with respect to time and take conjugates, turning it into the quasilinear equation⁴

$$(23) \quad \bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha} = -i\mathbf{a}_t\bar{z}_\alpha,$$

where we continue to have \bar{z}_t the boundary value of a periodic holomorphic function. This is the basic equation we will work with throughout the paper.

³We may solve for the velocity on $\Omega(t)$ from its boundary values (including the condition that it goes to zero as $y \rightarrow -\infty$), and then solve for the pressure from the velocity. The free surface equations are equivalent to the water wave system (1)-(5) in the smooth regime.

⁴We call it “quasilinear” because in the classical situation [33], this equation is quasilinear with the RHS the lower-order term. However, in our setting, due to the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ we do not know a priori that this is still the case; only by our proof do we show that the RHS is, indeed, lower-order and that (23) is in fact quasilinear. All references to (23) and related equations being “quasilinear” should be interpreted with this in mind.

The holomorphicity of \bar{z}_t implies that $i\frac{1}{|z_\alpha|}\partial_\alpha\bar{z}_t = \nabla_{\mathbf{n}}\bar{z}_t$, where $\nabla_{\mathbf{n}}$ is the Dirichlet-to-Neumann operator. We know $\nabla_{\mathbf{n}}$ is a positive operator.

In [34] and [35], coordinate-independent formulas for the RHS were derived, using the holomorphicity of \bar{z}_t and the invertibility of the double-layer potential. We will instead follow the original approach of [33], relying on the Riemann mapping version of the equation to derive the RHS. We do so in §3.3.3.

2.2. A special derivative

We introduce a special derivative

$$(24) \quad D_\alpha := \frac{1}{z_\alpha}\partial_\alpha.$$

If $g(\alpha, t) = G(z(\alpha, t), t)$, and G is holomorphic, then $\partial_\alpha g = (G_z \circ z)z_\alpha$, and

$$D_\alpha g = (\partial_z G) \circ z = -i(\partial_y G) \circ z.$$

So $D_\alpha^k g$ is the boundary value of holomorphic function $\partial_z^k G$, provided G is holomorphic. $D_\alpha^k g$ is in addition periodic for any $k \geq 1$, so long as G is periodic and holomorphic.

We may therefore conclude from the fact that \bar{z}_t is the boundary value of the periodic holomorphic function $\bar{\mathbf{v}}$ that $D_\alpha^k \bar{z}_t$ is the boundary value of the periodic holomorphic function $\partial_z^k \bar{\mathbf{v}}$ in $\Omega(t)$.

We will use D_α as the spatial derivative in constructing higher-order energies. In addition to preserving holomorphicity and periodic boundary behavior, it transforms well under the Riemann mapping, to be discussed in the next section.

3. The Riemann mapping version

We now analyze the water wave equations (21), (23) using the Riemann mapping that flattens out the curved free interface.⁵ The Riemann mapping version of the equations offers a key advantage, because the Hilbert

⁵To the best of our knowledge, [33] was the first paper that used the Riemann mapping to analyze the quantities A_1, \mathcal{A}_t and $h_t \circ h^{-1}$ in the water wave equations and prove the wellposedness of 2-d water waves in Sobolev spaces. Using the Riemann mapping, [19] later carried out a similar analysis as in [33, 35] and re-derived the formulas for the quantities $A_1, h_t \circ h^{-1}$. Here we follow the approach of [33, 35] to analyze the quantities A_1, \mathcal{A}_t and $h_t \circ h^{-1}$. The Riemann mapping is a common tool in the study of 2d potential flows. In water waves, Ovsjannikov [29] used the

transform associated to the periodic domain P^- is

$$(25) \quad \mathbb{H}f(\alpha') := \frac{1}{2i} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta'.$$

Since $\mathbb{H}f \in i\mathbb{R}$ for f real-valued, this allows us to invert the operator $(I - \mathbb{H})$ on purely real (resp., purely imaginary) functions by taking real (resp., imaginary) parts.

3.1. The Riemann mapping variables and notations

Let

$$(26) \quad \Phi : \Omega(t) \rightarrow P^- := \{(x, y) : x \in [-1, 1], y \leq 0\}.$$

be the unique Riemann mapping that takes the two upper corners of the interface at the walls $x = -1, 1$ to $(-1, 0)$ and $(1, 0)$, and ∞ to ∞ . We know Φ takes the free surface $\Sigma(t)$ to $I \times \{0\}$ and the wall to the wall; Φ_z is periodic: $\Phi_z(-1, y, t) = \Phi_z(1, y, t)$, and

$$(27) \quad \lim_{\Im z \rightarrow -\infty} \Phi_z(z, t) = 1.$$

Let

$$(28) \quad \alpha' = h(\alpha, t) := \Phi(z(\alpha, t), t) : I \rightarrow I$$

be the change of coordinates taking the Lagrangian variable α to the Riemann mapping variable α' , and let h^{-1} be the spatial inverse of h , defined by $h(h^{-1}(\alpha', t), t) = \alpha'$. We define

$$(29) \quad Z(\alpha', t) := z \circ h^{-1}(\alpha', t) = z(h^{-1}(\alpha', t), t).$$

$Z = Z(\alpha', t)$ is a parametrization of the free surface $\Sigma(t)$ in the Riemann mapping variable α' . We write

$$(30) \quad \begin{aligned} Z_t &:= z_t \circ h^{-1}; & Z_{tt} &:= z_{tt} \circ h^{-1}; \\ Z_{,\alpha'} &:= \partial_{\alpha'} Z; & Z_{t,\alpha'} &:= \partial_{\alpha'} Z_t; & Z_{tt,\alpha'} &:= \partial_{\alpha'} Z_{tt}; \text{ etc.} \end{aligned}$$

Riemann mapping to justify the shallow water equation from equations (1)-(4) in the analytic class; Zakharov et al. [40] used the Riemann mapping to carry out efficient numerical computations for the water waves.

and

$$(31) \quad \mathcal{A} := (\mathbf{a}h_\alpha) \circ h^{-1}; \quad \mathcal{A}_t := (\mathbf{a}_t h_\alpha) \circ h^{-1}.$$

Observe that $Z = z \circ h^{-1} = \Phi^{-1}$. Therefore

$$(32) \quad Z_{,\alpha'}(\alpha', t) = \partial_{\alpha'}(\Phi^{-1}(\alpha', t)), \quad \text{and} \quad \frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z,$$

so $Z_{,\alpha'}(\alpha', t)$ and $\frac{1}{Z_{,\alpha'}}(\alpha', t)$ are boundary values of the periodic holomorphic functions $(\Phi^{-1})_z(\cdot, t)$ and $\Phi_z(\cdot, t)$.

Observe also that under the change of variables $(D_\alpha f) \circ h^{-1} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}(f \circ h^{-1})$. We define

$$(33) \quad D_{\alpha'} := \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}.$$

3.2. An assumption at the spatial infinity

For the derivation of the water wave equations in the Riemann mapping variable, besides assuming all the quantities involved are sufficiently smooth, we assume that the Riemann mapping Φ satisfies

$$(34) \quad \lim_{\Im z \rightarrow -\infty} \Phi_t \circ \Phi^{-1}(z, t) = 0.$$

In [33] analogous assumptions were made to derive the quasilinear equation in the Riemann mapping variable for the whole line case; it was then shown that the quasilinear equation is well-posed in Sobolev spaces and the solutions of the quasilinear equation give rise to solutions of the water wave equation (1)-(4). Similar results can be proved for the periodic case as considered in this paper.

3.3. The water wave equations in the Riemann mapping variable

We now derive the water wave equations in Riemann mapping variable. We follow the approach of [33], although we work with the real and imaginary parts together instead of separating $Z_t = X_t + iY_t$ into real and imaginary parts.

Beginning with the conjugated form of our equation (21) and with (23), we precompose both sides with h^{-1} to get the free surface equations in the flattened Riemann mapping coordinate:

$$(35) \quad \bar{Z}_{tt} - i = -i\mathcal{A}\bar{Z}_{,\alpha'};$$

$$(36) \quad \bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'} = -i\mathcal{A}_t\bar{Z}_{,\alpha'},$$

where \bar{Z}_t is the boundary value of the periodic holomorphic function $\bar{\mathbf{v}} \circ \Phi^{-1}$. By the chain rule, the quantities \bar{Z}_{tt} , \bar{Z}_{ttt} are related to \bar{Z}_t by

$$\bar{Z}_{tt} = (\partial_t + \mathbf{b}\partial_{\alpha'})\bar{Z}_t, \quad \bar{Z}_{ttt} = (\partial_t + \mathbf{b}\partial_{\alpha'})^2\bar{Z}_t,$$

where

$$(37) \quad \mathbf{b} := h_t \circ h^{-1}.$$

The following proposition gives a characterization of the boundary value of a periodic holomorphic function on P^- .

Proposition 1. *a. Let $g \in L^1(I)$. Then g is the boundary value of a holomorphic function G on P^- satisfying $G(-1 + iy) = G(1 + iy)$ for all $y < 0$ and $G(x + iy) \rightarrow c_0$ as $y \rightarrow -\infty$ if and only if*

$$(38) \quad (I - \mathbb{H})g = c_0.$$

Moreover, $c_0 = \int_I g$.

b. Let $f \in L^1(I)$. Then $\mathbb{P}_H f := \frac{1}{2}(I + \mathbb{H})f$ is the boundary value of a periodic holomorphic function \mathcal{F} on P^- , with $\mathcal{F}(x + iy) \rightarrow \frac{1}{2}\int_I f$ as $y \rightarrow -\infty$.

Proposition 1 is a classical result, which can be proved by the Cauchy integral formula; see [22]. As a consequence, \bar{Z}_t satisfies

$$(39) \quad (I - \mathbb{H})\bar{Z}_t = 0,$$

and by (27), $\frac{1}{Z_{,\alpha'}}$ satisfies

$$(40) \quad (I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = 1.$$

We define the following projection operators:

$$(41) \quad \mathbb{P}_H f := \frac{(I + \mathbb{H})}{2}f; \quad \mathbb{P}_A f := \frac{(I - \mathbb{H})}{2}f.$$

We will refer to \mathbb{P}_H as the “holomorphic projection” and \mathbb{P}_A as the “antiholomorphic projection”. These operators are, indeed, proper projections when interpreted modulo a constant.

We now seek formulas for \mathcal{A} , \mathcal{A}_t and \mathfrak{b} . For derivations here we will rely on the technicalities given in appendix §A.

3.3.1. \mathcal{A} and the quantity \mathbf{A}_1 . We first derive a formula for \mathcal{A} . Historically, in [33], this derivation showed that the strong Taylor stability criterion would automatically hold. There is now [34] a direct proof of this via basic elliptic theory without using the Riemann mapping. For our purposes, though, this original derivation here will be crucial because it introduces a quantity, A_1 , that compares the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ directly with that of the Riemann mapping and therefore the geometry of the free surface.

We begin with (35). The key observation is that $Z_{,\alpha'}$ is periodic holomorphic and that $\overline{Z}_{tt} - i$ is in some sense close to periodic holomorphic, since \overline{Z}_t is periodic holomorphic; therefore $Z_{,\alpha'}(\overline{Z}_{tt} - i)$ is close to periodic holomorphic; and by (35), $Z_{,\alpha'}(\overline{Z}_{tt} - i)$ is purely imaginary. Applying $(I - \mathbb{H})$ to this, using Proposition 1 and taking imaginary parts, we shall get a formula for \mathcal{A} .

Multiplying both sides of (35) by $Z_{,\alpha'}$, we get

$$(42) \quad Z_{,\alpha'}(\overline{Z}_{tt} - i) = -i\mathcal{A} |Z_{,\alpha'}|^2.$$

We now expand out \overline{Z}_{tt} . Let

$$(43) \quad F(z(\alpha, t), t) := \overline{z}_t(\alpha, t),$$

where $F = \overline{\mathbf{v}}$, a periodic holomorphic function in $\Omega(t)$. We will use this expansion several times in the sequel, always with this definition of F . By the chain rule,

$$(44) \quad \overline{z}_{tt} = \frac{d}{dt} F(z(\alpha, t), t) = (F_z \circ z) z_t + (F_t \circ z).$$

Recall from §2.2 that $\partial_z = D_\alpha$ for holomorphic functions. Therefore, $F_z \circ z = \frac{\overline{z}_{t\alpha}}{z_\alpha}$, and thus $\overline{z}_{tt} = \frac{\overline{z}_{t\alpha}}{z_\alpha} z_t + F_t \circ z$. We precompose with h^{-1} :

$$(45) \quad \overline{Z}_{tt} = \left(\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}} \right) Z_t + F_t \circ Z.$$

We can now write our equation (42) as

$$(46) \quad \overline{Z}_{t,\alpha'} Z_t + Z_{,\alpha'} (F_t \circ Z) - i Z_{,\alpha'} = -i\mathcal{A} |Z_{,\alpha'}|^2.$$

We apply $(I - \mathbb{H})$ to both sides. By (304), (302) and (298), writing $(I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'}) = [Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'}$, we get

$$(47) \quad [Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} - i = (I - \mathbb{H}) \left(-i \mathcal{A} |Z_{,\alpha'}|^2 \right).$$

We now take imaginary parts of both sides. This gives us the new quantity A_1 :

$$(48) \quad A_1 := \mathcal{A} |Z_{,\alpha'}|^2 = \Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right) + 1.$$

This is the same A_1 as that in [33]. It's easy to see that $\Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right)$ is non-negative, by integration by parts. Indeed, if $Z_t = X_t + iY_t$, then

$$\begin{aligned} 2\Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right) &= -\Im \frac{1}{i} \int (Z_t(\alpha') - Z_t(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \bar{Z}_{t,\beta'} d\beta' = \\ &= \int \frac{1}{2} \left\{ -\partial_{\beta'} \left[(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2 \right] \right\} \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) d\beta' \\ &= \frac{\pi}{4} \int \frac{(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} d\beta' \\ &\geq 0. \end{aligned}$$

Here, there is no boundary term in the integration by parts because $\bar{Z}_t|_{\partial} = 0$. Therefore

$$(49) \quad A_1 \geq 1.$$

Combining (42) and (48) we get

$$(50) \quad \frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}.$$

3.3.2. Degenerate Taylor stability criterion and the singularity of the free surface. We can draw a few important conclusions from the derivations in §3.3.1. For the sake of exposition, we will in this section focus on the angle ν at the wall, and we will move the corner from ± 1 to 0; angled crests and other singularities in the middle of the surface can be handled similarly.

Let ν be the angle at the corner. By Christoffel-Schwarz Theorem, the Riemann mapping $\Phi(z) \approx z^r$ at the corner, where $r\nu = \frac{\pi}{2}$. By (32), $Z_{,\alpha'} =$

$(\Phi^{-1})_{z'}$. Therefore,

$$(51) \quad Z_{,\alpha'} = \partial_{\alpha'} \Phi^{-1} \approx (\alpha')^{1/r-1}$$

at the corner.

We observe from (49)-(50) that if the acceleration $|Z_{tt}| < \infty$, then $Z_{,\alpha'} \neq 0$. This implies $r \geq 1$ and $\nu \leq \pi/2$. Similarly, this implies that an angled crest has interior angle $\leq \pi$.

In this paper we work in the regime where the acceleration $|Z_{tt}| < \infty$.

Now, because

$$(52) \quad A_1 \circ h = \frac{\mathbf{a} |z_\alpha|^2}{h_\alpha}$$

and

$$(53) \quad -\frac{\partial P}{\partial \mathbf{n}} = |z_\alpha| \mathbf{a} = \frac{A_1 \circ h}{|Z_{,\alpha'} \circ h|},$$

$-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ always holds. In the regime where the free surface is $C^{1,\gamma}$ and makes a right angle at the corner ($\nu = \pi/2$), $0 < c_0 \leq |(\Phi^{-1})_{z'}| \leq C_0 < \infty$. This together with the estimate $A_1 \geq 1$ gave [33] a strictly positive lower bound for the Taylor coefficient $-\frac{\partial P}{\partial \mathbf{n}}$. In our situation, $\frac{1}{Z_{,\alpha'}} \rightarrow 0$ at the corner if $\nu < \frac{\pi}{2}$; similarly, $\frac{1}{Z_{,\alpha'}} \rightarrow 0$ at an angled crest if the interior angle is $< \pi$. If A_1 is in addition bounded from above—which will be true when our energy is finite—we know that the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ corresponds precisely to the degeneracy of $\frac{1}{Z_{,\alpha'}}$.

We note that our spatial derivative $D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}$ is less singular, in a sense, than $\partial_{\alpha'}$. We know $\frac{1}{Z_{,\alpha'}} \rightarrow 0$ at singularities, which indicates that the weight function $\frac{1}{Z_{,\alpha'}}$ has some “regularizing” effect. Indeed, we have $D_{\alpha'} \bar{Z}_t, D_{\alpha'} \bar{Z}_{tt} \in L^\infty$, but $\partial_{\alpha'} \bar{Z}_t, \partial_{\alpha'} \bar{Z}_{tt}$ are only in L^2 in our energy space, for example; see §5.1.

3.3.3. The quantities \mathcal{A}_t and $\frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}$. Now we seek a formula for the quantity on the RHS of (36), \mathcal{A}_t . As in §3.3.1, we start by multiplying both sides of (36) by $Z_{,\alpha'}$ to get a purely imaginary and almost holomorphic quantity, then apply $(I - \mathbb{H})$ to both sides and take imaginary parts to get a formula for \mathcal{A}_t . We have

$$(54) \quad Z_{,\alpha'} (\bar{Z}_{ttt} + i \mathcal{A} \bar{Z}_{t,\alpha'}) = -i \mathcal{A}_t |Z_{,\alpha'}|^2.$$

We once again carefully expand the LHS. As before, let $F(z(\alpha, t), t) = \bar{z}_t(\alpha, t)$. Again, we have

$$(55) \quad \bar{z}_{tt} = (F_z \circ z)z_t + F_t \circ z,$$

so

$$(56) \quad \bar{z}_{ttt} = (F_{zz} \circ z)z_t^2 + 2(F_{tz} \circ z)z_t + (F_z \circ z)z_{tt} + F_{tt} \circ z.$$

We now solve for $F_z \circ z$, $F_{zz} \circ z$ and $F_{tz} \circ z$. Since $\partial_z = D_\alpha$ on holomorphic functions,

$$(57) \quad F_z \circ z = D_\alpha \bar{z}_t, \quad F_{zz} \circ z = D_\alpha^2 \bar{z}_t.$$

We solve for $F_{tz} \circ z$ by applying $\partial_z = D_\alpha$ to (55):

$$(58) \quad F_{tz} \circ z = D_\alpha (\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t).$$

Therefore, by substituting (57) and (58) into (56), we get

$$(59) \quad \bar{z}_{ttt} = (D_\alpha^2 \bar{z}_t)z_t^2 + 2z_t D_\alpha (\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t) + (D_\alpha \bar{z}_t)z_{tt} + F_{tt} \circ z.$$

Precomposing with h^{-1} , we have

$$(60) \quad \bar{Z}_{ttt} = (D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z.$$

We now go back to (54), substituting in (60) to get

$$(61) \quad \begin{aligned} Z_{,\alpha'} ((D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z) \\ + Z_{,\alpha'} (i\mathcal{A}\bar{Z}_{t,\alpha'}) = -i\mathcal{A}_t |Z_{,\alpha'}|^2. \end{aligned}$$

We simplify, distributing the $Z_{,\alpha'}$ and then using the identity $Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}$ on the last term:

$$(62) \quad \begin{aligned} (\partial_{\alpha'} D_{\alpha'} \bar{Z}_t)Z_t^2 + 2Z_t \partial_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + 2\bar{Z}_{t,\alpha'} Z_{tt} + i\bar{Z}_{t,\alpha'} \\ + Z_{,\alpha'} (F_{tt} \circ Z) = -i\mathcal{A}_t |Z_{,\alpha'}|^2. \end{aligned}$$

We now apply $(I - \mathbb{H})$ to both sides. Various terms will disappear on the LHS and others will turn into commutators, due to holomorphicity; specifically, we use (304), (305), and (303). We get

$$(63) \quad \begin{aligned} [Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \bar{Z}_t + 2[Z_t, \mathbb{H}] \partial_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha'} \\ = (I - \mathbb{H}) \left\{ -i\mathcal{A}_t |Z_{,\alpha'}|^2 \right\}. \end{aligned}$$

We could continue working with this equation, but two integrations by parts will give us a nicer equation to work with. We take the first term and the second part of the second term and integrate by parts both terms, noting that we have no boundary terms. We get

$$(64) \quad \begin{aligned} & [Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \bar{Z}_t - 2[Z_t, \mathbb{H}] \partial_{\alpha'} ((D_{\alpha'} \bar{Z}_t) Z_t) \\ &= -\frac{\pi}{4i} \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t(\beta') d\beta'. \end{aligned}$$

This is a type of higher-order Calderon commutator, which we write as $-[Z_t, Z_t; D_{\alpha'} \bar{Z}_t]$ (see (14)). We therefore can rewrite (63) as

$$(65) \quad -i(I - \mathbb{H}) \left\{ \mathcal{A}_t |Z_{,\alpha'}|^2 \right\} = 2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t].$$

Taking imaginary parts, we get

$$(66) \quad \mathcal{A}_t |Z_{,\alpha'}|^2 = -\Im \left(2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t] \right).$$

Observe that dividing (66) by (48) we have

$$(67) \quad \frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1} = \frac{\mathcal{A}_t}{\mathcal{A}} = \frac{-\Im \left(2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t] \right)}{A_1}.$$

3.3.4. The quantity $\mathbf{b} := \mathbf{h}_t \circ \mathbf{h}^{-1}$. Here we derive a formula for $\mathbf{b} := h_t \circ h^{-1}$, following [33]. We recall (28): $h(\alpha, t) = \Phi(z(\alpha, t), t) = \Phi \circ z$. Therefore $h_\alpha = (\Phi_z \circ z) z_\alpha$, and

$$(68) \quad h_t = (\Phi_t \circ z) + (\Phi_z \circ z) z_t = (\Phi_t \circ z) + \frac{h_\alpha}{z_\alpha} z_t.$$

We precompose with h^{-1} :

$$(69) \quad (h_t \circ h^{-1})(\alpha', t) = \Phi_t \circ Z + \frac{1}{Z_{,\alpha'}} Z_t.$$

Apply $(I - \mathbb{H})$ to both sides, then take the real parts. By (299), we get

$$(70) \quad \mathbf{b} := (h_t \circ h^{-1})(\alpha', t) = \Re(I - \mathbb{H}) \left(\frac{1}{Z_{,\alpha'}} Z_t \right).$$

In what follows we will also use the following evolution equation for $\frac{1}{z_\alpha}$, or equivalently in the Riemann mapping variable, $\frac{1}{Z_{,\alpha'}}$. We have

$$(71) \quad \partial_t \frac{1}{z_\alpha} = -\frac{1}{z_\alpha} D_\alpha z_t, \quad \text{and}$$

$$(72) \quad \left(\partial_t \frac{h_\alpha}{z_\alpha} \right) \circ h^{-1} = (\partial_t + \mathbf{b} \partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \right) = \frac{1}{Z_{,\alpha'}} (\partial_{\alpha'} \mathbf{b} - D_{\alpha'} Z_t).$$

4. The main result

Observe that the system of free surface equations (35)-(36)-(48)-(67)-(70)-(39)-(40)-(72) is a closed system for the quantities \bar{Z}_t , \bar{Z}_{tt} and $\frac{1}{Z_{,\alpha'}}$. In what follows we will focus on this system and construct an energy that allows for $\frac{1}{Z_{,\alpha'}} = 0$, i.e., singularities on the interface and at the corner, and prove an a priori estimate.

4.1. Definition of the energy

We consider a general equation of the form

$$(73) \quad (\partial_t^2 + i\mathbf{a} \partial_\alpha) \theta = G_\theta$$

with the constraint that θ is the boundary value of a periodic holomorphic function on $\Omega(t)$. Precomposing with h^{-1} and using (52), we obtain the equation in the Riemann mapping variable

$$(74) \quad (\partial_t^2 \theta) \circ h^{-1} + i \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} (\theta \circ h^{-1}) = G_\theta \circ h^{-1}.$$

There are two mutually related basic energies. One is

$$(75) \quad \begin{aligned} E_{a,\theta}(t) := & \int_I \frac{|\theta_t \circ h^{-1}|^2}{A_1} d\alpha' \\ & + \int_I i \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} (\theta \circ h^{-1}) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} (\bar{\theta} \circ h^{-1}) \right) d\alpha' + \int_I \frac{|\theta \circ h^{-1}|^2}{A_1} d\alpha', \end{aligned}$$

for (74) in the Riemann mapping variables, another is

$$(76) \quad E_{b,\theta}(t) := \int_I \frac{1}{\mathbf{a}} |\theta_t|^2 d\alpha + \int_I (i \partial_\alpha \theta) \bar{\theta} d\alpha + \int_I \frac{(A_1 \circ h)}{\mathbf{a}} |\theta|^2 d\alpha$$

for (73). A form of $E_{b,\theta}(t)$ has appeared in [35]. Upon changing to the Riemann mapping coordinates, and by (52), we have

$$(77) \quad E_{b,\theta}(t) = \int_I \frac{|Z_{,\alpha'}\theta_t \circ h^{-1}|^2}{A_1} d\alpha' + \int_I i(\partial_{\alpha'}\theta \circ h^{-1})(\overline{\theta \circ h^{-1}})d\alpha' \\ + \int_I |Z_{,\alpha'}\theta \circ h^{-1}|^2 d\alpha'.$$

In $E_{a,\theta}$ and $E_{b,\theta}$, the first two terms are from the LHS of the equations, the last terms are of lower order. Because $\theta \circ h^{-1}$ and $\frac{1}{Z_{,\alpha'}}$ are holomorphic, the second terms in $E_{a,\theta}(t)$ and $E_{b,\theta}(t)$ are nonnegative and equal to $\|\frac{1}{Z_{,\alpha'}}(\theta \circ h^{-1})\|_{\dot{H}^{1/2}}^2$ and $\|\theta \circ h^{-1}\|_{\dot{H}^{1/2}}^2$ respectively; see §B.4.

Notice that the primary difference between the two basic energies $E_{a,\theta}$ and $E_{b,\theta}$ is either to multiply or to divide by the weight function $\frac{1}{Z_{,\alpha'}}$. In the classical case where $|Z_{,\alpha'}|$ is bounded away from 0 and ∞ , there is no real difference between the two energies. However it does make a difference if we want to allow $\frac{1}{Z_{,\alpha'}} \rightarrow 0$.

We now construct our energy functional, by applying the two basic energies to our equations. We begin with equation (23) (equivalently (36), by a change of the coordinates.):

$$(78) \quad (\partial_t^2 + ia\partial_\alpha)\bar{z}_t = -ia_t\bar{z}_\alpha.$$

Applying weighted derivatives D_α^k to (78) we get

$$(\partial_t^2 + ia\partial_\alpha)D_\alpha^k\bar{z}_t = G_{D_\alpha^k\bar{z}_t}$$

with

$$(79) \quad G_{D_\alpha^k\bar{z}_t} = D_\alpha^k(-ia_t\bar{z}_\alpha) + [\partial_t^2 + ia\partial_\alpha, D_\alpha^k]\bar{z}_t.$$

Our total energy consists primarily of $E_{a,D_\alpha^2\bar{z}_t}$ and $E_{b,D_\alpha\bar{z}_t}$. In addition, we will include one other term in our total energy: $|\bar{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Our total energy therefore is

$$(80) \quad E = E(t) := E_{a,D_\alpha^2\bar{z}_t}(t) + E_{b,D_\alpha\bar{z}_t}(t) + |\bar{z}_{tt}(\alpha_0, t) - i|.$$

We henceforth abbreviate notation and write $E_a := E_{a,D_\alpha^2\bar{z}_t}$ and $E_b :=$

$E_{b,D_\alpha \bar{z}_t}$. We know

$$(81) \quad \begin{aligned} E_a &:= E_{a,D_\alpha^2 \bar{z}_t} \\ &= \|(\partial_t D_\alpha^2 \bar{z}_t) \circ h^{-1}\|_{L^2(1/A_1)}^2 + \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2(1/A_1)}^2 \end{aligned}$$

and

$$(82) \quad E_b := E_{b,D_\alpha \bar{z}_t} = \|\partial_t D_\alpha \bar{z}_t\|_{L^2(\frac{1}{\alpha})}^2 + \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}^2 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2.$$

As we will see in §5.1, the first term in E_b , $\|\partial_t D_\alpha \bar{z}_t\|_{L^2(\frac{1}{\alpha})} \approx \|\bar{Z}_{tt,\alpha'}\|_{L^2}$; this together with the last term $|\bar{z}_{tt}(\alpha_0, t) - i|$ in the energy E gives us control of $\|\bar{Z}_{tt}(t) - i\|_{L^\infty}$. The inclusion of the last term in E_b gives us an upper bound for A_1 ; see (103).

After developing all necessary tools, in §11 we will show that our energy E is equivalent to the following

$$(83) \quad \begin{aligned} \mathcal{E}(t) &= \|\bar{Z}_{t,\alpha'}\|_{L^2}^2 + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 + \|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 + \|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 \\ &+ \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}^2 + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}^2; \end{aligned}$$

that is, there are universal polynomials P_1 and P_2 , such that $E(t) \leq P_1(\mathcal{E}(t))$, and $\mathcal{E}(t) \leq P_2(E(t))$. One may now get a glimpse of this fact by (50). Notice that there is no control of $\|Z_{,\alpha'}\|_{L^\infty}$ by our energy E or \mathcal{E} . So in the regime where $E < \infty$, we do allow $\frac{1}{Z_{,\alpha'}} \rightarrow 0$. We will discuss what types of singularities are allowed by a finite energy E in §11.2. In particular we will show that our energy class $E(t) < \infty$ allows for non-right angles at the corner with angle $< \frac{\pi}{4}$ as well as angled crest type interfaces with interior angles at the singularities $< \frac{\pi}{2}$, which coincides with the angles of the self-similar solutions constructed in [37]. The Stokes extreme waves have interior angle $= \frac{2\pi}{3}$ at the singularity, hence they are not in our energy class $E(t) < \infty$.

The lack of control of $\|Z_{,\alpha'}\|_{L^\infty}$ in E , or equivalently the lack of a positive lower bound for $-\frac{\partial P}{\partial \mathbf{n}}$, is an obstacle we need to circumvent in the proof of our a priori estimate, Theorem 2.

4.2. The main result

We now state our main result.

Theorem 2. *There exists a polynomial $p = p(x)$ with universal coefficients, such that for any solution of water wave equations (35)-(36)-(48)-(67)-(70)-(39)-(40)-(72), with $(Z_t, Z_{tt}) \in C^l([0, T], H^{k-l}(S^1) \times H^{k-l}(S^1))$ for $l = 0, 1$ and $k \geq 4$,*

$$(84) \quad \frac{d}{dt} E(t) \leq p(E(t))$$

for all $t \in [0, T]$.

Observe that the quantities \mathcal{A} , \mathcal{A}_t and $h_t \circ h^{-1}$ in equations (35)-(36) are given by formulas (48), (67) and (70). By (318), (322) and (50), the assumption in Theorem 2 implies, for $l = 0, 1$, $k \geq 4$,

$$(85) \quad \frac{1}{Z_{,\alpha'}}, A_1, h_t \circ h^{-1}, \frac{\mathcal{A}_t}{\mathcal{A}} \in C^l([0, T], H^{k-l}(S^1)).$$

Remark: 1. Observe that in Theorem 2 no regularity assumptions are made on Z . This is because by substituting (50), (48), (67) and (70) into (36), we see that the quasilinear equation (36) is an equation of the velocity Z_t and acceleration Z_{tt} ; the quantity Z itself doesn't appear explicitly. The assumption of Theorem 2 is consistent with this fact.

2. It appears that there is an $\infty \cdot 0$ ambiguity in the definition of E_b if we allow $Z_{\alpha'} \rightarrow \infty$. This can be resolved by expanding out (82),

$$(86) \quad E_b = \int_I \left(\frac{|\overline{Z}_{tt,\alpha'} - Z_{t,\alpha'} D_{\alpha'} \overline{Z}_t|^2}{A_1} + i \partial_{\alpha'} (D_{\alpha'} \overline{Z}_t) \overline{D_{\alpha'} \overline{Z}_t} + |\overline{Z}_{t,\alpha'}|^2 \right) d\alpha',$$

using the RHS of (86) as the definition for E_b and directly taking derivative to t to the RHS of (86). We opt for the current version (82) for the clarity of the origin and the more intuitive proof associated with this definition. The same remark applies to §5.1, §9, §10 and §11, where the $\frac{\infty}{\infty}$ ambiguity can be resolved by using some different algebraic identities.

3. The existence of solutions to the Cauchy problem in the class where $E(t) < \infty$ is obtained by mollifying the initial data and taking the limit of the sequence of approximating solutions. In the proof for the existence theorem we will apply Theorem 2 only to the approximating sequence which satisfies in addition that $Z_{,\alpha'} \in L^\infty$; see [38]. What's important is that the inequality (84) doesn't depend on any $\|Z_{,\alpha'}\|_{L^\infty}$ bound.

4.3. The proof

In the proof we will switch freely between Lagrangian and Riemann mapping variables: we will use Lagrangian coordinates when we need to take a time derivative, but use Riemann mapping variables when we need to estimate terms, since that gives us access to the easily invertible $(I - \mathbb{H})$ operator. We choose the initial parametrization of the interface such that $h(\alpha, 0) - \alpha \in C^1(S^1)$.⁶ By basic ODE theory and (85): $h_t \circ h^{-1} \in C^1([0, T], H^2(S^1))$, we know $h(\alpha, t) - \alpha \in C^1([0, T], C^1(S^1))$.

We start with writing the energy $E_{a,\theta}$ in Lagrangian coordinates. We use (52) to calculate

$$\begin{aligned}
 & \int_I i \left(\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} (\theta \circ h^{-1}) \right) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} (\bar{\theta} \circ h^{-1}) \right) d\alpha' \\
 &= \Re \int_I i \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \left(\bar{\theta} \frac{h_\alpha}{\bar{z}_\alpha} \right) d\alpha \\
 (87) \quad &= \Re \int_I i \frac{h_\alpha^2}{|z_\alpha|^2} (\partial_\alpha \theta) \bar{\theta} d\alpha + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
 &= \Re \int_I (i a \partial_\alpha \theta) \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha,
 \end{aligned}$$

so

$$\begin{aligned}
 (88) \quad E_{a,\theta}(t) &= \int_I |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I (i a \partial_\alpha \theta) \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha \\
 &\quad + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha + \int_I |\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha.
 \end{aligned}$$

We prove (84) by differentiating each component of $E(t)$ in time. For the two main energies, E_a and E_b , we then integrate by parts to arrive at a term $\partial_t^2 \theta + i a \partial_\alpha \theta$ and use the basic equation $\partial_t^2 \theta + i a \partial_\alpha \theta = G_\theta$ to replace it with G_θ . What remain to be estimated will be G_θ , along with several ancillary terms. We control those quantities in §5 through §10 in terms of a polynomial of the energy.

4.3.1. The estimate for E_a . We begin by differentiating E_a with respect to t .

⁶Notice that the a priori estimate (84) is independent of the initial parametrization.

We will work initially with general θ satisfying $\theta|_{\partial} = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the basic equation (73), and then we will specialize to the $\theta = D_{\alpha}^2 \bar{z}_t$ in our energies. The periodicity ensures that there is no boundary term in the integration by parts.

We differentiate (88) with respect to t and use the fact that $\frac{ah_{\alpha}}{(A_1 \circ h)} = \frac{h_{\alpha}^2}{|z_{\alpha}|^2}$ (by (52) or equivalently the definition for \mathcal{A} and A_1) in the following calculation.

$$\begin{aligned}
\frac{d}{dt} E_{a,\theta}(t) &= \int (\theta_{tt} \bar{\theta}_t + \theta_t \bar{\theta}_{tt}) \frac{h_{\alpha}}{A_1 \circ h} d\alpha + \int |\theta_t|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha \\
&- \int |\theta_t|^2 \frac{h_{\alpha}}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha + \Re \int i \left(\frac{h_{\alpha}^2}{|z_{\alpha}|^2} \right)_t \theta_{\alpha} \bar{\theta} d\alpha \\
&+ \underbrace{\Re \int i \frac{h_{\alpha}^2}{|z_{\alpha}|^2} \theta_{t\alpha} \bar{\theta} d\alpha}_{-\Re \int i \left(\frac{h_{\alpha}^2}{|z_{\alpha}|^2} \right)_{\alpha} \theta_t \bar{\theta} d\alpha + \Re \int \frac{h_{\alpha}^2}{|z_{\alpha}|^2} \theta_t i \bar{\theta}_{\alpha} d\alpha} + \Re \int i \frac{h_{\alpha}^2}{|z_{\alpha}|^2} \theta_{\alpha} \bar{\theta}_t d\alpha \\
&+ \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right)_t |\theta|^2 h_{\alpha} d\alpha + \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right) (\theta_t \bar{\theta} + \theta \bar{\theta}_t) h_{\alpha} d\alpha \\
&+ \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right) |\theta|^2 h_{t\alpha} d\alpha + \int (\theta_t \bar{\theta} + \bar{\theta}_t \theta) \frac{h_{\alpha}}{A_1 \circ h} \\
(89) \quad &+ \int |\theta|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha - \int |\theta|^2 \frac{h_{\alpha}}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha \\
&= \int 2\Re \left((\theta_{tt} + i\mathbf{a}\theta_{\alpha}) \bar{\theta}_t \right) \frac{h_{\alpha}}{A_1 \circ h} d\alpha \\
&+ \int \left(|\theta_t|^2 + |\theta|^2 \right) \left(\frac{h_{t\alpha}}{h_{\alpha}} - \frac{(A_1 \circ h)_t}{A_1 \circ h} \right) \frac{h_{\alpha}}{A_1 \circ h} d\alpha \\
&+ \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right)_t |\theta|^2 h_{\alpha} d\alpha + \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right) (\theta_t \bar{\theta} + \theta \bar{\theta}_t) h_{\alpha} d\alpha \\
&+ \Re \int i \left(\frac{1}{\bar{z}_{\alpha}} \partial_{\alpha} \frac{h_{\alpha}}{z_{\alpha}} \right) |\theta|^2 \frac{h_{t\alpha}}{h_{\alpha}} h_{\alpha} d\alpha + 2\Re \int \theta_t \bar{\theta} \frac{h_{\alpha}}{A_1 \circ h} d\alpha \\
&- \Re \int i \left(\frac{h_{\alpha}^2}{|z_{\alpha}|^2} \right)_{\alpha} \theta_t \bar{\theta} d\alpha + \Re \int i \left(\frac{h_{\alpha}^2}{|z_{\alpha}|^2} \right)_t \theta_{\alpha} \bar{\theta} d\alpha.
\end{aligned}$$

Now we show how we control each of these terms.

For the first, we replace $\theta_{tt} + i\mathbf{a}\theta_{\alpha}$ with the RHS G_{θ} by the main equation

(73) and then use the Cauchy-Schwarz inequality:

$$(90) \quad \begin{aligned} & \int 2\Re((\theta_{tt} + i\alpha\theta_\alpha)\bar{\theta}_t) \frac{h_\alpha}{A_1 \circ h} d\alpha \\ & \lesssim \left(\int |G_\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} \left(\int |\bar{\theta}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}. \end{aligned}$$

The first factor, which involves the RHS of the basic equation, is the main term to control. For $\theta = D_\alpha^2 \bar{z}_t$,

$$(91) \quad G_\theta = D_\alpha^2(-i\alpha_t \bar{z}_\alpha) + [\partial_t^2 + i\alpha \partial_\alpha, D_\alpha^2] \bar{z}_t.$$

We estimate these terms in §10.

In §6, we will control

$$(92) \quad \left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha \right| \lesssim (164).$$

Because $\frac{1}{z_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} = D_\alpha \frac{h_\alpha}{z_\alpha} = \left(D_{\alpha'} \frac{1}{Z, \alpha'} \right) \circ h$, we estimate

$$(93) \quad \left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \theta \bar{\theta}_t) h_\alpha d\alpha \right| \lesssim \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a, \theta}.$$

Similarly, we estimate

$$\left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 \frac{h_{t\alpha}}{h_\alpha} h_\alpha d\alpha \right| \lesssim \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a, \theta}.$$

We observe that

$$(94) \quad \left\| \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \right\|_{L^\infty} = \left\| \partial_{\alpha'} \frac{1}{|Z, \alpha'|^2} \right\|_{L^\infty} \leq 2 \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty},$$

so, using the Cauchy-Schwarz inequality, we have

$$(95) \quad \left| -\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha \right| \lesssim \|A_1\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} E_{a, \theta}.$$

In §7 we control

$$(96) \quad \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \lesssim (187).$$

We estimate the remaining two terms of (89) by the Cauchy-Schwarz inequality and Hölder's inequality.

We now combine these estimates and specialize to $\theta = D_\alpha^2 \bar{z}_t$. Each of the remaining factors we will control in §5; we list the location of the final estimate for each quantity of the following in the subscripts. We get

$$\begin{aligned}
 (97) \quad \left| \frac{d}{dt} E_a \right| &\lesssim \underbrace{\|G_{D_\alpha^2 \bar{z}_t}\|_{L^2(\frac{h_\alpha}{A_1 \circ h})}}_{\lesssim(249)} E_a^{1/2} + \underbrace{\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}}_{\lesssim(136)} E_a + \underbrace{\left\| \frac{(A_1 \circ h)_t}{A_1 \circ h} \right\|_{L^\infty}}_{\lesssim(138)} E_a \\
 &+ \underbrace{\left(1 + \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \right)}_{\lesssim 1 + (136)} \|A_1\|_{L^\infty} \underbrace{\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}}_{\lesssim(148)} E_a + E_a \\
 &+ \underbrace{\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha}_{\lesssim(164)} + \underbrace{\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha}_{\lesssim(187)}.
 \end{aligned}$$

4.3.2. The estimate for E_b . Now we consider our second term E_b . Once again, we work first with general θ satisfying $\theta|_\partial = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the main equation (73). Then we specialize to $\theta = D_\alpha \bar{z}_t$. The periodicity ensures there is no boundary term when we integrate by parts. We differentiate (76) with respect to t :

$$\begin{aligned}
 &\frac{d}{dt} E_{b,\theta}(t) \\
 &= \int \frac{1}{\mathbf{a}} (\theta_{tt} \bar{\theta}_t + \theta_t \bar{\theta}_{tt}) d\alpha - \int \frac{\mathbf{a}_t}{\mathbf{a}} \frac{1}{\mathbf{a}} |\theta_t|^2 d\alpha + \underbrace{\int i \theta_{t\alpha} \bar{\theta} d\alpha}_{= \int i \bar{\theta}_\alpha \theta_t d\alpha} + \int i \theta_\alpha \bar{\theta}_t d\alpha \\
 &+ \int \frac{(A_1 \circ h)}{\mathbf{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \frac{(A_1 \circ h)_t}{\mathbf{a}} |\theta|^2 d\alpha - \int \frac{\mathbf{a}_t}{\mathbf{a}} \frac{(A_1 \circ h)}{\mathbf{a}} |\theta|^2 d\alpha \\
 &= 2\Re \int \frac{G_\theta}{\mathbf{a}} \bar{\theta}_t d\alpha - \int \frac{\mathbf{a}_t}{\mathbf{a}} \frac{1}{\mathbf{a}} |\theta_t|^2 d\alpha \\
 &+ \int \frac{A_1 \circ h}{\mathbf{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \left(\frac{(A_1 \circ h)_t}{(A_1 \circ h)} - \frac{\mathbf{a}_t}{\mathbf{a}} \right) \frac{(A_1 \circ h)}{\mathbf{a}} |\theta|^2 d\alpha.
 \end{aligned}$$

By Hölder and the Cauchy-Schwarz inequalities, we conclude that

$$\begin{aligned}
 (98) \quad &\left| \frac{d}{dt} E_{b,\theta}(t) \right| \\
 &\lesssim \left\| \frac{G_\theta}{\sqrt{\mathbf{a}}} \right\|_{L^2} E_{b,\theta}^{1/2} + \left(\|A_1\|_{L^\infty}^{1/2} + \left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty} + \left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty} \right) E_{b,\theta}.
 \end{aligned}$$

For $\theta = D_\alpha \bar{z}_t$, we control $\left\| \frac{G_\theta}{\sqrt{a}} \right\|_{L^2}$ in §9, at (228). We control $\|A_1\|_{L^\infty}$ at (103), $\left\| \frac{a_t}{a} \right\|_{L^\infty}$ at (123) and $\left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty}$ at (138).

4.3.3. The estimate for $|z_{tt}(\alpha_0, t) - i|$. Finally, we show that we can control $\frac{d}{dt} |\bar{z}_{tt}(\alpha_0, t) - i|$. By differentiating with respect to t , we have, by the basic equations (36)-(35),

$$(99) \quad \begin{aligned} \frac{d}{dt} |\bar{z}_{tt}(\alpha_0) - i| &\leq |\bar{z}_{ttt}(\alpha_0)| = \left| \frac{\mathcal{A}_t}{\mathcal{A}} \circ h(\alpha_0) + \overline{D_\alpha z_t}(\alpha_0) \right| |\bar{z}_{tt}(\alpha_0) - i| \\ &\lesssim \left(\left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} + \|D_\alpha \bar{z}_t\|_{L^\infty} \right) |\bar{z}_{tt}(\alpha_0) - i|. \end{aligned}$$

We control $\left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}$ below at (123) and $\|D_\alpha \bar{z}_t\|_{L^\infty}$ at (105).

4.4. Outline of the remainder of the proof

In sections §5 through §10, we complete the proof of the a priori inequality (84).

In §5, we control various quantities that are necessary for our proof. In §5.1, we carefully list the basic quantities controlled by our energy. In §5.2-§5.6, we estimate various other quantities that are listed above in §4.3. In appendix §D, we list and give references to all the quantities controlled in §5, which we then use, sometimes without citation, in §6 through §10.

In §6 and §7 we estimate the terms from (92) and (96) in the estimate of $\frac{d}{dt} E_a$ above. Finally, in §9 and §10 we conclude the estimates for $\frac{d}{dt} E_b$ and $\frac{d}{dt} E_a$, respectively, by controlling the G_θ terms, completing the proof of Theorem 2.

The basic approach for many of the estimates is to try and use the fact that certain quantities are purely real-valued and others are holomorphic to express the terms in question as commutators involving the Hilbert transform, and then use the commutator estimates from §B.3 to avoid loss of derivatives. Because our estimates are very tight, we have to take care in using different estimates for different terms, including treating certain terms as commutators while keeping others in $(I - \mathbb{H})$ form. Very often we have to carefully expand the quantities, and then decompose the factors and regroup the terms to make sure no further cancellations are possible and the desired estimates can be obtained. We will give enough details to facilitate reading.

We use $C(E)$ to indicate a universal polynomial of E , which may differ from line to line.

Throughout the remaining derivations, we will repeatedly rely on the identity (52).

5. Quantities controlled by our energy

Here we collect together many quantities that are controlled by our energy.

5.1. Basic quantities controlled by the energy

In this section, we present a list of basic quantities controlled by our energy. Because conjugations and commutations of ∂_t with D_α add complexity, we take care to list some of those estimates as well. We list all of the basic terms controlled here at (121) below.

We start with (81) and (82). E_a and E_b directly control

$$(100) \quad \|(\partial_t + \mathfrak{b}\partial_{\alpha'})D_{\alpha'}^2\bar{Z}_t\|_{L^2}, \quad \|D_{\alpha'}^2\bar{z}_t\|_{L^2(\frac{h_\alpha}{A_1\circ h}d\alpha)} \leq E_a^{1/2},$$

$$(101) \quad \|D_{\alpha'}^2\bar{Z}_t\|_{L^2(\frac{1}{A_1}d\alpha')}, \quad \left\| \frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\bar{Z}_t \right\|_{\dot{H}^{1/2}} \leq E_a^{1/2},$$

$$(102) \quad \left(\int |D_{\alpha'}\bar{z}_t|^2 \frac{d\alpha}{\mathfrak{a}} \right)^{1/2} \leq \|\bar{Z}_{t,\alpha'}\|_{L^2} \leq E_b^{1/2},$$

where we used $A_1 \geq 1$ (49) in (102). By (48), using the commutator estimate (331), and then (102), we have

$$(103) \quad \|A_1\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 + 1 \lesssim E_b + 1.$$

Thanks to (103), we can now control $\|D_{\alpha'}^2\bar{z}_t\|_{L^2(h_\alpha d\alpha)} = \|D_{\alpha'}^2\bar{Z}_t\|_{L^2}$ by a polynomial of our energy E .

Now we control $\|D_{\alpha'}\bar{z}_t\|_{L^\infty} = \|D_{\alpha'}\bar{Z}_t\|_{L^\infty}$. We work in Riemann mapping variables and use the weighted Sobolev inequality (314) with weight $\omega = \frac{1}{|Z_{,\alpha'}|^2}$ (and $\varepsilon = 1$). Note that $f(D_{\alpha'}\bar{Z}_t)^2 = 0$ by (312). This gives

$$(104) \quad \begin{aligned} & \|D_{\alpha'}\bar{Z}_t\|_{L^\infty} \\ & \lesssim \left(\int |D_{\alpha'}\bar{Z}_t|^2 |Z_{,\alpha'}|^2 d\alpha' \right)^{1/2} + \left(\int |\partial_{\alpha'}D_{\alpha'}\bar{Z}_t|^2 \frac{1}{|Z_{,\alpha'}|^2} d\alpha' \right)^{1/2} \\ & = \|\bar{Z}_{t,\alpha'}\|_{L^2} + \|D_{\alpha'}^2\bar{Z}_t\|_{L^2} \leq E_b^{1/2} + \|A_1\|_{L^\infty}^{1/2} E_a^{1/2} \\ & \lesssim C(E). \end{aligned}$$

We conclude that

$$(105) \quad \|D_\alpha z_t\|_{L^\infty} = \|D_{\alpha'} Z_t\|_{L^\infty} = \|D_\alpha \bar{z}_t\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim C(E).$$

Now we use the commutator identity (337) to move the ∂_t inside the first term in E_b :

$$(106) \quad \begin{aligned} & \left(\int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ & \leq \left(\int |\partial_t D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} + \left(\int |[\partial_t, D_\alpha] \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ & \leq E_b^{1/2} + \|D_\alpha z_t\|_{L^\infty} \left(\int |D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ & \lesssim C(E) \end{aligned}$$

by (105) and (102). By changing variables and by (52), we conclude from (106) and (103) that

$$(107) \quad \begin{aligned} \|\bar{Z}_{tt, \alpha'}\|_{L^2} & \lesssim \|A_1\|_{L^\infty}^{1/2} \left(\int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ & \lesssim C(E). \end{aligned}$$

Note that $|D_\alpha^2 f| \neq |D_\alpha^2 \bar{f}|$. Nevertheless, for generic f , we can control $D_\alpha^2 f$ by $D_\alpha^2 \bar{f}$ in $L^2(h_\alpha d\alpha)$ norm, at the expense of some lower-order terms. For notational convenience, we define here

$$(108) \quad |D_\alpha| := \frac{1}{|z_\alpha|} \partial_\alpha = \frac{z_\alpha}{|z_\alpha|} D_\alpha.$$

We expand:

$$(109) \quad \begin{aligned} D_\alpha^2 f & = \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 f + \frac{|z_\alpha|}{z_\alpha} \left(|D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right) |D_\alpha| f \\ D_\alpha^2 \bar{f} & = \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 \bar{f} + \frac{|z_\alpha|}{z_\alpha} \left(|D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right) |D_\alpha| \bar{f}. \end{aligned}$$

Therefore,

$$(110) \quad |D_\alpha^2 f| \leq |D_\alpha^2 \bar{f}| + 2 \left| |D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right| |D_\alpha \bar{f}|$$

and so

$$(111) \quad \left(\int |D_\alpha^2 f|^2 h_\alpha d\alpha \right)^{1/2} \leq \left(\int |D_\alpha^2 \bar{f}|^2 h_\alpha d\alpha \right)^{1/2} + 2 \|D_\alpha \bar{f}\|_{L^\infty} \left(\int \left| D_\alpha \frac{|z_\alpha|}{z_\alpha} \right|^2 h_\alpha d\alpha \right)^{1/2}.$$

By (316) and then (315), (21), (52), and the fact $A_1 \geq 1$,

$$(112) \quad \left| D_\alpha \frac{|z_\alpha|}{z_\alpha} \right|^2 h_\alpha = \left| D_\alpha \frac{\bar{z}_{tt} - i}{|\bar{z}_{tt} - i|} \right|^2 h_\alpha \leq |D_\alpha \bar{z}_{tt}|^2 \frac{h_\alpha}{|\bar{z}_{tt} - i|^2} = |D_\alpha \bar{z}_{tt}|^2 \frac{h_\alpha}{a^2 |z_\alpha|^2} \leq |D_\alpha \bar{z}_{tt}|^2 \frac{1}{a}.$$

Plugging this into (111), and using (106), we get

$$(113) \quad \left(\int |D_\alpha^2 f|^2 h_\alpha d\alpha \right)^{1/2} \lesssim \left(\int |D_\alpha^2 \bar{f}|^2 h_\alpha d\alpha \right)^{1/2} + \|D_\alpha \bar{f}\|_{L^\infty} C(E).$$

We now apply (113) to $f = z_t$, using (105) to control $\|D_\alpha \bar{z}_t\|_{L^\infty}$ and (101) and (103) to control $\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}$:

$$(114) \quad \|D_{\alpha'}^2 Z_t\|_{L^2} = \left(\int |D_\alpha^2 z_t|^2 h_\alpha d\alpha \right)^{1/2} \lesssim C(E).$$

We now control $\|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)}$. We use the commutator identity (338) to get

$$(115) \quad \begin{aligned} \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} &= \|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)} \\ &\leq \|\partial_t D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + 2 \|(D_\alpha z_t) D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ &\quad + \|(D_\alpha^2 z_t) D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ &\leq \|A_1\|_{L^\infty}^{1/2} E_a^{1/2} + 2 \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ &\quad + \|D_\alpha^2 z_t\|_{L^2(h_\alpha d\alpha)} \|D_\alpha \bar{z}_t\|_{L^\infty} \lesssim C(E). \end{aligned}$$

We will also need to control $D_{\alpha'}^2 Z_{tt}$; we delay doing this until later, after we control $\|D_\alpha \bar{z}_{tt}\|_{L^\infty}$.

We will also at one point need to control

$$\begin{aligned} \|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} &\leq \|\partial_t D_\alpha D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|[\partial_t, D_\alpha] D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ &\leq \|A_1\|_{L^\infty}^{1/2} E_a^{1/2} + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ &\lesssim C(E). \end{aligned}$$

We now control $\|\bar{z}_{tt} - i\|_{L^\infty} = \|\bar{Z}_{tt} - i\|_{L^\infty}$. Recall from our definition of the energy (80) that the energy includes $|\bar{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Let $\alpha'_0 = h(\alpha_0, t)$. Then, by the fundamental theorem of calculus, for arbitrary $\alpha' \in I$,

$$\begin{aligned} (116) \quad |\bar{Z}_{tt}(\alpha', t) - i| &\lesssim |\bar{Z}_{tt}(\alpha'_0, t) - i| + \|\bar{Z}_{tt, \alpha'}\|_{L^1(I)} \\ &\lesssim |\bar{Z}_{tt}(\alpha'_0, t) - i| + (107). \end{aligned}$$

We conclude that

$$(117) \quad \|z_{tt} + i\|_{L^\infty} = \|Z_{tt} + i\|_{L^\infty} = \|\bar{z}_{tt} - i\|_{L^\infty} = \|\bar{Z}_{tt} - i\|_{L^\infty} \lesssim C(E).$$

Because of (50) and (49), we can also conclude that

$$(118) \quad \left\| \frac{1}{Z_{, \alpha'}} \right\|_{L^\infty} \lesssim C(E).$$

We use this to control $\|D_\alpha \bar{z}_{tt}\|_{L^\infty}$, using the weighted Sobolev inequality (313) in Riemann mapping variables with weight $\omega = \frac{1}{|Z_{, \alpha'}|^2}$ (and $\varepsilon = 1$):⁷

$$\begin{aligned} (119) \quad \|D_\alpha z_{tt}\|_{L^\infty} &= \|D_{\alpha'} Z_{tt}\|_{L^\infty} = \|D_\alpha \bar{z}_{tt}\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} \\ &\lesssim \|\bar{Z}_{tt, \alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} + \left(\int |D_{\alpha'} \bar{Z}_{tt}|^2 d\alpha' \right)^{1/2} \\ &\lesssim (1 + \|1/Z_{, \alpha'}\|_{L^\infty}) \|\bar{Z}_{tt, \alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} \\ &\lesssim (1 + (118)) (107) + (115) \\ &\lesssim C(E). \end{aligned}$$

⁷Note that unlike our proof for $\|D_{\alpha'} \bar{Z}_t\|_{L^\infty}$ at (104) above, we don't necessarily have that $f(D_{\alpha'} \bar{Z}_{tt})^2$ is zero, so we get a third term in the Sobolev inequality.

Finally, we use (113), (115), and (119) to control $D_\alpha^2 z_{tt}$ and $D_{\alpha'}^2 Z_{tt}$:

$$\begin{aligned}
 (120) \quad & \|D_{\alpha'}^2 Z_{tt}\|_{L^2} = \|D_\alpha^2 z_{tt}\|_{L^2(h_\alpha d\alpha)} \\
 & \lesssim \|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)} + \|D_\alpha \bar{z}_{tt}\|_{L^\infty} C(E) \\
 & \lesssim (115) + (119) C(E) \lesssim C(E).
 \end{aligned}$$

To sum up, we have the following quantities and their counterparts in Lagrangian coordinates controlled by universal polynomials of the energy E :

$$\begin{aligned}
 (121) \quad & \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \|D_{\alpha'}^2 Z_t\|_{L^2}, \\
 & \|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)}, \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}, \|D_{\alpha'} Z_{tt}\|_{L^\infty}, \\
 & \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}, \|D_{\alpha'} Z_t\|_{L^\infty}, \|\bar{Z}_{tt,\alpha'}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \int |D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}}, \\
 & \int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}}, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}, \|Z_{tt} + i\|_{L^\infty}, \|A_1\|_{L^\infty} \lesssim C(E).
 \end{aligned}$$

5.2. Controlling $\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty}$

We now show that we can control $\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty}$, using (67). Because $A_1 \geq 1$ (49), it suffices to control

$$(122) \quad \left\| 2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t] \right\|_{L^\infty}.$$

We control the first two terms by (331), and the last term by Hölder's inequality and then Hardy's inequality (317). We have

$$(123) \quad \left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|\bar{Z}_{tt,\alpha'}\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2}^2 \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim C(E).$$

5.3. Controlling $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$

Recall from (50) that $\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$. Therefore,

$$(124) \quad \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt,\alpha'}}{A_1} - i \frac{\bar{Z}_{tt} - i}{A_1^2} \partial_{\alpha'} A_1.$$

Because $A_1 \geq 1$ (49), we can control the first term by $\|\bar{Z}_{tt,\alpha'}\|_{L^2}$. Now we address the second term.

We recall that $A_1 = \Im(-[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}) + 1$ (48). Therefore,

$$\begin{aligned}
 \partial_{\alpha'} A_1 &= \partial_{\alpha'} \Im \frac{-1}{2i} \int (Z_t(\alpha') - Z_t(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \overline{Z}_{t,\beta'} d\beta' \\
 (125) \quad &= -\Im Z_{t,\alpha'} \mathbb{H} \overline{Z}_{t,\alpha'} + \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \overline{Z}_{t,\beta'}(\beta') d\beta' \\
 &= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \overline{Z}_{t,\beta'}(\beta') d\beta',
 \end{aligned}$$

where the first term disappears because $\mathbb{H}\overline{Z}_{t,\alpha'} = \overline{Z}_{t,\alpha'}$ and so $Z_{t,\alpha'} \mathbb{H}\overline{Z}_{t,\alpha'}$ is purely real. Therefore, multiplying (125) by $|\overline{Z}_{tt}(\alpha) - i|$ and splitting into two parts, we have

$$\begin{aligned}
 |\overline{Z}_{tt} - i| \partial_{\alpha'} A_1 &= \\
 (126) \quad &\Im \int \frac{\pi}{4i} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} (|\overline{Z}_{tt}(\alpha') - i| - |\overline{Z}_{tt}(\beta') - i|) \overline{Z}_{t,\beta'}(\beta') d\beta' \\
 &+ \Im \int \frac{\pi}{4i} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} |\overline{Z}_{tt}(\beta') - i| \overline{Z}_{t,\beta'}(\beta') d\beta' \\
 &= I + II.
 \end{aligned}$$

We need to control $\|I\|_{L^2}$ and $\|II\|_{L^2}$. By (328),

$$(127) \quad \|I\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|\overline{Z}_{tt,\alpha'}\|_{L^2} \|\overline{Z}_{t,\alpha'}\|_{L^2} = \|\overline{Z}_{t,\alpha'}\|_{L^2}^2 \|\overline{Z}_{tt,\alpha'}\|_{L^2}.$$

For II , we replace $|\overline{Z}_{tt}(\beta') - i|$ by $\left|\frac{-iA_1(\beta')}{Z_{,\beta'}}\right|$ (50) and use estimate (321), noticing that $\frac{1}{Z_{,\beta'}} \overline{Z}_{t,\beta'} = D_{\beta'} \overline{Z}_t$:

$$(128) \quad \|II\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|A_1 D_{\alpha'} \overline{Z}_t\|_{L^\infty} \leq \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'} \overline{Z}_t\|_{L^\infty} \|A_1\|_{L^\infty}.$$

We conclude that

$$\begin{aligned}
 (129) \quad &\|(\overline{Z}_{tt} - i) \partial_{\alpha'} A_1\|_{L^2} \\
 &\lesssim \|\overline{Z}_{tt,\alpha'}\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 + \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'} \overline{Z}_t\|_{L^\infty} \|A_1\|_{L^\infty} \lesssim C(E)
 \end{aligned}$$

and

$$\begin{aligned}
 (130) \quad &\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim \|\overline{Z}_{tt,\alpha'}\|_{L^2} + \|(\overline{Z}_{tt} - i) \partial_{\alpha'} A_1\|_{L^2} \\
 &\lesssim C(E).
 \end{aligned}$$

5.4. Controlling $\left\| \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z, \alpha'} \right\|_{L^\infty}$, $\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}$ and related quantities

Observe that by the assumption of Theorem 2, the quantities $\frac{Z_t}{Z, \alpha'}$, Z_t and $\frac{1}{Z, \alpha'}$ are in $C^1([0, T], C^2(S^1))$. We begin with computing

$$(131) \quad \begin{aligned} \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z, \alpha'} &= (I - \mathbb{H}) D_{\alpha'} Z_t + (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\} \\ &= 2D_{\alpha'} Z_t - (I + \mathbb{H}) D_{\alpha'} Z_t + (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\}. \end{aligned}$$

Now we use (306), (308), and the fact that \mathbb{H} is purely imaginary to rewrite the second and third terms on the RHS above into commutators, and use (331) to control

$$(132) \quad \|(I + \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty} = \left\| \left[\frac{1}{Z, \alpha'}, \mathbb{H} \right] Z_{t, \alpha'} \right\|_{L^\infty} \lesssim \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2} \|Z_{t, \alpha'}\|_{L^2},$$

$$(133) \quad \begin{aligned} &\left\| (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\} \right\|_{L^\infty} \\ &= \left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} \lesssim \|Z_{t, \alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2}. \end{aligned}$$

Therefore

$$(134) \quad \left\| \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z, \alpha'} \right\|_{L^\infty} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2} \|Z_{t, \alpha'}\|_{L^2} \lesssim C(E).$$

Observe that

$$(135) \quad \frac{h_{t\alpha}}{h_\alpha} \circ h^{-1} = \partial_{\alpha'}(h_t \circ h^{-1}) = \partial_{\alpha'} \mathbf{b},$$

so by (70) and (134),

$$(136) \quad \|\partial_{\alpha'} \mathbf{b}\|_{L^\infty} = \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \lesssim C(E).$$

5.5. Controlling $\left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty}$

Recall that $A_1 \circ h = \frac{\mathfrak{a}|z_\alpha|^2}{h_\alpha}$ (52). Therefore,

$$(137) \quad \frac{d}{dt}(A_1 \circ h) = \frac{\mathfrak{a}_t}{\mathfrak{a}} - \frac{h_{t\alpha}}{h_\alpha} + 2\Re D_\alpha z_t.$$

We have controlled each of the terms on the RHS in L^∞ in the previous sections. We conclude that

$$(138) \quad \left\| \frac{d}{dt}(A_1 \circ h) \right\|_{L^\infty} \lesssim \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \lesssim C(E).$$

5.6. Controlling $\left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^\infty}$ and related quantities

Recall from (50) that $\frac{1}{Z_{\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$. Therefore,

$$(139) \quad D_{\alpha'} \frac{1}{Z_{\alpha'}} = i \frac{D_{\alpha'} \bar{Z}_{tt}}{A_1} - i \frac{\bar{Z}_{tt} - i}{A_1^2} D_{\alpha'} A_1 = i \frac{D_{\alpha'} \bar{Z}_{tt}}{A_1} + \frac{(\bar{Z}_{tt} - i)^2}{A_1^3} \partial_{\alpha'} A_1.$$

Because we can control the first term on the RHS by $\|D_{\alpha'} Z_{tt}\|_{L^\infty}$, it suffices to focus on the second term. We start from (125), and then use a similar idea to (126):

$$(140) \quad \begin{aligned} & |\bar{Z}_{tt} - i|^2 \partial_{\alpha'} A_1 = \\ & \Im \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{4i \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (|\bar{Z}_{tt}(\alpha') - i|^2 - |\bar{Z}_{tt}(\beta') - i|^2) \bar{Z}_{t,\beta'}(\beta') d\beta' \\ & + \Im \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{4i \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} |\bar{Z}_{tt}(\beta') - i|^2 \bar{Z}_{t,\beta'}(\beta') d\beta' \\ & = I + II. \end{aligned}$$

To control $\|I\|_{L^\infty}$, we use the mean value theorem and the periodicity of Z_{tt} to estimate

$$(141) \quad \begin{aligned} & \left| \frac{|\bar{Z}_{tt}(\alpha') - i|^2 - |\bar{Z}_{tt}(\beta') - i|^2}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \right| \\ & \lesssim \|(\bar{Z}_{tt} - i) \partial_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} \lesssim \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}. \end{aligned}$$

From the Cauchy-Schwarz inequality and Hardy's inequality (317), we get

$$(142) \quad \|I\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}.$$

For II , observe that $|\bar{Z}_{tt}(\alpha') - i|^2 \bar{Z}_{t,\alpha'}(\alpha') = \frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t$ and $\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \Big|_{\partial} = 0$. We integrate by parts as in (319):

$$(143) \quad \begin{aligned} II &= \Im \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} |\bar{Z}_{tt}(\beta') - i|^2 \bar{Z}_{t,\beta'}(\beta') d\beta' \\ &= \Im \mathbb{H} \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) - \Im [Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right). \end{aligned}$$

We estimate the second term by (331):

$$(144) \quad \left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2}.$$

We expand the first term, using the conjugate of (308), noticing that

$$\Im \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) = 0.$$

$$(145) \quad \begin{aligned} &\Im \mathbb{H} \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) \\ &= -\Im \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t + \left[\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'} \right) \\ &= -\Im \left[\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'}. \end{aligned}$$

We estimate the RHS by (331):

$$(146) \quad \left\| \left[\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2}.$$

Now,

$$(147) \quad \begin{aligned} &\left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z},\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} \lesssim \|A_1\|_{L^\infty}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \\ &+ \|A_1\|_{L^\infty}^2 \left\| \partial_{\alpha'} \frac{1}{\bar{Z},\alpha'} \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} + \left\| \frac{A_1}{\bar{Z},\alpha'} \partial_{\alpha'} A_1 \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}. \end{aligned}$$

Combining the above calculations and using (129), (130), the estimates in §5.1, and the fact $A_1 \geq 1$, we conclude

$$\begin{aligned}
 (148) \quad & \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \leq \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|(\bar{Z}_{tt} - i)^2 \partial_{\alpha'} A_1\|_{L^\infty} \\
 & \lesssim \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2}^2 \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} \\
 & \quad + \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} \lesssim C(E).
 \end{aligned}$$

We record here the estimate for two related quantities, which we will use in later sections:

$$(149) \quad \left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \leq \|A_1\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim C(E);$$

and

$$(150) \quad \left\| \partial_{\alpha'} \frac{Z_{tt} + i}{Z_{,\alpha'}} \right\|_{L^\infty} \leq \|D_{\alpha'} Z_{tt}\|_{L^\infty} + \left\| (Z_{tt} + i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim C(E).$$

6. Controlling $\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha$

In this section, we control from (97) the term

$$\begin{aligned}
 (151) \quad & \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha = -\Re \int i \frac{\bar{z}_{t\alpha}}{\bar{z}_\alpha} \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
 & + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha
 \end{aligned}$$

for $\theta = D_\alpha^2 \bar{z}_t$. We can control the first of these terms by

$$\begin{aligned}
 (152) \quad & \left| -\Re \int i \frac{\bar{z}_{t\alpha}}{\bar{z}_\alpha} \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \right| \\
 & \lesssim \|D_\alpha \bar{z}_t\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2.
 \end{aligned}$$

Therefore, it suffices to focus on the second term on the RHS of (151). We expand it out:

$$\begin{aligned}
 & \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
 &= \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \left(\frac{h_\alpha}{z_\alpha} \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) \right) |\theta|^2 h_\alpha d\alpha \\
 (153) \quad &= \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\
 &\quad + \Re \int i \left(\frac{h_\alpha}{|\bar{z}_\alpha|^2} \partial_\alpha \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha.
 \end{aligned}$$

We can estimate the first term on the RHS by

$$\begin{aligned}
 (154) \quad & \left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \right| \\
 & \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2.
 \end{aligned}$$

Therefore, it suffices to focus on the second term on the RHS of (153). Observe that because h is real-valued,

$$\begin{aligned}
 (155) \quad & \Re \int i \left(\frac{h_\alpha}{|\bar{z}_\alpha|^2} \partial_\alpha \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha \\
 &= -\Re \int i \left(\frac{h_\alpha}{|\bar{z}_\alpha|^2} \partial_\alpha \left(\frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha.
 \end{aligned}$$

We now drop \Re and the i , write $D_\alpha^2 \bar{z}_t = \theta$, and switch to Riemann mapping variables. For consistency with the quantities we've controlled elsewhere, we will take a conjugate. We have

$$(156) \quad \int \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha'.$$

We want to take advantage of the holomorphicity and antiholomorphicity of various of these factors. To do this, we first use the identity

$$\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) = \frac{1}{\bar{Z}_{,\alpha'}^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{Z_{,\alpha'}} \right) + \overline{D_{\alpha'} Z_t} \left(D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} - \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right)$$

to replace the first factor to make it closer to holomorphic:

$$\begin{aligned}
 & \int \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \\
 (157) \quad & = \int \left(\frac{1}{\bar{Z}_{,\alpha'}^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \\
 & \quad + \int \overline{D_{\alpha'} \bar{Z}_t} \left(D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} - \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha'.
 \end{aligned}$$

We can estimate the second term by

$$\begin{aligned}
 (158) \quad & \left| \int \overline{D_{\alpha'} \bar{Z}_t} \left(D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} - \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \right| \\
 & \lesssim \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 \lesssim C(E).
 \end{aligned}$$

It therefore remains only to control the first term on the RHS of (157). Now we take advantage of holomorphicity. We rewrite this as

$$\begin{aligned}
 (159) \quad & \int \left(\frac{1}{\bar{Z}_{,\alpha'}^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \\
 & = \int \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \overline{\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right)} \mathbb{P}_H D_{\alpha'}^2 \bar{Z}_t d\alpha',
 \end{aligned}$$

where we have used (307) to insert \mathbb{P}_H in front of the $D_{\alpha'}^2 \bar{Z}_t$ and decomposed the first factor into the antiholomorphic and holomorphic projections. Now we use the adjoint property (291) to turn the \mathbb{P}_H into a \mathbb{P}_A on the opposite factors, and control using the Cauchy-Schwarz inequality:

$$\begin{aligned}
 (160) \quad & \left| \int \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \overline{\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right)} \mathbb{P}_H D_{\alpha'}^2 \bar{Z}_t d\alpha' \right| \lesssim \\
 & \left\| \mathbb{P}_A \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \overline{\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right)} \right\} \right\|_{L^2} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}.
 \end{aligned}$$

It now remains only to control this first factor.

First we consider the term with the \mathbb{P}_H . In this case, we can rewrite this

as a commutator:

$$\begin{aligned}
 (161) \quad & \mathbb{P}_A \left\{ \left(\mathbb{P}_H \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t} \right) \right\} \\
 &= \frac{1}{2} \left[\frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t, \mathbb{H} \right] \mathbb{P}_H \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \\
 &\quad + \frac{1}{4} \overline{\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t} \left(\int \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right).
 \end{aligned}$$

Here, the mean term appears because of (290). We now use commutator estimate (327) for the first term and Hölder's inequality for the second term, to conclude that

$$\begin{aligned}
 (162) \quad & \left\| \mathbb{P}_A \left\{ \left(\mathbb{P}_H \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \right) \left(\overline{\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t} \right) \right\} \right\|_{L^2} \\
 &\lesssim \left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} \left\| \mathbb{P}_H \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \right\|_{L^2} \\
 &\quad + \left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \left\| \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^1} \\
 &\lesssim \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2} \left(\left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2} \right).
 \end{aligned}$$

Finally, we consider the \mathbb{P}_A term in the first factor on the RHS of (160). By the L^2 boundedness of \mathbb{P}_A , it suffices to control

$$\begin{aligned}
 (163) \quad & \left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t (I - \mathbb{H}) \left(\frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right) \right\|_{L^2} \\
 &\lesssim \left\| D_{\alpha'}^2 \overline{Z}_t \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^2} \\
 &\quad + \left\| D_{\alpha'}^2 \overline{Z}_t (I - \mathbb{H}) \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \\
 &\lesssim \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2} \left\| \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_{L^\infty} \\
 &\quad + \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2} \left\| \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] D_{\alpha'}^2 \overline{Z}_t \right\|_{L^\infty} \\
 &\lesssim \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2},
 \end{aligned}$$

where we've used (307) to get the second commutator and used commutator estimate (331).

We now combine our estimates, concluding that

$$\begin{aligned}
 (164) \quad & \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha \lesssim (152) + (153) \\
 & \lesssim (152) + (154) + (158) + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \cdot ((162) + (163)) \\
 & \lesssim C(E).
 \end{aligned}$$

7. Controlling $\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha$

We now show that we can control the following term from the RHS of (97):

$$(165) \quad \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha = \Re \int i \left(2 \frac{h_{t\alpha}}{h_\alpha} - 2\Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha.$$

Here, all results will be expressed in terms of general energy $E_{a,\theta}$ for θ satisfying $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$ and $\theta|_\partial = 0$, rather than specifying $\theta = D_\alpha^2 \bar{z}_t$.⁸

We begin by rewriting this as

$$\begin{aligned}
 (166) \quad & \Re \int i \left(2 \frac{h_{t\alpha}}{h_\alpha} - 2\Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha \\
 & = \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \bar{\theta} \frac{h_\alpha}{z_\alpha} d\alpha \\
 & \quad - \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\frac{h_\alpha}{z_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \theta \bar{\theta} d\alpha \\
 & = I + II.
 \end{aligned}$$

II is easy to control, via Hölder's inequality and change of variables to Riemann mapping variables:

$$(167) \quad |II| \lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}.$$

Therefore, we can focus on I from (166). We introduce the following nota-

⁸ $D_\alpha^2 \bar{z}_t$ satisfies $(I - \mathbb{H})D_\alpha^2 \bar{z}_t \circ h^{-1} = 0$, $D_\alpha^2 \bar{z}_t|_\partial = 0$; see (307).

tions:

$$(168) \quad \begin{aligned} \psi &:= \begin{pmatrix} h_\alpha \\ z_\alpha \end{pmatrix} \theta \circ h^{-1}; & \Theta &:= \theta \circ h^{-1}; \\ B &:= \begin{pmatrix} h_{t\alpha} \\ h_\alpha \end{pmatrix} - \Re D_\alpha z_t \circ h^{-1} = (h_t \circ h^{-1})_{\alpha'} - \Re D_{\alpha'} Z_t. \end{aligned}$$

We know $(I - \mathbb{H})\Theta = 0$, $\Theta|_\partial = 0$, and

$$(169) \quad B|_\partial = 0, \quad \psi|_\partial = 0,$$

$$(170) \quad (I - \mathbb{H})\psi = 0,$$

$$(171) \quad \|\psi\|_{\dot{H}^{1/2}} + \|A_1\|_{L^\infty}^{-1/2} \|\Theta\|_{L^2} \lesssim E_{a,\theta}^{1/2},$$

where (169) follows from the assumption of Theorem 2 and (85); (170) follows from (40), $(I - \mathbb{H})\Theta = 0$ and principle no.2 in §A.2 (and for $\theta = D_\alpha^2 \bar{z}_t$ specifically from (309)); and (171) is immediate from the definition of $E_{a,\theta}$ and change of variables. Upon changing variables, we can write $I = \Re \int 2iB(\partial_{\alpha'}\psi)\bar{\psi}d\alpha'$.

Step 1. Green's identity. We now show that we can control I from (166). The main idea is to use Green's identity to move the derivative from $\psi := \begin{pmatrix} h_\alpha \\ z_\alpha \end{pmatrix} \theta \circ h^{-1}$ onto B . We note that $i\partial_{\alpha'}\psi = i\partial_{\alpha'}\mathbb{H}\psi$ by (170), and that the operator $i\partial_{\alpha'}\mathbb{H} = \nabla_n$, here ∇_n is the Dirichlet-Neumann operator.⁹ Letting ψ^{\hbar} and B^{\hbar} be the (periodic) harmonic extension of ψ and B to P^- respectively, we have

$$(172) \quad I = \Re \int 2iB(\partial_{\alpha'}\psi)\bar{\psi}d\alpha' = \Re \int 2B(\nabla_n\psi)\bar{\psi}dS = \int B\nabla_n(|\psi^{\hbar}|^2)dS.$$

By Green's identity,¹⁰

$$(173) \quad \begin{aligned} \int B\nabla_n(|\psi^{\hbar}|^2)dS &= \int (\nabla_n B)|\psi|^2 dS + \int_{P^-} B^{\hbar}\Delta(|\psi^{\hbar}|^2)dV \\ &= I_1 + I_2. \end{aligned}$$

⁹Recall that the Dirichlet-Neumann operator is defined by $\nabla_n f := \nabla_n f^{\hbar}$, the outward-facing normal derivative of f^{\hbar} , where f^{\hbar} is the extension of f that is harmonic and periodic in P^- and tending to a constant at infinity. For f real-valued, we can derive this by noting that $(I + \mathbb{H})f$ is holomorphic, so $i\partial_{\alpha'}(I + \mathbb{H})f = \nabla_n(I + \mathbb{H})f$. Taking real parts gives the identity.

¹⁰Here, to justify Green's identity, we can map (biholomorphically) the space P^- to the unit disk minus the slit, and then use the periodicity of all of the functions involved to consider the harmonic extensions of these functions to the whole unit disk.

We control the second term, I_2 , by

$$\begin{aligned}
 |I_2| &= \left| \int_{P^-} B^{\hbar} \Delta(|\psi^{\hbar}|^2) dV \right| = 2 \left| \int_{P^-} B^{\hbar} |\nabla \psi^{\hbar}|^2 dV \right| \\
 &\leq 2 \|B^{\hbar}\|_{L^\infty} \int_{P^-} |\nabla \psi^{\hbar}|^2 dV = \|B^{\hbar}\|_{L^\infty} \int_{P^-} \Delta(|\psi^{\hbar}|^2) dV \\
 (174) \quad &= \|B\|_{L^\infty} \int \nabla_n(|\psi^{\hbar}|^2) dS = 2 \|B\|_{L^\infty} \Re \int i(\partial_{\alpha'} \psi) \bar{\psi} d\alpha', \\
 &= 2 \|B\|_{L^\infty} \|\psi\|_{\dot{H}^{1/2}}^2 \lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty} \right) \|\psi\|_{\dot{H}^{1/2}}^2
 \end{aligned}$$

by the maximum principle and another application of Green's identity.

Step 2. *Controlling I_1 .* We are left from Step 1 with controlling

$$\begin{aligned}
 (175) \quad I_1 &= \int (\nabla_n B) |\psi|^2 dS = \Re \int (i \partial_{\alpha'} \mathbb{H} B) |\psi|^2 d\alpha' \\
 &= \Re \int (i \mathbb{H} \partial_{\alpha'} B) |\psi|^2 d\alpha' = \Re \int \frac{1}{Z_{,\alpha'}} (i \mathbb{H} \partial_{\alpha'} B) \Theta \bar{\psi} d\alpha'.
 \end{aligned}$$

We commute the $\frac{1}{Z_{,\alpha'}}$ factor inside the \mathbb{H} , and then apply the adjoint property (291):

$$\begin{aligned}
 (176) \quad I_1 &= \\
 &\Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' + \Re \int i \left(\mathbb{H} \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \right) \Theta \bar{\psi} d\alpha' \\
 &= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' - \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \mathbb{H} (\Theta \bar{\psi}) d\alpha' \\
 &= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' + \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) [\bar{\psi}, \mathbb{H}] \Theta d\alpha' \\
 &\quad - \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \bar{\psi} \mathbb{H} \Theta d\alpha' = I_{11} + I_{12} + I_{13}.
 \end{aligned}$$

Observe that because $\mathbb{H}\Theta = \Theta$,

$$(177) \quad I_{13} = -\Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \bar{\psi} \Theta d\alpha' = -\Re \int i (\partial_{\alpha'} B) |\psi|^2 d\alpha' = 0,$$

since $B \in \mathbb{R}$. It remains to control I_{11} and I_{12} .

We use the Cauchy-Schwarz inequality and then the $\dot{H}^{1/2} \times L^2$ commutator estimate (327) to control I_{12} :

$$(178) \quad |I_{12}| \leq \|D_{\alpha'} B\|_{L^2} \|[\bar{\psi}, \mathbb{H}]\Theta\|_{L^2} \lesssim \|D_{\alpha'} B\|_{L^2} \|\bar{\psi}\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2}.$$

We have controlled $\|\bar{\psi}\|_{\dot{H}^{1/2}}$ and $\|\Theta\|_{L^2}$ at (171), we will control $\|D_{\alpha'} B\|_{L^2}$ by (186) in Step 3 below.

It remains to control I_{11} from (176). Here we use Proposition 5, identity (293). Because $(I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = 1$ (40) and $f \partial_{\alpha'} B = 0$ by (169), we can rewrite

$$(179) \quad \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B = \mathbb{P}_H \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B \right) - \frac{1}{2} f D_{\alpha'} B.$$

We plug (179) into I_{11} , and then use adjoint property (291):

$$\begin{aligned} I_{11} &= \\ &\Re \int i \left\{ \mathbb{P}_H \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B \right) \right\} \Theta \bar{\psi} d\alpha' - \Re \left\{ \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right\} \\ &= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B \right) \mathbb{P}_A (\Theta \bar{\psi}) d\alpha' - \Re \left\{ \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right\}. \end{aligned}$$

To control the first term, we use the Cauchy-Schwarz inequality, and then control the first factor with the $L^2 \times L^\infty$ estimate (318) and control the second factor by rewriting it as a commutator by (170) and then using the $\dot{H}^{1/2} \times L^2$ estimate (327). We use (50) to rewrite $\frac{1}{Z_{,\alpha'}} = -i \frac{Z_{tt} + i}{A_1}$ in the second term.

$$\begin{aligned} (180) \quad |I_{11}| &\leq \\ &\left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B \right\|_{L^2} \|\mathbb{P}_A (\Theta \bar{\psi})\|_{L^2} + \left| \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right| \\ &\leq \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'} B \right\|_{L^2} \left\| \frac{1}{2} [\bar{\psi}, \mathbb{H}] \Theta \right\|_{L^2} \\ &\quad + \|D_{\alpha'} B\|_{L^2} \int \left| \frac{Z_{tt} + i}{A_1} |\Theta|^2 \right| d\alpha' \\ &\lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|B\|_{L^\infty} \|\bar{\psi}\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2} \\ &\quad + \|D_{\alpha'} B\|_{L^2} \|Z_{tt} + i\|_{L^\infty} E_{\alpha,\theta}. \end{aligned}$$

We have controlled all the quantities on the last line, except for $\|D_{\alpha'} B\|_{L^2}$.

Step 3. *Controlling $\|D_{\alpha'} B\|_{L^2}$.* We must control $\|D_{\alpha'} B\|_{L^2}$, where B is as defined by (168). By (70) and (131), rewriting the second and third terms in (131) as commutators, we have

$$(181) \quad \begin{aligned} B &= (h_t \circ h^{-1})_{\alpha'} - \Re D_{\alpha'} Z_t \\ &= \Re D_{\alpha'} Z_t + \Re \left\{ \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] Z_{t,\alpha'} + [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\}. \end{aligned}$$

Therefore, noting that $|\partial_{\alpha'} \Re f| \leq |\partial_{\alpha'} f|$ and so $|D_{\alpha'} \Re f| \leq |D_{\alpha'} f|$,

$$(182) \quad \begin{aligned} \|D_{\alpha'} B\|_{L^2} &\leq \|D_{\alpha'}^2 Z_t\|_{L^2} \\ &\quad + \left\| D_{\alpha'} \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] Z_{t,\alpha'} \right\|_{L^2} + \left\| D_{\alpha'} [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}. \end{aligned}$$

We've controlled $\|D_{\alpha'}^2 Z_t\|_{L^2}$, so it suffices to focus on the second and third terms. In what follows, we work on $D_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g$ for general functions f and g satisfying $f|_{\partial} = g|_{\partial} = 0$. Once we have an appropriate estimate, we will apply it to $f = \frac{1}{Z_{,\alpha'}}$, $g = Z_t$ for the second term, and $f = Z_t$, $g = \frac{1}{Z_{,\alpha'}}$ for the third term.

We know

$$(183) \quad \begin{aligned} &D_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g \\ &= \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{2i} \int (f(\alpha') - f(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \partial_{\beta'} g(\beta') d\beta' \\ &= \frac{1}{Z_{,\alpha'}} (\partial_{\alpha'} f) \mathbb{H} \partial_{\alpha'} g - \frac{1}{Z_{,\alpha'}} \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \partial_{\beta'} g(\beta') d\beta'. \end{aligned}$$

Via the boundedness of the Hilbert transform, we control the first of these terms by $\|D_{\alpha'} f\|_{L^\infty} \|\partial_{\alpha'} g\|_{L^2}$. Therefore, it suffices to focus on the second term. We commute the $\frac{1}{Z_{,\alpha'}}$ inside, getting

$$\begin{aligned} &-\frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} D_{\beta'} g(\beta') d\beta' \\ &\quad - \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta')) \left(\frac{1}{Z_{,\alpha'}}(\alpha') - \frac{1}{Z_{,\beta'}}(\beta') \right)}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right) \sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \partial_{\beta'} g(\beta') d\beta'. \end{aligned}$$

We control the first term by (321):

$$(184) \quad \left\| \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} D_{\beta'} g(\beta') d\beta' \right\|_{L^2} \lesssim \|\partial_{\alpha'} f\|_{L^2} \|D_{\alpha'} g\|_{L^\infty}.$$

We control the second term by (328):

$$\begin{aligned} & \left\| \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta')) \left(\frac{1}{Z_{,\alpha'}}(\alpha') - \frac{1}{Z_{,\beta'}}(\beta') \right)}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \partial_{\beta'} g(\beta') d\beta' \right\|_{L^2} \\ & \lesssim \|\partial_{\alpha'} f\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'} g\|_{L^2}. \end{aligned}$$

We conclude that

$$\begin{aligned} (185) \quad & \|D_{\alpha'}[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \lesssim \|D_{\alpha'} f\|_{L^\infty} \|\partial_{\alpha'} g\|_{L^2} + \|D_{\alpha'} g\|_{L^\infty} \|\partial_{\alpha'} f\|_{L^2} \\ & + \|\partial_{\alpha'} f\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'} g\|_{L^2}. \end{aligned}$$

We can conclude from (182) and (185) that

$$\begin{aligned} (186) \quad & \|D_{\alpha'} B\|_{L^2} \lesssim \|D_{\alpha'}^2 Z_t\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|D_{\alpha'} Z_t\|_{L^\infty} \\ & + \|Z_{t,\alpha'}\|_{L^2} \left(\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}^2 \right). \end{aligned}$$

Step 4. Conclusion. We now combine our various estimates. We have

$$\begin{aligned} (187) \quad & \left| \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \right| \leq |I| + |II| \leq |I_1| + |I_2| + (167) \\ & \leq |I_{11}| + |I_{12}| + (174) + (167) \\ & \leq (180) + (178) + (174) + (167), \end{aligned}$$

where we use (186) to control $\|D_{\alpha'} B\|_{L^2}$.

In particular, by specifying $\theta = D_\alpha^2 \bar{z}_t$, we have

$$(188) \quad \left| \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \right| \lesssim C(E).$$

8. Controlling $(I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2 \Theta + i\mathcal{A}\partial_{\alpha'} \Theta)$

We are left with controlling the G_θ terms in E_a and E_b . Before we do so, we first study the quantity $(I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2 \Theta + i\mathcal{A}\partial_{\alpha'} \Theta)$ for a general Θ satisfying $(I - \mathbb{H})\Theta = 0$. We have

Lemma 3. *Assume that $\Theta, (\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta \in C^l([0, T], H^{k-l}(S^1))$ for $l = 0, 1$, $k \geq 2$, and $(I - \mathbb{H})\Theta = 0$. Assume further that the assumptions of Theorem 2 hold. Then*

$$(189) \quad (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta) = 0;$$

$$(190) \quad \begin{aligned} (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta) &= [Z_t^2, \mathbb{H}]D_{\alpha'}^2\Theta \\ &+ 2[Z_t, \mathbb{H}]D_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta) + 2[Z_{tt}, \mathbb{H}]D_{\alpha'}\Theta; \end{aligned}$$

and

$$(191) \quad \begin{aligned} &\|(I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta)\|_{L^2} \\ &\leq C(E) \left(\|\Theta\|_{L^2} + \|(\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta\|_{L^2} + \left\| \frac{\Theta}{Z_{,\alpha'}} \right\|_{\dot{H}^{1/2}} \right). \end{aligned}$$

Proof. By (70),

$$(192) \quad \mathfrak{b} = \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) + \mathbb{P}_H \left(\frac{\bar{Z}_t}{\bar{Z}_{,\alpha'}} \right) = \frac{Z_t}{Z_{,\alpha'}} + \mathbb{P}_H \left(\frac{\bar{Z}_t}{\bar{Z}_{,\alpha'}} - \frac{Z_t}{Z_{,\alpha'}} \right),$$

so

$$(\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta = \partial_t\Theta + Z_t D_{\alpha'}\Theta + \mathbb{P}_H \left(\frac{\bar{Z}_t}{\bar{Z}_{,\alpha'}} - \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'}\Theta;$$

as a consequence of the dominated convergence theorem and principles no.1 and no.2 in §A.2, we have

$$(193) \quad (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta) = 0.$$

Now since $\Psi := (\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta$ satisfies $(I - \mathbb{H})\Psi = 0$, applying (193) yields

$$(194) \quad (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})\Psi - Z_t D_{\alpha'}\Psi) = 0.$$

We compute

$$(195) \quad \begin{aligned} &(\partial_t + \mathfrak{b}\partial_{\alpha'})\Psi - Z_t D_{\alpha'}\Psi = (\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta \\ &\quad - (\partial_t + \mathfrak{b}\partial_{\alpha'})(Z_t D_{\alpha'}\Theta) - Z_t D_{\alpha'}(\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta + Z_t D_{\alpha'}(Z_t D_{\alpha'}\Theta) \\ &= (\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta - Z_{tt}D_{\alpha'}\Theta \\ &\quad - 2Z_t D_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta) - Z_t^2 D_{\alpha'}^2\Theta, \end{aligned}$$

and by (35),

$$(196) \quad i\mathcal{A}\partial_{\alpha'}\Theta = (Z_{tt} + i)D_{\alpha'}\Theta;$$

and we know by principles no.1 and no.2 in §A.2 that $(I - \mathbb{H})D_{\alpha'}\Theta = 0$. (194)-(196) then gives

$$(197) \quad \begin{aligned} & (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta) = \\ & (I - \mathbb{H})(2Z_{tt}D_{\alpha'}\Theta + 2Z_tD_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_tD_{\alpha'}\Theta) + Z_t^2D_{\alpha'}^2\Theta); \end{aligned}$$

using the holomorphicity of the factors to rewrite the right hand side of (197) as commutators yields (190).

While (190) is sufficient to give us control of the G_θ term in E_b , we need (191), which is the result of some further analysis of (197), to control the G_θ term in E_a .¹¹

We begin with (197), rewriting, using the product rule and the identity $\frac{Z_t}{Z_{,\alpha'}} = \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) + \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)$,

$$(198) \quad \begin{aligned} & Z_t^2D_{\alpha'}^2\Theta = \\ & \left(\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) + \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)^2\partial_{\alpha'}^2\Theta + \frac{Z_t^2}{Z_{,\alpha'}}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta, \end{aligned}$$

and

$$(199) \quad \begin{aligned} & Z_tD_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_tD_{\alpha'}\Theta) \\ & = \left(\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) + \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)\partial_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_tD_{\alpha'}\Theta) \\ & = \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\left((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - \left(\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) + \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)\partial_{\alpha'}\Theta\right) \\ & + \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - Z_tD_{\alpha'}\Theta). \end{aligned}$$

Observe that, by (193) and principles no.1 and no.2 in §A.2, the last quantity on the RHS of (199) is holomorphic with mean zero. Expanding further the right hand sides of (198) and (199) and summing up, observe that certain terms cancel out with others, and the quantity $\left(\mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)^2\partial_{\alpha'}^2\Theta$ in (198)

¹¹We can also use (191) to control the G_θ term in E_b .

is holomorphic with mean zero, by principles no.1 and no.2 in §A.2. We have

$$\begin{aligned}
& (I - \mathbb{H})(2Z_t D_{\alpha'}((\partial_t + \mathbf{b}\partial_{\alpha'})\Theta - Z_t D_{\alpha'}\Theta) + Z_t^2 D_{\alpha'}^2 \Theta) \\
&= (I - \mathbb{H}) \left(2\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \left((\partial_t + \mathbf{b}\partial_{\alpha'})\Theta - \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right) \right) \\
(200) \quad & - (I - \mathbb{H}) \left(2\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \left(\mathbb{P}_H \left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} \right) + \mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \partial_{\alpha'} \Theta \right) \\
& + (I - \mathbb{H}) \left(\left(\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \right)^2 \partial_{\alpha'}^2 \Theta + \frac{Z_t^2}{Z_{,\alpha'}} \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right).
\end{aligned}$$

We analyze further the RHS of (200). Observe that

$$(I - \mathbb{H}) \left(2\mathbb{P}_H \left(\frac{Z_t}{Z_{,\alpha'}} \right) \mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right) = 0$$

by principles no.1 and no.2 of §A.2; we add this to the second term on the RHS of (200). Observe further that

$$(201) \quad -2 \frac{Z_t}{Z_{,\alpha'}} \mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) + \frac{Z_t^2}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = -\frac{Z_t}{Z_{,\alpha'}} \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right),$$

and

$$\begin{aligned}
(202) \quad & -\frac{Z_t}{Z_{,\alpha'}} \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta = \\
& -Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} \left(\frac{\Theta}{Z_{,\alpha'}} \right) + \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \Theta;
\end{aligned}$$

and by straightforward expansion,

$$(203) \quad [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = -2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) + (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right),$$

and

$$\begin{aligned}
(204) \quad & Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) = \\
& \left\{ \mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}^2 - \left\{ \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}^2.
\end{aligned}$$

Now beginning with (197), combining the calculations in (200)-(204) and using principles no.1 and no.2 in §A.2, we get

$$\begin{aligned}
& (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta) = (I - \mathbb{H})\left(2\left(\frac{Z_{tt} + i}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta\right) \\
& + (I - \mathbb{H})\left(2\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\left((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta\right)\right) \\
(205) \quad & - (I - \mathbb{H})\left(2\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\mathbb{P}_H\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta\right) \\
& + (I - \mathbb{H})\left(\left(\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)^2\partial_{\alpha'}^2\Theta - \left\{\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}^2\Theta\right) \\
& + (I - \mathbb{H})\left(\frac{1}{2}\left\{[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\}\partial_{\alpha'}\left(\frac{\Theta}{Z_{,\alpha'}}\right)\right).
\end{aligned}$$

Using the holomorphicity of the factors to rewrite all, except for two, terms on the right hand side of (205) as commutators gives

$$\begin{aligned}
& (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta) = 2\left[\frac{Z_{tt} + i}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\Theta \\
& + 2\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}\left((\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta - \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta\right) \\
(206) \quad & - (I - \mathbb{H})\left(2\mathbb{P}_H(D_{\alpha'}Z_t)\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\partial_{\alpha'}\Theta\right) \\
& + \left[\left(\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right)^2, \mathbb{H}\right]\partial_{\alpha'}^2\Theta - (I - \mathbb{H})\left(\left\{\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}^2\Theta\right) \\
& + \left[\frac{1}{2}\left\{[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\}, \mathbb{H}\right]\partial_{\alpha'}\left(\frac{\Theta}{Z_{,\alpha'}}\right).
\end{aligned}$$

By the identity¹²

$$(207) \quad -2[g_1, \mathbb{H}]\partial_{\alpha'}(g_1g_2) + [g_1^2, \mathbb{H}]\partial_{\alpha'}g_2 = -[g_1, g_1; g_2],$$

we combine part of the second term and the fourth term on the RHS of (206):

¹²This identity is an easy consequence of integration by parts.

$$(208) \quad -2 \left[\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} \left(\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right) + \left[\left(\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \right)^2, \mathbb{H} \right] \partial_{\alpha'}^2 \Theta = - \left[\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right); \partial_{\alpha'} \Theta \right].$$

Observe that, using the identity $\mathbb{P}_A + \mathbb{P}_H = I$ and the fact that $(I - \mathbb{H})\partial_{\alpha'}\Theta = 0$,

$$(209) \quad \begin{aligned} & (I - \mathbb{H}) \left(2\mathbb{P}_H (D_{\alpha'} Z_t) \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right) \\ &= (I - \mathbb{H}) \left(\mathbb{P}_H (D_{\alpha'} Z_t) \left[\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} \Theta \right) \\ &+ (I - \mathbb{H}) \left(2\mathbb{P}_H (D_{\alpha'} Z_t) \mathbb{P}_H \left\{ \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right\} \right); \end{aligned}$$

by (292) the second term equals the mean

$$(210) \quad \begin{aligned} & (I - \mathbb{H}) \left(2\mathbb{P}_H (D_{\alpha'} Z_t) \mathbb{P}_H \left\{ \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta \right\} \right) \\ &= \frac{1}{2} \left(\int D_{\alpha'} Z_t d\alpha' \right) \left(\int \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} \Theta d\alpha' \right) \\ &= -\frac{1}{2} \left(\int D_{\alpha'} Z_t d\alpha' \right) \left(\int \partial_{\alpha'} \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \Theta d\alpha' \right). \end{aligned}$$

We can now conclude, from (206)-(210) and using (322), (323) and (334), that

$$(211) \quad \begin{aligned} & \left\| (I - \mathbb{H})((\partial_t + \mathfrak{b}\partial_{\alpha'})^2 \Theta + i\mathcal{A}\partial_{\alpha'}\Theta) \right\|_{L^2} \lesssim \left\| \partial_{\alpha'} \frac{Z_{tt} + i}{Z_{,\alpha'}} \right\|_{L^\infty} \|\Theta\|_{L^2} \\ &+ \left\| \partial_{\alpha'} \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty} \|(\partial_t + \mathfrak{b}\partial_{\alpha'})\Theta\|_{L^2} \\ &+ \left\| \partial_{\alpha'} \left\{ [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^2} \left\| \frac{\Theta}{Z_{,\alpha'}} \right\|_{\dot{H}^{1/2}} \\ &+ \left(\left\| \partial_{\alpha'} \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2 + \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2 \right) \|\Theta\|_{L^2} \\ &+ (\|\mathbb{P}_H D_{\alpha'} Z_t\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty}) \left\| \partial_{\alpha'} \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty} \|\Theta\|_{L^2}. \end{aligned}$$

We have estimated all the factors on the RHS of (211), except for

$$\left\| \partial_{\alpha'} \left\{ [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^2},$$

which can be controlled by (332):

$$(212) \quad \left\| \partial_{\alpha'} \left\{ [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim C(E).$$

This proves (191). □

9. Controlling G_θ of E_b

By (98), we must control

$$(213) \quad \left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a} \partial_\alpha, D_\alpha] \bar{z}_t|^2 d\alpha \right)^{1/2}.$$

We control the commutator via (341):

$$(214) \quad \begin{aligned} & \left(\int \frac{1}{\mathfrak{a}} |[\partial_t^2 + i\mathfrak{a} \partial_\alpha, D_\alpha] \bar{z}_t|^2 d\alpha \right)^{1/2} \\ & \lesssim (\|D_\alpha z_{tt}\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty}^2) \|D_\alpha \bar{z}_t\|_{L^2(\frac{1}{\mathfrak{a}} d\alpha)}, \end{aligned}$$

where we have controlled all the quantities on the RHS in §5.1. We are left with the term $\left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2}$. Since $\mathfrak{a} |z_\alpha|^2 = (A_1 \circ h) h_\alpha$ (52), $A_1 \geq 1$ (49), and $(\mathfrak{a}_t \bar{z}_\alpha) \circ h^{-1} = \mathcal{A}_t \bar{Z}_{,\alpha'}$ we have

$$(215) \quad \begin{aligned} \left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2} & \leq \left(\int \frac{1}{h_\alpha} |\partial_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2} \\ & = \left(\int |\partial_{\alpha'}(-i\mathcal{A}_t \bar{Z}_{,\alpha'})|^2 d\alpha' \right)^{1/2}, \end{aligned}$$

where in the second step we changed to Riemann mapping variables. We write $-i\mathcal{A}_t \bar{Z}_{,\alpha'}$ as $\frac{\mathcal{A}_t}{\mathcal{A}}(-i\mathcal{A} \bar{Z}_{,\alpha'})$ and apply $\partial_{\alpha'}$. Since $\bar{Z}_{tt} - i = -i\mathcal{A} \bar{Z}_{,\alpha'}$ (35), we have

$$(216) \quad \partial_{\alpha'}(-i\mathcal{A}_t \bar{Z}_{,\alpha'}) = (-i\mathcal{A} \bar{Z}_{,\alpha'}) \partial_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) + \frac{\mathcal{A}_t}{\mathcal{A}} \bar{Z}_{tt,\alpha'}.$$

Therefore,

$$(217) \quad \begin{aligned} & \left(\int |\partial_{\alpha'}(-i\mathcal{A}_t\bar{\mathcal{Z}}_{,\alpha'})|^2 d\alpha' \right)^{1/2} \\ & \leq \left(\int \left| \mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right) \right|^2 d\alpha' \right)^{1/2} + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|\bar{\mathcal{Z}}_{tt,\alpha'}\|_{L^2}. \end{aligned}$$

We controlled the factors in the second term on the RHS in (107) and (123). We can therefore concentrate on the first term.

We seek a way of writing $\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)$. The idea is to take advantage of the fact that $\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)$ is real, $|f| \leq |(I - \mathbb{H})f|$ for f real, and $\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'} = i(\bar{\mathcal{Z}}_{tt} - i)$ is controllable to bound the term $\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)$ by a sum of controllable terms and commutators.

Starting from (216), we replace the LHS by the derivative of the LHS of our quasilinear equation (36), and then apply $(I - \mathbb{H})$ to the equation. We get

$$(218) \quad \begin{aligned} & (I - \mathbb{H}) \left\{ (-i\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right\} \\ & = (I - \mathbb{H})\partial_{\alpha'}(\bar{\mathcal{Z}}_{ttt} + i\mathcal{A}\bar{\mathcal{Z}}_{t,\alpha'}) - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{\mathcal{Z}}_{tt} \right\}. \end{aligned}$$

We next commute the factor $-i\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}$ outside of $(I - \mathbb{H})$ on the LHS,

$$(219) \quad \begin{aligned} & (-i\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'})(I - \mathbb{H})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} = (I - \mathbb{H})\partial_{\alpha'}(\bar{\mathcal{Z}}_{ttt} + i\mathcal{A}\bar{\mathcal{Z}}_{t,\alpha'}) \\ & - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{\mathcal{Z}}_{tt} \right\} + [i\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}. \end{aligned}$$

Now $\frac{\mathcal{A}_t}{\mathcal{A}}$ is real and \mathbb{H} is purely imaginary, so $|\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}| \leq |(I - \mathbb{H})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}|$. Taking absolute value on both sides, we have

$$(220) \quad \begin{aligned} & \left| \mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right| \leq |(I - \mathbb{H})\partial_{\alpha'}(\bar{\mathcal{Z}}_{ttt} + i\mathcal{A}\bar{\mathcal{Z}}_{t,\alpha'})| \\ & + \left| (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{\mathcal{Z}}_{tt} \right\} - [i\mathcal{A}\bar{\mathcal{Z}}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right|. \end{aligned}$$

We can easily control the L^2 norm of the second and third terms. By the L^2 boundedness of \mathbb{H} and Hölder's inequality for the second term and

estimate (318) for the third term, and since $i\mathcal{A}\bar{Z}_{,\alpha'} = -(\bar{Z}_{tt} - i)$,

$$(221) \quad \left\| (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}} \bar{Z}_{tt,\alpha'} \right\} - [i\mathcal{A}\bar{Z}_{,\alpha'}, \mathbb{H}] \partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}.$$

We can now focus on controlling

$$(222) \quad \|(I - \mathbb{H}) \partial_{\alpha'} (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} = \|\partial_{\alpha'} (I - \mathbb{H}) (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2},$$

where we used $(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) \in C^2(S^1)$, which follows from equations (36), (35), (67) and the assumption of Theorem 2, to commute $\partial_{\alpha'}$ outside $(I - \mathbb{H})$.

By (190), taking $\Theta = \bar{Z}_t$, we have

$$(223) \quad \begin{aligned} & (I - \mathbb{H}) (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) \\ &= [Z_t^2, \mathbb{H}] D_{\alpha'}^2 \bar{Z}_t + 2[Z_t, \mathbb{H}] D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2[Z_{tt}, \mathbb{H}] D_{\alpha'} \bar{Z}_t. \end{aligned}$$

Therefore, by (222) and (223), we have to control the L^2 norm of

$$(224) \quad \begin{aligned} & \partial_{\alpha'} [Z_t^2, \mathbb{H}] D_{\alpha'}^2 \bar{Z}_t + 2\partial_{\alpha'} [Z_t, \mathbb{H}] D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \\ & \quad + 2\partial_{\alpha'} [Z_{tt}, \mathbb{H}] D_{\alpha'} \bar{Z}_t. \end{aligned}$$

We use the identity

$$(225) \quad \partial_{\alpha'} [f, \mathbb{H}] g = f' \mathbb{H} g - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta'$$

to expand out each term in (224), and use (307) and (311) to remove the \mathbb{H} s from the RHS. We get

$$\begin{aligned} (224) &= 2Z_t Z_{t,\alpha'} D_{\alpha'}^2 \bar{Z}_t + 2Z_{t,\alpha'} D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2Z_{tt,\alpha} D_{\alpha'} \bar{Z}_t \\ & - \frac{\pi}{4i} \int \frac{Z_t^2(\alpha') - Z_t^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'}^2 \bar{Z}_t d\beta' - \frac{\pi}{2i} \int \frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t d\beta' \\ & - \frac{\pi}{2i} \int \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} (\bar{Z}_{tt} - (D_{\beta'} \bar{Z}_t) Z_t) d\beta'. \end{aligned}$$

We expand out the RHS. We note that certain terms cancel out with others,

and we further observe the following identity:

$$\begin{aligned} & \int \frac{f^2(\alpha') - f^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' - 2 \int \frac{(f(\alpha') - f(\beta'))f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \\ &= \int \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta'. \end{aligned}$$

We have

$$\begin{aligned} (224) &= 2Z_{t,\alpha'}(D_{\alpha'}\bar{Z}_{tt} - (D_{\alpha'}\bar{Z}_t)D_{\alpha'}Z_t) + 2Z_{tt,\alpha}D_{\alpha'}\bar{Z}_t \\ &- \frac{\pi}{4i} \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'}^2 \bar{Z}_t d\beta' - \frac{\pi}{2i} \int \frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t d\beta' \\ &- \frac{\pi}{2i} \int \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (D_{\beta'} \bar{Z}_{tt} - (D_{\beta'} \bar{Z}_t)D_{\beta'} Z_t) d\beta'. \end{aligned}$$

We now apply Hölder's inequality to the first two terms, (328) to the third term, and (321) to the last two terms. We get

$$\begin{aligned} (226) \quad & \|(I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} = \|(224)\|_{L^2} \\ & \lesssim \|Z_{t,\alpha'}\|_{L^2} (\|D_{\alpha'}\bar{Z}_{tt}\|_{L^\infty} + \|D_{\alpha'}\bar{Z}_t\|_{L^\infty}^2) \\ & \quad + \|Z_{tt,\alpha'}\|_{L^2} \|D_{\alpha'}\bar{Z}_t\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}. \end{aligned}$$

We now combine our various estimates. We have

$$(227) \quad \left\| \mathcal{A}\bar{Z}_{,\alpha'} \partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \leq (221) + (226) \lesssim C(E)$$

and

$$\begin{aligned} (228) \quad & \left(\int \frac{1}{\mathfrak{a}} |D_{\alpha}(-i\mathfrak{a}_t \bar{z}_{\alpha}) + [\partial_t^2 + i\mathfrak{a}\partial_{\alpha}, D_{\alpha}]\bar{z}_t|^2 d\alpha \right)^{1/2} \leq (214) + (217) \\ & \leq (214) + (227) + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|\bar{Z}_{tt,\alpha'}\|_{L^2} \lesssim C(E). \end{aligned}$$

We can now conclude that $\frac{d}{dt}E_b$ is bounded by a polynomial of E .

We record here the estimate

$$(229) \quad \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \leq \left\| \mathcal{A}\bar{Z}_{,\alpha'} \partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim C(E),$$

which holds because $|\mathcal{A}\bar{Z}_{,\alpha'}| = \frac{\mathcal{A}_1}{|Z_{,\alpha'}|} \geq \frac{1}{|Z_{,\alpha'}|}$; we will use this in §10.

10. Controlling G_θ of E_a

From (97), we must control

$$(230) \quad \left(\int |D_\alpha^2(-i\mathbf{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathbf{a} \partial_\alpha, D_\alpha^2] \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}.$$

Recall that $A_1 \geq 1$ (49). We control the commutator via (342) and Hölder's inequality:

$$(231) \quad \begin{aligned} & \| [\partial_t^2 + i\mathbf{a} \partial_\alpha, D_\alpha^2] \bar{z}_t \|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \leq \| [\partial_t^2 + i\mathbf{a} \partial_\alpha, D_\alpha^2] \bar{z}_t \|_{L^2(h_\alpha d\alpha)} \\ & \lesssim \| D_\alpha z_{tt} \|_{L^\infty} \| D_\alpha^2 \bar{z}_t \|_{L^2(h_\alpha d\alpha)} + \| D_\alpha z_t \|_{L^\infty}^2 \| D_\alpha^2 \bar{z}_t \|_{L^2(h_\alpha d\alpha)} \\ & + \| D_\alpha z_t \|_{L^\infty} \| D_\alpha \partial_t D_\alpha \bar{z}_t \|_{L^2(h_\alpha d\alpha)} + \| D_\alpha^2 z_{tt} \|_{L^2(h_\alpha d\alpha)} \| D_\alpha \bar{z}_t \|_{L^\infty} \\ & + \| D_\alpha z_t \|_{L^\infty}^2 \| D_\alpha^2 z_t \|_{L^2(h_\alpha d\alpha)} + \| D_\alpha^2 z_t \|_{L^2(h_\alpha d\alpha)} \| D_\alpha \bar{z}_{tt} \|_{L^\infty} \\ & + \| D_\alpha z_t \|_{L^\infty} \| D_\alpha^2 \bar{z}_{tt} \|_{L^2(h_\alpha d\alpha)}. \end{aligned}$$

We have controlled all quantities on the RHS in §5.1. We are left with the term $\left(\int |D_\alpha^2(\mathbf{a}_t \bar{z}_\alpha)|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}$.

We know

$$(232) \quad \begin{aligned} \left(\int |D_\alpha^2(\mathbf{a}_t \bar{z}_\alpha)|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} & \leq \left(\int |D_\alpha^2(\mathbf{a}_t \bar{z}_\alpha)|^2 h_\alpha d\alpha \right)^{1/2} \\ & = \left(\int |D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'})|^2 d\alpha' \right)^{1/2}, \end{aligned}$$

where we changed to Riemann mapping coordinate in the second step. We will now focus on estimating

$$(233) \quad \left(\int |D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'})|^2 d\alpha' \right)^{1/2}.$$

Our plan is to first turn the task of controlling $\|D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'})\|_{L^2}$ to controlling $\|(I - \mathbb{H})(D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'}))\|_{L^2}$. We will use the same idea as in the previous section, §9, that is, to take advantage of the fact that $\frac{A_t}{A}$ is real-valued and $\Re(I - \mathbb{H})f = f$ for real valued f . We will then use (191) to control $\|(I - \mathbb{H})(D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'}))\|_{L^2}$.

We begin by writing $\mathcal{A}_t \bar{Z}_{,\alpha'} = \left(\frac{\mathcal{A}_t}{\mathcal{A}}\right) \mathcal{A} \bar{Z}_{,\alpha'}$. By the product rule,

$$(234) \quad \begin{aligned} & D_{\alpha'}^2 (\mathcal{A}_t \bar{Z}_{,\alpha'}) \\ &= \left(D_{\alpha'}^2 \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) \right) \mathcal{A} \bar{Z}_{,\alpha'} + 2 D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) D_{\alpha'} (\mathcal{A} \bar{Z}_{,\alpha'}) + \frac{\mathcal{A}_t}{\mathcal{A}} D_{\alpha'}^2 (\mathcal{A} \bar{Z}_{,\alpha'}). \end{aligned}$$

We can handle the second and third terms directly, using $\mathcal{A} \bar{Z}_{,\alpha'} = i(\bar{Z}_{tt} - i)$ (35):

$$(235) \quad \begin{aligned} & \left\| 2 D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) D_{\alpha'} (\mathcal{A} \bar{Z}_{,\alpha'}) + \frac{\mathcal{A}_t}{\mathcal{A}} D_{\alpha'}^2 (\mathcal{A} \bar{Z}_{,\alpha'}) \right\|_{L^2} \\ & \leq 2 \left\| D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \end{aligned}$$

where we have controlled all the quantities on the RHS in §5.1 and in (123) and (229). It therefore suffices to focus on the first term on the RHS of (234), $(D_{\alpha'}^2 \left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)) \mathcal{A} \bar{Z}_{,\alpha'} = i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)$.

We now rearrange this term so that we can apply $(I - \mathbb{H})$ in a way so that we will be able to invert the operator by taking real parts. Note that $\frac{\mathcal{A}_t}{\mathcal{A}}$ is purely real. However, our derivative $D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}$ is not purely real. To get around this, we factor the derivative into a real derivative and a complex modulus-one weight. Recall our notation $|D_{\alpha'}| = \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$. Since

$$D_{\alpha'} = \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) |D_{\alpha'}|, \text{ we rewrite}$$

$$(236) \quad D_{\alpha'}^2 = \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 |D_{\alpha'}|^2 + \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right) |D_{\alpha'}|.$$

Therefore,

$$(237) \quad \begin{aligned} i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} &= i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 |D_{\alpha'}|^2 \frac{\mathcal{A}_t}{\mathcal{A}} \\ &\quad + i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}. \end{aligned}$$

We use

$$(238) \quad e := i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}$$

to denote the second term, which we will control directly, below at (243). We now apply $(I - \mathbb{H})$ to both sides of (237):

$$\begin{aligned}
 (239) \quad & (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \\
 & = (I - \mathbb{H}) \left\{ i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\} + (I - \mathbb{H})e.
 \end{aligned}$$

Observe that the first term on the RHS is purely real, except for the controllable factor $i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2$. We commute that part outside the $(I - \mathbb{H})$. We get

$$\begin{aligned}
 (240) \quad & (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \\
 & = i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 (I - \mathbb{H}) \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \\
 & \quad + \left[i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) + (I - \mathbb{H})e.
 \end{aligned}$$

Because \mathbb{H} is purely imaginary, $|\partial_{\alpha'} (|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}})| \leq |(I - \mathbb{H}) \partial_{\alpha'} (|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}})|$. By taking absolute values, we have

$$\begin{aligned}
 (241) \quad & \left| i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right| \leq \left| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right| \\
 & \quad + \left| \left[i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) + (I - \mathbb{H})e \right|.
 \end{aligned}$$

Now we may begin controlling these terms. Recall that what we needed to control was the L^2 norm of (237). We can estimate this by

$$\begin{aligned}
 (242) \quad & \left\| -i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim \left\| i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2} \\
 & \lesssim \|(241)\|_{L^2} + \|e\|_{L^2} \\
 & \lesssim \left\| (I - \mathbb{H}) \left\{ i(\overline{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2} \\
 & \quad + \left\| \left[i \frac{(\overline{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2}.
 \end{aligned}$$

Thus, it suffices to focus on these three terms.

First we check the error term, e (238). We control

$$(243) \quad \begin{aligned} \|e\|_{L^2} &\leq \left\| (\bar{Z}_{tt} - i) \left(|D_{\alpha'}| \frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \\ &\lesssim \left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}, \end{aligned}$$

where in the second step we used $\left| \partial_{\alpha'} \frac{f}{|f|} \right| \leq \left| \frac{f'}{|f|} \right|$ (315). We have controlled both factors on the RHS in (149) and (229).

Now we estimate the second term on the RHS of (242). We have, by $L^\infty \times L^2$ commutator estimate (322),

$$(244) \quad \begin{aligned} &\left\| \left[i \frac{(\bar{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \\ &\lesssim \left\| \partial_{\alpha'} \left(\frac{(\bar{Z}_{tt} - i)}{|Z_{,\alpha'}|} \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \right) \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \\ &\lesssim \left(\|D_{\alpha'} Z_{tt}\|_{L^\infty} + \left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}, \end{aligned}$$

where in the second step we used (315). We have controlled all factors on the RHS in (119), (149) and (229).

We're left with the first, main term of the RHS of (242). Observe that by (234), our main equation $\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'} = -i\mathcal{A}_t\bar{Z}_{,\alpha'}$ (36), and the L^2 boundedness of \mathbb{H} ,

$$(245) \quad \begin{aligned} &\left\| (I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2} \\ &\lesssim \|(I - \mathbb{H}) (D_{\alpha'}^2 (\mathcal{A}_t \bar{Z}_{,\alpha'}))\|_{L^2} + (235) \\ &= \|(I - \mathbb{H}) D_{\alpha'}^2 (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} + (235). \end{aligned}$$

We have then reduced things to controlling $\|(I - \mathbb{H}) D_{\alpha'}^2 (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2}$.

Observe that

$$(246) \quad \begin{aligned} &D_{\alpha'}^2 (\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) \\ &= ((\partial_t + \mathfrak{b}\partial_{\alpha'})^2 + i\mathcal{A}\partial_{\alpha'}) D_{\alpha'}^2 \bar{Z}_t + [D_{\alpha'}^2, (\partial_t + \mathfrak{b}\partial_{\alpha'})^2 + i\mathcal{A}\partial_{\alpha'}] \bar{Z}_t. \end{aligned}$$

We have controlled the second term on the RHS in (231). Applying $(I - \mathbb{H})$ and using (191) on the first term then gives

$$(247) \quad \left\| (I - \mathbb{H})D_{\alpha'}^2(\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'}) \right\|_{L^2} \lesssim C(E).$$

We can now conclude that

$$(248) \quad \left\| D_{\alpha'}^2(\mathcal{A}_t Z_{\alpha'}) \right\|_{L^2} \lesssim C(E).$$

and therefore

$$(249) \quad \left\| G_{D_{\alpha'}^2 \overline{z}_t} \right\|_{L^2(\frac{h_{\alpha'}}{A_1 \sigma h})} \leq (231) + (232) \leq (231) + (248) \lesssim C(E).$$

We have now shown that $\frac{d}{dt}E_a$ is bounded by a polynomial of E . This completes the proof of Theorem 2. \square

11. A characterization of the energy

Our energy is expressed in terms of not only the free surface Z , the velocity Z_t , and their spatial derivatives, but also time derivatives of these quantities. In this section, we give a characterization of our energy in terms of the free surface Z , the velocity Z_t , and their spatial derivatives. In §11.2, we discuss which crest angles are allowed by a finite energy E .

11.1. A characterization of the energy in terms of position and velocity

In this section, we translate the terms of our energy involving time derivatives into terms depending only on the free surface Z , the velocity Z_t , and their spatial derivatives. We do this using the basic equation (50), (48), (70) and the holomorphicity of \overline{Z}_t and $\frac{1}{Z_{,\alpha'}}$. These basic equations allow us to show that quantities involving \overline{Z}_{tt} can be controlled by analogous quantities involving $\frac{1}{Z_{,\alpha'}}$, along with various lower-order terms.¹³

The estimate we prove is

$$(250) \quad E(t) \leq C \left(\left\| \overline{Z}_{t,\alpha'} \right\|_{L^2}, \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \right. \\ \left. \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}, \left\| D_{\alpha'} \overline{Z}_t \right\|_{\dot{H}^{1/2}}, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \right),$$

¹³We remark that for these estimates we *do not* ever rely on (high order) $\dot{H}^{1/2}$ parts of the energies.

where the constant depends polynomially on its terms. We remark that this inequality can be reversed: each of the factors on the RHS of (250) is controlled by the energy. That is,

$$(251) \quad \left\| \bar{Z}_{t,\alpha'} \right\|_{L^2}, \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \\ \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \left\| D_{\alpha'} \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim C(E(t)).$$

Therefore, these quantities fully characterize our energy. In the proof of our a priori estimate, we have shown (251) for every term except $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$, which we never had a need to control. One can adapt the argument in §11.1.3 below to show that $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ can be controlled by the energy.¹⁴

We remark because both \bar{Z}_t and $\frac{1}{Z_{,\alpha'}}$ are the boundary values of periodic holomorphic functions, the weighted derivative $D_{\alpha'}$ corresponds to the complex derivative ∂_z , or the gradient of the corresponding quantities in the spatial domain P^- . We also note that $Z_{,\alpha'} = (\Phi^{-1})_z$ is a natural geometric quantity well-suited to this problem: it captures the geometry of the free surface directly through the Riemann mapping $\Phi^{-1} : P^- \rightarrow \Omega(t)$.

¹⁴To do this, it comes down once again to estimating $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2}$, except this time we need to do this without the dependence on $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$. That dependence comes from estimate (275). (It also comes from using $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ in the Sobolev inequality for $\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$; this is not a problem, since $\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ is controlled by the energy.) To handle (275), we take advantage of the fact that $(I - \mathbb{H}) \left\{ \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} = 0$ (this is due to (40) and the second principle in §A.2) to rewrite the term in question as a commutator and then use commutator estimate (318):

$$(252) \quad \left\| (I - \mathbb{H}) \left\{ \frac{A_1}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^2} = \left\| [\bar{Z}_{tt}, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \\ \lesssim \left\| \bar{Z}_{tt,\alpha'} \right\|_{L^2} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty},$$

both of which are controlled by the energy.

11.1.1. The proof. Throughout the following proof we will rely on the fact that $A_1 \geq 1$ (49), the estimate (103)

$$(253) \quad \|A_1\|_{L^\infty} \lesssim 1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2,$$

the Sobolev estimate (104)

$$(254) \quad \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim \|\bar{Z}_{t,\alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2},$$

and the estimate

$$(255) \quad \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2},$$

which holds by Sobolev inequality (314).¹⁵

We begin by noting that it suffices to control only the first terms of E_a and E_b , since the remaining terms of the energy are (up to a factor of A_1) already on the RHS of (250).

For the first term of E_a , by the commutator identity (338),

$$(256) \quad \begin{aligned} & \int |\partial_t D_{\alpha'}^2 \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ & \lesssim \int |D_{\alpha'}^2 \bar{z}_{tt}|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |[\partial_t, D_{\alpha'}^2] \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ & \lesssim \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}^2 + \int |2(D_\alpha z_t) D_{\alpha'}^2 \bar{z}_t + (D_{\alpha'}^2 z_t) D_\alpha \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ & \lesssim \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 (\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 + \|D_{\alpha'}^2 Z_t\|_{L^2}^2). \end{aligned}$$

By (110) and (315),

$$(257) \quad \|D_{\alpha'}^2 Z_t\|_{L^2} \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}.$$

We conclude that

$$(258) \quad \begin{aligned} & \int |\partial_t D_{\alpha'}^2 \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ & \lesssim C \left(\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right). \end{aligned}$$

¹⁵Note that $f\left(D_{\alpha'} \frac{1}{Z_{,\alpha'}}\right)^2 = 0$ by the same argument that was used at (312) to show $f(D_{\alpha'} \bar{Z}_t)^2 = 0$.

For the first term of E_b , we use the commutator identity (337) to get

$$\begin{aligned}
 (259) \quad & \int |\partial_t D_\alpha \bar{z}_t|^2 \frac{1}{\mathbf{a}} d\alpha \lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{1}{\mathbf{a}} d\alpha + \int |[\partial_t, D_\alpha] \bar{z}_t|^2 \frac{1}{\mathbf{a}} d\alpha \\
 & \lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{(A_1 \circ h)}{\mathbf{a}} d\alpha + \int |(D_\alpha z_t) D_\alpha \bar{z}_t|^2 \frac{(A_1 \circ h)}{\mathbf{a}} d\alpha \\
 & \lesssim \|\bar{Z}_{tt, \alpha'}\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 \|\bar{Z}_{t, \alpha'}\|_{L^2}^2 \\
 & \leq C(\|\bar{Z}_{tt, \alpha'}\|_{L^2}, \|\bar{Z}_{t, \alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}).
 \end{aligned}$$

All that remains to do from (258) and (259) is to estimate $\|\bar{Z}_{tt, \alpha'}\|_{L^2}$ and $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$ in terms of Z_t and $\frac{1}{Z_{, \alpha'}}$, which we now do, in §11.1.2 and §11.1.3.

11.1.2. Controlling $\|\bar{Z}_{tt, \alpha'}\|_{L^2}$. Using (50), we estimate

$$(260) \quad \|\partial_{\alpha'} \bar{Z}_{tt}\|_{L^2} \lesssim \|A_1\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^2} + \|D_{\alpha'} A_1\|_{L^2}.$$

To control $\|D_{\alpha'} A_1\|_{L^2}$, we follow a similar procedure to what we did in (125)-(126), except instead of using $\bar{Z}_{tt} - i$, we use $\frac{1}{Z_{, \alpha'}}$ and estimate things in terms of $\frac{1}{Z_{, \alpha'}}$. We get

$$(261) \quad \|D_{\alpha'} A_1\|_{L^2} \lesssim \|\bar{Z}_{t, \alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^2} + \|Z_{t, \alpha'}\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}.$$

Combining (260) and (261) we conclude that

$$(262) \quad \|\partial_{\alpha'} \bar{Z}_{tt}\|_{L^2} \leq C \left(\|\bar{Z}_{t, \alpha'}\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \right).$$

11.1.3. Controlling $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$. From (50), we have

$$(263) \quad i D_{\alpha'}^2 \bar{Z}_{tt} = \underbrace{A_1 D_{\alpha'}^2 \frac{1}{Z_{, \alpha'}} + 2(D_{\alpha'} A_1) D_{\alpha'} \frac{1}{Z_{, \alpha'}}}_{e_1} + \frac{1}{Z_{, \alpha'}} D_{\alpha'}^2 A_1.$$

We estimate $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$ through the following procedure. First we note that the only challenging term to control on the RHS of (263) is the last one, $\frac{1}{Z_{, \alpha'}} D_{\alpha'}^2 A_1$. We observe that this is almost real, modulo factors of $\frac{1}{Z_{, \alpha'}}$ and

its derivatives. Therefore, we will be able to use the $\Re(I - \mathbb{H})$ trick and, through a series of commutators, reduce the estimate for $\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1$ to an estimate of $(I - \mathbb{H})(D_{\alpha'}^2 \frac{A_1}{\bar{Z}_{,\alpha'}}) = (I - \mathbb{H})(iD_{\alpha'}^2 \bar{Z}_{tt})$. Since \bar{Z}_t is holomorphic, we will be able to rewrite $(I - \mathbb{H})(iD_{\alpha'}^2 \bar{Z}_{tt})$ in terms of commutators and obtain desirable estimates. We now give the details.

We first estimate the error term e_1 in (263):

$$\begin{aligned}
 \|e_1\|_{L^2} &\lesssim \|A_1\|_{L^\infty} \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} + \|D_{\alpha'} A_1\|_{L^2} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \\
 (264) \quad &\lesssim (1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2) \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \\
 &\quad + (261) \left(\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \right).
 \end{aligned}$$

It remains to control $\left\| \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2}$. We want to use $(I - \mathbb{H})$ to turn our quantity into commutators, but to do so we need to factor $D_{\alpha'}$ into a real-weighted derivative $|D_{\alpha'}| := \frac{1}{|\bar{Z}_{,\alpha'}|} \partial_{\alpha'}$ so that we may invert $(I - \mathbb{H})$. From (236), we have

$$\begin{aligned}
 \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1 &= \frac{1}{\bar{Z}_{,\alpha'}} \left(\frac{|\bar{Z}_{,\alpha'}|}{\bar{Z}_{,\alpha'}} \right)^2 |D_{\alpha'}|^2 A_1 \\
 (265) \quad &\quad + \underbrace{\frac{1}{\bar{Z}_{,\alpha'}} \frac{|\bar{Z}_{,\alpha'}|}{\bar{Z}_{,\alpha'}} \left(|D_{\alpha'}| \frac{|\bar{Z}_{,\alpha'}|}{\bar{Z}_{,\alpha'}} \right)}_{e_2} |D_{\alpha'}| A_1.
 \end{aligned}$$

We multiply both sides by $\left(\frac{\bar{Z}_{,\alpha'}}{|\bar{Z}_{,\alpha'}|} \right)^3$ so that the first term on the RHS is purely real:

$$\begin{aligned}
 (266) \quad &\left(\frac{\bar{Z}_{,\alpha'}}{|\bar{Z}_{,\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1 \\
 &= \left(\frac{\bar{Z}_{,\alpha'}}{|\bar{Z}_{,\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{,\alpha'}} \left(\frac{|\bar{Z}_{,\alpha'}|}{\bar{Z}_{,\alpha'}} \right)^2 |D_{\alpha'}|^2 A_1 + \left(\frac{\bar{Z}_{,\alpha'}}{|\bar{Z}_{,\alpha'}|} \right)^3 e_2.
 \end{aligned}$$

Now we apply $\Re(I - \mathbb{H})$ to each side, and conclude from the fact that $A_1 \in \mathbb{R}$ that

$$(267) \quad \left| \frac{1}{Z, \alpha'} |D_{\alpha'}|^2 A_1 \right| \lesssim \left| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} D_{\alpha'}^2 A_1 \right\} \right| + \left| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 e_2 \right\} \right|.$$

We conclude from (265) and (267) that

$$(268) \quad \left\| \frac{1}{Z, \alpha'} D_{\alpha'}^2 A_1 \right\|_{L^2} \lesssim \|e_2\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}.$$

By (315) and (261) we estimate

$$(269) \quad \|e_2\|_{L^2} \lesssim \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \lesssim (255)(261).$$

It remains to estimate $\left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}$. To get the right commutator estimate, we first rewrite this as

$$(270) \quad \begin{aligned} & \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2} \\ & \lesssim \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} \partial_{\alpha'} \left(\frac{1}{Z, \alpha'} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \\ & \quad + \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} \left(\partial_{\alpha'} \frac{1}{Z, \alpha'} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2}. \end{aligned}$$

We estimate the second term on the RHS of (270) directly:

$$(271) \quad \begin{aligned} & \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3 \frac{1}{Z, \alpha'} \left(\partial_{\alpha'} \frac{1}{Z, \alpha'} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2} \\ & \lesssim \left\| D_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \lesssim (255)(261). \end{aligned}$$

For the first term on the RHS of (270), we commute the factor $\left(\frac{Z, \alpha'}{|Z, \alpha'|} \right)^3$ outside the $(I - \mathbb{H})$, bringing along $\frac{1}{Z, \alpha'}$ to ensure that the commutator is controllable, and then bringing the $\frac{1}{Z, \alpha'}$ back inside:

$$\begin{aligned}
 (272) \quad & \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \\
 & \lesssim \left\| \left[\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\
 & \quad + \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\
 & \quad + \left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2}.
 \end{aligned}$$

We estimate the first two terms on the RHS of (272) using commutator estimate (318):

$$\begin{aligned}
 (273) \quad & \left\| \left[\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right)^3 \frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\
 & \quad + \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\
 & \lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}.
 \end{aligned}$$

We will postpone estimating $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}$ until the end of this series of calculations. For the moment, we take the last term from the RHS of (272):

$$\begin{aligned}
 (274) \quad & \left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \lesssim \\
 & \left\| (I - \mathbb{H}) D_{\alpha'}^2 \left(\frac{1}{Z_{,\alpha'}} A_1 \right) \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2}.
 \end{aligned}$$

We estimate the second term by

$$\begin{aligned}
 (275) \quad & \left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2} \\
 & \lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|A_1\|_{L^\infty} + \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \\
 & \lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} (1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2) + (255)(261).
 \end{aligned}$$

Finally, for the first term on the RHS of (274), we use (50) to replace $\frac{1}{Z_{,\alpha'}} A_1$ by $i(\bar{Z}_{tt} - i)$, and apply $(I - \mathbb{H}) D_{\alpha'}^2$ to this, and then use (189): $(I - \mathbb{H})(\bar{Z}_{tt} - Z_t D_{\alpha'} \bar{Z}_t) = 0$, (40) and principles no.1 and no.2 of §A.2. We get

$$\begin{aligned}
& \left\| (I - \mathbb{H}) D_{\alpha'}^2 \left(\frac{1}{Z_{,\alpha'}} A_1 \right) \right\|_{L^2} = \left\| (I - \mathbb{H}) D_{\alpha'}^2 (Z_t D_{\alpha'} \bar{Z}_t) \right\|_{L^2} \\
(276) \quad & \lesssim \left\| (I - \mathbb{H}) \left\{ (D_{\alpha'}^2 Z_t) D_{\alpha'} \bar{Z}_t \right\} \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ (D_{\alpha'} Z_t) (D_{\alpha'}^2 \bar{Z}_t) \right\} \right\|_{L^2} \\
& + \left\| (I - \mathbb{H}) \left\{ \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2}.
\end{aligned}$$

We estimate the first two terms directly by

$$\begin{aligned}
& \left\| (I - \mathbb{H}) \left\{ (D_{\alpha'}^2 Z_t) D_{\alpha'} \bar{Z}_t \right\} \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ (D_{\alpha'} Z_t) (D_{\alpha'}^2 \bar{Z}_t) \right\} \right\|_{L^2} \\
(277) \quad & \lesssim \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} (\|D_{\alpha'}^2 Z_t\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}) \\
& \lesssim (254) (\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + (257)).
\end{aligned}$$

We are left with the last term on the RHS of (276). We first decompose $\frac{Z_t}{Z_{,\alpha'}}$ into its holomorphic and antiholomorphic projections. The term with the holomorphic projection disappears by (310). With what remains, we use (308) to get a commutator, which we control by commutator estimate (322):

$$\begin{aligned}
& \left\| (I - \mathbb{H}) \left\{ \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2} \\
& = \left\| (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2} \\
(278) \quad & = \left\| \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \lesssim \left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \\
& \lesssim \left(\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \left(1 + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right) \right) \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}
\end{aligned}$$

by (134) and (254).

We now give the estimate for $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty} = \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_{L^\infty}$ in (273). We do so using (125). We have

$$\begin{aligned}
& \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \\
(279) \quad & = \Im \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \left(\frac{1}{|Z_{,\alpha'}|^2} - \frac{1}{|Z_{,\beta'}|^2} \right) \bar{Z}_{t,\beta'}(\beta') d\beta' \\
& + \Im \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \frac{1}{\bar{Z}_{,\beta'}} D_{\beta'} \bar{Z}_t(\beta') d\beta' \\
& = I + II.
\end{aligned}$$

From Hölder's inequality, Hardy's inequality (317) and the mean value theorem,¹⁶ we have

$$(280) \quad \|I\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \quad (255).$$

We rewrite II using integration-by-parts identity (319):

$$(281) \quad \begin{aligned} & \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \frac{1}{\overline{Z}_{,\beta'}} D_{\beta'} \overline{Z}_t(\beta') d\beta' \\ &= -[Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'} \overline{Z}_t \right) + \mathbb{H} \left(Z_{t,\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'} \overline{Z}_t \right) \\ &= -[Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'} \overline{Z}_t \right) - \left[\frac{1}{\overline{Z}_{,\alpha'}} \overline{D_{\alpha'} \overline{Z}_t}, \mathbb{H} \right] \overline{Z}_{t,\alpha'} + |D_{\alpha'} \overline{Z}_t|^2. \end{aligned}$$

Using (331) on the first two terms on the RHS of (281), we get

$$(282) \quad \begin{aligned} \|II\|_{L^\infty} &\lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'} \overline{Z}_t \right) \right\|_{L^2} + \|D_{\alpha'} \overline{Z}_t\|_{L^\infty}^2 \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2} \left(\|D_{\alpha'}^2 \overline{Z}_t\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2} \|D_{\alpha'} \overline{Z}_t\|_{L^\infty} \right) \\ &\quad + \|D_{\alpha'} \overline{Z}_t\|_{L^\infty}^2. \end{aligned}$$

Combining (279), (280), (282), (255), and (254), we have

$$(283) \quad \begin{aligned} \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_{L^\infty} &\leq \|I\|_{L^\infty} + \|II\|_{L^\infty} \\ &\leq C \left(\|\overline{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2} \right). \end{aligned}$$

We now sum up these estimates. From (268), and using (283) to estimate $\left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}$, we have

$$(284) \quad \begin{aligned} \left\| \frac{1}{\overline{Z}_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2} &\lesssim (269) + (271) + (273) + (275) + (277) + (278) \\ &\lesssim C \left(\left\| D_{\alpha'}^2 \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2}, \|\overline{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2} \right). \end{aligned}$$

¹⁶Note that $\frac{1}{|Z_{,\alpha'}|^2}$ is periodic.

Combining (264) and (284) we conclude that

$$(285) \quad \begin{aligned} & \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} \\ & \lesssim C \left(\left\| D_{\alpha'}^2 \frac{1}{Z, \alpha'} \right\|_{L^2}, \left\| \bar{Z}_{t, \alpha'} \right\|_{L^2}, \left\| D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2} \right). \end{aligned}$$

11.2. Singularities and the angle of the crest

In §11.1 at (250) and (251) we characterized our energy in terms of various L^2 and $\dot{H}^{1/2}$ norms of quantities in Riemann mapping variables (as well as a single quantity, $\frac{1}{Z, \alpha'}$, in L^∞). When there is a non-right angle ν at the corner, or when there is a singularity in the middle of the free surface, the Riemann mapping will have a singularity. In this section, we discuss what this suggests about the angle ν or the interior angle of an angled crest in the middle of the free surface, continuing the discussion from §3.3.2.

As in §3.3.2, we will move the corner at the wall to 0, and phrase our discussion in terms of a singularity at the corner, but it applies more broadly to singularities in the middle of the free surface.¹⁷ We thus henceforth focus on the angle ν at the corner.

We first observe that our energy is finite in the regime when the interface and velocity are smooth and the angle $\nu = \frac{\pi}{2}$, so we can focus on the case where $\nu < \frac{\pi}{2}$.¹⁸

If ν is the angle the water wave makes with the wall, the Riemann mapping $\Phi(z)$ should behave like z^r at the corner, where $r\nu = \frac{\pi}{2}$. For $\nu < \frac{\pi}{2}$, we have $r > 1$.

Recall from (250) and (251) that among the quantities that characterize the energy, $\left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2}$, $\left\| D_{\alpha'}^2 \frac{1}{Z, \alpha'} \right\|_{L^2}$ and $\left\| \frac{1}{Z, \alpha'} \right\|_{L^\infty}$ are the terms purely related to the surface. We will see through the following calculation what non-right angles ν are allowed if $\left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_{L^2}$, $\left\| D_{\alpha'}^2 \frac{1}{Z, \alpha'} \right\|_{L^2}$ and $\left\| \frac{1}{Z, \alpha'} \right\|_{L^\infty}$ are finite.

Note that $Z(\alpha') = \Phi^{-1}(\alpha') \approx (\alpha')^{1/r}$ so $Z, \alpha' = \partial_{\alpha'}(\Phi^{-1}) \approx (\alpha')^{1/r-1}$, and

$$(286) \quad \frac{1}{Z, \alpha'} \approx (\alpha')^{1-1/r}, \quad \partial_{\alpha'} \frac{1}{Z, \alpha'} \approx (\alpha')^{-1/r} \quad (r \neq 1).$$

¹⁷The angled crests in the middle of the free surface don't have to be symmetric.

¹⁸Recall from the discussion in §3.3.2 that we cannot have $\nu > \frac{\pi}{2}$ in our energy regime.

Therefore, assuming $r > 1$, $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \in L^2$ if and only if $r > 2$ if and only if $\nu < \frac{\pi}{4}$. Similarly, $D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \in L^2$ so long as $r > 2$.¹⁹

We conclude from this discussion that our energy will be finite only when $\nu < \frac{\pi}{4}$ (or when $\nu = \frac{\pi}{2}$). This coincides precisely with the angles in the self-similar solutions of [37].²⁰ For singularities in the middle of the free surface, this suggests that the interior angle must be less than $\frac{\pi}{2}$.²¹

Now let's consider the behavior of the angle ν over time. This angle is determined by $\Im \ln z_\alpha$ at the corner. Therefore, the behavior of $D_\alpha z_t = \partial_t (\ln z_\alpha)$ at the corner should determine how the angle changes. Since $Z_{t,\alpha'} = (D_\alpha z_t) \circ h^{-1} Z_{,\alpha'}$ and $Z_{,\alpha'} \approx (\alpha')^{1/r-1}$, we must have $D_\alpha z_t \rightarrow 0$ at the corner if $Z_{t,\alpha'} \in L^2$, as our energy assumes. This suggests that if initially $\nu < \frac{\pi}{4}$, the angle would not change while the energy remained finite. This holds true also for interior angles at the angled crests.

Appendix A. Holomorphicity and mean

A.1. The Hilbert transform \mathbb{H}

Recall that in §3 we introduced the Hilbert transform \mathbb{H} associated to the periodic domain P^- :

$$(287) \quad \mathbb{H}f(\alpha') := \frac{1}{2i} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta', \quad \text{for } \alpha' \in [-1, 1].$$

We know from Proposition 1 that a function $g \in L^p$ is the boundary value of a periodic holomorphic function on P^- if and only if $(I - \mathbb{H})g = f_I g$, and that for any function $f \in L^p$, $(I + \mathbb{H})f$ is the boundary value of a periodic holomorphic function on P^- , with $(I - \mathbb{H})(I + \mathbb{H})f = f f$. Recall also that we defined at (41) the holomorphic and antiholomorphic projection operators

$$(288) \quad \mathbb{P}_A f := \frac{(I - \mathbb{H})}{2} f; \quad \mathbb{P}_H f := \frac{(I + \mathbb{H})}{2} f.$$

¹⁹We remark that, even though E_a (which roughly includes $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$) is higher-order than E_b (which roughly includes $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$) in terms of the number of derivatives, the two energies are comparable in the sense that they allow precisely the same angles.

²⁰We recall that our energy is finite for these solutions.

²¹We note that our energy does not apply to Stokes waves of maximum height (interior angle = $\frac{2\pi}{3}$).

Here we gather some basic properties of the Hilbert transform \mathbb{H} that will be used in this paper:

Proposition 4. *a. Let $1 < p < \infty$. Then there exists $C_p < \infty$ such that for all $f \in L^p$*

$$(289) \quad \|\mathbb{H}f\|_{L^p} \leq C_p \|f\|_{L^p}.$$

b. Let $f \in L^p$ for some $p > 1$. Then

$$(290) \quad \mathbb{H}^2 f = f - \int f; \quad \mathbb{P}_A \mathbb{P}_H f = \mathbb{P}_H \mathbb{P}_A f = \frac{1}{4} \int f.$$

c. Let $f \in L^p, g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty$. Then

$$(291) \quad \int f(\mathbb{H}g) = - \int (\mathbb{H}f)g; \quad \int (\mathbb{P}_A f)g = \int f(\mathbb{P}_H g).$$

d. Let $f \in L^p, g \in L^q$ for $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} < 1$. Then

$$(292) \quad \mathbb{P}_A \{(\mathbb{P}_H f)(\mathbb{P}_H g)\} = \frac{1}{8} \left(\int f \right) \left(\int g \right).$$

Parts a., b., and c. are classical results. We know the product of periodic holomorphic functions is periodic holomorphic. Part d. is an easy consequence of Proposition 1, parts b. and c., and the fact that $(\mathbb{P}_H f)(\mathbb{P}_H g)$ is the boundary value of a periodic holomorphic function on P^- .

We will also need the following proposition.

Proposition 5. *Let $f \in L^\infty, g \in L^p$ for some $p > 1$. Suppose $(I - \mathbb{H})f = \int f$. Then*

$$(293) \quad [f, \mathbb{H}]g = \frac{1}{2}(I + \mathbb{H})([f, \mathbb{H}]g) - \frac{1}{2} \int (fg) + \frac{1}{2} \left(\int f \right) \left(\int g \right).$$

Proof. We begin by observing that

$$(294) \quad [f, \mathbb{H}]g = f(I + \mathbb{H})g - (I + \mathbb{H})(fg).$$

Because $(I - \mathbb{H})f = \int f$, by Proposition 1, $[f, \mathbb{H}]g$ is the boundary value of

a periodic holomorphic function on P^- . Therefore,

$$\begin{aligned}
 (295) \quad (I - \mathbb{H})([f, \mathbb{H}]g) &= \int [f, \mathbb{H}]g = \int f(I + \mathbb{H})g - \int fg \\
 &= \int ((I - \mathbb{H})f)g - \int (fg) = \left(\int f \right) \left(\int g \right) - \int (fg)
 \end{aligned}$$

by (291). □

A.2. Periodic holomorphic functions

In this section, we note which of the functions we are dealing with are the boundary values of periodic holomorphic functions—and which, further, have mean zero. From Proposition 1 we know that, to show that $(I - \mathbb{H})$ of various functions disappears, assuming they are in L^p , $p \geq 1$, it suffices to show that they are boundary values of periodic holomorphic functions and that their means are zero.

We start with the following basic facts. Assume that the quantities involved are sufficiently smooth, and that the assumption (34) holds.

First, we have that the complex conjugate velocity is periodic holomorphic, and goes to zero as $y \rightarrow -\infty$ by (9), so

$$(296) \quad (I - \mathbb{H})\bar{Z}_t = 0.$$

Then we have three identities about the Riemann mapping. Recall that

$$(297) \quad Z_{,\alpha'} = \partial_{\alpha'} \Phi^{-1}(\alpha', t); \quad \frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z.$$

Both of these are clearly periodic holomorphic. Therefore by (27) we have

$$(298) \quad (I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = 1, \quad (I - \mathbb{H})Z_{,\alpha'} = 1.$$

The mean $\int Z_{,\alpha'} = 1$ can also be checked directly by the fundamental theorem of calculus, since $Z(1, t) = 1, Z(-1, t) = -1$ for all time.

Finally, we have

$$(299) \quad (I - \mathbb{H})\{\Phi_t \circ Z\} = 0$$

by (34). Here, we have $\Phi_t \circ \Phi^{-1}$ is holomorphic because it is the limit of holomorphic functions, and we know that Φ_t is periodic by the periodicity of our domain $\Omega(t)$.

From these facts, we will be able to deduce everything else that we need in the derivation of the free surface equations, from the following principles:

1. Assume that $f \in C^0(S^1)$ with $\partial_{\alpha'} f \in L^p$, $p \geq 1$. If $(I - \mathbb{H})f = c$, then $(I - \mathbb{H})\partial_{\alpha'} f = 0$.

This is straightforward by taking derivatives. One may also use the correspondence between periodic holomorphic functions on P_- and holomorphic functions on the unit disc, and take a derivative to conclude.

2. Assume that $f \in L^p$, $g \in L^q$ and $fg \in L^r$ with $p, q, r \geq 1$. If $(I - \mathbb{H})f = 0$ and $(I - \mathbb{H})g = c$ then $(I - \mathbb{H})(fg) = 0$.

This is because the product of periodic holomorphic functions is periodic holomorphic. If one of the factors goes to 0 as $y \rightarrow -\infty$, then the product goes to 0 as $y \rightarrow -\infty$.

3. Assume $G(z, t)$ is a periodic holomorphic function on $\Omega(t)$ going to zero as $y \rightarrow -\infty$, G is continuous differentiable with respect to t , and $G_t \circ Z \in L^q$, $G \circ Z \in C^0(S^1)$, $\partial_{\alpha'}(G \circ Z) \in L^p$, $q, p \geq 1$, so $(I - \mathbb{H})(G \circ Z) = 0$. Then $(I - \mathbb{H})G_t \circ Z = 0$.²²

It is clear that $G_t \circ Z$ is periodic holomorphic, since it is the limit of periodic holomorphic functions. It remains to show that $\int G_t \circ Z = 0$. Note that $\int G \circ Z = 0$ for all time. Also, since $\Phi(\Phi^{-1}(\alpha', t), t) = \alpha'$, we have $(\Phi_z \circ Z) \cdot (\Phi^{-1})_t + \Phi_t \circ Z = 0$, and therefore

$$(300) \quad (\Phi^{-1})_t = (-Z_{,\alpha'})\Phi_t \circ Z.$$

Therefore, using (300) and the fact $Z = \Phi^{-1}$, we have

$$(301) \quad \begin{aligned} 0 &= \frac{d}{dt} \int G(Z(\alpha', t), t) = \int G_t(Z(\alpha', t), t) + \int G_z \circ Z (\Phi^{-1})_t(\alpha', t) \\ &= \int G_t \circ Z - \int (\partial_{\alpha'}(G \circ Z))\Phi_t \circ \Phi^{-1}. \end{aligned}$$

The second integral is zero by principles no.1 and no.2 above.

Let $F = \bar{v}$ be the complex conjugate velocity. Note that $F_t \circ Z = \bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t$ as shown in §3.3.3. As immediate consequences of the basic facts and the three principles above, we have the following statements:

$$(302) \quad (I - \mathbb{H}) \{ (Z_{,\alpha'})^j (F_t \circ Z) \} = 0, \quad \text{for } j = 0, 1$$

$$(303) \quad (I - \mathbb{H}) \{ (Z_{,\alpha'})^j (F_{tt} \circ Z) \} = 0, \quad \text{for } j = 0, 1$$

²²Note that this argument does not apply to $\Phi_t \circ Z$ itself, because Φ is not periodic.

$$(304) \quad (I - \mathbb{H})(\partial_{\alpha'}^j D_{\alpha'}^k \bar{Z}_t) = 0, \quad \text{for } j, k = 0, 1$$

$$(305) \quad (I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) = 0.$$

A.2.1. Some identities used in the proof of Theorem 2. In this and the following subsection §A.2.2, we assume only that the assumption in Theorem 2 and its consequence (85) hold. From (39) and (40):

$$(I - \mathbb{H})\bar{Z}_t = 0, \quad (I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = 1,$$

and the above principles no.1 and no.2, we have

$$(306) \quad (I - \mathbb{H})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = 0$$

$$(307) \quad (I - \mathbb{H})D_{\alpha'}^k \bar{Z}_t = 0, \quad \text{for } 0 \leq k \leq 3$$

$$(308) \quad (I - \mathbb{H})\partial_{\alpha'} D_{\alpha'}^k \bar{Z}_t = 0, \quad \text{for } 0 \leq k \leq 2$$

$$(309) \quad (I - \mathbb{H})\left\{ \frac{1}{Z_{,\alpha'}} D_{\alpha'}^k \bar{Z}_t \right\} = 0, \quad \text{for } 0 \leq k \leq 2$$

$$(310) \quad (I - \mathbb{H})\left((\partial_{\alpha'}^j D_{\alpha'}^2 \bar{Z}_t) \partial_{\alpha'}^l \left(\left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right)^k \right) \right) = 0, \\ \text{for } j = 1, 2, k = 1, 2, l = 0, 1.$$

By (189), $(I - \mathbb{H})(\bar{Z}_{tt} - Z_t D_{\alpha'} \bar{Z}_t) = 0$. This then gives, by an application of principles no.1 and no.2 above,

$$(311) \quad (I - \mathbb{H})D_{\alpha'}^k(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) = 0, \quad \text{for } 0 \leq k \leq 3.$$

A.2.2. Mean conditions. We have implicitly in the preceding section shown that various quantities are mean-zero, but we don't use that fact other than in those identities. We will at one point require an explicit mean-zero condition, to use a variant of the Sobolev inequality. This is that

$$(312) \quad \int (D_{\alpha'} \bar{Z}_t)^2 d\alpha' = 0.$$

We note that because $(I - \mathbb{H})D_{\alpha'} \bar{Z}_t = 0$ (307), $D_{\alpha'} \bar{Z}_t$ is the boundary value of a periodic holomorphic function going to 0 as $y \rightarrow -\infty$, so its square will also be the boundary value of a periodic holomorphic function going to 0 as $y \rightarrow -\infty$.

Appendix B. Useful inequalities and identities

We present here assorted inequalities and identities that we need in our paper. None of the results here are original, so we omit the proofs in most cases. We refer the reader to [23] for details of the proofs.

B.1. Sobolev inequalities

We present here the one-dimensional Sobolev inequality we use in our proof.

Proposition 6 (Weighted Sobolev Inequality with ε). *Let $\varepsilon > 0$. Then for all $f \in C^1(-1, 1) \cap L^2(-1, 1)$,*

$$(313) \quad \|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{\cdot}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)} + \|f\|_{L^2}$$

for any weight $\omega \geq 0$.

Furthermore

$$(314) \quad \int f^2 = 0 \Rightarrow \|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{\cdot}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)}.$$

We omit the proof since it is fairly standard.

B.2. Derivatives and complex-valued functions

Because our functions will be complex-valued, and we will often be looking at derivatives of angular and modular parts of these functions, we note here a few elementary facts about such functions.

Let $f(\alpha) = r(\alpha)e^{i\theta(\alpha)}$, where r, θ are real-valued functions. Then

$$(315) \quad |\partial_\alpha |f|| \leq |f'|; \quad \left| \partial_\alpha \frac{f}{|f|} \right| \leq \left| \frac{f'}{|f|} \right|.$$

From $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$, we have $\mathbf{a} \geq 0$ and

$$(316) \quad \frac{z_{tt} + i}{|z_{tt} + i|} = i \frac{z_\alpha}{|z_\alpha|}.$$

We will use this fact in this paper to replace the angular part of the spatial derivative z_α with the angular part of the time derivative $z_{tt} + i$.

B.3. Hardy’s inequality and commutator estimates

We present here the basic estimates we will rely on for this paper. Several of these estimates control quantities of the form $[f, \mathbb{H}]g'$ by something involving f' and g ; they thus reduce the amount of regularity required on g , at the expense of further regularity on f .

For many of these estimates, we must pay close attention to the boundary conditions. Recall that $f \in C^0(S^1)$ implies that $f|_{\partial} = 0$. Many of these estimates do not hold if this periodic boundary condition is removed. To save space, we have not always explicitly cited these boundary conditions when we quote these estimates, but they are always met, by our assumption of Theorem 2 and its consequence (85).

Proposition 7 (Hardy’s Inequality). *Let $f \in C^0(S^1) \cap C^1(-1, 1)$, with $f' \in L^2$. Then there exists $C > 0$ independent of f such that for any $\alpha' \in I$,*

$$(317) \quad \left| \int_I \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\beta' \right| \leq C \|f'\|_{L^2}^2.$$

Proposition 8 ($L^2 \times L^\infty$ Estimate). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in C^0[-1, 1]$ with $g' \in L^p$ for some $p > 1$ ²³ (so $f|_{\partial} = 0$, though possibly $g|_{\partial} \neq 0$),*

$$(318) \quad \|[f, \mathbb{H}]\partial_{\alpha'}g\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{L^\infty}.$$

Proof. This result is the periodic modification of a result from [35], which in turn is a consequence of the $T(b)$ theorem [15].

We begin by integrating by parts,

$$(319) \quad \begin{aligned} \partial_{\alpha'}g &= \mathbb{H}(f'g) - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta \\ &\quad + \frac{1}{2i} (f(\alpha') - f(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta')) g(\beta') \Big|_{\partial}, \end{aligned}$$

where we have a boundary term because we didn’t place periodic boundary assumption on g . We control the first term by the L^2 boundedness of \mathbb{H} and Hölder. We control the last term by Hardy’s inequality (317).

We handle the second term by using the identity

$$(320) \quad \left(\frac{\pi}{2}\right)^2 \frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = \sum_{l \in \mathbb{Z}} \frac{1}{(\alpha' + 2l)^2}$$

²³We require this only to ensure that $[f, \mathbb{H}]g'$ is well-defined.

and reducing it to the real line version, Proposition 3.2 in [35]. For details, see [23]. \square

Proposition 9 ($L^2 \times L^\infty$ Estimate Variant). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in L^\infty[-1, 1]$ (and so $f|_\partial = 0$ but possibly $g|_\partial \neq 0$),*

$$(321) \quad \left\| \int \frac{(f(\alpha') - f(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \right\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{L^\infty}.$$

This is just the second term on the RHS of (319).

Proposition 10 ($L^\infty \times L^2$ Estimate [8]). *There exists a constant $C > 0$ such that for any $f \in C^1[-1, 1] \cap C^0(S^1)$, $g \in C^0(S^1) \cap C^1(-1, 1)$ with $g' \in L^p$ for some $p > 1$ ²⁴,*

$$(322) \quad \|[f, \mathbb{H}] \partial_{\alpha'} g\|_{L^2} \leq C \|f'\|_{L^\infty} \|g\|_{L^2}.$$

To prove Proposition 10, we begin with the integration-by-part formula (319), noting that the third term, the boundary term, is zero since $f|_\partial = g|_\partial = 0$. We handle the first term by the L^2 boundedness of \mathbb{H} and Hölder, and the second term by identity (320), reducing it to the classical result on \mathbb{R} by [8].²⁵

Proposition 11. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$ and $g' \in L^p$ for some $p > 1$,*

$$(323) \quad \|[f, \mathbb{H}] \partial_{\alpha'} g\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{\dot{H}^{1/2}}.$$

Proof. We integrate by parts, first rewriting $\partial_{\beta'} g(\beta') = \partial_{\beta'}(g(\beta') - g(\alpha'))$:

$$(324) \quad \begin{aligned} \partial_{\alpha'} g &= \frac{1}{2i} \int f_{\beta'}(\beta') \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) (g(\beta') - g(\alpha')) d\beta' \\ &\quad - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (g(\beta') - g(\alpha')) d\beta', \end{aligned}$$

where there is no boundary term because of the periodic boundary conditions.

²⁴We assume $g' \in L^p$ only to ensure $[f, \mathbb{H}]g'$ is well-defined.

²⁵The result was later extended by [11].

For the first term, we apply the Cauchy-Schwarz inequality:

$$(325) \quad \left| \int f'(\beta') \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right)(g(\beta') - g(\alpha'))d\beta' \right| \leq \|f'\|_{L^2} \left(\int |g(\alpha') - g(\beta')|^2 \left| \cot^2\left(\frac{\pi}{2}(\alpha' - \beta')\right) \right| d\beta' \right)^{1/2}.$$

Taking L^2 of this in α' and using the boundedness of cosine to replace \cot^2 with $\frac{1}{\sin^2}$, we get the desired estimate.

For the second term, we use the Cauchy-Schwarz inequality:

$$(326) \quad \left\| \int \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)}(g(\beta') - g(\alpha'))d\beta' \right\|_{L^2_{\alpha'}} \leq \left(\int \int \frac{|f(\alpha') - f(\beta')|^2}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} d\beta' \int \frac{|g(\alpha') - g(\beta')|^2}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} d\beta' d\alpha' \right)^{1/2},$$

and then we use Hardy's inequality (317) to f to get our inequality (323). \square

Proposition 12. *There exists a constant $C > 0$ such that for any $f \in \dot{H}^{1/2}, g \in L^2$,*

$$(327) \quad \|[f, \mathbb{H}]g\|_{L^2} \leq C \|f\|_{\dot{H}^{1/2}} \|g\|_{L^2}.$$

Proof. This is immediate by the Cauchy-Schwarz inequality and the boundedness of cosine. \square

Proposition 13. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f', g' \in L^2$ and $h \in L^2$,*

$$(328) \quad \|[f, g; h]\|_{L^2} := \left\| \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \frac{g(\alpha') - g(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} h(\beta') d\beta' \right\|_{L^2} \leq C \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}.$$

Proof. By the Cauchy-Schwarz inequality,

$$(329) \quad \left| \int \frac{f(\alpha') - f(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \frac{g(\alpha') - g(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} h(\beta') d\beta' \right| \leq \left(\int \left| \frac{f(\alpha') - f(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \right|^2 d\beta' \right)^{1/2} \left(\int \left| \frac{g(\alpha') - g(\beta')}{\sin\left(\frac{\pi}{2}(\alpha' - \beta')\right)} h(\beta') \right|^2 d\beta' \right)^{1/2}.$$

Now we take the L^2 of this in the α' variable. By Hardy's inequality (317), we control the f factor by $\|f'\|_{L^2}$, and are left with

$$(330) \quad \left(\int \int \left| \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') \right|^2 d\beta' d\alpha' \right)^{1/2}.$$

Applying Fubini's Theorem and then using Hardy's inequality (317) once more gives the result. \square

Proposition 14. *There exists a constant $C > 0$ such that for any $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$, $g \in L^2$,*

$$(331) \quad \|[f, \mathbb{H}]g\|_{L^\infty} \leq C \|f'\|_{L^2} \|g\|_{L^2}.$$

Proof. Estimate (331) holds by the Cauchy-Schwarz inequality and Hardy's inequality (317). \square

Proposition 15. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f', g' \in L^2$, and $h \in L^2$,*

$$(332) \quad \|\partial_{\alpha'}[f, [g, \mathbb{H}]]h\|_{L^2} \leq C \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}.$$

Proof. We differentiate:

$$(333) \quad \begin{aligned} & \partial_{\alpha'} \frac{1}{2i} \int (f(\alpha') - f(\beta'))(g(\alpha') - g(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) h(\beta') d\beta' \\ &= f'[g, \mathbb{H}]h + g'[f, \mathbb{H}]h - \frac{1}{2i} \int \frac{\pi (f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta'. \end{aligned}$$

We control the first two terms by Hölder's inequality and then (331). We control the last term by (328). \square

Proposition 16 (Higher-Order Calderon Commutator [11]). *There exists a constant $C > 0$ such that for any $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^\infty$, $h \in C^0(S^1) \cap C^1(-1, 1)$ with $h' \in L^p$ for some $p > 1$,*

$$(334) \quad \|[f, f; \partial_{\alpha'} h]\|_{L^2} \leq C \|f'\|_{L^\infty}^2 \|h\|_{L^2}.$$

Proof. The proof is entirely analogous to the proof of (322), and now follows from the work of [11], which extends the original result of [8] used for (322) to allow two difference quotient factors, instead of one. To move from \mathbb{R} to our compact domain, we do the same infinite summation argument, with

the only difference being that instead of (320), we use $(\frac{\pi}{2})^2 \partial_{\alpha'} \frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = -2 \sum \frac{1}{(\alpha'+2l)^3}$. \square

B.4. The $\dot{H}^{1/2}$ norm

We present here the following proposition.

Proposition 17. *Let $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$. Then*

$$(335) \quad (I - \mathbb{H})f = \rlap{-}\int f \Rightarrow \|f\|_{\dot{H}^{1/2}}^2 = \int i(\partial_{\alpha'} f) \bar{f} d\alpha'.$$

(335) holds because for f satisfying the assumption of Proposition 17,

$$(336) \quad \int i(\partial_{\alpha'} f) \bar{f} d\alpha' = \int i(\partial_{\alpha'} \mathbb{H}f) \bar{f} d\alpha',$$

and $i \partial_{\alpha'} \mathbb{H}f = |D|f$, where $|D|$ is the positive operator satisfying $|D|^2 = -\partial_{\alpha'}^2$. It is easy to see that $\int i(\partial_{\alpha'} f) \bar{f} d\alpha'$ is real-valued by integration by parts.

B.5. Commutator identities

We include here for reference the various commutator identities that are necessary.

$$(337) \quad [\partial_t, D_\alpha] = -(D_\alpha z_t) D_\alpha;$$

$$(338) \quad [\partial_t, D_\alpha^2] = [\partial_t, D_\alpha] D_\alpha + D_\alpha [\partial_t, D_\alpha] = -2(D_\alpha z_t) D_\alpha^2 - (D_\alpha^2 z_t) D_\alpha;$$

$$(339) \quad \begin{aligned} [\partial_t^2, D_\alpha] &= \partial_t [\partial_t, D_\alpha] + [\partial_t, D_\alpha] \partial_t \\ &= (-D_\alpha z_{tt}) D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t) D_\alpha \partial_t. \end{aligned}$$

To calculate $[i\mathbf{a}\partial_\alpha, D_\alpha]$, we use $i\mathbf{a}z_\alpha = z_{tt} + i$ (21) to rewrite $i\mathbf{a}\partial_\alpha = i\mathbf{a}z_\alpha D_\alpha = (z_{tt} + i)D_\alpha$. Therefore

$$(340) \quad [i\mathbf{a}\partial_\alpha, D_\alpha] = [(z_{tt} + i)D_\alpha, D_\alpha] = -(D_\alpha z_{tt}) D_\alpha.$$

Adding (339) and (340), we conclude that

$$(341) \quad [\partial_t^2 + i\mathbf{a}\partial_\alpha, D_\alpha] = (-2D_\alpha z_{tt}) D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t) D_\alpha \partial_t.$$

Because $[(\partial_t^2 + ia\partial_\alpha), D_\alpha^2] = [(\partial_t^2 + ia\partial_\alpha), D_\alpha]D_\alpha + D_\alpha[(\partial_t^2 + ia\partial_\alpha), D_\alpha]$, we have

$$(342) \quad \begin{aligned} & [(\partial_t^2 + ia\partial_\alpha), D_\alpha^2] \\ &= (-4D_\alpha z_{tt})D_\alpha^2 + 4(D_\alpha z_t)^2 D_\alpha^2 - 2(D_\alpha z_t)D_\alpha \partial_t D_\alpha - (2D_\alpha^2 z_{tt})D_\alpha \\ & \quad + 4(D_\alpha z_t)(D_\alpha^2 z_t)D_\alpha - 2(D_\alpha^2 z_t)D_\alpha \partial_t - 2(D_\alpha z_t)D_\alpha^2 \partial_t. \end{aligned}$$

Appendix C. Summary of notation

We list here the various notations we've introduced in the paper. See also §1.3 for a discussion of the conventions used.

- $f|_\partial := f(1, t) - f(-1, t)$.
- ν is the angle the water wave makes with the wall. See Figure 2.
- $I := [-1, 1]$ (except when it's used for the identity or as an abbreviation for a quantity to be controlled).
- $\Re z, \Im z$ are the real and imaginary parts of a complex number z .
- Function spaces $C^k(-1, 1), C^k[-1, 1], C^k(S^1), H^k(S^1), L^p$, and $\dot{H}^{1/2}$ are defined in §1.3. We define $\|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta'\right)^{\frac{1}{2}}$.
- $f f = f_I f := \frac{1}{2} \int_I f(\beta') d\beta'$.
- $z(\alpha, t)$ is the Lagrangian parametrization, and $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity. $z_{tt} = \partial_t z_t$, etc.
- $\mathbf{a} = \frac{|\bar{z}_{tt} - i|}{|z_\alpha|} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|}$. We refer to $-\frac{\partial P}{\partial \mathbf{n}}$ as the Taylor coefficient; \mathbf{n} is the outward-facing normal to $\Sigma(t)$.
- $h : \alpha \mapsto \alpha'$ is defined by $h(\alpha) = \Phi(z(\alpha, t), t)$ and gives the Riemann mapping variables, where Φ is the Riemann mapping defined in §3.1. $\mathfrak{b} := h_t \circ h^{-1}$. $\partial_{\alpha'}(f \circ h^{-1}) = \frac{\partial_\alpha f}{h_\alpha} \circ h^{-1}$. $d\alpha' = h_\alpha d\alpha$. Full details are in §3.1 and §3.3.
- α and β are our variables in Lagrangian coordinates; α' and β' are our variables in Riemann mapping coordinates.
- \mathbb{H} is the Hilbert transform in Riemann mapping variables, defined by

$$(343) \quad \mathbb{H}f(\alpha') := \frac{1}{2i} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta'.$$

- We define $\mathbb{P}_A := \frac{(I - \mathbb{H})}{2}$ and $\mathbb{P}_H := \frac{(I + \mathbb{H})}{2}$ as the antiholomorphic and holomorphic projections.
- $[f, g; h](\alpha') := \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta'$, the higher order Calderon commutator.

- We use $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$ at several points (and do not use F for any other purpose).
- $Z := z \circ h^{-1}$, $Z_t := z_t \circ h^{-1}$, $Z_{tt} := z_{tt} \circ h^{-1}$, $Z_{,\alpha'} = \partial_{\alpha'}(z \circ h^{-1})$, $Z_{t,\alpha'} = \partial_{\alpha'}(z_t \circ h^{-1})$, etc.
- Compositions and inverses are always with respect to the spatial variable.
- $\mathcal{A} := (\mathbf{a}h_\alpha) \circ h^{-1}$, $\mathcal{A}_t := (\mathbf{a}_t h_\alpha) \circ h^{-1}$.
- $D_\alpha := \frac{1}{z_\alpha} \partial_\alpha$, $|D_\alpha| := \frac{1}{|z_\alpha|} \partial_\alpha$, $D_{\alpha'} := \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$, $|D_{\alpha'}| := \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$ and $\overline{D_{\alpha'}} := \frac{1}{\overline{Z_{,\alpha'}}} \partial_{\alpha'}$.
- $A_1 = \mathcal{A} |Z_{,\alpha'}|^2 = i Z_{,\alpha'} (\overline{Z}_{tt} - i) \in \mathbb{R}$ (48). On changing variables, we have

$$(344) \quad A_1 \circ h = \frac{\mathbf{a} |z_\alpha|^2}{h_\alpha},$$

originally derived at (52); we use this repeatedly without citation. We often use $A_1 \geq 1$ (49), and also use $\frac{1}{Z_{,\alpha'}} = i \frac{\overline{Z}_{tt} - i}{A_1}$ (50).

- We define our energies in §4.1. We define generic energies $E_{a,\theta}$ and $E_{b,\theta}$, and then specialize to $E_a := E_{a,D_\alpha^2 \bar{z}_t}$ and $E_b := E_{b,D_\alpha \bar{z}_t}$. We use G_θ to describe the RHS of the equation $(\partial_t^2 + i\mathbf{a}\partial_\alpha)\theta = G_\theta$. For $\theta = D_\alpha^k \bar{z}_t$, $G_\theta = D_\alpha^k (-i\mathbf{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathbf{a}\partial_\alpha, D_\alpha^k] \bar{z}_t$.
- $\Theta := \theta \circ h^{-1}$; $B := \left(\frac{h_t \alpha}{h_\alpha} - \Re D_{\alpha'} z_t\right) \circ h^{-1}$; $\psi := \left(\frac{h_\alpha \theta}{z_\alpha}\right) \circ h^{-1}$ (168).
- See §1.3 for a discussion of how broadly I, II, I_1, I_{12} , etc., are defined. In short, they are unambiguous within each section, but ambiguous between sections.
- We use $C(E)$ to represent a polynomial of the energy E .

Appendix D. Main quantities controlled

We list here the various quantities that are controlled by our energy, for ease of reference. We don't list every single quantity we have controlled, but we do include any quantities that we give at the end of a concluding inequality without further explanation.

- In §5.1, we controlled

$$\begin{aligned} & \|(\partial_t + \mathbf{b}\partial_{\alpha'}) D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \\ & \|D_{\alpha'}^2 Z_t\|_{L^2}, \left\| \frac{1}{\overline{Z_{,\alpha'}}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)}, \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}, \end{aligned}$$

$$\|D_{\alpha'} Z_{tt}\|_{L^\infty}, \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}, \|D_{\alpha'} Z_t\|_{L^\infty}, \|\bar{Z}_{tt, \alpha'}\|_{L^2}, \|\bar{Z}_{t, \alpha'}\|_{L^2},$$

$$\int |D_{\alpha'} \bar{z}_t|^2 \frac{d\alpha}{\mathfrak{a}}, \int |D_{\alpha'} \bar{z}_{tt}|^2 \frac{d\alpha}{\mathfrak{a}}, \left\| \frac{1}{Z_{, \alpha'}} \right\|_{L^\infty}, \|Z_{tt} + i\|_{L^\infty}, \|A_1\|_{L^\infty}.$$

- $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} = \left\| \frac{A_t}{\mathcal{A}} \right\|_{L^\infty}$ is controlled at (123) in §5.2.
- $\left\| \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^2}$ is controlled at (130) in §5.3.
- $\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}$ is controlled at (136), $\|(I + \mathbb{H})D_{\alpha'} Z_t\|_{L^\infty}$ is controlled at (132) in §5.4.
- $\left\| D_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^\infty}$ is controlled at (148) in §5.6. $\left\| (Z_{tt} + i) \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right\|_{L^\infty}$ and $\left\| \partial_{\alpha'} \frac{Z_{tt} + i}{Z_{, \alpha'}} \right\|_{L^\infty}$ are also estimated there.
- $\left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{, \alpha'}} \right\|_{L^\infty}$ is controlled at (134) in §5.4. We estimated at (133) the related term $\left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{, \alpha'}} \right) \right\|_{L^\infty}$.
- $\|D_{\alpha'} B\|_{L^2}$ is controlled at (186).

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