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Wellposedness of the 2D full water wave equation in a regime that allows for non-C^1 interfaces

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Abstract We consider the two dimensional gravity water wave equation in a regime where the free interface is allowed to be non-C^1. In this regime, only a degenerate Taylor inequality $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds, with degeneracy at the singularities. In Kinsey and Wu (Camb J Math 6(2):93–181, 2018) an energy functional $\mathcal{E}(t)$ was constructed and an a-priori estimate was proved. The energy functional $\mathcal{E}(t)$ is not only finite for interfaces and velocities in Sobolev spaces, but also finite for a class of non-C^1 interfaces with angled crests. In this paper we prove the existence, uniqueness and stability of the solution of the 2d gravity water wave equation in the class where $\mathcal{E}(t) < \infty$, locally in time, for any given data satisfying $\mathcal{E}(0) < \infty$.

1 Introduction

A class of water wave problems concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in

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the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{k} - \nabla P & \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 0, & \text{on } \Omega(t),\ t \geq 0, \\ P = 0, & \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases} \tag{1.1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. There is an important condition for these problems:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0 \tag{1.2}$$

pointwise on the interface, where $\mathbf{n}$ is the outward unit normal to the fluid interface $\Sigma(t)$ [35]; it is well known that when surface tension is neglected and the Taylor sign condition (1.2) fails, the water wave motion can be subject to the Taylor instability [7,9,17,35].

The study on water waves dates back centuries. Early mathematical works include Newton [31], Stokes [34], Levi-Civita [26], and Taylor [35]. Nalimov [28], Yosihara [45] and Craig [15] proved local in time existence and uniqueness of solutions for the 2d water wave equation (1.1) for small and smooth initial data. In [38,39], we showed that for dimensions $n \geq 2$, the strong Taylor sign condition

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{1.3}$$

always holds for the infinite depth water wave problem (1.1), as long as the interface is in $C^{1+\epsilon}$, $\epsilon > 0$; and the initial value problem of Eq. (1.1) is locally well-posed in Sobolev spaces H^s, $s \geq 4$ for arbitrary given data. Since then, local wellposedness for water waves with additional effects such as the surface tension, bottom and non-zero vorticity, under the assumption (1.3),[1] were obtained, c.f. [6,11,12,21,25,27,29,33,46]. Alazard et al. [3,4] proved local wellposedness of (1.1) in low regularity Sobolev spaces where the interfaces are only in $C^{3/2}$. Hunter, Ifrim and Tararu [19] obtained a low regularity result for the 2d water waves that improves on [3]. The author [40,41], Germain et al. [18], Ionescu and Pusateri [22] and Alazard and Delort [5] obtained almost global and global existence for two and three dimensional water wave equation

[1] When there is surface tension, or bottom, or vorticity, (1.3) does not always hold, it needs to be assumed.

(1.1) for small, smooth and localized data; see [8,16,19,20,36,37] for some additional developments. Furthermore in [10], Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano proved that for the 2d water wave equation (1.1), there exist initially non-self-intersecting interfaces that become self-intersecting at a later time; and as was shown in [13], the same result holds in 3d.

All these work either prove or assume the strong Taylor sign condition (1.3), and the lowest regularity considered are $C^{3/2}$ interfaces.

A common phenomena we observe in the ocean are waves with angled crests, with the interface possibly non-C^1. A natural question is: is the water wave equation (1.1) well-posed in any class that includes non-C^1 interfaces?

We focus on the two dimensional case in this paper.

As was explained in [24], the main difficulty in allowing for non-C^1 interfaces with angled crests is that in this case, both the quantity $-\frac{\partial P}{\partial \mathbf{n}}$ and the Dirichlet-to-Neumann operator $\nabla_{\mathbf{n}}$ degenerate, with degeneracy at the singularities on the interface[2]; and only a weak Taylor inequality $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds. From earlier work [3,6,25,33,38,39], we know the problem of solving the water wave equation (1.1) can be reduced to solving a quasilinear equation of the interface $z = z(\alpha, t)$, of type

$$\partial_t^2 \mathfrak{u} + a \nabla_{\mathbf{n}} \mathfrak{u} = f(\mathfrak{u}, \partial_t \mathfrak{u}) \tag{1.4}$$

where $a = -\frac{\partial P}{\partial \mathbf{n}}$. When the strong Taylor sign condition (1.3) holds and $\nabla_{\mathbf{n}}$ is non-degenerate, Eq. (1.4) is of the hyperbolic type with the right hand side consisting of lower order terms, and the Cauchy problem can be solved using classical tools. In the case where the solution dependent quantity $a = -\frac{\partial P}{\partial \mathbf{n}}$ and the operator $\nabla_{\mathbf{n}}$ degenerate, Eq. (1.4) losses its hyperbolicity, classical tools do not apply. New ideas are required to solve the problem.

In [24], R. Kinsey and the author constructed an energy functional $\mathcal{E}(t)$ and proved an a-priori estimate, which states that for solutions of the water wave equation (1.1), if $\mathcal{E}(0) < \infty$, then $\mathcal{E}(t)$ remains finite for a time period that depends only on $\mathcal{E}(0)$. The energy functional $\mathcal{E}(t)$ is finite for interfaces and velocities in Sobolev classes, and most importantly, it is also finite for a class of non-C^1 interfaces with angled crests.[3] In this paper, we show that for any given data satisfying $\mathcal{E}(0) < \infty$, there is a $T > 0$, depending only on $\mathcal{E}(0)$, such that the 2d water wave equation (1.1) has a unique solution in the class

[2] We assume the acceleration is finite.

[3] In particular, the class where $\mathcal{E}(t) < \infty$ allows for angled crest type interfaces with interior angle at the crest $< \frac{\pi}{2}$, which coincides with the range of the angles of the self-similar solutions in [42], see § 11.2 of [24]. See [1] for some further discussions of the type of singularities allowed by the class $\mathcal{E}(t) < \infty$ and the properties of the singularities. Stokes extreme waves is not in the class where $\mathcal{E}(t) < \infty$.

where $\mathcal{E}(t) < \infty$ for time $0 \le t \le T$, and the solution is stable. We will work on the free surface equations that were derived in [38,39]. The novelty of this paper is that we study the degenerate case, and solve the equation in a class[4] that includes non-C^1 interfaces.[5]

1.1 Outline of the paper

In Sect. 1.2 we introduce some basic notations and conventions; further notations will be introduced throughout the paper. In Sect. 2 we recall the results in [38,39], and derive the free surface equation and its quasi-linearization, from system (1.1), in both the Lagrangian and Riemann mapping variables, for interfaces and velocities in Sobolev spaces. We derived the quasilinear equation in terms of the horizontal component in the Riemann mapping variable in [38], and in terms of full components in the Lagrangian coordinates in [39]. Here we re-derive the equations for the sake of coherence. In Sect. 2.5 we will recover the water wave equation (1.1) from the interface equation (2.9)–(2.11)–(2.15)–(2.14), showing the equivalence of the two systems for smooth and non-self-intersecting interfaces. In Sect. 3.1, we present the energy functional $\mathcal{E}(t)$ constructed and the a-priori estimate proved in [24]. In Sect. 3.2, we give a blow-up criteria in terms of the energy functional $\mathcal{E}(t)$ and a stability inequality for solutions of the interface equation (2.9)–(2.11)–(2.15)–(2.14) with a bound depending only on $\mathcal{E}(t)$. In Sect. 3.3 we present the main result, that is, the local in time wellposedness of the Cauchy problem for the water wave equation (1.1) in the class where $\mathcal{E}(t) < \infty$. In Sect. 4, we give the proof for the blow-up criteria, Theorem 3.6 and in Sect. 5, the stability inequality, Theorem 3.7. For the sake of completeness, we will also provide a proof for the a-priori estimate of [24] in the current setting in Sect. 4. In Sect. 6, we will prove the main result, Theorem 3.9.

[4] The class where $\mathcal{E}(t) < \infty$ allows for interfaces that is piecewise H^3 with angled crest type singularities. In this work and in [24], we do not intend to optimize the number of derivatives required in the energy functional $\mathcal{E}(t)$. For example in [2], an energy functional that involves $1/2$ less spatial derivatives than that of $\mathcal{E}(t)$ and allows for non-C^1 interfaces is constructed and an a-priori estimate is proved.

[5] In § 11.2 of [24] and in [1], it is shown that the angles of the crests at the interface will not change as long as the solution remains in the class $\mathcal{E}(t) < \infty$. A natural question is then how does the angle form in the first place? One possibility for the formation of the angles in the first place is the presence of the wind. It is well-known that once a wave forms it can travel long distances. Observe that in our model (1.1), the wind is not present since the air is assumed to have density 0. So one possible explanation is that once the angle form (due to the wind), it will not change as long as there is no wind. Of course this is a speculation that requires further work to clarify. One should also keep in mind that in our Eq. (1.1), besides the wind, the effect of the surface tension, viscosity, bottom and coast are all been neglected.

Some basic preparatory results are given in Appendix A; various identities that are useful for the paper are derived in Appendix B. And in Appendix C, we list the quantities that are controlled by $\mathcal{E}$. A majority of these are already shown in [24].

Remark The blow-up criteria and the proof for the existence part of Theorem 3.9 are from the unpublished manuscript of the author [44], with some small modifications.

1.2 Notation and convention

We consider solutions of the water wave equation (1.1) in the setting where the fluid domain $\Omega(t)$ is simply connected, with the free interface $\Sigma(t) := \partial\Omega(t)$ being a Jordan curve,[6]

$$\mathbf{v}(z,t) \to 0, \qquad \text{as } |z| \to \infty$$

and the interface $\Sigma(t)$ tending to horizontal lines at infinity.[7]

We use the following notations and conventions: $[A, B] := AB - BA$ is the commutator of operators A and B. $H^s = H^s(\mathbb{R})$ is the Sobolev space with norm $\|f\|_{H^s} := (\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi)^{1/2}$, $\dot{H}^s = \dot{H}^s(\mathbb{R})$ is the Sobolev space with norm $\|f\|_{\dot{H}^s} := c(\int |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi)^{1/2}$, $L^p = L^p(\mathbb{R})$ is the L^p space with $\|f\|_{L^p} := (\int |f(x)|^p\, dx)^{1/p}$ for $1 \le p < \infty$, and $f \in L^\infty$ if $\|f\|_{L^\infty} := \sup |f(x)| < \infty$. When not specified, all the norms $\|f\|_{H^s}$, $\|f\|_{\dot{H}^s}$, $\|f\|_{L^p}$, $1 \le p \le \infty$ are in terms of the spatial variable only, and $\|f\|_{H^s(\mathbb{R})}$, $\|f\|_{\dot{H}^s(\mathbb{R})}$, $\|f\|_{L^p(\mathbb{R})}$, $1 \le p \le \infty$ are in terms of the spatial variables. We say $f \in C^j([0,T], H^s)$ if the mapping $f = f(t) := f(\cdot, t) : t \in [0,T] \to H^s$ is j-times continues differentiable, with $\sup_{[0,T],\ 0\le k\le j} \|\partial_t^k f(t)\|_{H^s} < \infty$; we say $f \in L^\infty([0,T], H^s)$ if $\sup_{[0,T]} \|f(t)\|_{H^s} < \infty$. $C^j(X)$ is the space of j-times continuously differentiable functions on the set X; $C_0^j(\mathbb{R})$ is the space of j-times continuously differentiable functions that decays at the infinity.

Compositions are always in terms of the spatial variables and we write for $f = f(\cdot,t)$, $g = g(\cdot,t)$, $f(g(\cdot,t),t) := f \circ g(\cdot,t) := U_g f(\cdot,t)$. We identify (x, y) with the complex number $x + iy$; $\operatorname{Re} z$, $\operatorname{Im} z$ are the real and imaginary parts of z; $\overline{z} = \operatorname{Re} z - i \operatorname{Im} z$ is the complex conjugate of z. $\overline{\Omega}$ is the closure of the domain Ω, $\partial\Omega$ is the boundary of Ω, $\mathscr{P}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ is the lower half plane. We write

[6] That is, $\Sigma(t)$ is homeomorphic to the line $\mathbb{R}$.

[7] The problem with velocity $\mathbf{v}(z,t) \to (c,0)$ as $|z| \to \infty$ can be reduced to the one with $\mathbf{v} \to 0$ at infinity by studying the solutions in a moving frame. $\Sigma(t)$ may tend to two different lines at $+\infty$ and $-\infty$.

$$[f, g; h] := \frac{1}{\pi i} \int \frac{(f(x) - f(y))(g(x) - g(y))}{(x - y)^2} h(y)\, dy. \tag{1.5}$$

We use c, C to denote universal constants. $c(a_1, \ldots), C(a_1, \ldots), M(a_1, \ldots)$ are constants depending on $a_1, \ldots$; constants appearing in different contexts need not be the same. We write $f \lesssim g$ if there is a universal constant c, such that $f \leq cg$.

2 Preliminaries

Equation (1.1) is a nonlinear equation defined on moving domains, it is difficult to study it directly. A classical approach is to reduce from (1.1) to an equation on the interface, and study solutions of the interface equation. Then use the incompressibility and irrotationality of the velocity field to recover the velocity in the fluid domain by solving a boundary value problem for the Laplace equation.

In what follows we derive the interface equations from (1.1), and vice versa; we assume that the interface, velocity and acceleration are in Sobolev spaces.

2.1 The equation for the free surface in Lagrangian variable

Let the free interface $\Sigma(t) : z = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be given by Lagrangian parameter α, so $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t); t)$ is the velocity of the fluid particles on the interface, $z_{tt}(\alpha, t) = \mathbf{v_t} + (\mathbf{v} \cdot \nabla)\mathbf{v}(z(\alpha, t); t)$ is the acceleration. Notice that $P = 0$ on $\Sigma(t)$ implies that ∇P is normal to $\Sigma(t)$, therefore $\nabla P = -i\mathfrak{a} z_\alpha$, where

$$\mathfrak{a} = -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial \mathbf{n}}; \tag{2.1}$$

and the first and third equation of (1.1) gives

$$z_{tt} + i = i\mathfrak{a} z_\alpha. \tag{2.2}$$

The second equation of (1.1): $\operatorname{div} \mathbf{v} = \operatorname{curl} \mathbf{v} = 0$ implies that $\overline{\mathbf{v}}$ is holomorphic in the fluid domain $\Omega(t)$, hence $\overline{z}_t$ is the boundary value of a holomorphic function in $\Omega(t)$.

Let $\Omega \subset \mathbb{C}$ be a domain with boundary $\Sigma : z = z(\alpha)$, $\alpha \in I$, oriented clockwise. Let $\mathfrak{H}$ be the Hilbert transform associated to Ω:

$$\mathfrak{H} f(\alpha) = \frac{1}{\pi i} \operatorname{pv.} \int \frac{z_\beta(\beta)}{z(\alpha) - z(\beta)} f(\beta)\, d\beta \tag{2.3}$$

We have the following characterization of the trace of a holomorphic function on Ω.

Proposition 2.1 [23] *a. Let $g \in L^p$ for some $1 < p < \infty$. Then g is the boundary value of a holomorphic function G on Ω with $G(z) \to 0$ at infinity if and only if*

$$(I - \mathfrak{H})g = 0. \tag{2.4}$$

b. Let $f \in L^p$ for some $1 < p < \infty$. Then $\frac{1}{2}(I + \mathfrak{H})f$ is the boundary value of a holomorphic function $\mathfrak{G}$ on Ω, with $\mathfrak{G}(z) \to 0$ at infinity.

c. $\mathfrak{H}1 = 0$.

By Proposition 2.1 the second equation of (1.1) is equivalent to $\overline{z}_t = \mathfrak{H}\overline{z}_t$. So the motion of the fluid interface $\Sigma(t) : z = z(\alpha, t)$ is given by

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \overline{z}_t = \mathfrak{H}\overline{z}_t. \end{cases} \tag{2.5}$$

(2.5) is a fully nonlinear equation. In [38], Riemann mapping was introduced to analyze Eq. (2.5) and to derive the quasilinear equation.

2.2 The free surface equation in Riemann mapping variable

Let $\Phi(\cdot, t) : \Omega(t) \to \mathscr{P}_-$ be the Riemann mapping taking $\Omega(t)$ to the lower half plane $\mathscr{P}_-$, satisfying $\Phi(z(0, t), t) = 0$ and $\lim_{z\to\infty} \Phi_z(z, t) = 1$. Let

$$h(\alpha, t) := \Phi(z(\alpha, t), t), \tag{2.6}$$

so $h(0, t) = 0$ and $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Let h^{-1} be defined by

$$h(h^{-1}(\alpha', t), t) = \alpha', \quad \alpha' \in \mathbb{R};$$

and

$$\begin{aligned} Z(\alpha', t) &:= z \circ h^{-1}(\alpha', t), \quad Z_t(\alpha', t) := z_t \circ h^{-1}(\alpha', t), \\ Z_{tt}(\alpha', t) &:= z_{tt} \circ h^{-1}(\alpha', t) \end{aligned} \tag{2.7}$$

be the reparametrization of the position, velocity and acceleration of the interface in the Riemann mapping variable α'. Let

$$\begin{aligned} Z_{,\alpha'}(\alpha', t) &:= \partial_{\alpha'} Z(\alpha', t), \quad Z_{t,\alpha'}(\alpha', t) := \partial_{\alpha'} Z_t(\alpha', t), \\ Z_{tt,\alpha'}(\alpha', t) &:= \partial_{\alpha'} Z_{tt}(\alpha', t), \text{ etc.} \end{aligned} \tag{2.8}$$

Notice that $\overline{\mathbf{v}} \circ \Phi^{-1} : \mathscr{P}_- \to \mathbb{C}$ is holomorphic in the lower half plane $\mathscr{P}_-$ with $\overline{\mathbf{v}} \circ \Phi^{-1}(\alpha', t) = \overline{Z}_t(\alpha', t)$. Precomposing (2.2) with h^{-1} and applying Proposition 2.1 to $\overline{\mathbf{v}} \circ \Phi^{-1}$ in $\mathscr{P}_-$, we have the free surface equation in the Riemann mapping variable:

$$\begin{cases} Z_{tt} + i = i\mathcal{A} Z_{,\alpha'} \\ \overline{Z}_t = \mathbb{H}\overline{Z}_t \end{cases} \tag{2.9}$$

where $\mathcal{A} \circ h = \mathfrak{a} h_\alpha$ and $\mathbb{H}$ is the Hilbert transform associated with the lower half plane $\mathscr{P}_-$:

$$\mathbb{H} f(\alpha') = \frac{1}{\pi i} \text{pv.} \int \frac{1}{\alpha' - \beta'} f(\beta')\, d\beta'. \tag{2.10}$$

Observe that $\Phi^{-1}(\alpha', t) = Z(\alpha', t)$ and $(\Phi^{-1})_{z'}(\alpha', t) = Z_{,\alpha'}(\alpha', t)$. So $Z_{,\alpha'}$, $\dfrac{1}{Z_{,\alpha'}}$ are boundary values of the holomorphic functions $(\Phi^{-1})_{z'}$ and $\dfrac{1}{(\Phi^{-1})_{z'}}$, tending to 1 at the spatial infinity. By Proposition 2.1,[8]

$$\frac{1}{Z_{,\alpha'}} - 1 = \mathbb{H}\left(\frac{1}{Z_{,\alpha'}} - 1\right). \tag{2.11}$$

By the chain rule, we know for any function f, $U_h^{-1} \partial_t U_h f = (\partial_t + b\partial_{\alpha'}) f$, where

$$b := h_t \circ h^{-1}.$$

So $Z_{tt} = (\partial_t + b\partial_{\alpha'}) Z_t$, and $Z_t = (\partial_t + b\partial_{\alpha'}) Z$.

2.2.1 *Some additional notations*

We will often use the fact that $\mathbb{H}$ is purely imaginary, and decompose a function into the sum of its holomorphic and antiholomorphic parts. We define the projections to the space of holomorphic functions in the lower, and respectively, upper half planes by

$$\mathbb{P}_H := \frac{1}{2}(I + \mathbb{H}), \qquad \text{and} \quad \mathbb{P}_A := \frac{1}{2}(I - \mathbb{H}). \tag{2.12}$$

[8] We work in the regime where $\frac{1}{Z_{,\alpha'}} - 1 \in L^2(\mathbb{R})$.

We also define

$$D_\alpha = \frac{1}{z_\alpha}\partial_\alpha, \quad \text{and} \quad D_{\alpha'} = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}. \tag{2.13}$$

We know by the chain rule that $(D_\alpha f)\circ h^{-1} = D_{\alpha'}\left(f\circ h^{-1}\right)$; and for any holomorphic function G on $\Omega(t)$ with boundary value $g(\alpha,t) := G(z(\alpha,t),t)$, $D_\alpha g = G_z \circ z$, and $D_{\alpha'}(g\circ h^{-1}) = G_z \circ Z$. Hence D_α, $D_{\alpha'}$ preserves the holomorphicity of g, $g\circ h^{-1}$.

2.2.2 *The formulas for A_1 and b*

Let $A_1 := \mathcal{A}|Z_{,\alpha'}|^2$. Notice that $\mathcal{A}\circ h = \mathfrak{a}h_\alpha = -\frac{\partial P}{\partial \mathbf{n}}\frac{h_\alpha}{|z_\alpha|}$, so A_1 is related to the important quantity $-\frac{\partial P}{\partial \mathbf{n}}$ by

$$-\frac{\partial P}{\partial \mathbf{n}}\circ Z = \frac{A_1}{|Z_{,\alpha'}|}.$$

Using Riemann mapping we analyzed the quantities A_1 and b in [38] and we gave a re-derivation of the formulas in [43]. We mention that in [19], using the formulation of Ovsjannikov [30], the authors also re-derived the formulas (2.14) and (2.15). Here we record the results in [38] and [43].

Proposition 2.2 (Lemma 3.1 and (4.7) of [38], or Proposition 2.2 and (2.18) of [43]) *We have*

$$b := h_t\circ h^{-1} = \mathrm{Re}(I-\mathbb{H})\left(\frac{Z_t}{Z_{,\alpha'}}\right); \tag{2.14}$$

$$A_1 = 1 - \mathrm{Im}[Z_t,\mathbb{H}]\overline{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi}\int \frac{|Z_t(\alpha',t)-Z_t(\beta',t)|^2}{(\alpha'-\beta')^2}\,d\beta' \geq 1; \tag{2.15}$$

$$-\frac{\partial P}{\partial \mathbf{n}}\Big|_{Z=Z(\cdot,t)} = \frac{A_1}{|Z_{,\alpha'}|}. \tag{2.16}$$

In particular, if the interface $\Sigma(t) \in C^{1+\epsilon}$ for some $\epsilon > 0$, then the strong Taylor sign condition (1.3) holds.

By the definition of A_1 we have

$$Z_{,\alpha'}(\overline{Z}_{tt} - i) = -i\mathcal{A}|Z_{,\alpha'}|^2 := -iA_1. \tag{2.17}$$

2.3 The quasilinear equation

In [38,39] we showed that the quasi-linearization of the free surface equation (2.5) can be accomplished by just taking one time derivative to Eq. (2.2).

Taking derivative to t to (2.2) we get

$$\overline{z}_{ttt} + i\mathfrak{a}\overline{z}_{t\alpha} = -i\mathfrak{a}_t\overline{z}_\alpha = \frac{\mathfrak{a}_t}{\mathfrak{a}}(\overline{z}_{tt} - i). \tag{2.18}$$

Precomposing with h^{-1} on both sides of (2.18), we have the equation in the Riemann mapping variable

$$\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\overline{Z}_{tt} - i) \tag{2.19}$$

We compute $\dfrac{\mathfrak{a}_t}{\mathfrak{a}}$ by the identities $\mathfrak{a}h_\alpha = \mathcal{A} \circ h$, and $\mathcal{A} \circ h = \dfrac{A_1}{|Z_{,\alpha'}|^2} \circ h = A_1 \circ h \dfrac{h_\alpha^2}{|z_\alpha|^2}$, so

$$\mathfrak{a} = A_1 \circ h \frac{h_\alpha}{|z_\alpha|^2}; \tag{2.20}$$

and we obtain, by taking derivative to t to (2.20),

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} = \frac{\partial_t (A_1 \circ h)}{A_1 \circ h} + \frac{h_{t\alpha}}{h_\alpha} - 2\,\mathrm{Re}\,\frac{z_{t\alpha}}{z_\alpha}.$$

Notice that $\frac{h_{t\alpha}}{h_\alpha} \circ h^{-1} = (h_t \circ h^{-1})_{\alpha'} := b_{\alpha'}$. So

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{(\partial_t + b\partial_{\alpha'})A_1}{A_1} + b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t; \tag{2.21}$$

where we calculate from (2.14) that

$$\begin{aligned} b_{\alpha'} &= \mathrm{Re}\left((I - \mathbb{H})\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} + (I - \mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right) \\ &= 2\,\mathrm{Re}\,D_{\alpha'}Z_t + \mathrm{Re}\left((-I - \mathbb{H})\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} + (I - \mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right) \\ &= 2\,\mathrm{Re}\,D_{\alpha'}Z_t + \mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right), \end{aligned} \tag{2.22}$$

here in the last step we used the fact that $(I+\mathbb{H})Z_{t,\alpha'}=0$ and $(I-\mathbb{H})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}=0$[9] to rewrite the terms as commutators; and we compute, by (2.15) and (B.19),

$$(\partial_t+b\partial_{\alpha'})A_1=-\operatorname{Im}\left([Z_{tt},\mathbb{H}]\,\overline{Z}_{t,\alpha'}+[Z_t,\mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{tt}-[Z_t,b;\,\overline{Z}_{t,\alpha'}]\right). \tag{2.23}$$

We now sum up the above calculations and write the quasilinear system in the Riemann mapping variable. We have

$$\begin{cases}(\partial_t+b\partial_{\alpha'})^2\overline{Z}_t+i\dfrac{A_1}{\left|Z_{,\alpha'}\right|^2}\partial_{\alpha'}\overline{Z}_t=\dfrac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i)\\ \overline{Z}_t=\mathbb{H}\overline{Z}_t\end{cases} \tag{2.24}$$

where

$$\begin{cases}b:=h_t\circ h^{-1}=\operatorname{Re}(I-\mathbb{H})\left(\dfrac{Z_t}{Z_{,\alpha'}}\right)\\ A_1=1-\operatorname{Im}[Z_t,\mathbb{H}]\overline{Z}_{t,\alpha'}=1+\dfrac{1}{2\pi}\displaystyle\int\frac{|Z_t(\alpha',t)-Z_t(\beta',t)|^2}{(\alpha'-\beta')^2}\,d\beta'\\ \dfrac{1}{Z_{,\alpha'}}=i\dfrac{\overline{Z}_{tt}-i}{A_1}\\ \dfrac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}=\dfrac{(\partial_t+b\partial_{\alpha'})A_1}{A_1}+b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\end{cases} \tag{2.25}$$

Here the third equation in (2.25) is obtained by rearranging the terms of the Eq. (2.17). Using it to replace $\frac{1}{Z_{,\alpha'}}$ by $i\frac{\overline{Z}_{tt}-i}{A_1}$, we get a system for the complex conjugate velocity and acceleration $(\overline{Z}_t,\overline{Z}_{tt})$. The initial data for the system (2.24)–(2.25) is set up as follows.

2.3.1 *The initial data*

Without loss of generality, we choose the parametrization of the initial interface $\Sigma(0):Z(\cdot,0):=Z(0)$ by the Riemann mapping variable, so $h(\alpha,0)=\alpha$ for $\alpha\in\mathbb{R}$; we take the initial velocity $Z_t(0)$, such that it satisfies $\overline{Z}_t(0)=\mathbb{H}\overline{Z}_t(0)$. And we take the initial acceleration $Z_{tt}(0)$ so that it solves the Eq. (2.17) or the third equation in (2.25).

[9] These follows from $(I-\mathbb{H})\overline{Z}_t=0$ and (2.11).

2.4 Local wellposedness in Sobolev spaces

By (2.16) and (2.15), if $Z_{,\alpha'} \in L^\infty$, then the strong Taylor stability criterion (1.3) holds. In this case, the system (2.24)–(2.25) is quasilinear of the hyperbolic type, with the left hand side of the first equation in (2.24) consisting of the higher order terms.[10] In [38] we showed that the Cauchy problem of (2.24)–(2.25), and equivalently of (2.9)–(2.11)–(2.14)–(2.15), is uniquely solvable in Sobolev spaces H^s, $s \geq 4$.

Let the initial data be given as in Sect. 2.3.1.

Theorem 2.3 (Local wellposedness in Sobolev spaces, cf. Theorem 5.11, §6 of [38]) *Let $s \geq 4$. Assume that $Z_t(0) \in H^{s+1/2}(\mathbb{R})$, $Z_{tt}(0) \in H^s(\mathbb{R})$ and $Z_{,\alpha'}(0) \in L^\infty(\mathbb{R})$. Then there is $T > 0$, such that on $[0, T]$, the initial value problem of (2.24)–(2.25), or equivalently of (2.9)–(2.11)–(2.14)–(2.15), has a unique solution $Z = Z(\cdot, t)$, satisfying $(Z_t, Z_{tt}) \in C^l([0, T], H^{s+1/2-l}(\mathbb{R}) \times H^{s-l}(\mathbb{R}))$, and $Z_{,\alpha'} - 1 \in C^l([0, T], H^{s-l}(\mathbb{R}))$, for $l = 0, 1$.*

Moreover if T^* is the supremum over all such times T, then either $T^* = \infty$, or $T^* < \infty$, but

$$\sup_{[0,T^*)} (\|Z_{,\alpha'}(t)\|_{L^\infty} + \|Z_{tt}(t)\|_{H^3} + \|Z_t(t)\|_{H^{3+1/2}}) = \infty. \tag{2.26}$$

Remark 2.4 1. Let $h = h(\alpha, t)$ be the solution of the ODE

$$\begin{cases} h_t = b(h, t), \\ h(\alpha, 0) = \alpha \end{cases} \tag{2.27}$$

where b is as given by (2.14). Then $z = Z \circ h$ satisfies Eq. (2.2), cf. §6 of [38].

2. (2.24)–(2.25) is a system for the complex conjugate velocity and acceleration $(\overline{Z}_t, \overline{Z}_{tt})$, the interface doesn't appear explicitly, so a solution can exist even if $Z = Z(\cdot, t)$ becomes self-intersecting. Similarly, Eqs. (2.9)–(2.11)–(2.14)–(2.15) makes sense even if $Z = Z(\cdot, t)$ self-intersects. To obtain the solution of the water wave equation (1.1) from the solution of the quasilinear equation (2.24)–(2.25) as given in Theorem 2.3 above, in §6 of [38], an additional chord–arc condition is assumed for the initial interface, and it was shown that the solution $Z = Z(\cdot, t)$ remains non-self-intersecting for a time period depending only on the initial chord–arc constant and the initial Sobolev norms.[11]

[10] $i\partial_{\alpha'} = |\partial_{\alpha'}|$ when acting on holomorphic functions. The Dirichlet-to-Neumann operator $\nabla_{\mathbf{n}} = \frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}|$.

[11] (2.9)–(2.11)–(2.14)–(2.15) is equivalent to the water wave equation (1.1) only when the interface is non-self-intersecting, see Sect. 2.5.

3. Observe that we arrived at (2.24)–(2.25) from (1.1) using only the following properties of the domain: 1. there is a conformal mapping taking the fluid region $\Omega(t)$ to $\mathscr{P}_-$; 2. $P = 0$ on $\Sigma(t)$. We note that $z \to z^{1/2}$ is a conformal map that takes the region $\mathbb{C}\backslash\{z = x + i0, x > 0\}$ to the upper half plane; so a domain with its boundary self-intersecting at the positive real axis can be mapped conformally onto the lower half plane $\mathscr{P}_-$. Taking such a domain as the initial fluid domain, assuming $P = 0$ on $\Sigma(t)$ even when $\Sigma(t)$ self-intersects,[12] one can still solve Eqs. (2.24)–(2.25) for a short time, by Theorem 2.3. Indeed this is one of the main ideas in the work of [10]. Using this idea and the time reversibility of the water wave equation, by choosing an appropriate initial velocity field that pulls the initial domain apart, Castro, Cordoba et. al. [10] proved the existence of "splash" and "splat" singularities starting from a smooth non-self-intersecting fluid interface.

2.5 Recovering the water wave equation (1.1) from the interface equations

In this section, we derive the equivalent system in the lower half plane $\mathscr{P}_-$ for the interface equations (2.9)–(2.11)–(2.15)–(2.14), and show how to recover from here the water wave equation (1.1). Although the derivation is quite straight-forward, to the best knowledge of the author, it has not been done before.

Let $Z = Z(\cdot, t)$ be a solution of (2.9)–(2.11)–(2.15)–(2.14), satisfying the regularity properties of Theorem 2.3; let $U(\cdot, t) : \mathscr{P}_- \to \mathbb{C}$, $\Psi(\cdot, t) : \mathscr{P}_- \to \mathbb{C}$ be the holomorphic functions, continuous on $\overline{\mathscr{P}}_-$, such that

$$U(\alpha', t) = \overline{Z}_t(\alpha', t), \qquad \Psi(\alpha', t) = Z(\alpha', t), \qquad \Psi_{z'}(\alpha', t) = Z_{,\alpha'}(\alpha', t), \tag{2.28}$$

and $\lim_{z' \to \infty} U(z', t) = 0$, $\lim_{z' \to \infty} \Psi_{z'}(z', t) = 1$.[13] From $Z_t = (\partial_t + b\partial_{\alpha'})Z = \Psi_t(\alpha', t) + b\Psi_{z'}(\alpha', t)$, we have

$$b = \frac{Z_t}{\Psi_{z'}} - \frac{\Psi_t}{\Psi_{z'}} = \frac{\overline{U}}{\Psi_{z'}} - \frac{\Psi_t}{\Psi_{z'}}, \qquad \text{on } \partial\mathscr{P}_-, \tag{2.29}$$

[12] We note that when $\Sigma(t)$ self-intersects, the condition $P = 0$ on $\Sigma(t)$ is unphysical.

[13] We know $U(z', t) = K_{y'} * \overline{Z}_t$, $\Psi_{z'} = K_{y'} * Z_{,\alpha'}$ and by the Maximum principle, $\frac{1}{\Psi_{z'}} = K_{y'} * \frac{1}{Z_{,\alpha'}}$, here $K_{y'}$ is the Poisson kernel defined by (3.12). By (2.17), $\frac{1}{Z_{,\alpha'}} - 1 \in C([0, T], H^s(\mathbb{R}))$ for $s \geq 4$, so $\Psi_{z'} \neq 0$ on $\overline{\mathscr{P}}_-$.

and substituting in we get

$$\overline{Z}_{tt} = (\partial_t + b\partial_{\alpha'})\overline{Z}_t = U_t + \left(\frac{\overline{U}}{\Psi_{z'}} - \frac{\Psi_t}{\Psi_{z'}}\right) U_{z'} \qquad \text{on } \partial\mathscr{P}_-;$$

so $\overline{Z}_{tt}$ is the trace of the function $U_t - \frac{\Psi_t}{\Psi_{z'}} U_{z'} + \frac{\overline{U}}{\Psi_{z'}} U_{z'}$ on $\partial\mathscr{P}_-$; and $Z_{,\alpha'}(\overline{Z}_{tt} - i)$ is then the trace of the function $\Psi_{z'}U_t - \Psi_t U_{z'} + \overline{U}U_{z'} - i\Psi_{z'}$ on $\partial\mathscr{P}_-$. This gives, from (2.17) that

$$\Psi_{z'}U_t - \Psi_t U_{z'} + \overline{U}U_{z'} - i\Psi_{z'} = -iA_1, \qquad \text{on } \partial\mathscr{P}_-. \tag{2.30}$$

Observe that on the left hand side of (2.30), $\Psi_{z'}U_t - \Psi_t U_{z'} - i\Psi_{z'}$ is holomorphic on $\mathscr{P}_-$, while $\overline{U}U_{z'} = \partial_{z'}(\overline{U}U)$. So there is a real valued function $\mathfrak{P} : \mathscr{P}_- \to \mathbb{R}$, such that

$$\Psi_{z'}U_t - \Psi_t U_{z'} + \overline{U}U_{z'} - i\Psi_{z'} = -2\partial_{z'}\mathfrak{P} = -(\partial_{x'} - i\partial_{y'})\mathfrak{P}, \qquad \text{on } \mathscr{P}_-; \tag{2.31}$$

and by (2.30), because iA_1 is purely imaginary,

$$\mathfrak{P} = c, \qquad \text{on } \partial\mathscr{P}_-. \tag{2.32}$$

where $c \in \mathbb{R}$ is a constant. Applying $\partial_{x'} + i\partial_{y'} := 2\overline{\partial}_{z'}$ to (2.31) yields

$$\Delta\mathfrak{P} = -2|U_{z'}|^2 \qquad \text{on } \mathscr{P}_-. \tag{2.33}$$

It is easy to check that for $y' \le 0$ and $t \in [0, T]$, $\left(U, U_t, U_{z'}, \Psi_{z'} - 1, \frac{1}{\Psi_{z'}} - 1, \Psi_t\right)(\cdot + iy', t) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $(U, \Psi, \frac{1}{\Psi_{z'}}, \mathfrak{P}) \in C^1(\overline{\mathscr{P}}_- \times [0, T])$.

It is clear that the above process is reversible. From a solution $(U, \Psi, \mathfrak{P}) \in C^1(\overline{\mathscr{P}}_- \times [0, T])$ of the system (2.31)–(2.33)–(2.32)–(2.29), with $\left(U, U_t, U_{z'}, \Psi_{z'} - 1, \frac{1}{\Psi_{z'}} - 1, \Psi_t\right)(\cdot + iy', t) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for $y' \le 0$, $t \in [0, T]$, $U(\cdot, t)$, $\Psi(\cdot, t)$ holomorphic in $\mathscr{P}_-$, and b real valued, the boundary value $(Z(\alpha', t), Z_t(\alpha', t)) := (\Psi(\alpha', t), \overline{U}(\alpha', t))$ satisfies the interface equation (2.9)–(2.11)–(2.14)–(2.15). Therefore the systems (2.9)–(2.11)–(2.14)–(2.15) and (2.31)–(2.33)–(2.32)–(2.29), with $U(\cdot, t)$, $\Psi(\cdot, t)$ holomorphic in $\mathscr{P}_-$, and $\Psi_{z'}(\cdot, t) \neq 0$, b real valued, are equivalent in the smooth regime.

Assume $(U, \Psi) \in C(\overline{\mathscr{P}}_- \times [0, T]) \cap C^1(\mathscr{P}_- \times (0, T))$ is a solution of the system (2.31)–(2.33)–(2.32), with $U(\cdot, t)$, $\Psi(\cdot, t)$ holomorphic in $\mathscr{P}_-$, assume

in addition that $\Sigma(t) = \{Z = Z(\alpha', t) := \Psi(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is a Jordan curve with

$$\lim_{|\alpha'|\to\infty} Z_{,\alpha'}(\alpha', t) = 1.$$

Let $\Omega(t)$ be the domain bounded by $Z = Z(\cdot, t)$ from the above, then $Z = Z(\alpha', t)$, $\alpha' \in \mathbb{R}$ winds the boundary of $\Omega(t)$ exactly once. By the argument principle, $\Psi : \overline{\mathscr{P}}_- \to \overline{\Omega}(t)$ is one-to-one and onto, $\Psi^{-1} : \Omega(t) \to \mathscr{P}_-$ exists and is a holomorphic function; and by Eq. (2.31) and the chain rule,

$$(U \circ \Psi^{-1})_t + \overline{U} \circ \Psi^{-1}(U \circ \Psi^{-1})_z + (\partial_x - i\partial_y)(\mathfrak{P} \circ \Psi^{-1}) = i, \qquad \text{on } \Omega(t). \tag{2.34}$$

Let $\overline{\mathbf{v}} = U \circ \Psi^{-1}$, $P = \mathfrak{P} \circ \Psi^{-1}$. Observe that $\mathbf{v}\overline{\mathbf{v}}_z = (\mathbf{v} \cdot \nabla)\overline{\mathbf{v}}$. So $(\mathbf{v}, P)$ satisfies the water wave equation (1.1) in the domain $\Omega(t)$.

2.6 Non-C^1 interfaces

Assume that the interface $Z = Z(\cdot, t)$ has an angled crest at α'_0 with interior angle ν, we know from the discussion in §3.3.2 of [24] that if the acceleration is finite, then it is necessary that $\nu \le \pi$; and if $\nu < \pi$ then $\frac{1}{Z_{,\alpha'}}(\alpha'_0, t) = 0$. We henceforth call those points at which $\frac{1}{Z_{,\alpha'}} = 0$ the singularities.

If the interface is allowed to be non-C^1 with interior angles at the crests $< \pi$, then the coefficient $\frac{A_1}{|Z_{,\alpha'}|^2}$ of the second term on the left hand side of the first equation in (2.24) can be degenerate, and in this case it is not clear if Eq. (2.24) is still hyperbolic. In order to handle this situation, we need to understand how the singularities propagate. In what follows we derive the evolution equation for $\frac{1}{Z_{,\alpha'}}$. We will also give the evolution equations for the basic quantities $\overline{Z}_t$ and $\overline{Z}_{tt}$.[14]

2.7 Some basic evolution equations

We begin with

$$\frac{1}{Z_{,\alpha'}} \circ h = \frac{h_\alpha}{z_\alpha},$$

[14] In Lagrangian coordinates, the first equation in (2.24) is of the form $(\partial_t^2 + a\nabla_{\mathbf{n}})\overline{z}_t = f$, where $a = -\frac{\partial P}{\partial \mathbf{n}} = \frac{A_1}{|Z_{,\alpha'}|} \circ h$, and the Dirichilet-Neumann operator $\nabla_{\mathbf{n}} \circ h^{-1} = \frac{i}{|Z_{,\alpha'}|}\partial_{\alpha'}$. So at the singularities both a and $\nabla_{\mathbf{n}}$ are degenerate.

taking derivative to t yields,

$$\partial_t \left(\frac{1}{Z_{,\alpha'}} \circ h \right) = \frac{1}{Z_{,\alpha'}} \circ h \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right);$$

precomposing with h^{-1} gives

$$(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \right) = \frac{1}{Z_{,\alpha'}} \left(b_{\alpha'} - D_{\alpha'} Z_t \right). \tag{2.35}$$

The evolution equations for $\overline{Z}_t$ and $\overline{Z}_{tt}$ can be obtained from (2.17) and (2.18).

We have, by (2.17),

$$(\partial_t + b\partial_{\alpha'})\overline{Z}_t := \overline{Z}_{tt} = -i \frac{A_1}{Z_{,\alpha'}} + i. \tag{2.36}$$

Using (2.2) to replace $i\mathfrak{a}$ by $-\frac{\overline{z}_{tt} - i}{\overline{z}_\alpha}$ in equation (2.18) yields

$$\overline{z}_{ttt} = (\overline{z}_{tt} - i) \left(\overline{D_\alpha z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \right);$$

precomposing with h^{-1} gives

$$(\partial_t + b\partial_{\alpha'})\overline{Z}_{tt} = (\overline{Z}_{tt} - i) \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right). \tag{2.37}$$

Equations (2.35), (2.36) and (2.37) describe the time evolution of the basic quantities $\frac{1}{Z_{,\alpha'}}$, $\overline{Z}_t$ and $\overline{Z}_{tt}$. In fact, Eqs. (2.35)–(2.36) together with (2.14), (2.15) and (2.22) give a complete evolutionary system for the holomorphic quantities $\frac{1}{Z_{,\alpha'}}$ and $\overline{Z}_t$, which characterize the fluid domain $\Omega(t)$ and the complex conjugate velocity $\overline{\mathbf{v}}$. We will explore this evolution system in our future work. These equations give a first indication that it is natural to study the water wave problem in a setting where bounds are only imposed on $\frac{1}{Z_{,\alpha'}}$, $\overline{Z}_t$ and their derivatives.

2.8 An important equation

Here we record an important equation, which is obtained by rearranging the terms of (2.17).

$$\frac{1}{Z_{,\alpha'}} = i \frac{\overline{Z}_{tt} - i}{A_1}. \tag{2.38}$$

3 Well-posedness in a broader class that includes non-C^1 interfaces

We are now ready to study the Cauchy problem for the water wave equation (1.1) in a regime that allows for non-C^1 interfaces. We begin with an a-priori estimate.

3.1 A-priori estimate for water waves with angled crests

Motivated by the question of the interaction of the free interface with a fixed vertical boundary, in [24], Kinsey and the author studied the water wave equation (1.1) in a regime that includes non-C^1 interfaces with angled crests in a periodic setting, constructed an energy functional and proved an a-priori estimate which does not require a positive lower bound for $\frac{1}{|Z_{,\alpha'}|}$. A similar result holds for the whole line case. While a similar proof as that in [24] applies to the whole line, for the sake of completeness, we will provide a slightly different argument in Sect. 4.1. In the first proof in [24], we expanded and then re-organized the terms to ensure that there is no further cancelations and the estimates can be closed. Here instead we will rely on the estimates for the quantities $b_{\alpha'}$, A_1 and their derivatives.[15]

Let

$$\mathbf{E}_a(t) = \int \frac{1}{A_1} |Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) D_{\alpha'} \overline{Z}_t|^2 \, d\alpha' + \left\| D_{\alpha'} \overline{Z}_t(t) \right\|^2_{\dot{H}^{1/2}}, \quad (3.1)$$

and

$$\mathbf{E}_b(t) = \int \frac{1}{A_1} \left| Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t \right) \right|^2 d\alpha' + \left\| \frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t(t) \right\|^2_{\dot{H}^{1/2}}. \quad (3.2)$$

Let

$$\mathfrak{E}(t) = \mathbf{E}_a(t) + \mathbf{E}_b(t) + \|\overline{Z}_{t,\alpha'}(t)\|^2_{L^2} + \|D^2_{\alpha'} \overline{Z}_t(t)\|^2_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}(t) \right\|^2_{L^2} + \left| \frac{1}{Z_{,\alpha'}}(0,t) \right|^2. \quad (3.3)$$

Theorem 3.1 (cf. Theorem 2 of [24] for the periodic version) *Let $Z = Z(\cdot, t)$, $t \in [0, T]$ be a solution of the system* (2.9)–(2.11)–(2.14)–(2.15), *satisfying*

[15] These estimates become available in the work [44]. The same results in the current paper hold in the periodic setting.

$(Z_t, Z_{tt}) \in C^l([0,T], H^{s+1/2-l}(\mathbb{R}) \times H^{s-l}(\mathbb{R}))$, $l = 0, 1$ *for some* $s \geq 4$. *There is a polynomial* C *with universal nonnegative coefficients, such that*

$$\frac{d}{dt}\mathfrak{E}(t) \leq C(\mathfrak{E}(t)), \qquad \text{for } t \in [0,T]. \tag{3.4}$$

For the sake of completeness we will give a proof of Theorem 3.1 in Sect. 4.

Remark 3.2 It appears that there is an $\infty \cdot 0$ ambiguity in the definition of $\mathbf{E}_a$ and $\mathbf{E}_b$. This can be resolved by replacing the ambiguous quantities by the right hand sides of (3.5) and (3.6). The same remark applies to Lemmas 4.1, 4.5, 5.1. We opt for the current version for the clarity of the origins of the definitions and the more intuitive proofs.[16]

By (2.35) and product rules,

$$\begin{aligned} Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_t &= (b_{\alpha'} - D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'} + (\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'} \\ &= \overline{Z}_{tt,\alpha'} - (D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'}, \end{aligned} \tag{3.5}$$

and

$$Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right) = (b_{\alpha'} - D_{\alpha'}Z_t)D^2_{\alpha'}\overline{Z}_t + (\partial_t + b\partial_{\alpha'})D^2_{\alpha'}\overline{Z}_t. \tag{3.6}$$

Let

$$\begin{aligned} \mathfrak{e}(t) = &\left\|\overline{Z}_{tt,\alpha'}(t)\right\|^2_{L^2} + \left\|D_{\alpha'}\overline{Z}_t(t)\right\|^2_{\dot{H}^{1/2}} + \left\|D^2_{\alpha'}\overline{Z}_{tt}(t)\right\|^2_{L^2} + \left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t(t)\right\|^2_{\dot{H}^{1/2}} \\ &+\|\overline{Z}_{t,\alpha'}(t)\|^2_{L^2} + \|D^2_{\alpha'}\overline{Z}_t(t)\|^2_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|^2_{L^2} + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2. \end{aligned} \tag{3.7}$$

It is easy to check that the argument in Sect. 4.1.1 gives

$$\mathfrak{E}(t) \lesssim c_1(\mathfrak{e}(t)), \qquad \text{and} \qquad \mathfrak{e}(t) \lesssim c_2(\mathfrak{E}(t)). \tag{3.8}$$

for some universal polynomials $c_1 = c_1(x)$ and $c_2 = c_2(x)$.

In fact, as was shown in §10 in [24], we have the following characterization, which is essentially a consequence of (3.8) and Eq. (2.38), of the energy func-

[16] The assumptions in Theorems 3.1, 3.4, 3.7 and Proposition 3.3 is consistent with the completeness of the evolutionary equations (2.35)–(2.36). We mention that to obtain the wellposed-ness result, Theorem 3.9, we only apply Theorems 3.1, 3.4, 3.7 and Proposition 3.3 to solutions that satisfy in addition that $Z_{,\alpha'} \in L^\infty$.

tional $\mathfrak{E}$ in terms of the holomorphic quantities $\frac{1}{\overline{Z}_{,\alpha'}}$ and $\overline{Z}_t$. Since the proof in [24] applies to the current setting, we omit the proof.

Let

$$\mathcal{E}(t) = \|\overline{Z}_{t,\alpha'}(t)\|_{L^2}^2 + \|D_{\alpha'}^2\overline{Z}_t(t)\|_{L^2}^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^2}^2 + \left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^2}^2 + \left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\overline{Z}_t(t)\right\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'}\overline{Z}_t(t)\|_{\dot{H}^{1/2}}^2 + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2. \tag{3.9}$$

Proposition 3.3 *(*A characterization of $\mathfrak{E}$ via $\mathcal{E}$, cf. §10 of [24]*) There are polynomials $C_1 = C_1(x)$ and $C_2 = C_2(x)$, with nonnegative universal coefficients, such that for any solution Z of* (2.9)–(2.11)–(2.14)–(2.15)*, satisfying the assumption of Theorem* 3.1*,*

$$\mathcal{E}(t) \le C_1(\mathfrak{E}(t)), \qquad \text{and} \quad \mathfrak{E}(t) \le C_2(\mathcal{E}(t)). \tag{3.10}$$

A corollary of Theorem 3.1 and Proposition 3.3 is the following

Theorem 3.4 (A-priori estimate [24]) *Let $Z = Z(\cdot,t)$, $t \in [0,T']$ be a solution of the system* (2.9)–(2.11)–(2.14)–(2.15)*, satisfying the assumption of Theorem* 3.1*. There are constants $T = T(\mathcal{E}(0)) > 0$, $C = C(\mathcal{E}(0)) > 0$ that depend only on $\mathcal{E}(0)$, and with $-T(e)$, $C(e)$ increasing with respect to e, such that*

$$\sup_{[0,\min\{T,T'\}]} \mathcal{E}(t) \le C(\mathcal{E}(0)) < \infty. \tag{3.11}$$

Remark 3.5 1. Let t be fixed, $s \ge 2$, and assume $Z_t(t) \in H^s(\mathbb{R})$. By Proposition A.7 and Sobolev embeddings, $A_1(t) - 1 = -\operatorname{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} \in H^s(\mathbb{R})$; and by (2.38), $Z_{tt}(t) \in H^s(\mathbb{R})$ is equivalent to $\frac{1}{Z_{,\alpha'}}(t) - 1 \in H^s(\mathbb{R})$.

2. Assume that $\left(Z_t(t), \frac{1}{Z_{,\alpha'}}(t) - 1\right) \in (H^{s+1/2}(\mathbb{R}), H^s(\mathbb{R}))$, $s \ge 2$, or equivalently $(Z_t(t), Z_{tt}(t)) \in (H^{s+1/2}(\mathbb{R}), H^s(\mathbb{R}))$. It is easy to check that $\mathcal{E}(t) < \infty$. So in the class where $\mathcal{E}(t) < \infty$, it allows for interfaces and velocities in Sobolev classes; it is clear that in the class where $\mathcal{E}(t) < \infty$ it also allows for $\frac{1}{Z_{,\alpha'}} = 0$, that is, singularities on the interface.

3.1.1 A description of the class $\mathcal{E} < \infty$ in $\mathscr{P}_-$

We give here an equivalent description of the class $\mathcal{E} < \infty$ in the lower half plane $\mathscr{P}_-$.

Let $1 < p \le \infty$, and

$$K_y(x) = \frac{-y}{\pi(x^2+y^2)}, \qquad y < 0 \tag{3.12}$$

be the Poisson kernel. We know for any holomorphic function G on P_-,

$$\sup_{y<0} \|G(x+iy)\|_{L^p(\mathbb{R},dx)} < \infty$$

if and only if there exists $g \in L^p(\mathbb{R})$ such that $G(x+iy) = K_y * g(x)$. In this case, $\sup_{y<0} \|G(x+iy)\|_{L^p(\mathbb{R},dx)} = \|g\|_{L^p}$. Moreover, if $g \in L^p(\mathbb{R})$, $1 < p < \infty$, then $\lim_{y\to 0-} K_y * g(x) = g(x)$ in $L^p(\mathbb{R})$ and if $g \in L^\infty \cap C(\mathbb{R})$, then $\lim_{y\to 0-} K_y * g(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $Z = Z(\cdot,t)$ be a solution of (2.9)–(2.11)–(2.15)–(2.14), satisfying the assumption of Theorem 3.1; let Ψ, U be the holomorphic functions as given in Sect. 2.5, so

$$\begin{aligned} &U(x'+iy',t) = K_{y'} * \overline{Z}_t(x',t), \\ &\frac{1}{\Psi_{z'}}(x'+iy',t) = K_{y'} * \frac{1}{Z_{,\alpha'}}(x',t), \qquad \text{for } y' < 0. \end{aligned}$$

Let $z' = x' + iy'$. We have

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2, \tag{3.13}$$

where

$$\begin{aligned} \mathcal{E}_1(t) := &\sup_{y'<0} \left\|U_{z'}(\cdot+iy',t)\right\|^2_{L^2(\mathbb{R})} + \sup_{y'<0} \left\|\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}U_{z'}\right)(\cdot+iy',t)\right\|^2_{L^2(\mathbb{R})} \\ &+ \sup_{y'<0} \left\|\frac{1}{\{\Psi_{z'}\}^2}\partial_{z'}\left(\frac{1}{\Psi_{z'}}U_{z'}\right)(\cdot+iy',t)\right\|^2_{\dot{H}^{1/2}(\mathbb{R})} \\ &+ \sup_{y'<0} \left\|\frac{1}{\Psi_{z'}}U_{z'}(\cdot+iy',t)\right\|^2_{\dot{H}^{1/2}(\mathbb{R})} \\ &+ \sup_{y'<0} \left\|\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)\right)(\cdot+iy',t)\right\|^2_{L^2(\mathbb{R})} \\ &+ \sup_{y'<0} \left\|\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)(\cdot+iy',t)\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.14}$$

3.2 A blow-up criteria and a stability inequality

The main objective of this paper is to show the unique solvability of the Cauchy problem for the water wave equation (1.1) in the class where $\mathcal{E} < \infty$. We will build on the existing result, Theorem 2.3, by mollifying the initial data, constructing an approximating sequence and passing to the limit. However the existence time of the solution as given in Theorem 2.3 depends on the Sobolev norm of the initial data. In order to have an approximating sequence defined on a time interval that has a uniform positive lower bound, we need a blow-up criteria; a uniqueness and stability theorem will allow us to prove the convergence of the sequence, and the uniqueness and stability of the solutions obtained by this process.

Let the initial data be as given in Sect. 2.3.1.

Theorem 3.6 *(*A blow-up criteria via $\mathcal{E}$*) Let $s \geq 4$. Assume $Z_{,\alpha'}(0) \in L^\infty(\mathbb{R})$, $Z_t(0) \in H^{s+1/2}(\mathbb{R})$ and $Z_{tt}(0) \in H^s(\mathbb{R})$. Then there is $T > 0$, such that on $[0, T]$, the initial value problem of (2.9)–(2.11)–(2.14)–(2.15) has a unique solution $Z = Z(\cdot, t)$, satisfying $(Z_t, Z_{tt}) \in C^l([0, T], H^{s+1/2-l}(\mathbb{R}) \times H^{s-l}(\mathbb{R}))$ for $l = 0, 1$, and $Z_{,\alpha'} - 1 \in C([0, T], H^s(\mathbb{R}))$.*

Moreover if T^* is the supremum over all such times T, then either $T^* = \infty$, or $T^* < \infty$, but

$$\sup_{[0,T^*)} \mathcal{E}(t) = \infty \tag{3.15}$$

The proof for Theorem 3.6 will be given in Sect. 4. We now give the uniqueness and stability theorem.

Let $Z = Z(\alpha', t)$, $\mathfrak{Z} = \mathfrak{Z}(\alpha', t)$ be solutions of the system (2.9)–(2.11)–(2.14)–(2.15), with $z = z(\alpha, t)$, $\mathfrak{z} = \mathfrak{z}(\alpha, t)$ being their re-parametrizations in Lagrangian coordinates, and their initial data as given in Sect. 2.3.1; let

$$Z_t,\ Z_{tt},\ Z_{,\alpha'},\ z_\alpha,\ h,\ A_1,\ \mathcal{A},\ b,\ \mathfrak{a},\ D_{\alpha'},\ D_\alpha,\ \mathfrak{H},\ \mathfrak{E}(t),\ \mathcal{E}(t),\quad etc.$$

be the quantities associated with Z, z as defined in Sects. 2, 3.1, and

$$\mathfrak{Z}_t,\ \mathfrak{Z}_{tt},\ \mathfrak{Z}_{,\alpha'},\ \mathfrak{z}_\alpha,\ \tilde{h},\ \tilde{A}_1,\ \tilde{\mathcal{A}},\ \tilde{b},\ \tilde{\mathfrak{a}},\ \tilde{D}_{\alpha'},\ \tilde{D}_\alpha,\ \tilde{\mathfrak{H}},\ \tilde{\mathfrak{E}}(t),\ \tilde{\mathcal{E}}(t),\quad etc.$$

be the corresponding quantities for $\mathfrak{Z}$, $\mathfrak{z}$. Define

$$l = \tilde{h} \circ h^{-1}. \tag{3.16}$$

so $l(\alpha', 0) = \alpha'$, for $\alpha' \in \mathbb{R}$.

Theorem 3.7 *(Uniqueness and Stability in $\mathcal{E} < \infty$) Assume that Z, $\mathfrak{Z}$ are solutions of Eqs. (2.9)–(2.11)–(2.15)–(2.14), satisfying $(Z_t, Z_{tt}), (\mathfrak{Z}_t, \mathfrak{Z}_{tt}) \in C^l([0,T], H^{s+1/2-m}(\mathbb{R}) \times H^{s-m}(\mathbb{R}))$ for $m = 0, 1$, $s \geq 4$. There is a constant C, depending only on T, $\sup_{[0,T]} \mathcal{E}(t)$ and $\sup_{[0,T]} \tilde{\mathcal{E}}(t)$, such that*

$$
\begin{aligned}
&\sup_{[0,T]} \Big(\| \left(\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l \right)(t) \|_{\dot{H}^{1/2}} + \| \left(\overline{Z}_{tt} - \overline{\mathfrak{Z}}_{tt} \circ l \right)(t) \|_{\dot{H}^{1/2}} \\
&\quad + \left\| \left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l \right)(t) \right\|_{\dot{H}^{1/2}} \Big) \\
&\quad + \sup_{[0,T]} \Big(\| \left(l_{\alpha'} - 1 \right)(t) \|_{L^2} + \| D_{\alpha'} Z_t - (\tilde{D}_{\alpha'} \mathfrak{Z}_t) \circ l \|_{L^2} \\
&\quad + \| (A_1 - \tilde{A}_1 \circ l)(t) \|_{L^2} + \| (b_{\alpha'} - \tilde{b}_{\alpha'} \circ l)(t) \|_{L^2} \Big) \\
&\leq C \Big(\| \left(\overline{Z}_t - \overline{\mathfrak{Z}}_t \right)(0) \|_{\dot{H}^{1/2}} + \| \left(\overline{Z}_{tt} - \overline{\mathfrak{Z}}_{tt} \right)(0) \|_{\dot{H}^{1/2}} \\
&\quad + \left\| \left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \right)(0) \right\|_{\dot{H}^{1/2}} \Big) \\
&\quad + C \Big(\| \left(D_{\alpha'} Z_t - (\tilde{D}_{\alpha'} \mathfrak{Z}_t) \right)(0) \|_{L^2} + \left\| \left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \right)(0) \right\|_{L^\infty} \Big)
\end{aligned}
\tag{3.17}
$$

By precomposing with h, we see that inequality (3.17) effectively gives control of the differences, $z_t - \mathfrak{z}_t$, $z_{tt} - \mathfrak{z}_{tt}$ etc, in Lagrangian coordinates.

Notice that in the stability inequality (3.17), we control the $\dot{H}^{1/2}$ norms of the differences of Z_t and $\mathfrak{Z}_t \circ l$, Z_{tt} and $\mathfrak{Z}_{tt} \circ l$, and $\frac{1}{Z_{,\alpha'}}$ and $\frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l$, and the L^2 norms of the differences of $D_{\alpha'} Z_t$ and $(\tilde{D}_{\alpha'} \mathfrak{Z}_t) \circ l$, and A_1 and $\tilde{A}_1 \circ l$, while the energy functional $\mathfrak{E}(t)$, or equivalently $\mathcal{E}(t)$, gives us control of the L^2 norms of $Z_{t,\alpha'}$, $Z_{tt,\alpha'}$ and $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}$, and the L^∞ and $\dot{H}^{1/2}$ norms[17] of $D_{\alpha'} Z_t$ and A_1. Indeed, because the coefficient $\frac{A_1}{|Z_{,\alpha'}|^2}$ in Eq. (2.24) is solution dependent and possibly degenerate, for given solutions $Z = Z(\alpha', t)$, $\mathfrak{Z} = \mathfrak{Z}(\alpha', t)$ of Eqs. (2.24)–(2.25), the sets of zeros in $\frac{1}{Z_{,\alpha'}}(t)$ and $\frac{1}{\mathfrak{Z}_{,\alpha'}}(t)$ are different and move with the solutions, hence one cannot simply subtract the two solutions and perform energy estimates, as is usually done in classical cases. Our approach is to first get a good understanding of the evolution of the degenerate factor $\frac{1}{Z_{,\alpha'}}$ via Eq. (2.35), this allows us to construct a series of model equations that capture the key degenerate features of the Eq. (2.24) to get some ideas of what would work. We then tailor the ideas to the specific structure of our equations. We give the proof for Theorem 3.7 in Sect. 5.

[17] See Sects. 4.1.1 and 5.2.3.

3.3 The wellposedness of the water wave equation (1.1) in $\mathcal{E} < \infty$

Since it can be tricky to define solutions for the interface equation (2.9) when the interface is allowed to have singularities, we will directly solve the water wave equation (1.1) via the system (2.31)–(2.33)–(2.32)–(2.29). As we know from the discussions in Sects. 2.5 and 3.1.1, Eq. (1.1) is equivalent to (2.31)–(2.33)–(2.32), for $(U, \Psi) \in C(\overline{\mathscr{P}_-} \times [0, T]) \cap C^1(\mathscr{P}_- \times (0, T))$ with $U(\cdot, t)$, $\Psi(\cdot, t)$ holomorphic, provided $\Psi(\cdot, t)$ is a Jordan curve; and the energy functionals $\mathcal{E} = \mathcal{E}_1 + |\frac{1}{Z_{,\alpha'}}(0, t)|^2$. Observe that the energy functional $\mathcal{E}(t)$ does not give direct control of the lower order norms $\|Z_t(t)\|_{L^2}$, $\|Z_{tt}(t)\|_{L^2}$ and $\left\|\frac{1}{Z_{,\alpha'}}(t) - 1\right\|_{L^2(\mathbb{R})}$; in the class where we want to solve the water wave equation we require in addition that $Z_t(t) \in L^2(\mathbb{R})$ and $\frac{1}{Z_{,\alpha'}}(t) - 1 \in L^2(\mathbb{R})$. This is consistent with the decay assumption made in Sect. 1.2.

3.3.1 The initial data

Let $\Omega(0)$ be the initial fluid domain, with the interface $\Sigma(0) := \partial\Omega(0)$ being a Jordan curve that tends to horizontal lines at the infinity, and let $\Psi(\cdot, 0) : \mathscr{P}_- \to \Omega(0)$ be the Riemann Mapping such that $\lim_{z' \to \infty} \partial_{z'} \Psi(z', 0) = 1$. We know $\Psi(\cdot, 0) : \mathscr{P}_- \to \Omega(0)$ is a homeomorphism. Let $Z(\alpha', 0) := \Psi(\alpha', 0)$ for $\alpha' \in \mathbb{R}$, so $Z = Z(\cdot, 0) : \mathbb{R} \to \Sigma(0)$ is the parametrization of $\Sigma(0)$ in the Riemann Mapping variable. Let $\mathbf{v}(\cdot, 0) : \Omega(0) \to \mathbb{C}$ be the initial velocity field, and $U(z', 0) = \overline{\mathbf{v}}(\Psi(z', 0), 0)$. Assume $\overline{\mathbf{v}}(\cdot, 0)$ is holomorphic on $\Omega(0)$, so $U(\cdot, 0)$ is holomorphic on $\mathscr{P}_-$. Assume that the energy functional $\mathcal{E}_1(0)$ for $(U(\cdot, 0), \Psi(\cdot, 0))$ as given in (3.14) satisfy $\mathcal{E}_1(0) < \infty$. Assume in addition that[18]

$$c_0 := \sup_{y' < 0} \|U(\cdot + iy', 0)\|_{L^2(\mathbb{R})} + \sup_{y' < 0} \left\| \frac{1}{\Psi_{z'}(\cdot + iy', 0)} - 1 \right\|_{L^2(\mathbb{R})} < \infty. \tag{3.18}$$

In light of the discussion in Sect. 2.5 and the uniqueness and stability Theorem 3.7, we define solutions for the Cauchy problem of the system (2.31)–(2.33)–(2.32) as follows.

Definition 3.8 Let the data be as given in Sect. 3.3.1, and $(U, \Psi, \mathfrak{P}) \in C(\overline{\mathscr{P}_-} \times [0, T])$, with $(U, \Psi) \in C^1(\mathscr{P}_- \times (0, T))$, $\lim_{z' \to \infty}(U, \Psi_{z'} - 1)(z', t) = (0, 0)$ and $U(\cdot, t)$, $\Psi(\cdot, t)$ holomorphic in the lower half plane $\mathscr{P}_-$ for $t \in [0, T]$. We say $(U, \Psi, \mathfrak{P})$ is a solution of the Cauchy problem

[18] This is equivalent to $\|U(\cdot + i0, 0)\|_{L^2(\mathbb{R})} + \left\|\frac{1}{Z_{,\alpha'}}(0) - 1\right\|_{L^2(\mathbb{R})} < \infty$, see Sect. 3.1.1.

of the system (2.31)–(2.33)–(2.32), if it satisfies the system (2.31)–(2.33)–(2.32) on $\mathscr{P}_- \times [0,T]$, and if there is a sequence $Z_n = Z_n(\alpha', t)$, $(\alpha', t) \in \mathbb{R} \times [0,T]$, which are solutions of the system (2.9)–(2.11)–(2.15)–(2.14), satisfying $(Z_{n,t}, \frac{1}{\partial_{\alpha'} Z_n} - 1, \partial_{\alpha'} Z_n - 1) \in C^j([0,T], H^{s+1/2-l}(\mathbb{R}) \times H^{s-l}(\mathbb{R}) \times H^{s-l}(\mathbb{R}))$ for some $s \geq 4$, $l = 0, 1$, $\sup_{n,t\in[0,T]} \mathcal{E}_n(t) < \infty$ and $\sup_{n,t\in[0,T]}(\|Z_{n,t}(t)\|_{L^2} + \left\|\frac{1}{\partial_{\alpha'} Z_n}(t) - 1\right\|_{L^2}) < \infty$, and the holomorphic extension (U_n, Ψ_n) in $\mathscr{P}_-$ of $(\overline{Z}_{n,t}, Z_n)$, with $\lim_{z'\to\infty}(U_n, \partial_{z'}\Psi_n - 1)(z', t) = (0,0)$, and the function $\mathfrak{P}_n$ defined by (2.31)–(2.32)–(2.33), such that $\lim_{n\to\infty} U_n = U$, $\lim_{n\to\infty} \Psi_n = \Psi$, $\lim_{n\to\infty} \mathfrak{P}_n = \mathfrak{P}$ and $\lim_{n\to\infty} \frac{1}{\partial_{z'}\Psi_n} = \frac{1}{\partial_{z'}\Psi}$, uniformly on compact subsets of $\overline{\mathscr{P}}_- \times [0,T]$, and the data $(Z_n(\cdot,0), Z_{n,t}(\cdot,0))$ converges in the topology of the right hand side of the inequality (3.17) to the trace $(\Psi(\cdot + i0, 0), \overline{U}(\cdot + i0, 0))$.

Let $\mathcal{E}(0) = \mathcal{E}_1(0) + |\frac{1}{Z_{,\alpha'}}(0,0)|^2$.

Theorem 3.9 *(*Local wellposedness in the $\mathcal{E} < \infty$ regime*) 1. There exists $T > 0$, depending only on $\mathcal{E}(0)$, such that on $[0,T]$, the initial value problem of the system (2.31)–(2.33)–(2.32) has a unique solution $(U, \Psi, \mathfrak{P})$, with the properties that $U(\cdot,t)$, $\Psi(\cdot,t)$ are holomorphic on $\mathscr{P}_-$ for each fixed $t \in [0,T]$, U, Ψ, $\frac{1}{\Psi_{z'}}$, $\mathfrak{P}$ are continuous on $\overline{\mathscr{P}}_- \times [0,T]$, U, Ψ, $\mathfrak{P}$ are continuous differentiable on $\mathscr{P}_- \times [0,T]$, $\sup_{[0,T]} \mathcal{E}_1(t) < \infty$ and*

$$\sup_{[0,T]} \sup_{y'<0} \left(\|U(\cdot + iy', t)\|_{L^2(\mathbb{R})} + \|\frac{1}{\Psi_{z'}(\cdot + iy', t)} - 1\|_{L^2(\mathbb{R})} \right) < \infty. \tag{3.19}$$

The solution $(U, \Psi, \mathfrak{P})$ gives rise to a solution $(\overline{\mathbf{v}}, P) = (U \circ \Psi^{-1}, \mathfrak{P} \circ \Psi^{-1})$ of the water wave equation (1.1) so long as $\Sigma(t) = \{Z = \Psi(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is a Jordan curve.

2. If in addition that the initial interface is chord–arc, that is, $Z_{,\alpha'}(\cdot,0) \in L^1_{loc}(\mathbb{R})$ and there is $0 < \delta < 1$, such that

$$\begin{aligned} \delta \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma,0)|\, d\gamma &\leq |Z(\alpha',0) - Z(\beta',0)| \\ &\leq \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma,0)|\, d\gamma, \quad \forall -\infty < \alpha' < \beta' < \infty. \end{aligned}$$

Then there is $T > 0, T_1 > 0$, T, T_1 depend only on $\mathcal{E}(0)$, such that on $[0, \min\{T, \frac{\delta}{T_1}\}]$, the initial value problem of the water wave equation (1.1) has a unique solution, satisfying $\mathcal{E}_1(t) < \infty$ and (3.19), and the interface $Z = Z(\cdot,t)$ remains chord–arc.

We prove Theorem 3.9 in Sect. 6.

4 The proof of Theorem 3.1 and Theorem 3.6

We need the following basic inequalities in the proof of Theorems 3.1 and 3.6. The basic energy inequality in Lemma 4.1 has already appeared in [40]. We give a proof nevertheless.

Lemma 4.1 (Basic energy inequality I, cf. [40], lemma 4.1) *Assume* $\Theta = \Theta(\alpha', t)$, $\alpha' \in \mathbb{R}$, $t \in [0, T)$ *is smooth, decays fast at the spatial infinity, satisfying* $(I - \mathbb{H})\Theta = 0$ *and*

$$(\partial_t + b\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta = G_\Theta. \tag{4.1}$$

Let

$$E_\Theta(t) := \int \frac{1}{\mathcal{A}}|(\partial_t + b\partial_{\alpha'})\Theta|^2\, d\alpha' + i\int (\partial_{\alpha'}\Theta)\overline{\Theta}\, d\alpha'. \tag{4.2}$$

Then

$$\frac{d}{dt}E_\Theta(t) \le \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} E_\Theta(t) + 2E_\Theta(t)^{1/2}\left(\int \frac{|G_\Theta|^2}{\mathcal{A}}\, d\alpha'\right)^{1/2}. \tag{4.3}$$

Remark 4.2 By $\Theta = \mathbb{H}\Theta$ and (A.2),

$$i\int (\partial_{\alpha'}\Theta)\overline{\Theta}\, d\alpha' = \int (i\partial_{\alpha'}\mathbb{H}\Theta)\overline{\Theta}\, d\alpha' = \|\Theta\|^2_{\dot{H}^{1/2}} \ge 0. \tag{4.4}$$

Proof By a change of the variables in (4.1), we have

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)(\Theta \circ h) = G_\Theta \circ h$$

where $\mathfrak{a}h_\alpha = \mathcal{A} \circ h$; and in (4.2),

$$E_\Theta(t) = \int \frac{1}{\mathfrak{a}}|\partial_t(\Theta \circ h)|^2\, d\alpha + \int i\,\partial_\alpha(\Theta \circ h)\overline{\Theta \circ h}\, d\alpha.$$

So

$$\begin{aligned}\frac{d}{dt}E_\Theta(t) &= \int 2\,\mathrm{Re}\left\{\frac{1}{\mathfrak{a}}\partial_t^2(\Theta\circ h)\partial_t(\overline{\Theta\circ h})\right\} - \frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\partial_t(\Theta\circ h)|^2 \\ &\quad + 2\,\mathrm{Re}\left\{i\int \partial_\alpha(\Theta\circ h)\partial_t(\overline{\Theta\circ h})\,d\alpha\right\} \\ &= 2\,\mathrm{Re}\int \frac{1}{\mathfrak{a}}G_\Theta\circ h\partial_t(\overline{\Theta\circ h})\,d\alpha - \int\frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\partial_t(\Theta\circ h)|^2\,d\alpha,\end{aligned} \tag{4.5}$$

where we used integration by parts in the first step. Changing back to the Riemann mapping variable, applying Cauchy–Schwarz inequality and (4.4) yields (4.3). □

We also need the following simple energy inequality.

Lemma 4.3 (Basic energy inequality II) *Assume $\Theta = \Theta(\alpha', t)$ is smooth and decays fast at the spatial infinity. And assume*

$$(\partial_t + b\partial_{\alpha'})\Theta = g_\Theta. \tag{4.6}$$

Then

$$\frac{d}{dt}\|\Theta(t)\|_{L^2}^2 \le 2\,\|g_\Theta(t)\|_{L^2}\,\|\Theta(t)\|_{L^2} + \|b_{\alpha'}(t)\|_{L^\infty}\,\|\Theta(t)\|_{L^2}^2 \tag{4.7}$$

Proof We have, upon changing variables,

$$\int |\Theta(\alpha', t)|^2\,d\alpha' = \int |\Theta(h(\alpha, t), t)|^2 h_\alpha\,d\alpha,$$

so

$$\begin{aligned}\frac{d}{dt}\int |\Theta(\alpha', t)|^2\,d\alpha' &= \int 2\,\mathrm{Re}\,\partial_t(\Theta\circ h)\overline{\Theta\circ h}\,h_\alpha + |\Theta\circ h|^2 h_{t\alpha}\,d\alpha \\ &= \int 2\,\mathrm{Re}\,((\partial_t + b\partial_{\alpha'})\Theta)\,\overline{\Theta}(\alpha', t) + b_{\alpha'}|\Theta(\alpha', t)|^2\,d\alpha';\end{aligned} \tag{4.8}$$

here in the second step we changed back to the Riemann mapping variable, and used the fact that $\frac{h_{t\alpha}}{h_\alpha} = b_{\alpha'}\circ h$. Inequality (4.7) follows from Cauchy–Schwarz inequality. □

Let

$$\mathcal{P} = (\partial_t + b\partial_{\alpha'})^2 + i\mathcal{A}\partial_{\alpha'}. \tag{4.9}$$

We need two more basic inequalities.

Lemma 4.4 (Basic inequality III) *Assume that $\Theta = \Theta(\alpha', t)$ is smooth and decays fast at the spatial infinity, and assume $\Theta = \mathbb{H}\Theta$. Then*

$$\begin{aligned}\|(I-\mathbb{H})(\mathcal{P}\Theta)(t)\|_{L^2} &\le \|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b\|_{L^\infty}\|\Theta(t)\|_{L^2}\\ &+ \|b_{\alpha'}\|_{L^\infty}\|(\partial_t + b\partial_{\alpha'})\Theta(t)\|_{L^2} + \|b_{\alpha'}\|_{L^\infty}^2\|\Theta(t)\|_{L^2} + \|\mathcal{A}_{\alpha'}\|_{L^\infty}\|\Theta(t)\|_{L^2}. \quad (4.10)\end{aligned}$$

Proof Because $\Theta = \mathbb{H}\Theta$, we have

$$(I-\mathbb{H})(\mathcal{P}\Theta) = [\mathcal{P}, \mathbb{H}]\,\Theta;$$

and by (B.22),

$$\begin{aligned}[\mathcal{P}, \mathbb{H}]\,\Theta &= [(\partial_t + b\partial_{\alpha'})b, \mathbb{H}]\,\partial_{\alpha'}\Theta + 2\,[b, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\Theta\\ &-[b, b;\, \partial_{\alpha'}\Theta] + [i\mathcal{A}, \mathbb{H}]\,\partial_{\alpha'}\Theta.\end{aligned}$$

Inequality (4.10) follows from (A.14). □

Lemma 4.5 (Basic inequality IV) *Assume f is smooth and decays fast at the spatial infinity. Then*

$$\begin{aligned}&\left\| Z_{,\alpha'}\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right] f\right\|_{L^2}\\ &\lesssim \|(\partial_t + b\partial_{\alpha'})(b_{\alpha'} - D_{\alpha'}Z_t)\|_{L^\infty}\|f\|_{L^2}\\ &\quad + \|(b_{\alpha'} - D_{\alpha'}Z_t)\|_{L^\infty}^2\|f\|_{L^2} + \|(b_{\alpha'} - D_{\alpha'}Z_t)\|_{L^\infty}\|(\partial_t + b\partial_{\alpha'})f\|_{L^2}\\ &+ \|A_1\|_{L^\infty}\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\|f\|_{L^2}. \quad (4.11)\end{aligned}$$

Proof Lemma 4.5 is straightforward from the commutator relation (B.25), identities (B.26), (B.27) and the definition $A_1 := \mathcal{A}\left|Z_{,\alpha'}\right|^2$. □

Let $Z = Z(\cdot, t)$ be a solution of the system (2.9)–(2.11)–(2.14)–(2.15), satisfying the assumption of Theorem 3.1. By (2.19) and (B.15), we have

$$\begin{aligned}\mathcal{P}\overline{Z}_{t,\alpha'} &= -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt} - i\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_t\\ &+ \partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt} - i)\right) \quad (4.12)\end{aligned}$$

Equation (4.12) is our base equation in the proof of Theorems 3.1 and 3.6.

4.1 The proof of Theorem 3.1

We begin with computing a few evolutionary equations. We have

$$\mathcal{P}D_{\alpha'}\overline{Z}_t = \left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} + \frac{1}{Z_{,\alpha'}}\mathcal{P}\overline{Z}_{t,\alpha'}; \tag{4.13}$$

$$\mathcal{P}\left(\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right) = \left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]D^2_{\alpha'}\overline{Z}_t + \frac{1}{Z_{,\alpha'}}\left[\mathcal{P}, D^2_{\alpha'}\right]\overline{Z}_t + \frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\mathcal{P}\overline{Z}_t. \tag{4.14}$$

And, by the commutator identity (B.12) and the fact that $(\partial_t + b\partial_{\alpha'})\overline{Z}_t = \overline{Z}_{tt}$,

$$(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'} = \overline{Z}_{tt,\alpha'} - b_{\alpha'}\overline{Z}_{t,\alpha'}; \tag{4.15}$$

and by (2.35) and (B.12)

$$\begin{aligned}
&(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \\
&= \partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} + [(\partial_t + b\partial_{\alpha'}), \partial_{\alpha'}]\frac{1}{Z_{,\alpha'}} \\
&= \left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(b_{\alpha'} - D_{\alpha'}Z_t) + D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t) - b_{\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \\
&= -D_{\alpha'}Z_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t).
\end{aligned} \tag{4.16}$$

We know from the definition of $\mathbf{E}_a(t)$, $\mathbf{E}_b(t)$, and $A_1 := \mathcal{A}|Z_{,\alpha'}|^2$,

$$\mathbf{E}_a(t) := E_{D_{\alpha'}\overline{Z}_t}(t), \qquad \text{and} \quad \mathbf{E}_b(t) := E_{\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t}(t),$$

where $E_\Theta(t)$ is the basic energy as defined in (4.2). Notice that the quantities $D_{\alpha'}\overline{Z}_t$ and $\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t$ are holomorphic. So the energy functional

$$\begin{aligned}
\mathfrak{E}(t) = {} & E_{D_{\alpha'}\overline{Z}_t}(t) + E_{\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t}(t) + \|\overline{Z}_{t,\alpha'}(t)\|^2_{L^2} \\
& + \|D^2_{\alpha'}\overline{Z}_t(t)\|^2_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|^2_{L^2} + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2.
\end{aligned} \tag{4.17}$$

Our goal is to show that there is a universal polynomial $C = C(x)$, such that

$$\frac{d}{dt}\mathfrak{E}(t) \le C(\mathfrak{E}(t)). \tag{4.18}$$

We begin with a list of quantities controlled by $\mathfrak{E}(t)$.

4.1.1 *Quantities controlled by* $\mathfrak{E}(t)$

It is clear that $\mathfrak{E}(t)$ controls the following quantities:

$$\|D_{\alpha'}\overline{Z}_t\|_{\dot{H}^{1/2}}, \quad \left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{1/2}}, \quad \|\overline{Z}_{t,\alpha'}\|_{L^2}, \quad \|D^2_{\alpha'}\overline{Z}_t\|_{L^2},$$
$$\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \quad \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|. \tag{4.19}$$

By (A.18) and (2.15),

$$1 \le A_1, \qquad \textit{and} \quad \|A_1\|_{L^\infty} \lesssim 1 + \|\overline{Z}_{t,\alpha'}\|^2_{L^2} \le 1 + \mathfrak{E}. \tag{4.20}$$

We also have, by (2.22) and (A.18), that

$$\|b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'}Z_t\|_{L^\infty} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|\overline{Z}_{t,\alpha'}\|_{L^2} \le \mathfrak{E}. \tag{4.21}$$

We now estimate $\|D_{\alpha'}Z_t\|_{L^\infty}$. We have, by the fundamental Theorem of calculus,

$$\left(D_{\gamma'}\overline{Z}_t\right)^2 - \int_0^1 \left(D_{\beta'}\overline{Z}_t\right)^2 d\beta' = 2\int_0^1\int_{\beta'}^{\gamma'} D_{\alpha'}\overline{Z}_t\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\, d\alpha' d\beta'$$
$$= 2\int_0^1\int_{\beta'}^{\gamma'} \partial_{\alpha'}\overline{Z}_t D^2_{\alpha'}\overline{Z}_t\, d\alpha' d\beta', \tag{4.22}$$

where in the last equality, we moved $\frac{1}{Z_{,\alpha'}}$ from the first to the second factor. So for any $\gamma' \in \mathbb{R}$,

$$\left|\left(D_{\gamma'}\overline{Z}_t(\gamma',t)\right)^2 - \int_0^1 \left(D_{\beta'}\overline{Z}_t(\beta',t)\right)^2 d\beta'\right| \le 2\|\overline{Z}_{t,\alpha'}\|_{L^2}\|D^2_{\alpha'}\overline{Z}_t\|_{L^2} \le 2\mathfrak{E}. \tag{4.23}$$

Now by the fundamental Theorem of calculus and Cauchy–Schwarz inequality we have, for $\beta' \in [0,1]$,

$$\left|\frac{1}{Z_{,\beta'}}(\beta',t) - \frac{1}{Z_{,\alpha'}}(0,t)\right| \le \int_0^1 \left|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right| d\alpha' \le \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.24}$$

so

$$\left|\int_0^1 \left(D_{\beta'}\overline{Z}_t\right)^2 d\beta'\right| \le \left\|\frac{1}{Z_{,\alpha'}}\right\|^2_{L^\infty[0,1]} \|\overline{Z}_{t,\alpha'}\|^2_{L^2}$$
$$\le \left(\left|\frac{1}{Z_{,\alpha'}}(0,t)\right| + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)^2 \|\overline{Z}_{t,\alpha'}\|^2_{L^2} \lesssim \mathfrak{E}^2.$$

Combining the above argument, we get

$$\|D_{\alpha'}Z_t\|_{L^\infty} = \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{4.25}$$

This together with (4.21) gives us

$$\|b_{\alpha'}\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{4.26}$$

We now explore the remaining terms in $\mathbf{E}_a(t)$ and $\mathbf{E}_b(t)$. We know

$$\mathbf{E}_a(t) = \int \frac{1}{A_1}|Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_t|^2\, d\alpha' + \left\|D_{\alpha'}\overline{Z}_t\right\|^2_{\dot{H}^{1/2}}. \tag{4.27}$$

Now by (2.35), product rules and (4.15), [19]

$$Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_t = (b_{\alpha'} - D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'} + (\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'}$$
$$= \overline{Z}_{tt,\alpha'} - (D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'}; \tag{4.28}$$

so

$$\|Z_{tt,\alpha'}\|_{L^2} \le \|D_{\alpha'}Z_t\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2} + \|Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_t\|_{L^2}$$
$$\le \|D_{\alpha'}Z_t\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2} + (\|A_1\|_{L^\infty}\mathbf{E}_a)^{1/2} \lesssim C(\mathfrak{E}). \tag{4.29}$$

Similarly,

$$\mathbf{E}_b(t) = \int \frac{1}{A_1}\left|Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right)\right|^2 d\alpha' + \left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\overline{Z}_t\right\|^2_{\dot{H}^{1/2}}, \tag{4.30}$$

[19] One can also compute by changing to the Lagrangian coordinate and using the commutator relation (B.1).

and by product rule and (2.35),

$$Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t\right) = (b_{\alpha'} - D_{\alpha'} Z_t) D^2_{\alpha'} \overline{Z}_t + (\partial_t + b\partial_{\alpha'}) D^2_{\alpha'} \overline{Z}_t; \tag{4.31}$$

so

$$\begin{aligned} &\left\| (\partial_t + b\partial_{\alpha'}) D^2_{\alpha'} \overline{Z}_t \right\|_{L^2} \\ &\le \left\| Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} D^2_{\alpha'} \overline{Z}_t\right) \right\|_{L^2} \\ &\quad + \|b_{\alpha'} - D_{\alpha'} Z_t\|_{L^\infty} \|D^2_{\alpha'} \overline{Z}_t\|_{L^2} \lesssim C(\mathfrak{E}). \end{aligned} \tag{4.32}$$

Now from

$$D^2_{\alpha'} Z_t = \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} D_{\alpha'} Z_t + \frac{1}{Z^2_{,\alpha'}} \partial^2_{\alpha'} Z_t, \tag{4.33}$$

$$D^2_{\alpha'} \overline{Z}_t = \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} D_{\alpha'} \overline{Z}_t + \frac{1}{Z^2_{,\alpha'}} \partial^2_{\alpha'} \overline{Z}_t, \tag{4.34}$$

we have

$$\|D^2_{\alpha'} Z_t\|_{L^2} \le 2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|D_{\alpha'} Z_t\|_{L^\infty} + \|D^2_{\alpha'} \overline{Z}_t\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.35}$$

Commuting $\partial_t + b\partial_{\alpha'}$ with $D^2_{\alpha'}$ by (B.7), we get

$$D^2_{\alpha'} \overline{Z}_{tt} = (\partial_t + b\partial_{\alpha'}) D^2_{\alpha'} \overline{Z}_t + 2(D_{\alpha'} Z_t) D^2_{\alpha'} \overline{Z}_t + (D^2_{\alpha'} Z_t) D_{\alpha'} \overline{Z}_t; \tag{4.36}$$

by (4.19), (4.25), (4.35) and (4.32), we have

$$\|D^2_{\alpha'} \overline{Z}_{tt}\|_{L^2} \le C(\mathfrak{E}). \tag{4.37}$$

From (4.37) and (4.29), we can work through the same argument as from (4.22) to (4.25) and get

$$\|D_{\alpha'} Z_{tt}\|_{L^\infty} = \|D_{\alpha'} \overline{Z}_{tt}\|_{L^\infty} \lesssim C(\mathfrak{E}); \tag{4.38}$$

and then by a similar calculation as in (4.33)–(4.34) and (4.37), (4.38),

$$\|D_{\alpha'}^2 Z_{tt}\|_{L^2} \le C(\mathfrak{E}). \tag{4.39}$$

Additionally, by (4.15),

$$\|(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'}\|_{L^2} \le \|Z_{tt,\alpha'}\|_{L^2} + \|b_{\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.40}$$

Sum up the estimates from (4.19) through (4.40), we have that the following quantities are controlled by $\mathfrak{E}$:

$$\begin{aligned} &\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{1/2}}, \quad \left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\overline{Z}_t\right\|_{\dot{H}^{1/2}}, \quad \|\overline{Z}_{t,\alpha'}\|_{L^2}, \quad \|D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \\ &\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}, \quad \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|, \\ &\|A_1\|_{L^\infty}, \quad \|b_{\alpha'}\|_{L^\infty}, \quad \|D_{\alpha'}Z_t\|_{L^\infty}, \quad \|D_{\alpha'}Z_{tt}\|_{L^\infty}, \quad \|(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'}\|_{L^2} \\ &\|Z_{tt,\alpha'}\|_{L^2}, \quad \|D_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}, \quad \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \quad \|(\partial_t + b\partial_{\alpha'})D_{\alpha'}^2\overline{Z}_t\|_{L^2}, \quad \|D_{\alpha'}^2 Z_t\|_{L^2}. \end{aligned} \tag{4.41}$$

We will use Lemmas 4.1–4.5 to do estimates. Hence we need to control the quantities that appear on the right hand sides of the inequalities in these Lemmas.

4.1.2 Controlling $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right\|_{L^\infty}$ *and* $\|(\partial_t + b\partial_{\alpha'})A_1\|_{L^\infty}$

By (2.21),

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{(\partial_t + b\partial_{\alpha'})A_1}{A_1} + b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'}Z_t.$$

We have controlled $\|b_{\alpha'}\|_{L^\infty}$ and $\|D_{\alpha'}Z_t\|_{L^\infty}$ in Sect. 4.1.1. We are left with the quantity $\|(\partial_t + b\partial_{\alpha'})A_1\|_{L^\infty}$. By (2.23),

$$(\partial_t + b\partial_{\alpha'})A_1 = -\,\mathrm{Im}\left([Z_{tt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{tt} - [Z_t, b; \overline{Z}_{t,\alpha'}]\right).$$

Applying (A.18) to the first two terms and (A.22) to the last we get

$$\|(\partial_t + b\partial_{\alpha'})A_1\|_{L^\infty} \lesssim \|Z_{tt,\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2} + \|b_{\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}^2 \lesssim C(\mathfrak{E}); \tag{4.42}$$

consequently

$$\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} \leq \|(\partial_t + b\partial_{\alpha'})A_1\|_{L^\infty} + \|b_{\alpha'}\|_{L^\infty} + 2\|D_{\alpha'} Z_t\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{4.43}$$

4.1.3 Controlling $\|\mathcal{A}_{\alpha'}\|_{L^\infty}$ and $\left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$

By (2.9), we have

$$i\mathcal{A} = \frac{Z_{tt} + i}{Z_{,\alpha'}}. \tag{4.44}$$

Differentiating with respect to α' yields

$$i\mathcal{A}_{\alpha'} = (Z_{tt} + i)\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} + D_{\alpha'} Z_{tt}. \tag{4.45}$$

Apply $I - \mathbb{H}$ to both sides of the equation and use the fact that $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = \mathbb{H}\left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right)$ to rewrite the first term on the right hand side as a commutator, we get

$$i(I - \mathbb{H})\mathcal{A}_{\alpha'} = [Z_{tt}, \mathbb{H}]\, \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} + (I - \mathbb{H}) D_{\alpha'} Z_{tt}. \tag{4.46}$$

Notice that $\mathcal{A}_{\alpha'}$ is purely real, so $\mathrm{Im}\,(i(I - \mathbb{H})\mathcal{A}_{\alpha'}) = \mathcal{A}_{\alpha'}$, and $|\mathcal{A}_{\alpha'}| \leq |\,(i(I - \mathbb{H})\mathcal{A}_{\alpha'})\,|$. Therefore,

$$|\mathcal{A}_{\alpha'}| \leq \left| [Z_{tt}, \mathbb{H}]\, \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right| + 2|D_{\alpha'} Z_{tt}| + |(I + \mathbb{H}) D_{\alpha'} Z_{tt}|. \tag{4.47}$$

We estimate the first term by (A.18),

$$\left\| [Z_{tt}, \mathbb{H}]\, \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{tt,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2},$$

and the second term has been controlled in Sect. 4.1.1. We are left with the third term, $(I + \mathbb{H}) D_{\alpha'} Z_{tt}$. We rewrite it by commuting out $\frac{1}{Z_{,\alpha'}}$:

$$(I + \mathbb{H}) D_{\alpha'} Z_{tt} = D_{\alpha'} (I + \mathbb{H}) Z_{tt} - \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] Z_{tt,\alpha'}, \tag{4.48}$$

where we can estimate the second term by (A.18). For the first term, we know $(I+\mathbb{H})Z_t = 0$ because $(I-\mathbb{H})\overline{Z}_t = 0$ and $\mathbb{H}$ is purely imaginary; and $Z_{tt} = (\partial_t + b\partial_{\alpha'})Z_t$. So

$$(I+\mathbb{H})Z_{tt} = -[\partial_t + b\partial_{\alpha'}, \mathbb{H}]Z_t = -[b, \mathbb{H}]Z_{t,\alpha'}. \tag{4.49}$$

We further rewrite it by (2.14):

$$b = \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) + \mathbb{P}_H\left(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right) = \frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} + \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}} - \frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right), \tag{4.50}$$

Prop A.1, the fact that $(I+\mathbb{H})Z_{t,\alpha'} = 0$ and $(I+\mathbb{H})\overline{D_{\alpha'}\overline{Z}_t} = 0$. We have

$$(I+\mathbb{H})Z_{tt} = -\left[\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} = -\left[\overline{Z}_t, \mathbb{H}\right]\overline{D_{\alpha'}\overline{Z}_t}. \tag{4.51}$$

We have reduced the task of estimating $D_{\alpha'}(I+\mathbb{H})Z_{tt}$ to estimating $D_{\alpha'}\left[\overline{Z}_t, \mathbb{H}\right]\overline{D_{\alpha'}\overline{Z}_t}$. We compute, for general functions f and g,

$$\partial_{\alpha'}[f, \mathbb{H}]g = f_{\alpha'}\mathbb{H}g - \frac{1}{\pi i}\int \frac{(f(\alpha') - f(\beta'))}{(\alpha'-\beta')^2} g(\beta')\, d\beta' \tag{4.52}$$

therefore

$$\begin{aligned} D_{\alpha'}[f, \mathbb{H}]g \;=\; & \frac{1}{Z_{,\alpha'}} f_{\alpha'}\mathbb{H}g \\ & -\frac{1}{\pi i}\int \frac{\left(f(\alpha') - f(\beta')\right)\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\beta'}}\right)}{(\alpha'-\beta')^2} g(\beta')\, d\beta' \\ & -\frac{1}{\pi i}\int \frac{(f(\alpha') - f(\beta'))}{(\alpha'-\beta')^2}\frac{1}{Z_{,\beta'}} g(\beta')\, d\beta'. \end{aligned} \tag{4.53}$$

Now using (4.51), (4.53), and the fact that $(I+\mathbb{H})\overline{D_{\alpha'}\overline{Z}_t} = 0$, we have

$$\begin{aligned} D_{\alpha'}(I+\mathbb{H})Z_{tt} = & \left|D_{\alpha'}\overline{Z}_t\right|^2 \\ & + \frac{1}{\pi i}\int \frac{\left(\overline{Z}_t(\alpha') - \overline{Z}_t(\beta')\right)\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\beta'}}\right)}{(\alpha'-\beta')^2}\overline{D_{\beta'}\overline{Z}_t}\, d\beta' \\ & + \frac{1}{\pi i}\int \frac{(\overline{Z}_t(\alpha') - \overline{Z}_t(\beta'))}{(\alpha'-\beta')^2}\frac{1}{Z_{,\beta'}}\overline{D_{\beta'}\overline{Z}_t}\, d\beta', \end{aligned} \tag{4.54}$$

where we rewrite the third term further

$$\begin{aligned}
&\frac{1}{\pi i}\int \frac{(\overline{Z}_t(\alpha')-\overline{Z}_t(\beta'))}{(\alpha'-\beta')^2}\frac{1}{Z_{,\beta'}}\overline{D_{\beta'}Z_t}\,d\beta' \\
&\quad=\frac{1}{\pi i}\int \frac{(\overline{Z}_t(\alpha')-\overline{Z}_t(\beta'))}{(\alpha'-\beta')^2}\left(\frac{1}{Z_{,\beta'}}\overline{D_{\beta'}Z_t}-\frac{1}{Z_{,\alpha'}}\overline{D_{\alpha'}Z_t}\right)d\beta' \\
&\quad+\frac{1}{Z_{,\alpha'}}\overline{D_{\alpha'}\overline{Z}_t}\,\overline{Z}_{t,\alpha'};
\end{aligned} \tag{4.55}$$

here we simplified the second term on the right hand side by the fact that $\overline{Z}_t=\mathbb{H}\overline{Z}_t$.

We can now estimate $\|D_{\alpha'}(I+\mathbb{H})Z_{tt}\|_{L^\infty}$. We apply (A.23) to the second term on the right side of (4.54); for the third term we use (4.55), and apply (A.23) to the first term on the right hand side of (4.55), and notice that

$$\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\overline{D_{\alpha'}\overline{Z}_t}\right)=\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\right)\overline{D_{\alpha'}\overline{Z}_t}+D_{\alpha'}\overline{D_{\alpha'}\overline{Z}_t}; \tag{4.56}$$

we have

$$\begin{aligned}
\|D_{\alpha'}(I+\mathbb{H})Z_{tt}\|_{L^\infty}&\lesssim \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}^2 \\
&+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}+\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}.
\end{aligned} \tag{4.57}$$

Sum up the calculations from (4.47) through (4.57), and use the estimates in Sect. 4.1.1, we conclude

$$\|\mathcal{A}_{\alpha'}\|_{L^\infty}\lesssim C(\mathfrak{E}). \tag{4.58}$$

Observe that the same argument also gives, by taking the real parts in (4.46),

$$\|\mathbb{H}\mathcal{A}_{\alpha'}\|_{L^\infty}\lesssim C(\mathfrak{E}). \tag{4.59}$$

Now from (4.45) and (2.38),

$$\frac{iA_1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}=i\mathcal{A}_{\alpha'}-D_{\alpha'}Z_{tt}; \tag{4.60}$$

Because $A_1\geq 1$, we have

$$\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\leq\|\mathcal{A}_{\alpha'}\|_{L^\infty}+\|D_{\alpha'}Z_{tt}\|_{L^\infty}\lesssim C(\mathfrak{E}). \tag{4.61}$$

4.1.4 Controlling $\left\| \partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ *and* $\left\| (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \right\|_{L^2}$

We begin with (4.16), and rewrite the second term on the right hand side to get

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} &= -D_{\alpha'}Z_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + D_{\alpha'}\left(b_{\alpha'} - D_{\alpha'}Z_t\right)\\ &= -D_{\alpha'}Z_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + D_{\alpha'}\left(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t\right) + D_{\alpha'}\overline{D_{\alpha'}Z_t}.\end{aligned} \tag{4.62}$$

We control the first and third terms by

$$\left\| D_{\alpha'}Z_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{L^2} \le \|D_{\alpha'}Z_t\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim C(\mathfrak{E}) \tag{4.63}$$

and

$$\left\|D_{\alpha'}\overline{D_{\alpha'}Z_t}\right\|_{L^2} = \left\|D_{\alpha'}^2 Z_t\right\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.64}$$

We are left with the term $D_{\alpha'}\left(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t\right)$. We begin with (2.22):

$$b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t = \mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right). \tag{4.65}$$

Notice that the right hand side consists of $\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'}$, $[Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$ and their complex conjugates. We use (4.53) to compute

$$\begin{aligned}D_{\alpha'}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} &= -\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}D_{\alpha'}Z_t\\ &-\frac{1}{\pi i}\int \frac{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\beta'}}\right)^2}{(\alpha'-\beta')^2}Z_{t,\beta'}\,d\beta' - \frac{1}{\pi i}\int \frac{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\beta'}}\right)}{(\alpha'-\beta')^2}D_{\beta'}Z_t\,d\beta'.\end{aligned} \tag{4.66}$$

Applying (A.21) to the second term and (A.12) to the third term yields

$$\left\|D_{\alpha'}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'}\right\|_{L^2} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|D_{\alpha'}Z_t\|_{L^\infty} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\|Z_{t,\alpha'}\|_{L^2}. \tag{4.67}$$

Similarly

$$D_{\alpha'}[Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = Z_{t,\alpha'}\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - \frac{1}{\pi i}\int \frac{\big(Z_t(\alpha') - Z_t(\beta')\big)\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{Z_{,\beta'}}\right)}{(\alpha'-\beta')^2}\partial_{\beta'}\frac{1}{Z_{,\beta'}}\,d\beta' - \frac{1}{\pi i}\int \frac{\big(Z_t(\alpha') - Z_t(\beta')\big)}{(\alpha'-\beta')^2}\frac{1}{Z_{,\beta'}}\partial_{\beta'}\frac{1}{Z_{,\beta'}}\,d\beta', \quad (4.68)$$

and applying (A.21) to the second term and (A.12) to the third term yields

$$\left\| D_{\alpha'}[Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \left\|Z_{t,\alpha'}\right\|_{L^2} + \left\|Z_{t,\alpha'}\right\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}. \quad (4.69)$$

The estimate of the complex conjugate terms is similar, we omit. This concludes, with an application of the results in Sects. 4.1.1 and 4.1.3, that

$$\|D_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\|_{L^2} \lesssim C(\mathfrak{E}), \quad (4.70)$$

therefore

$$\left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim C(\mathfrak{E}). \quad (4.71)$$

Now by

$$\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + b_{\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}, \quad (4.72)$$

we also have

$$\left\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim C(\mathfrak{E}). \quad (4.73)$$

4.1.5 Controlling $\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b\|_{L^\infty}$, $\|(\partial_t + b\partial_{\alpha'})b_{\alpha'}\|_{L^\infty}$ and $\|(\partial_t + b\partial_{\alpha'})D_{\alpha'}Z_t\|_{L^\infty}$

We apply (B.19) to (4.65) and get

$$\begin{aligned}
&(\partial_t + b\partial_{\alpha'})\,(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\\
&\quad = \mathrm{Re}\left(\left[(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} + \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{tt,\alpha'} - \left[\frac{1}{Z_{,\alpha'}}, b; Z_{t,\alpha'}\right]\right)\\
&\qquad + \mathrm{Re}\left([Z_{tt}, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} - \left[Z_t, b; \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right]\right);
\end{aligned} \tag{4.74}$$

using (A.18), (A.22) and results from previous subsections we obtain

$$\begin{aligned}
&\|(\partial_t + b\partial_{\alpha'})\,(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\|_{L^\infty}\\
&\lesssim \left\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \left\|Z_{t,\alpha'}\right\|_{L^2}\\
&\quad + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \left\|Z_{tt,\alpha'}\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|b_{\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\\
&\lesssim C(\mathfrak{E}).
\end{aligned} \tag{4.75}$$

We now compute $(\partial_t + b\partial_{\alpha'})D_{\alpha'}Z_t$. By (B.6),

$$(\partial_t + b\partial_{\alpha'})D_{\alpha'}Z_t = D_{\alpha'}Z_{tt} - (D_{\alpha'}Z_t)^2\,. \tag{4.76}$$

So by the estimates in Sect. 4.1.1, we have

$$\|(\partial_t + b\partial_{\alpha'})D_{\alpha'}Z_t\|_{L^\infty} \le \|D_{\alpha'}Z_{tt}\|_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty}^2 \lesssim C(\mathfrak{E}). \tag{4.77}$$

This combine with (4.75) yields

$$\|(\partial_t + b\partial_{\alpha'})b_{\alpha'}\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{4.78}$$

From $\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b = (\partial_t + b\partial_{\alpha'})b_{\alpha'} + (b_{\alpha'})^2$,

$$\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b\|_{L^\infty} \le \|(\partial_t + b\partial_{\alpha'})b_{\alpha'}\|_{L^\infty} + \|b_{\alpha'}\|_{L^\infty}^2 \lesssim C(\mathfrak{E}). \tag{4.79}$$

We are now ready to estimate $\frac{d}{dt}\mathfrak{E}$.

4.1.6 Controlling $\frac{d}{dt}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}^2$, $\frac{d}{dt}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}^2$, $\frac{d}{dt}\left\|\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{L^2}^2$ *and* $\frac{d}{dt}\left|\frac{1}{\overline{Z}_{,\alpha'}}(0,t)\right|^2$

We use Lemma 4.3 to control $\frac{d}{dt}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}^2$, $\frac{d}{dt}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}^2$ and $\frac{d}{dt}\left\|\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{L^2}^2$. Notice that when we substitute

$$\Theta = Z_{t,\alpha'}, \qquad \Theta = D_{\alpha'}^2\overline{Z}_t, \qquad \text{and} \quad \Theta = \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$$

in (4.7), all the terms on the right hand sides are already controlled in Sects. 4.1.1 and 4.1.4. So we have

$$\frac{d}{dt}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}^2 + \frac{d}{dt}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}^2 + \frac{d}{dt}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \lesssim C(\mathfrak{E}). \tag{4.80}$$

To estimate $\frac{d}{dt}\left|\frac{1}{\overline{Z}_{,\alpha'}}(0,t)\right|^2$, we start with (2.35) and compute

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})\left|\frac{1}{Z_{,\alpha'}}\right|^2 &= 2\,\mathrm{Re}\left(\frac{1}{\overline{Z}_{,\alpha'}}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right)\\ &= \left|\frac{1}{Z_{,\alpha'}}\right|^2 (2b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\,.\end{aligned} \tag{4.81}$$

Recall we chose the Riemann mapping so that $h(0,t) = 0$ for all t. So $h_t \circ h^{-1}(0,t) = b(0,t) = 0$ and

$$\frac{d}{dt}\left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2 = \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|^2 (2b_{\alpha'}(0,t) - 2\,\mathrm{Re}\,D_{\alpha'}Z_t(0,t)) \lesssim C(\mathfrak{E}). \tag{4.82}$$

We use Lemma 4.1 to estimate the two main terms $\frac{d}{dt}\mathbf{E}_a(t)$ and $\frac{d}{dt}\mathbf{E}_b(t)$.

4.1.7 Controlling $\frac{d}{dt}\mathbf{E}_a(t)$

We begin with $\frac{d}{dt}\mathbf{E}_a(t)$. Apply Lemma 4.1 to $\Theta = D_{\alpha'}\overline{Z}_t$ we get

$$\frac{d}{dt}\mathbf{E}_a(t) \le \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^\infty}\mathbf{E}_a(t) + 2\mathbf{E}_a(t)^{1/2}\left(\int\frac{|\mathcal{P}\mathcal{D}_{\alpha'}\overline{Z}_t|^2}{\mathcal{A}}\,d\alpha'\right)^{1/2}. \tag{4.83}$$

By (4.43), we know the first term is controlled by $C(\mathfrak{E})$. We need to estimate the factor $\left(\int \frac{|\mathcal{P}D_{\alpha'}\overline{Z}_t|^2}{\mathcal{A}}\,d\alpha'\right)^{1/2}$ in the second term. By (4.13):

$$\mathcal{P}D_{\alpha'}\overline{Z}_t = \left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} + \frac{1}{Z_{,\alpha'}}\mathcal{P}\overline{Z}_{t,\alpha'}, \tag{4.84}$$

and we have

$$\begin{aligned}\int \frac{|\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\,d\alpha' &= \int \frac{|Z_{,\alpha'}\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}|^2}{A_1}\,d\alpha' \\ &\le \left\|Z_{,\alpha'}\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_{L^2} \lesssim C(\mathfrak{E}),\end{aligned} \tag{4.85}$$

here in the last step we used (4.11), notice that all the terms on the right hand side of (4.11) with $f = \overline{Z}_{t,\alpha'}$ are controlled in Sects. 4.1.1–4.1.5. We are left with the term $\int \frac{|\frac{1}{Z_{,\alpha'}}\mathcal{P}\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\,d\alpha'$. Because $A_1 \ge 1$,

$$\int \frac{|\frac{1}{Z_{,\alpha'}}\mathcal{P}\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\,d\alpha' \le \int |\mathcal{P}\overline{Z}_{t,\alpha'}|^2\,d\alpha'.$$

By the base Eq. (4.12),

$$\begin{aligned}\mathcal{P}\overline{Z}_{t,\alpha'} = &-(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt} \\ &-i\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_t + \partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\overline{Z}_{tt} - i)\right);\end{aligned} \tag{4.86}$$

we expand the last term by product rules,

$$\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\overline{Z}_{tt} - i)\right) = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\overline{Z}_{tt,\alpha'} + \partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)(\overline{Z}_{tt} - i). \tag{4.87}$$

It is clear that the first three terms in (4.86) are controlled by $\mathfrak{E}$, by the results of Sects. 4.1.1–4.1.5:

$$\|-(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt} - i\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_t\|_{L^2} \lesssim C(\mathfrak{E}), \tag{4.88}$$

and the first term in (4.87) satisfies

$$\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \overline{Z}_{tt,\alpha'} \right\|_{L^2} \leq \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} \|\overline{Z}_{tt,\alpha'}\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.89}$$

We are left with one last term $\partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) (\overline{Z}_{tt} - i)$ in (4.86). We write

$$\mathcal{P} \overline{Z}_{t,\alpha'} = \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) (\overline{Z}_{tt} - i) + \mathcal{R} \tag{4.90}$$

where $\mathcal{R} = -(\partial_t + b\partial_{\alpha'})(b_{\alpha'} \partial_{\alpha'} \overline{Z}_t) - b_{\alpha'} \partial_{\alpha'} \overline{Z}_{tt} - i \mathcal{A}_{\alpha'} \partial_{\alpha'} \overline{Z}_t + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \overline{Z}_{tt,\alpha'}$. We want to take advantage of the fact that $\partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right)$ is purely real; notice that we have control of $\|(I - \mathbb{H}) \mathcal{P} \overline{Z}_{t,\alpha'}\|_{L^2}$ and $\|\mathcal{R}\|_{L^2}$, by Lemma 4.4 and Sect. 4.1.1–4.1.5, and by (4.88) and (4.89).

Apply $(I - \mathbb{H})$ to both sides of Eq. (4.90), we get

$$\begin{aligned} &(I - \mathbb{H}) \mathcal{P} \overline{Z}_{t,\alpha'} \\ &= (I - \mathbb{H}) \left(\partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) (\overline{Z}_{tt} - i) \right) + (I - \mathbb{H}) \mathcal{R} \\ &= (\overline{Z}_{tt} - i)(I - \mathbb{H}) \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) + \left[\overline{Z}_{tt}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) + (I - \mathbb{H}) \mathcal{R} \end{aligned} \tag{4.91}$$

where we commuted $\overline{Z}_{tt} - i$ out in the second step. Now because $\partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right)$ is purely real,

$$\left| (\overline{Z}_{tt} - i) \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right| \leq \left| (\overline{Z}_{tt} - i)(I - \mathbb{H}) \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right|, \tag{4.92}$$

so by (4.91),

$$\begin{aligned} \left| (\overline{Z}_{tt} - i) \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right| \leq &\left| (I - \mathbb{H}) \mathcal{P} \overline{Z}_{t,\alpha'} \right| \\ &+ \left| \left[\overline{Z}_{tt}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right| + \left| (I - \mathbb{H}) \mathcal{R} \right|. \end{aligned} \tag{4.93}$$

We estimate the L^2 norm of the first term by Lemma 4.4, the second term by (A.15), and the third term by (4.88) and (4.89). We obtain

$$\begin{aligned} &\left\| (\overline{Z}_{tt} - i) \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^2} \lesssim C(\mathfrak{E}) + \left\| Z_{tt,\alpha'} \right\|_{L^2} \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} \\ &\quad + C(\mathfrak{E}) \lesssim C(\mathfrak{E}). \end{aligned} \tag{4.94}$$

This concludes

$$\frac{d}{dt}\mathbf{E}_a(t) \lesssim C(\mathfrak{E}(t)). \tag{4.95}$$

We record here the following estimate that will be used later. By (4.94), $\overline{Z}_{tt} - i = -\frac{iA_1}{Z_{,\alpha'}}$ and $A_1 \geq 1$, we have

$$\left\| D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)\right\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.96}$$

4.1.8 Controlling $\frac{d}{dt}\mathbf{E}_b(t)$

Taking $\Theta = \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t$ in Lemma 4.1, we have,

$$\frac{d}{dt}\mathbf{E}_b(t) \leq \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right\|_{L^\infty} \mathbf{E}_b(t) + 2\mathbf{E}_b(t)^{1/2}\left(\int \frac{|\mathcal{P}\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right)|^2}{\mathcal{A}}\, d\alpha'\right)^{1/2}. \tag{4.97}$$

By (4.43), the first term is controlled by $\mathfrak{E}$. We consider the second term. We know

$$\mathcal{P}\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\right) = \left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right] D_{\alpha'}^2 \overline{Z}_t + \frac{1}{Z_{,\alpha'}}\left[\mathcal{P}, D_{\alpha'}^2\right] \overline{Z}_t + \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \mathcal{P}\overline{Z}_t, \tag{4.98}$$

and because $A_1 \geq 1$,

$$\begin{aligned}\int \frac{|\mathcal{P}\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t|^2}{\mathcal{A}}\, d\alpha' &\lesssim \int \left| Z_{,\alpha'}\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right] D_{\alpha'}^2 \overline{Z}_t\right|^2 d\alpha' \\ &\quad + \int \left|\left[\mathcal{P}, D_{\alpha'}^2\right] \overline{Z}_t\right|^2 d\alpha' + \int \left|D_{\alpha'}^2 \mathcal{P}\overline{Z}_t\right|^2 d\alpha'.\end{aligned} \tag{4.99}$$

Now by Lemma 4.5 and the results of Sects. 4.1.1–4.1.5, the first term on the right hand side of (4.99) is controlled by $\mathfrak{E}$. For the second term, we compute, using (B.10),

$$\begin{aligned}\left[\mathcal{P}, D_{\alpha'}^2\right] \overline{Z}_t &= -4(D_{\alpha'} Z_{tt}) D_{\alpha'}^2 \overline{Z}_t + 6(D_{\alpha'} Z_t)^2 D_{\alpha'}^2 \overline{Z}_t - (2D_{\alpha'}^2 Z_{tt}) D_{\alpha'} \overline{Z}_t \\ &\quad + 6(D_{\alpha'} Z_t)(D_{\alpha'}^2 Z_t) D_{\alpha'} \overline{Z}_t - 2(D_{\alpha'}^2 Z_t) D_{\alpha'} \overline{Z}_{tt} - 4(D_{\alpha'} Z_t) D_{\alpha'}^2 \overline{Z}_{tt}.\end{aligned} \tag{4.100}$$

By results in Sect. 4.1.1, we have

$$\| \left[\mathcal{P}, D_{\alpha'}^2\right] \overline{Z}_t \|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.101}$$

We are left with the term $\int \left| D_{\alpha'}^2 \mathcal{P} \overline{Z}_t \right|^2 d\alpha'$, where

$$\mathcal{P}\overline{Z}_t := \overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\overline{Z}_{tt} - i).$$

We expand $D_{\alpha'}^2 \mathcal{P}\overline{Z}_t$ by product rules,

$$\begin{aligned} D_{\alpha'}^2 \mathcal{P}\overline{Z}_t = {} & D_{\alpha'}^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) (\overline{Z}_{tt} - i) \\ & + 2 D_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) D_{\alpha'} \overline{Z}_{tt} + \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) D_{\alpha'}^2 \overline{Z}_{tt}. \end{aligned} \tag{4.102}$$

We know how to handle the second and third terms, thanks to the work in the previous subsections. We want to use the same idea as in the previous subsection to control the first term, however $D_{\alpha'}$ is not purely real, so we go through the following slightly evoluted process.

First, we have

$$\begin{aligned} & \left\| 2 D_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) D_{\alpha'} \overline{Z}_{tt} + \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) D_{\alpha'}^2 \overline{Z}_{tt} \right\|_{L^2} \\ & \lesssim \| D_{\alpha'} \overline{Z}_{tt} \|_{L^\infty} \left\| D_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^2} \\ & \quad + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} \| D_{\alpha'}^2 \overline{Z}_{tt} \|_{L^2} \lesssim C(\mathfrak{E}); \end{aligned} \tag{4.103}$$

and by Lemma 4.4 and (4.101),

$$\begin{aligned} & \left\| (I - \mathbb{H}) D_{\alpha'}^2 \mathcal{P} \overline{Z}_t \right\|_{L^2} \leq \left\| (I - \mathbb{H}) \mathcal{P} \mathcal{D}_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \\ & \quad + \left\| (I - \mathbb{H}) \left[D_{\alpha'}^2, \mathcal{P} \right] \overline{Z}_t \right\|_{L^2} \lesssim C(\mathfrak{E}). \end{aligned} \tag{4.104}$$

So

$$\left\| (I - \mathbb{H}) \left(D_{\alpha'}^2 \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) (\overline{Z}_{tt} - i) \right) \right\|_{L^2} \lesssim C(\mathfrak{E}). \tag{4.105}$$

This gives, from

$$\begin{aligned}
&(I-\mathbb{H})\left(D_{\alpha'}^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt}-i)\right)\\
&\quad=\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}(I-\mathbb{H})\left(\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\qquad+\left[\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}},\mathbb{H}\right]\left(\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right),
\end{aligned}\tag{4.106}$$

and (A.14) that

$$\left\|\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}(I-\mathbb{H})\left(\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right\|_{L^2}\lesssim C(\mathfrak{E}).\tag{4.107}$$

Now we move the factor $\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|}$ back into $(I-\mathbb{H})$ to get

$$\begin{aligned}
&\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|}(I-\mathbb{H})\left(\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\quad=-\left[\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|},\mathbb{H}\right]\left(\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\qquad+(I-\mathbb{H})\left(\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|}\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right);
\end{aligned}\tag{4.108}$$

and observe that

$$\begin{aligned}
\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|}\partial_{\alpha'}D_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)&=\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&+\frac{(\overline{Z}_{tt}-i)}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\right)\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)-\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\right)\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right).
\end{aligned}\tag{4.109}$$

We know the L^2 norms of the last two terms on the right hand side of (4.109) are controlled by $C(\mathfrak{E})$; and by (A.14), the L^2 norm of the commutator in (4.108) is also controlled by $C(\mathfrak{E})$, therefore by (4.108), (4.107), (4.109),

$$\left\|(I-\mathbb{H})\left(\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right)\right\|_{L^2}\lesssim C(\mathfrak{E}).\tag{4.110}$$

Now we commute out the factor $\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}$ from $(I-\mathbb{H})$ to get

$$\begin{aligned}
&(I-\mathbb{H})\left(\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right)\\
&\quad=\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}(I-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\qquad+\left[\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)
\end{aligned}\tag{4.111}$$

Observe that the quantity the operator $(I-\mathbb{H})$ acts on in the first term on the right hand side of (4.111) is purely real. Applying (A.14) again to the commutator in (4.111) and using (4.110) and the fact that $|f|\le|(I-\mathbb{H})f|$ for f real, we obtain

$$\begin{aligned}
&\left\|\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right\|_{L^2}\\
&\le\left\|\frac{(\overline{Z}_{tt}-i)}{Z_{,\alpha'}}(I-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right\|_{L^2}\lesssim C(\mathfrak{E}).
\end{aligned}\tag{4.112}$$

Applying (4.112) to (4.109) yields,

$$\left\|D_{\alpha'}^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt}-i)\right\|_{L^2}\lesssim C(\mathfrak{E});$$

and by (4.103), (4.102),

$$\left\|D_{\alpha'}^2\mathcal{P}\overline{Z}_t\right\|_{L^2}\lesssim C(\mathfrak{E}).\tag{4.113}$$

This finishes the proof of

$$\frac{d}{dt}\mathbf{E}_b(t)\lesssim C(\mathfrak{E}(t)).\tag{4.114}$$

Sum up the results in Sect. 4.1.6–4.1.8, we obtain

$$\frac{d}{dt}\mathfrak{E}(t)\lesssim C(\mathfrak{E}(t)).\tag{4.115}$$

4.2 The proof of Theorem 3.6

Assume that the initial data satisfies the assumption of Theorem 3.6, we know by (2.17), Proposition A.7 and Sobolev embedding, that $\frac{1}{Z_{,\alpha'}}(0)-1$, $Z_{,\alpha'}(0)-1 \in H^s(\mathbb{R})$, with

$$\begin{aligned}
\left\| \frac{1}{Z_{,\alpha'}}(0) \right\|_{L^\infty} &\le \|Z_{tt}(0)\|_{L^\infty} + 1 \lesssim \|Z_{tt}(0)\|_{H^1} + 1 < \infty; \\
\left\| \frac{1}{Z_{,\alpha'}}(0) - 1 \right\|_{H^s} &\lesssim C\left(\|Z_t(0)\|_{H^s}, \|Z_{tt}(0)\|_{H^s}\right); \\
\left\| Z_{,\alpha'}(0) - 1 \right\|_{H^s} &\lesssim C\left(\|Z_t(0)\|_{H^s}, \|Z_{tt}(0)\|_{H^s}\right).
\end{aligned} \tag{4.116}$$

From Theorem 2.3 and Proposition 3.3, we know to prove the blow-up criteria, Theorem 3.6, it suffices to show that for any solution of (2.9)–(2.11)–(2.14)–(2.15), satisfying the regularity properties in Theorem 3.6, and for any $T_0 > 0$,

$$\begin{aligned}
&\sup_{[0,T_0)} \mathfrak{E}(t) < \infty \quad \text{implies} \quad \sup_{[0,T_0)} \left(\|Z_{,\alpha'}(t)\|_{L^\infty} + \|Z_t(t)\|_{H^{3+1/2}}\right. \\
&\left. + \|Z_{tt}(t)\|_{H^3}\right) < \infty.
\end{aligned}$$

We begin with the lower order norms. We first show that, as a consequence of Eq. (2.35), if $\|Z_{,\alpha'}(0)\|_{L^\infty} < \infty$, then

$$\sup_{[0,T_0)} \|Z_{,\alpha'}(t)\|_{L^\infty} < \infty \qquad \text{as long as} \qquad \sup_{[0,T_0)} \mathfrak{E}(t) < \infty. \tag{4.117}$$

Solving Eq. (2.35) we get, because $\partial_t + b\partial_{\alpha'} = U_h^{-1}\partial_t U_h$,

$$\frac{1}{Z_{,\alpha'}}(h(\alpha,t),t) = \frac{1}{Z_{,\alpha'}}(\alpha,0) e^{\int_0^t (b_{\alpha'}\circ h(\alpha,\tau) - D_\alpha z_t(\alpha,\tau))\, d\tau}; \tag{4.118}$$

so by (4.41) of Sect. 4.1.1,

$$\begin{aligned}
\sup_{[0,T]} \left\| Z_{,\alpha'}(t) \right\|_{L^\infty} &\le \left\| Z_{,\alpha'}(0) \right\|_{L^\infty} e^{\int_0^T \|b_{\alpha'}(\tau) - D_{\alpha'} Z_t(\tau)\|_{L^\infty}\, d\tau} \\
&\lesssim \left\| Z_{,\alpha'}(0) \right\|_{L^\infty} e^{T \sup_{[0,T]} C(\mathfrak{E}(t))},
\end{aligned} \tag{4.119}$$

hence (4.117) holds. Notice that from (4.118), we also have

$$\sup_{[0,T]} \left\| \frac{1}{Z_{,\alpha'}}(t) \right\|_{L^\infty} \lesssim \left\| \frac{1}{Z_{,\alpha'}}(0) \right\|_{L^\infty} e^{T \sup_{[0,T]} C(\mathfrak{E}(t))}. \tag{4.120}$$

Now by Lemma 4.3,

$$\frac{d}{dt}\|Z_t(t)\|_{L^2}^2 \lesssim \|Z_{tt}(t)\|_{L^2}\|Z_t(t)\|_{L^2} + \|b_{\alpha'}(t)\|_{L^\infty}\|Z_t(t)\|_{L^2}^2, \tag{4.121}$$

$$\frac{d}{dt}\|Z_{tt}(t)\|_{L^2}^2 \lesssim \|Z_{ttt}(t)\|_{L^2}\|Z_{tt}(t)\|_{L^2} + \|b_{\alpha'}(t)\|_{L^\infty}\|Z_{tt}(t)\|_{L^2}^2; \tag{4.122}$$

and from Eqs. (2.37) and (2.38),

$$\overline{Z}_{ttt} = (\overline{Z}_{tt} - i)\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} + \overline{D_{\alpha'} Z_t}\right) = -\frac{iA_1}{Z_{,\alpha'}}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} + \overline{D_{\alpha'} Z_t}\right), \tag{4.123}$$

so

$$\|\overline{Z}_{ttt}(t)\|_{L^2} \lesssim \|A_1(t)\|_{L^\infty}\left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^\infty}\left(\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(t)\right\|_{L^2} + \|D_{\alpha'} Z_t(t)\|_{L^2}\right). \tag{4.124}$$

We want to show that $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(t)\right\|_{L^2}$ and $\|D_{\alpha'} Z_t(t)\|_{L^2}$ can be controlled by $\mathfrak{E}$ and the initial data; by (2.21), it suffices to control $\|b_{\alpha'}(t)\|_{L^2}$, $\|D_{\alpha'} Z_t(t)\|_{L^2}$ and $\|(\partial_t + b\partial_{\alpha'})A_1(t)\|_{L^2}$.

Applying Hölder's inequality and (A.15) to (2.22) yields

$$\|b_{\alpha'}(t)\|_{L^2} + \|D_{\alpha'} Z_t(t)\|_{L^2} \lesssim \left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^\infty}\|Z_{t,\alpha'}(t)\|_{L^2}, \tag{4.125}$$

and applying Hölder's inequality, (A.15), (A.21) to (2.23) gives

$$\|(\partial_t + b\partial_{\alpha'})A_1(t)\|_{L^2} \lesssim \|Z_{tt}(t)\|_{L^\infty}\|Z_{t,\alpha'}(t)\|_{L^2} + \|b_{\alpha'}(t)\|_{L^2}\|Z_{t,\alpha'}(t)\|_{L^2}^2; \tag{4.126}$$

so by (2.21), using the fact (2.38), we have

$$\begin{aligned}&\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right\|_{L^2} + \|D_{\alpha'} Z_t\|_{L^2}\\ &\lesssim \left(\|A_1\|_{L^\infty}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} + 1\right)\|Z_{t,\alpha'}\|_{L^2} + \|b_{\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}^2.\end{aligned} \tag{4.127}$$

This gives, by further applying the estimates (4.41) in Sect. 4.1.1 and (4.120), that for $t \in [0, T]$,

$$\begin{aligned} \|Z_{ttt}(t)\|_{L^2} &\lesssim C(T, \sup_{[0,T]} \mathfrak{E}(t)) \left(1 + \left\| \frac{1}{Z_{,\alpha'}}(t) \right\|_{L^\infty}^2 \right) \\ &\lesssim C(T, \sup_{[0,T]} \mathfrak{E}(t)) \left(1 + \left\| \frac{1}{Z_{,\alpha'}}(0) \right\|_{L^\infty}^2 \right). \end{aligned} \tag{4.128}$$

We now apply Gronwall's inequality to (4.122). This yields

$$\sup_{[0,T]} \|Z_{tt}(t)\|_{L^2} \lesssim C \left(T, \sup_{[0,T]} \mathfrak{E}(t), \|Z_{tt}(0)\|_{L^2}, \left\| \frac{1}{Z_{,\alpha'}}(0) \right\|_{L^\infty} \right). \tag{4.129}$$

We then apply Gronwall's inequality to (4.121), using (4.129). We obtain

$$\sup_{[0,T]} \|Z_t(t)\|_{L^2} \lesssim C \left(T, \sup_{[0,T]} \mathfrak{E}(t), \|Z_t(0)\|_{L^2}, \|Z_{tt}(0)\|_{L^2}, \left\| \frac{1}{Z_{,\alpha'}}(0) \right\|_{L^\infty} \right). \tag{4.130}$$

Therefore the lower order norm $\sup_{[0,T]}(\|Z_t(t)\|_{L^2} + \|Z_{tt}(t)\|_{L^2})$ is controlled by $\sup_{[0,T]} \mathfrak{E}(t)$, the L^2 norm of $(Z_t(0), Z_{tt}(0))$ and the L^∞ norm of $\frac{1}{Z_{,\alpha'}(0)}$.[20]

We are left with proving

$$\sup_{[0,T_0)} \mathfrak{E}(t) < \infty \quad \text{implies} \quad \sup_{[0,T_0)} (\|\partial_{\alpha'}^3 Z_t(t)\|_{\dot{H}^{1/2}} + \|\partial_{\alpha'}^3 Z_{tt}(t)\|_{L^2}) < \infty. \tag{4.131}$$

We do so via two stronger results, Propositions 4.6 and 4.7.

Let

$$\begin{aligned} E_k(t) &:= E_{D_{\alpha'}\partial_{\alpha'}^{k-1}\overline{Z}_t}(t) + \|\partial_{\alpha'}^k \overline{Z}_t(t)\|_{L^2}^2 \\ &:= \int \frac{1}{A_1} \left| Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^k \overline{Z}_t \right) \right|^2 d\alpha' \\ &\quad + \left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^k \overline{Z}_t(t) \right\|_{\dot{H}^{1/2}}^2 + \|\partial_{\alpha'}^k \overline{Z}_t(t)\|_{L^2}^2, \end{aligned} \tag{4.132}$$

[20] $\left\| \frac{1}{Z_{,\alpha'}(0)} \right\|_{L^\infty}$ is controlled by the H^1 norm of $Z_{tt}(0)$, see (4.116).

where $k = 2, 3$. We have

Proposition 4.6 *There exists a polynomial $p_1 = p_1(x)$ with universal coefficients such that*

$$\frac{d}{dt}E_2(t) \le p_1\left(\mathfrak{E}(t)\right) E_2(t). \tag{4.133}$$

Proposition 4.7 *There exists a polynomial $p_2 = p_2(x, y, z)$ with universal coefficients such that*

$$\frac{d}{dt}E_3(t) \le p_2\left(\mathfrak{E}(t), E_2(t), \left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^\infty}\right)(E_3(t)+1). \tag{4.134}$$

By Gronwall's inequality, we have from (4.133) and (4.134) that

$$\begin{aligned} E_2(t) &\le E_2(0)e^{\int_0^t p_1(\mathfrak{E}(s))\,ds}; \qquad \text{and} \\ E_3(t) &\le \left(E_3(0) + \int_0^t p_2\left(\mathfrak{E}(s), E_2(s), \left\|\frac{1}{Z_{,\alpha'}}(s)\right\|_{L^\infty}\right) ds\right) \\ &\qquad e^{\int_0^t p_2\left(\mathfrak{E}(s), E_2(s), \left\|\frac{1}{Z_{,\alpha'}}(s)\right\|_{L^\infty}\right) ds}, \end{aligned} \tag{4.135}$$

so $\sup_{[0,T]} E_2(t)$ is controlled by $E_2(0)$ and $\sup_{[0,T]} \mathfrak{E}(t)$; and $\sup_{[0,T]} E_3(t)$ is controlled by $E_3(0)$, $\sup_{[0,T]} \mathfrak{E}(t)$, $\sup_{[0,T]} E_2(t)$ and $\sup_{[0,T]} \left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_{L^\infty}$. And by (4.120), $\sup_{[0,T]} E_3(t)$ is in turn controlled by $E_3(0)$, $\sup_{[0,T]} \mathfrak{E}(t)$, $E_2(0)$ and $\left\|\frac{1}{Z_{,\alpha'}}(0)\right\|_{L^\infty}$. We will prove Propositions 4.6 and 4.7 in the next two subsections. In Sect. 4.5 we will exam the relation between the energy functionals E_2, E_3 and the Sobolev norms $\|Z_t(t)\|_{H^{3+1/2}}$, $\|Z_{tt}(t)\|_{H^3}$ and complete the proof of Theorem 3.6.

4.3 The proof of Proposition 4.6

We begin with a list of quantities controlled by $E_2(t)$.

4.3.1 *Quantities controlled by $E_2(t)$*

It is clear by the definition that the following are controlled by $E_2(t)$.

$$\begin{aligned} &\|\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}^2 \le E_2, \quad \left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}^2 \\ &\le E_2, \quad \left\| Z_{,\alpha'} (\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right) \right\|_{L^2}^2 \le C(\mathfrak{E}) E_2, \end{aligned} \tag{4.136}$$

because $1 \le A_1 \le C(\mathfrak{E})$ by (4.20). We compute, by product rules and (2.35), that

$$Z_{,\alpha'} (\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right) = (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t + (b_{\alpha'} - D_{\alpha'} Z_t) \partial_{\alpha'}^2 \overline{Z}_t, \tag{4.137}$$

therefore, by estimates (4.41) in Sect. 4.1.1,

$$\left| \left\| (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} - \left\| Z_{,\alpha'} (\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right\|_{L^2} \right| \le C(\mathfrak{E}) \|\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \tag{4.138}$$

so

$$\left\| (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}^2 \le C(\mathfrak{E}) E_2. \tag{4.139}$$

Now by (B.12),

$$\partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) \overline{Z}_{t,\alpha'} = (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t + b_{\alpha'} \partial_{\alpha'}^2 \overline{Z}_t, \tag{4.140}$$

so by (4.41),

$$\|\partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) \overline{Z}_{t,\alpha'}\|_{L^2}^2 \le C(\mathfrak{E}) E_2. \tag{4.141}$$

Using Sobolev inequality (A.7) and (4.41), we obtain

$$\|Z_{t,\alpha'}\|_{L^\infty}^2 \le 2 \|Z_{t,\alpha'}\|_{L^2} \|\partial_{\alpha'}^2 Z_t\|_{L^2} \le C(\mathfrak{E}) E_2^{1/2}; \quad \text{and} \tag{4.142}$$

$$\|(\partial_t + b\partial_{\alpha'}) Z_{t,\alpha'}\|_{L^\infty}^2 \le 2 \|(\partial_t + b\partial_{\alpha'}) Z_{t,\alpha'}\|_{L^2} \|\partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) Z_{t,\alpha'}\|_{L^2} \le C(\mathfrak{E}) E_2^{1/2}. \tag{4.143}$$

We need the estimates for some additional quantities, which we give in the following subsections.

4.3.2 Controlling the quantity $\|\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2}$

We begin with Eq. (2.22), and differentiate with respect to α'. We get

$$\begin{aligned}
&\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t) \\
&\quad = \mathrm{Re}\left(\left[\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} + \left[Z_{t,\alpha'}, \mathbb{H}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\
&\qquad + \mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}^2 Z_t + [Z_t, \mathbb{H}]\,\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right);
\end{aligned} \tag{4.144}$$

using $\mathbb{H} Z_{t,\alpha'} = -Z_{t,\alpha'}$ to rewrite the first term,

$$\left[\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} = -(I + \mathbb{H})\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} Z_{t,\alpha'}\right) \tag{4.145}$$

and then applying (A.15) and (A.14) to the last two terms. We get, by (4.142) and (4.41),

$$\|\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \le C(\mathfrak{E}) E_2^{1/4}. \tag{4.146}$$

4.3.3 Controlling $\|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2}$

We start with $(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t$, and commute $\partial_t + b\partial_{\alpha'}$ with $\partial_{\alpha'}^2$; by (B.16), we have

$$\begin{aligned}
\partial_{\alpha'}^2 \overline{Z}_{tt} - (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t &= [\partial_{\alpha'}^2, (\partial_t + b\partial_{\alpha'})]\overline{Z}_t \\
&= 2b_{\alpha'}\partial_{\alpha'}^2 \overline{Z}_t + (\partial_{\alpha'} b_{\alpha'})\,\overline{Z}_{t,\alpha'},
\end{aligned} \tag{4.147}$$

We further expand the second term

$$\begin{aligned}
&(\partial_{\alpha'} b_{\alpha'})\,\overline{Z}_{t,\alpha'} \\
&\quad = (\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t))\,\overline{Z}_{t,\alpha'} + 2\,\mathrm{Re}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} Z_{t,\alpha'}\right)\overline{Z}_{t,\alpha'} \\
&\qquad + 2\,\mathrm{Re}\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2 Z_t\right)\overline{Z}_{t,\alpha'};
\end{aligned} \tag{4.148}$$

we get, by (4.147) and (4.148) that

$$\begin{aligned}\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2} &\lesssim \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}\\ &+\|b_{\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2} + \|\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\|_{L^2}\,\|Z_{t,\alpha'}\|_{L^\infty}\\ &+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^\infty}^2 + \|D_{\alpha'}Z_t\|_{L^\infty}\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}\end{aligned} \tag{4.149}$$

Therefore by the estimates in Sects. 4.3.1, 4.1.1 and (4.146),

$$\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}^2 \lesssim C(\mathfrak{E})E_2. \tag{4.150}$$

As a consequence of the Sobolev inequality (A.7), and estimates (4.41) in Sect. 4.1.1,

$$\|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^\infty}^2 \le 2\|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^2}\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.151}$$

We also have, by the L^2 boundedness of $\mathbb{H}$,

$$\begin{aligned}\|\mathbb{H}\overline{Z}_{tt,\alpha'}\|_{L^\infty}^2 &\le 2\|\partial_{\alpha'}\mathbb{H}\overline{Z}_{tt}\|_{L^2}\|\partial_{\alpha'}^2\mathbb{H}\overline{Z}_{tt}\|_{L^2}\\ &\lesssim \|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^2}\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}.\end{aligned} \tag{4.152}$$

4.3.4 Controlling the quantity $\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\|_{L^2}$

We begin with (4.144), replacing the first term by (4.145), then use (B.17) on the first term and use (B.19) to compute the remaining three terms,

$$\begin{aligned}&(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\\ &\quad= -\,\mathrm{Re}\left([b,\mathbb{H}]\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}Z_{t,\alpha'}\right) + (I+\mathbb{H})(\partial_t + b\partial_{\alpha'})\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}Z_{t,\alpha'}\right)\right)\\ &\qquad+\mathrm{Re}\left(\left[(\partial_t + b\partial_{\alpha'})Z_{t,\alpha'},\mathbb{H}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right.\\ &\qquad\left.+\left[Z_{t,\alpha'},\mathbb{H}\right]\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} - \left[Z_{t,\alpha'},b;\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right]\right)\\ &\qquad+\mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})Z_{t,\alpha'}\right.\\ &\qquad\left.+\left[(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}^2 Z_t - \left[\frac{1}{Z_{,\alpha'}},b;\,\partial_{\alpha'}^2 Z_t\right]\right)\\ &\qquad+\mathrm{Re}\left([Z_{tt},\mathbb{H}]\,\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} + [Z_t,\mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - \left[Z_t,b;\,\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right]\right).\end{aligned} \tag{4.153}$$

We have, by (A.14), (A.11), (A.15), (A.12), and estimates in Sects. 4.1.1, 4.1.4, 4.3.1–4.3.3,

$$\begin{aligned}\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2} &\lesssim \|b_{\alpha'}\|_{L^\infty} \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^\infty} \\ &+ \left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^\infty} + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|(\partial_t + b\partial_{\alpha'}) Z_{t,\alpha'}\|_{L^\infty} \\ &+ \left\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^\infty} + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{tt,\alpha'}\|_{L^\infty} \\ &\lesssim C(\mathfrak{E}) E_2^{1/4}. \end{aligned} \tag{4.154}$$

4.3.5 *Controlling the quantity* $\partial_{\alpha'} \mathcal{A}_{\alpha'}$

We begin with Eq. (4.45),

$$i\mathcal{A}_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} Z_{tt} + (Z_{tt} + i)\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \tag{4.155}$$

and differentiate with respect to α'. We get

$$i\partial_{\alpha'}\mathcal{A}_{\alpha'} = \frac{\partial_{\alpha'}^2 Z_{tt}}{Z_{,\alpha'}} + 2\partial_{\alpha'} Z_{tt} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} + (Z_{tt} + i)\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}. \tag{4.156}$$

Applying $(I - \mathbb{H})$ yields

$$\begin{aligned} i(I - \mathbb{H})\partial_{\alpha'}\mathcal{A}_{\alpha'} = (I - \mathbb{H}) \left(\frac{\partial_{\alpha'}^2 Z_{tt}}{Z_{,\alpha'}}\right) + 2(I - \mathbb{H}) \left(\partial_{\alpha'} Z_{tt} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \\ + (I - \mathbb{H}) \left((Z_{tt} + i)\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right). \end{aligned} \tag{4.157}$$

We rewrite the first term on the right by commuting out $\frac{1}{Z_{,\alpha'}}$, and use $(I - \mathbb{H})\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} = 0$ to rewrite the third term on the right of (4.157) as a commutator. We have

$$\begin{aligned} i(I - \mathbb{H})\partial_{\alpha'}\mathcal{A}_{\alpha'} - \frac{1}{Z_{,\alpha'}}(I - \mathbb{H})\partial_{\alpha'}^2 Z_{tt} = \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] \partial_{\alpha'}^2 Z_{tt} \\ + 2(I - \mathbb{H}) \left(\partial_{\alpha'} Z_{tt} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) + [Z_{tt}, \mathbb{H}]\, \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}. \end{aligned} \tag{4.158}$$

Taking imaginary parts, then applying (A.14), (A.15) and Hölder's inequality gives

$$\begin{aligned}&\left\|\partial_{\alpha'}\mathcal{A}_{\alpha'}-\operatorname{Im}\left\{\frac{1}{Z_{,\alpha'}}(I-\mathbb{H})(\partial_{\alpha'}^2 Z_{tt})\right\}\right\|_{L^2}\\&\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{tt,\alpha'}\|_{L^\infty}\lesssim C(\mathfrak{E})E_2^{1/4}.\end{aligned}\tag{4.159}$$

4.3.6 Controlling $\left\|\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^2}$

We begin with (2.21). We have controlled $\|\partial_{\alpha'}(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2}$ in (4.146), we are left with $\left\|\partial_{\alpha'}\left(\frac{(\partial_t+b\partial_{\alpha'})A_1}{A_1}\right)\right\|_{L^2}$.

We proceed with computing $\partial_{\alpha'}A_1$, using (2.15). We have

$$\begin{aligned}\partial_{\alpha'}A_1&=-\operatorname{Im}\left([Z_{t,\alpha'},\mathbb{H}]\overline{Z}_{t,\alpha'}+[Z_t,\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t\right)\\&=-\operatorname{Im}(-\mathbb{H}|\overline{Z}_{t,\alpha'}|^2+[Z_t,\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t);\end{aligned}\tag{4.160}$$

here we used the fact $\mathbb{H}\overline{Z}_{t,\alpha'}=\overline{Z}_{t,\alpha'}$ to expand the first term, then removed the term $\operatorname{Im}|Z_{t,\alpha'}|^2=0$. Applying (A.14), (4.142) and (4.41) gives

$$\|\partial_{\alpha'}A_1\|_{L^2}\lesssim\|Z_{t,\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/4}.\tag{4.161}$$

Now taking derivative $\partial_t+b\partial_{\alpha'}$ to (4.160), using (B.17) and (B.19), yields

$$\begin{aligned}(\partial_t+b\partial_{\alpha'})\partial_{\alpha'}A_1&=\operatorname{Im}\left([b,\mathbb{H}]\partial_{\alpha'}|\overline{Z}_{t,\alpha'}|^2+2\mathbb{H}\operatorname{Re}\left\{Z_{t,\alpha'}(\partial_t+b\partial_{\alpha'})\overline{Z}_{t,\alpha'}\right\}\right)\\&\quad-\operatorname{Im}([Z_{tt},\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t)+[Z_t,\mathbb{H}]\partial_{\alpha'}(\partial_t+b\partial_{\alpha'})\overline{Z}_{t,\alpha'}-[Z_t,b;\partial_{\alpha'}^2\overline{Z}_t]);\end{aligned}\tag{4.162}$$

By (A.14), then use (4.41), (4.142), (4.151) we get

$$\begin{aligned}\|(\partial_t+b\partial_{\alpha'})\partial_{\alpha'}A_1\|_{L^2}&\lesssim\|b_{\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^\infty}+\|Z_{tt,\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\\&\quad+\|Z_{t,\alpha'}\|_{L^\infty}\|(\partial_t+b\partial_{\alpha'})\overline{Z}_{t,\alpha'}\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/4}.\end{aligned}\tag{4.163}$$

Commuting $\partial_{\alpha'}$ with $\partial_t+b\partial_{\alpha'}$ gives

$$\|\partial_{\alpha'}(\partial_t+b\partial_{\alpha'})A_1\|_{L^2}\lesssim\|(\partial_t+b\partial_{\alpha'})\partial_{\alpha'}A_1\|_{L^2}+\|b_{\alpha'}\partial_{\alpha'}A_1\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/4}.\tag{4.164}$$

Combine (4.164) with (4.161) and (4.146), using (4.20), (4.42), we obtain

$$\left\| \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^2} \lesssim C(\mathfrak{E}) E_2^{1/4}. \tag{4.165}$$

Sum up the estimates obtained in Sects. 4.3.1–4.3.6, we have that the following quantities are controlled by $C(\mathfrak{E})E_2^{1/2}$:

$$\begin{gathered} \|\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \quad \left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}, \quad \left\| Z_{,\alpha'} (\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right) \right\|_{L^2}, \\ \left\| (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t \right\|_{L^2}, \quad \|\partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) \overline{Z}_{t,\alpha'}\|_{L^2}, \quad \|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} \\ \|Z_{t,\alpha'}\|_{L^\infty}^2, \quad \|(\partial_t + b\partial_{\alpha'}) Z_{t,\alpha'}\|_{L^\infty}^2, \quad \|\partial_{\alpha'} \overline{Z}_{tt}\|_{L^\infty}^2, \quad \|\mathbb{H} \overline{Z}_{tt,\alpha'}\|_{L^\infty}^2, \\ \|\partial_{\alpha'} (b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2}^2, \quad \|(\partial_t + b\partial_{\alpha'}) \partial_{\alpha'} (b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2}^2, \\ \left\| \partial_{\alpha'} \mathcal{A}_{\alpha'} - \mathrm{Im} \left\{ \frac{1}{Z_{,\alpha'}} (I - \mathbb{H})(\partial_{\alpha'}^2 Z_{tt}) \right\} \right\|_{L^2}^2, \quad \left\| \partial_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^2}^2, \quad \|\partial_{\alpha'} A_1\|_{L^2}^2 \end{gathered} \tag{4.166}$$

4.3.7 *Controlling* $\frac{d}{dt} E_2(t)$

We are now ready to estimate $\frac{d}{dt} E_2(t)$. We know

$$E_2(t) = E_{D_{\alpha'} \partial_{\alpha'} \overline{Z}_t}(t) + \|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2,$$

where $E_{D_{\alpha'} \partial_{\alpha'} \overline{Z}_t}(t)$ is as defined in (4.2). We use Lemma 4.1 on $E_{D_{\alpha'} \partial_{\alpha'} \overline{Z}_t}(t)$ and Lemma 4.3 on $\|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2$.

We start with $\|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2$. We know by Lemma 4.3 that

$$\begin{aligned} \frac{d}{dt} \|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2 \lesssim & \|(\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2} \|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2} \\ & + \|b_{\alpha'}\|_{L^\infty} \|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2 \end{aligned} \tag{4.167}$$

We have controlled $\|(\partial_t + b\partial_{\alpha'}) \partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}$ in (4.139), and $\|b_{\alpha'}\|_{L^\infty}$ in Sect. 4.1.1, therefore

$$\frac{d}{dt} \|\partial_{\alpha'}^2 \overline{Z}_t(t)\|_{L^2}^2 \lesssim C(\mathfrak{E}(t)) E_2(t). \tag{4.168}$$

We now estimate $\frac{d}{dt}E_{D_{\alpha'}\partial_{\alpha'}\overline{Z}_t}(t)$. Take $\Theta = D_{\alpha'}\overline{Z}_{t,\alpha'}$ in Lemma 4.1, we have

$$\frac{d}{dt}E_{D_{\alpha'}\partial_{\alpha'}\overline{Z}_t}(t) \le \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty} E_{D_{\alpha'}\partial_{\alpha'}\overline{Z}_t}(t) + 2E_{D_{\alpha'}\partial_{\alpha'}\overline{Z}_t}(t)^{1/2} \left(\int \frac{|\mathcal{P}D_{\alpha'}\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\, d\alpha' \right)^{1/2}. \quad (4.169)$$

We have controlled $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty}$ in (4.43). We need to control $\int \frac{|\mathcal{P}D_{\alpha'}\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\, d\alpha'$.

We know $\mathcal{A} = \frac{A_1}{|Z_{,\alpha'}|^2}$ and $A_1 \ge 1$, so

$$\int \frac{|\mathcal{P}D_{\alpha'}\overline{Z}_{t,\alpha'}|^2}{\mathcal{A}}\, d\alpha' \le \int |Z_{,\alpha'}\mathcal{P}D_{\alpha'}\overline{Z}_{t,\alpha'}|^2\, d\alpha'. \quad (4.170)$$

We compute

$$\mathcal{P}D_{\alpha'}\overline{Z}_{t,\alpha'} = [\mathcal{P}, D_{\alpha'}]\overline{Z}_{t,\alpha'} + \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'}; \quad (4.171)$$

further expanding $[\mathcal{P}, D_{\alpha'}]\overline{Z}_{t,\alpha'}$ by (B.9) yields

$$[\mathcal{P}, D_{\alpha'}]\overline{Z}_{t,\alpha'} = (-2D_{\alpha'}Z_{tt})D_{\alpha'}\overline{Z}_{t,\alpha'} - 2(D_{\alpha'}Z_t)(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_{t,\alpha'}; \quad (4.172)$$

and by (4.41) and (4.136),

$$\begin{aligned} \|Z_{,\alpha'}[\mathcal{P}, D_{\alpha'}]\overline{Z}_{t,\alpha'}\|_{L^2} &\lesssim \|D_{\alpha'}Z_{tt}\|_{L^\infty}\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2} \\ &\quad + \|D_{\alpha'}Z_t\|_{L^\infty}\left\|Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})D_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{L^2} \\ &\lesssim C(\mathfrak{E})E_2^{1/2}. \end{aligned} \quad (4.173)$$

We are left with controlling $\|\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'}\|_{L^2}$.

Taking derivative to α' to (4.12) yields

$$\begin{aligned} \partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'} &= -(\partial_t + b\partial_{\alpha'})((\partial_{\alpha'}b_{\alpha'})\partial_{\alpha'}\overline{Z}_t) - (\partial_{\alpha'}b_{\alpha'})\partial_{\alpha'}\overline{Z}_{tt} - i(\partial_{\alpha'}\mathcal{A}_{\alpha'})\partial_{\alpha'}\overline{Z}_t \\ &\quad -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}^2\overline{Z}_{tt} - i\mathcal{A}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t - b_{\alpha'}^2\partial_{\alpha'}^2\overline{Z}_t - b_{\alpha'}(\partial_{\alpha'}b_{\alpha'})\partial_{\alpha'}\overline{Z}_t \\ &\quad + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\partial_{\alpha'}^2\overline{Z}_{tt} + 2\left(\partial_{\alpha'}\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)\partial_{\alpha'}\overline{Z}_{tt} + \left(\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)(\overline{Z}_{tt} - i); \end{aligned} \quad (4.174)$$

we further expand the terms in the first line and the last term in the second line according to the available estimates in Sects. 4.3.2–4.3.6,

$$\begin{aligned}
&(\partial_t + b\partial_{\alpha'})((\partial_{\alpha'} b_{\alpha'})\,\partial_{\alpha'}\overline{Z}_t)\\
&\quad = (\partial_t + b\partial_{\alpha'})\left\{\partial_{\alpha'}\left(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t\right)\partial_{\alpha'}\overline{Z}_t\right\}\\
&\qquad +2\left\{\mathrm{Re}\,\partial_{\alpha'}\left(D_{\alpha'}Z_t\right)\right\}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\overline{Z}_t\\
&\qquad +2\left\{\mathrm{Re}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\left(D_{\alpha'}Z_t\right)\right\}\partial_{\alpha'}\overline{Z}_t
\end{aligned} \tag{4.175}$$

we expand the factors in the second line further by product rules,

$$\mathrm{Re}\,\partial_{\alpha'}\left(D_{\alpha'}Z_t\right) = \mathrm{Re}\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}Z_t + \mathrm{Re}\,\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}}, \tag{4.176}$$

$$\begin{aligned}
&\mathrm{Re}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\left(D_{\alpha'}Z_t\right)\\
&\quad = \mathrm{Re}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}Z_t + \mathrm{Re}\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}Z_t\\
&\qquad + \mathrm{Re}\,\frac{(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} + \mathrm{Re}\,\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'}Z_t),
\end{aligned} \tag{4.177}$$

here we used (2.35) in the last term; and by (B.12),

$$(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\overline{Z}_t = \overline{Z}_{tt,\alpha'} - b_{\alpha'}\overline{Z}_{t,\alpha'}. \tag{4.178}$$

We are now ready to conclude, by (4.146), (4.154), (4.142), (4.143), (4.41), (4.71), Sect. 4.3.1 and the expansions (4.175)–(4.178) that

$$\|(\partial_t + b\partial_{\alpha'})((\partial_{\alpha'} b_{\alpha'})\,\partial_{\alpha'}\overline{Z}_t)\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.179}$$

Similarly we can conclude, after expanding if necessary, with a similar estimate for all the terms on the right hand side of (4.174) except for $\left(\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt} - i)$. Let

$$\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'} = \mathcal{R}_1 + \left(\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt} - i); \tag{4.180}$$

where $\mathcal{R}_1$ is the sum of the remaining terms on the right hand side of (4.174). We have, by the argument above, that

$$\|\mathcal{R}_1\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.181}$$

We control the term $\left(\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt} - i)$ with a similar idea as that in Sect. 4.1.7, by taking advantage of the fact that $\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}$ is purely real.

Applying $(I-\mathbb{H})$ to both sides of (4.180), and commuting out $\overline{Z}_{tt}-i$ yields

$$(I-\mathbb{H})\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'} = (I-\mathbb{H})\mathcal{R}_1 + \left[\overline{Z}_{tt},\mathbb{H}\right]\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1} + (\overline{Z}_{tt}-i)(I-\mathbb{H})\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}; \quad (4.182)$$

Because $\mathbb{H}$ is purely imaginary, we have $\left|\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right| \le \left|(I-\mathbb{H})\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right|$, and

$$\left|(\overline{Z}_{tt}-i)\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right| \le \left|(I-\mathbb{H})\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'}\right| + |(I-\mathbb{H})\mathcal{R}_1| + \left|\left[\overline{Z}_{tt},\mathbb{H}\right]\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right|. \quad (4.183)$$

Now by (B.15),

$$[\mathcal{P},\partial_{\alpha'}]\overline{Z}_{t,\alpha'} = -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\overline{Z}_{t,\alpha'}) - b_{\alpha'}\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'} - i\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_{t,\alpha'}; \quad (4.184)$$

so

$$\|[\mathcal{P},\partial_{\alpha'}]\overline{Z}_{t,\alpha'}\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2} \quad (4.185)$$

by (4.41), Sects. 4.1.3 and 4.1.5. By Lemma 4.4, and (4.185),

$$\|(I-\mathbb{H})\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'}\|_{L^2} \le \|(I-\mathbb{H})\mathcal{P}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\|_{L^2} + \|(I-\mathbb{H})[\mathcal{P},\partial_{\alpha'}]\overline{Z}_{t,\alpha'}\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \quad (4.186)$$

Now we apply (A.14) to the commutator on the right hand side of (4.183). By (4.181) and (4.186), we have

$$\left\|(\overline{Z}_{tt}-i)\partial_{\alpha'}^2\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2} + \left\|\overline{Z}_{tt,\alpha'}\right\|_{L^\infty}\left\|\partial_{\alpha'}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \quad (4.187)$$

This together with (4.181) and (4.180) gives

$$\|\partial_{\alpha'}\mathcal{P}\overline{Z}_{t,\alpha'}\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \quad (4.188)$$

We can now conclude, by (4.170), (4.171), (4.173) and (4.188) that

$$\int \frac{|\mathcal{P} D_{\alpha'} \overline{Z}_{t,\alpha'}|^2}{\mathcal{A}} \, d\alpha' \lesssim C(\mathfrak{E}) E_2^{1/2}; \tag{4.189}$$

and consequently,

$$\frac{d}{dt} E_{D_{\alpha'} \partial_{\alpha'} \overline{Z}_t}(t) \lesssim C(\mathfrak{E}) E_2. \tag{4.190}$$

Combining (4.168) and (4.190) yields

$$\frac{d}{dt} E_2(t) \lesssim C(\mathfrak{E}(t)) E_2(t). \tag{4.191}$$

This concludes the proof for Proposition 4.6.

4.4 The proof of Proposition 4.7

We begin with discussing quantities controlled by E_3. Since the idea is similar to that in previous sections, when the estimates are straightforward, we don't always give the full details.

4.4.1 Quantities controlled by E_3 and a polynomial of $\mathfrak{E}$ and E_2

By the definition of E_3, and the fact that $1 \le A_1 \le C(\mathfrak{E})$, cf. (4.20),

$$\begin{aligned} \|\partial_{\alpha'}^3 \overline{Z}_t\|_{L^2}^2 \le E_3, \quad & \left\| Z_{,\alpha'} (\partial_t + b \partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^3 \overline{Z}_t \right) \right\|_{L^2}^2 \\ \le C(\mathfrak{E}) E_3, \quad & \left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^3 \overline{Z}_t \right\|_{\dot{H}^{1/2}}^2 \le E_3. \end{aligned} \tag{4.192}$$

By (2.35) and product rules,

$$Z_{,\alpha'} (\partial_t + b \partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^3 \overline{Z}_t \right) = (\partial_t + b \partial_{\alpha'}) \partial_{\alpha'}^3 \overline{Z}_t + (b_{\alpha'} - D_{\alpha'} Z_t) \partial_{\alpha'}^3 \overline{Z}_t \tag{4.193}$$

so by (4.41),

$$\left| \left\| (\partial_t + b \partial_{\alpha'}) \partial_{\alpha'}^3 \overline{Z}_t \right\|_{L^2} - \left\| Z_{,\alpha'} (\partial_t + b \partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^3 \overline{Z}_t \right) \right\|_{L^2} \right| \le C(\mathfrak{E}) \|\partial_{\alpha'}^3 \overline{Z}_t\|_{L^2}, \tag{4.194}$$

therefore

$$\left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^3 \overline{Z}_t\right\|_{L^2}^2 \leq C(\mathfrak{E})E_3. \tag{4.195}$$

We commute out $\partial_{\alpha'}$, by (B.12),

$$\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^3 \overline{Z}_t + b_{\alpha'}\partial_{\alpha'}^3 \overline{Z}_t, \tag{4.196}$$

so

$$\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}^2 \leq C(\mathfrak{E})E_3. \tag{4.197}$$

As a consequence of the Sobolev inequality (A.7), and (4.166),

$$\|\partial_{\alpha'}^2 \overline{Z}_t\|_{L^\infty}^2 \leq 2\|\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}\|\partial_{\alpha'}^3 \overline{Z}_t\|_{L^2} \lesssim C(\mathfrak{E}, E_2)E_3^{1/2}, \tag{4.198}$$

$$\begin{aligned}\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t\|_{L^\infty}^2 &\leq 2\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2}\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2} \\ &\lesssim C(\mathfrak{E}, E_2)E_3^{1/2}.\end{aligned} \tag{4.199}$$

Now we commute out $\partial_{\alpha'}^2$, by (B.16), and get

$$\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\partial_{\alpha'} \overline{Z}_t = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^3 \overline{Z}_t + (\partial_{\alpha'} b_{\alpha'})\,\partial_{\alpha'}^2 \overline{Z}_t + 2b_{\alpha'}\partial_{\alpha'}^3 \overline{Z}_t, \tag{4.200}$$

We expand the second term further according to the available estimate (4.146), as we did in (4.148); we get

$$\|\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\partial_{\alpha'} \overline{Z}_t\|_{L^2}^2 \leq C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3 + 1); \tag{4.201}$$

and consequently by Sobolev inequality (A.7) and (4.166),

$$\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'} \overline{Z}_t\|_{L^\infty}^2 \leq C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1). \tag{4.202}$$

We need to control some additional quantities.

4.4.2 Controlling $\|\partial_{\alpha'} A_1\|_{L^\infty}$, $\|\partial_{\alpha'}^2 A_1\|_{L^2}$ and $\left\|\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$

We begin with (4.160):

$$\partial_{\alpha'} A_1 = -\operatorname{Im}\left([Z_{t,\alpha'}, \mathbb{H}]\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\partial_{\alpha'}^2 \overline{Z}_t\right). \tag{4.203}$$

By (A.18), (4.41), (4.166),

$$\|\partial_{\alpha'} A_1\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|\partial_{\alpha'}^2 Z_t\|_{L^2} \lesssim C(\mathfrak{E}) E_2^{1/2}. \tag{4.204}$$

Differentiating (4.203) with respect to α' then apply (A.14), (A.16) and use $\mathbb{H}\overline{Z}_{t,\alpha'} = \overline{Z}_{t,\alpha'}$ gives

$$\|\partial_{\alpha'}^2 A_1\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^\infty} \|\partial_{\alpha'}^2 Z_t\|_{L^2} \leq C(\mathfrak{E}, E_2), \tag{4.205}$$

where in the last step we used (4.166). To estimate $\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}$ we begin with (2.17):

$$-i \frac{1}{Z_{,\alpha'}} = \frac{\overline{Z}_{tt} - i}{A_1}.$$

Taking two derivatives with respect to α' gives

$$-i\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} = \frac{\partial_{\alpha'}^2 \overline{Z}_{tt}}{A_1} - 2\overline{Z}_{tt,\alpha'} \frac{\partial_{\alpha'} A_1}{A_1^2} + (\overline{Z}_{tt} - i)\left(-\frac{\partial_{\alpha'}^2 A_1}{A_1^2} + 2\frac{(\partial_{\alpha'} A_1)^2}{A_1^3}\right); \tag{4.206}$$

therefore, because $A_1 \geq 1$, and (2.38), (4.41), (4.166), (4.204), (4.205),

$$\begin{aligned} \left\| \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} &\lesssim \|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} + \|\partial_{\alpha'} \overline{Z}_{tt}\|_{L^2} \|\partial_{\alpha'} A_1\|_{L^\infty} \\ &\quad + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} (\|\partial_{\alpha'}^2 A_1\|_{L^2} + \|\partial_{\alpha'} A_1\|_{L^2} \|\partial_{\alpha'} A_1\|_{L^\infty}) \\ &\leq C\left(\mathfrak{E}, E_2, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}\right), \end{aligned} \tag{4.207}$$

and consequently by Sobolev inequality (A.7), and (4.41),

$$\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \leq C\left(\mathfrak{E}, E_2, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}\right). \tag{4.208}$$

4.4.3 *Controlling* $\|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2}$ *and* $\|\partial_{\alpha'}^3 Z_{tt}\|_{L^2}$

We are now ready to give the estimates for $\|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2}$ and $\|\partial_{\alpha'}^3 \overline{Z}_{tt}\|_{L^2}$. We begin with (4.144), differentiating with respect to α', then use (A.14), (A.15), the fact that $\mathbb{H} Z_{t,\alpha'} = -Z_{t,\alpha'}$, $\mathbb{H}\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}$, and Hölder's inequality; we get

$$\begin{aligned}\|\partial_{\alpha'}^2(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\|_{L^2} &\lesssim \|\partial_{\alpha'}^2 Z_t\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\\ &\le C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right).\end{aligned} \tag{4.209}$$

It is easy to show, by product rules and Hölder's inequality that

$$\begin{aligned}\|\partial_{\alpha'}^2 D_{\alpha'} Z_t\|_{L^2} &\le c\|\partial_{\alpha'}^2 Z_t\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\\ &\quad + \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3 Z_t\right\|_{L^2}\\ &\le cC\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right) + \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} E_3^{1/2},\end{aligned} \tag{4.210}$$

so

$$\|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right) + \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} E_3^{1/2}. \tag{4.211}$$

Now starting from (4.15) and taking two derivatives to α' gives

$$\begin{aligned}\partial_{\alpha'}^3 \overline{Z}_{tt} &= \partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\overline{Z}_t + \partial_{\alpha'}^2(b_{\alpha'}\overline{Z}_{t,\alpha'})\\ &= \partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\overline{Z}_t + (\partial_{\alpha'}^2 b_{\alpha'})\overline{Z}_{t,\alpha'} + 2(\partial_{\alpha'} b_{\alpha'})\partial_{\alpha'}\overline{Z}_{t,\alpha'} + b_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t,\end{aligned} \tag{4.212}$$

so

$$\|\partial_{\alpha'}^3 \overline{Z}_{tt}\|_{L^2}^2 \le C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3 + 1), \tag{4.213}$$

and as a consequence of (A.7),

$$\|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^\infty}^2 \le C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1). \tag{4.214}$$

4.4.4 *Controlling* $\partial_{\alpha'}^2 \mathcal{A}_{\alpha'}$

We differentiate (4.158) with respect to α' then take the imaginary parts and use Hölder's inequality, (A.14), (A.15). We have,

$$\begin{aligned}\|\partial_{\alpha'}^2 \mathcal{A}_{\alpha'}\|_{L^2} \leq & \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \|\partial_{\alpha'}^3 \overline{Z}_{tt}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} \\ & + \left\|\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|\partial_{\alpha'} \overline{Z}_{tt}\|_{L^\infty} \leq C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1).\end{aligned} \tag{4.215}$$

4.4.5 *Controlling* $\left\|\partial_{\alpha'}^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right\|_{L^2}$

We begin with (2.21), and take two derivatives to α'.

$$\begin{aligned}\partial_{\alpha'}^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = & \frac{\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})A_1}{A_1} - 2\frac{\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})A_1 \partial_{\alpha'} A_1}{A_1^2} \\ & + (\partial_t + b\partial_{\alpha'})A_1\left(\frac{-\partial_{\alpha'}^2 A_1}{A_1^2} + 2\frac{(\partial_{\alpha'}A_1)^2}{A_1^3}\right) + \partial_{\alpha'}^2(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\end{aligned} \tag{4.216}$$

We have controlled $\left\|\partial_{\alpha'}^2(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t)\right\|_{L^2}$, $\left\|\partial_{\alpha'}^2 A_1\right\|_{L^2}$, $\|\partial_{\alpha'} A_1\|_{L^\infty}$, $\|\partial_{\alpha'} A_1\|_{L^2}$ and $\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})A_1\|_{L^2}$ etc. in (4.209), (4.204), (4.205), (4.161), (4.164) and (4.20), (4.42). We are left with $\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})A_1$.

We begin with (2.15), taking two derivatives to α', then one derivative to $\partial_t + b\partial_{\alpha'}$. We have

$$(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 A_1 = -\sum_{k=0}^{2} C_2^k\, \mathrm{Im}(\partial_t + b\partial_{\alpha'})\left(\left[\partial_{\alpha'}^k Z_t, \mathbb{H}\right]\partial_{\alpha'}^{2-k}\overline{Z}_{t,\alpha'}\right) \tag{4.217}$$

where $C_2^0 = 1$, $C_2^1 = 2$, $C_2^2 = 1$. Using (B.19) to expand the right hand side, then use (A.14), (A.15) and (A.16) to do the estimates. We have

$$\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 A_1\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/4}+1). \tag{4.218}$$

Now we use (B.16) to compute

$$\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})A_1 = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 A_1 + \partial_{\alpha'} b_{\alpha'}\partial_{\alpha'} A_1 + 2b_{\alpha'}\partial_{\alpha'}^2 A_1. \tag{4.219}$$

Therefore

$$\|\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})A_1\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/4}+1), \quad (4.220)$$

consequently

$$\left\|\partial_{\alpha'}^2 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1). \quad (4.221)$$

4.4.6 Controlling $\left\|(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$ *and* $\left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$

We begin with (2.35),

$$(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'}Z_t), \quad (4.222)$$

differentiating twice with respect to α'; we get

$$\begin{aligned}\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} &= \left(\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)(b_{\alpha'} - D_{\alpha'}Z_t)\\ &+2\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(\partial_{\alpha'}b_{\alpha'} - \partial_{\alpha'}D_{\alpha'}Z_t) + \frac{1}{Z_{,\alpha'}}(\partial_{\alpha'}^2 b_{\alpha'} - \partial_{\alpha'}^2 D_{\alpha'}Z_t). \quad (4.223)\end{aligned}$$

We further expand $\partial_{\alpha'}D_{\alpha'}Z_t$ and $\partial_{\alpha'}^2 D_{\alpha'}Z_t$ by product rules then use Hölder's inequality, (4.41), (4.166) and (4.207), (4.208), (4.210), (4.211). We have

$$\begin{aligned}\left\|\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^2} &\lesssim C(\mathfrak{E})\left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\|\partial_{\alpha'}b_{\alpha'} - \partial_{\alpha'}D_{\alpha'}Z_t\|_{L^2}\\ &+\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\|\partial_{\alpha'}^2 b_{\alpha'} - \partial_{\alpha'}^2 D_{\alpha'}Z_t\|_{L^2} \quad (4.224)\\ &\lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1).\end{aligned}$$

Now by (B.16),

$$(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} = \partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}} - (\partial_{\alpha'}b_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - 2b_{\alpha'}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \quad (4.225)$$

so

$$\left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1). \tag{4.226}$$

We apply (4.41) to (4.222), and obtain

$$\left\|(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \le C(\mathfrak{E})\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}. \tag{4.227}$$

4.4.7 Controlling $\|\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})b_{\alpha'}\|_{L^2}$

By (B.16),

$$\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})b_{\alpha'} = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 b_{\alpha'} + (\partial_{\alpha'}b_{\alpha'})^2 + 2b_{\alpha'}\partial_{\alpha'}^2 b_{\alpha'} \tag{4.228}$$

where by Hölder's and Sobolev inequalities (A.7),

$$\begin{aligned}
&\|(\partial_{\alpha'}b_{\alpha'})^2\|_{L^2} + \|b_{\alpha'}\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2} \lesssim \|\partial_{\alpha'}b_{\alpha'}\|_{L^2}\|\partial_{\alpha'}b_{\alpha'}\|_{L^\infty} + \|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^\infty}\\
&\lesssim \|\partial_{\alpha'}b_{\alpha'}\|_{L^2}^{3/2}\|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2}^{1/2} + \|\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^\infty}\\
&\le C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1).
\end{aligned} \tag{4.229}$$

Now we consider $(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 b_{\alpha'}$. We begin with (4.65), differentiating twice with respect to α', then to $\partial_t + b\partial_{\alpha'}$,

$$\begin{aligned}
&(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'}Z_t)\\
&= \sum_{k=0}^{2} C_2^k\,\mathrm{Re}(\partial_t + b\partial_{\alpha'})\left(\left[\partial_{\alpha'}^k \frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}^{2-k} Z_{t,\alpha'}\right)\\
&\quad + \sum_{k=0}^{2} C_2^k\,\mathrm{Re}(\partial_t + b\partial_{\alpha'})\left(\left[\partial_{\alpha'}^k Z_t, \mathbb{H}\right]\partial_{\alpha'}^{3-k}\frac{1}{Z_{,\alpha'}}\right).
\end{aligned} \tag{4.230}$$

where $C_2^0 = 1, C_2^1 = 2, C_2^2 = 1$. We expand $(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 D_{\alpha'}Z_t$ by product rules, and use (B.19) to further expand the right hand side of (4.230); we then

use (A.14), (A.15), (A.16), (A.21) and Hölder's inequality to do the estimates. We have

$$\begin{aligned}
\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2} &\lesssim \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^3 Z_t\|_{L^2} \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \\
&+ \|\partial_{\alpha'}^3 Z_t\|_{L^2} \left\|(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} + \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 Z_t\|_{L^\infty} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \\
&+ \|\partial_{\alpha'}^2 Z_t\|_{L^\infty} \left(\left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \left\|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \right) \\
&+ \|\partial_{\alpha'}^2 Z_t\|_{L^\infty}\|b_{\alpha'}\|_{L^\infty} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'} Z_t\|_{L^\infty} \left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \\
&+ \|\partial_{\alpha'} Z_t\|_{L^\infty}\|b_{\alpha'}\|_{L^\infty} \left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{tt,\alpha'}\|_{L^\infty} \left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \\
&+ \|Z_{t,\alpha'}\|_{L^\infty} \left\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.
\end{aligned} \tag{4.231}$$

This, together with (4.229) gives,

$$\begin{aligned}
&\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 b_{\alpha'}\|_{L^2} + \|\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})b_{\alpha'}\|_{L^2} \\
&\lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1).
\end{aligned} \tag{4.232}$$

4.4.8 *Controlling* $\frac{d}{dt}E_3(t)$

We know $E_3(t)$ consists of $E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}$ and $\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2$. We apply Lemma 4.1 to $E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}$ and Lemma 4.3 to $\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2$. We begin with $\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2$. We have, by Lemma 4.3,

$$\frac{d}{dt}\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2 \lesssim \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2} + \|b_{\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2 \tag{4.233}$$

We have controlled all the factors, in (4.41) and (4.195). We have

$$\frac{d}{dt}\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2 \lesssim C(\mathfrak{E})E_3(t). \tag{4.234}$$

We now consider $E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}$. Applying Lemma 4.1 to $\Theta = D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t$ yields

$$\frac{d}{dt}E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}(t) \le \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^\infty} E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}(t) + 2E_{D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t}(t)^{1/2}\left(\int \frac{|\mathcal{P}\mathcal{D}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t|^2}{\mathcal{A}}\,d\alpha'\right)^{1/2} \tag{4.235}$$

We have controlled the factor $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^\infty}$ in Sect. 4.1.2, we are left with the second term. We know, by $\mathcal{A}|Z_{,\alpha'}|^2 = A_1 \ge 1$, that

$$\int \frac{|\mathcal{P}\mathcal{D}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t|^2}{\mathcal{A}}\,d\alpha' \le \int |Z_{,\alpha'}\mathcal{P}\mathcal{D}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t|^2\,d\alpha'. \tag{4.236}$$

We compute

$$\mathcal{P}D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t = [\mathcal{P}, D_{\alpha'}]\,\partial_{\alpha'}^2\overline{Z}_t + D_{\alpha'}[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_{t,\alpha'} + D_{\alpha'}\partial_{\alpha'}[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_t + D_{\alpha'}\partial_{\alpha'}^2\mathcal{P}\overline{Z}_t; \tag{4.237}$$

and expand further by (B.9),

$$[\mathcal{P}, D_{\alpha'}]\,\partial_{\alpha'}^2\overline{Z}_t = (-2D_{\alpha'}Z_{tt})D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t - 2(D_{\alpha'}Z_t)(\partial_t + b\partial_{\alpha'})D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t; \tag{4.238}$$

by (B.15) and product rules,

$$\begin{aligned}
&\partial_{\alpha'}[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_{t,\alpha'} \\
&= -(\partial_t + b\partial_{\alpha'})(\partial_{\alpha'}b_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t) - \partial_{\alpha'}b_{\alpha'}\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'} - i\,\partial_{\alpha'}\mathcal{A}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t \\
&\quad -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})\overline{Z}_{t,\alpha'} - i\mathcal{A}_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t \\
&\quad -b_{\alpha'}^2\partial_{\alpha'}^3\overline{Z}_t - b_{\alpha'}\partial_{\alpha'}b_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t;
\end{aligned} \tag{4.239}$$

and by (B.15),

$$\partial_{\alpha'}^2[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_t = -\partial_{\alpha'}^2(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\overline{Z}_t) - \partial_{\alpha'}^2(b_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt}) - i\,\partial_{\alpha'}^2(\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_t), \tag{4.240}$$

and then expand (4.240) by product rules. We have controlled all the factors on the right hand sides of (4.238), (4.239) and (4.240) in (4.41), (4.166) and

Sect. 4.4.1–4.4.7. We have, by Hölder's inequality,

$$\int |Z_{,\alpha'}[\mathcal{P}, D_{\alpha'}]\partial_{\alpha'}^2\overline{Z}_t|^2 + |\partial_{\alpha'}[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_{t,\alpha'}|^2 + |\partial_{\alpha'}^2[\mathcal{P}, \partial_{\alpha'}]\overline{Z}_t|^2\, d\alpha' \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3+1). \tag{4.241}$$

We are left with the last term $\partial_{\alpha'}^3\mathcal{P}\overline{Z}_t$. We expand by product rules, starting from (2.19); we have

$$\partial_{\alpha'}^3\mathcal{P}\overline{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^3\overline{Z}_{tt} + 3\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\partial_{\alpha'}^2\overline{Z}_{tt} + 3\partial_{\alpha'}^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\overline{Z}_{tt,\alpha'} + \partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt}-i). \tag{4.242}$$

Let

$$\mathcal{R}_2 = \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^3\overline{Z}_{tt} + 3\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\partial_{\alpha'}^2\overline{Z}_{tt} + 3\partial_{\alpha'}^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\overline{Z}_{tt,\alpha'}. \tag{4.243}$$

We have controlled all the factors of the terms in $\mathcal{R}_2$, with

$$\|\mathcal{R}_2\|_{L^2} \le \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^\infty}\|\partial_{\alpha'}^3\overline{Z}_{tt}\|_{L^2} + 3\left\|\partial_{\alpha'}\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^2}\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^\infty} + 3\left\|\partial_{\alpha'}^2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^2}\|\overline{Z}_{tt,\alpha'}\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2}+1). \tag{4.244}$$

We are left with controlling $\left\|\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)(\overline{Z}_{tt}-i)\right\|_{L^2}$. We use a similar idea as that in Sect. 4.1.7, that is, to take advantage of the fact that $\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)$ is purely real.

Applying $(I-\mathbb{H})$ to both sides of (4.242), with the first three terms replaced by $\mathcal{R}_2$, and commuting out $\overline{Z}_{tt}-i$ yields,

$$(I-\mathbb{H})\partial_{\alpha'}^3\mathcal{P}\overline{Z}_t = (I-\mathbb{H})\mathcal{R}_2 + [\overline{Z}_{tt}, \mathbb{H}]\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) + (\overline{Z}_{tt}-i)(I-\mathbb{H})\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right). \tag{4.245}$$

Now

$$\partial_{\alpha'}^3\mathcal{P}\overline{Z}_t = \partial_{\alpha'}^2[\partial_{\alpha'}, \mathcal{P}]\overline{Z}_t + \partial_{\alpha'}[\partial_{\alpha'}, \mathcal{P}]\overline{Z}_{t,\alpha'} + [\partial_{\alpha'}, \mathcal{P}]\partial_{\alpha'}^2\overline{Z}_t + \mathcal{P}\partial_{\alpha'}^3\overline{Z}_t. \tag{4.246}$$

and by (B.15),

$$[\mathcal{P}, \partial_{\alpha'}]\partial_{\alpha'}^2 \overline{Z}_t = -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}\partial_{\alpha'}^2 \overline{Z}_t) - b_{\alpha'}\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^2 \overline{Z}_t \\ -i\mathcal{A}_{\alpha'}\partial_{\alpha'}\partial_{\alpha'}^2 \overline{Z}_t \tag{4.247}$$

so by Sects. 4.1.1–4.1.5, 4.4.1 and Hölder's inequality,

$$\|[\mathcal{P}, \partial_{\alpha'}]\partial_{\alpha'}^2 \overline{Z}_t\|_{L^2} \lesssim C(\mathfrak{E})E_3^{1/2}. \tag{4.248}$$

Applying Lemma 4.4 to the last term in (4.246). We have, by (4.241), (4.248) and Lemma 4.4, that

$$\|(I - \mathbb{H})\partial_{\alpha'}^3 \mathcal{P}\overline{Z}_t\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1). \tag{4.249}$$

This gives, by (4.245), that

$$\begin{aligned}
&\left\|\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)(\overline{Z}_{tt} - i)\right\|_{L^2} \\
&\quad \le \|(I - \mathbb{H})\partial_{\alpha'}^3 \mathcal{P}\overline{Z}_t\|_{L^2} + \|\mathcal{R}_2\|_{L^2} + \left\|[\overline{Z}_{tt}, \mathbb{H}]\partial_{\alpha'}^3\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right)\right\|_{L^2} \\
&\quad \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1);
\end{aligned} \tag{4.250}$$

here for the commutator, we used (A.14), (4.166), and (4.221). Combining with (4.242), (4.243), (4.244) yields

$$\left\|\partial_{\alpha'}^3 \mathcal{P}\overline{Z}_t\right\|_{L^2} \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3^{1/2} + 1). \tag{4.251}$$

Further combing with (4.236), (4.237), (4.241) gives

$$\int \frac{|\mathcal{P}\mathcal{D}_{\alpha'}\partial_{\alpha'}^2 \overline{Z}_t|^2}{\mathcal{A}}\, d\alpha' \lesssim C\left(\mathfrak{E}, E_2, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\right)(E_3 + 1). \tag{4.252}$$

By (4.235), this shows that Proposition 4.7 holds.

4.5 Completing the proof for Theorem 3.6

Now we continue the discussion in Sect. 4.2, assuming that the initial data satisfies the assumption of Theorem 3.6, and the solution Z satisfies the regularity property in Theorem 3.6. By (4.135) and the ensuing discussion, to

complete the proof of Theorem 3.6, it suffices to show that for the given data, $E_2(0) < \infty$ and $E_3(0) < \infty$; and $\sup_{[0,T]}(E_2(t) + E_3(t) + \mathfrak{E}(t))$ control the higher order Sobolev norm $\sup_{[0,T]}(\|\partial_{\alpha'}^3 Z_t(t)\|_{\dot{H}^{1/2}} + \|\partial_{\alpha'}^3 Z_{tt}(t)\|_{L^2})$.

By (4.137) and (4.147),

$$Z_{,\alpha'}(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2 \overline{Z}_t\right) = \partial_{\alpha'}^2 \overline{Z}_{tt} - (b_{\alpha'} + D_{\alpha'} Z_t)\partial_{\alpha'}^2 \overline{Z}_t - (\partial_{\alpha'} b_{\alpha'})\overline{Z}_{t,\alpha'}; \tag{4.253}$$

expanding further according to the available estimates in (4.21), (4.146), and using Hölder's inequality, we have

$$\begin{aligned} &\left\| Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) D_{\alpha'}\partial_{\alpha'}\overline{Z}_t \right\|_{L^2} \\ &\lesssim \|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'}^2 Z_t\|_{L^2} + \|D_{\alpha'} Z_t \partial_{\alpha'}^2 \overline{Z}_t\|_{L^2} \\ &\quad + \|Z_{t,\alpha'}\|_{L^\infty}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|(\partial_{\alpha'} D_{\alpha'} Z_t)\partial_{\alpha'}\overline{Z}_t\|_{L^2}; \end{aligned} \tag{4.254}$$

it is clear that we also have

$$\left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}} \le C\left(\left\| \frac{1}{Z_{,\alpha'}} - 1 \right\|_{H^1}, \|Z_t\|_{H^{2+1/2}} \right); \tag{4.255}$$

so for the given initial data, we have $E_2(0) < \infty$. A similar argument shows that we also have $E_3(0) < \infty$. This implies, by (4.135) and the ensuing discussion, that $\sup_{[0,T_0)}(E_2(t) + E_3(t)) < \infty$ provided $\sup_{[0,T_0)} \mathfrak{E}(t) < \infty$. On the other hand, we have shown in (4.213) that $\|\partial_{\alpha'}^3 \overline{Z}_{tt}(t)\|_{L^2}$ is controlled by $E_2(t)$, $E_3(t)$ and $\left\| \frac{1}{Z_{,\alpha'}}(t) \right\|_{L^\infty}$; and by (A.6),

$$\|\partial_{\alpha'}^3 \overline{Z}_t\|_{\dot{H}^{1/2}} \lesssim \|Z_{,\alpha'}\|_{L^\infty} \left(\left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^3 \overline{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'}^3 \overline{Z}_t\|_{L^2} \right),$$

so $\|\partial_{\alpha'}^3 \overline{Z}_t\|_{\dot{H}^{1/2}}$ is controlled by $E_3(t)$, $\mathfrak{E}(t)$ and $\|Z_{,\alpha'}(t)\|_{L^\infty}$. With a further application of (4.119) and (4.120), we have

$$\sup_{[0,T_0)} \|\partial_{\alpha'}^3 \overline{Z}_{tt}(t)\|_{L^2} + \|\partial_{\alpha'}^3 \overline{Z}_t\|_{\dot{H}^{1/2}} < \infty, \quad \text{provided} \quad \sup_{[0,T_0)} \mathfrak{E}(t) < \infty. \tag{4.256}$$

This, together with (4.119), (4.129), (4.130) and Theorem 2.3 shows that Theorem 3.6 holds.

5 The proof of Theorem 3.7

5.1 Some basic preparations

We begin with some basic preparatory analysis that will be used in the proof of Theorem 3.7.

In the first lemma we construct an energy functional for the difference of the solutions of an equation of the type (2.24). We will apply Lemma 5.1 to $\Theta = \overline{Z}_t$, $\frac{1}{Z_{,\alpha'}} - 1$ and $\overline{Z}_{tt}$.

Lemma 5.1 *Assume Θ, $\tilde{\Theta}$ are smooth and decay at the spatial infinity, and satisfy*

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})^2\Theta + i\mathcal{A}\partial_{\alpha'}\Theta = G,\\ (\partial_t + \tilde{b}\partial_{\alpha'})^2\tilde{\Theta} + i\tilde{\mathcal{A}}\partial_{\alpha'}\tilde{\Theta} = \tilde{G}.\end{aligned} \tag{5.1}$$

Let

$$\begin{aligned}\mathfrak{F}(t) = \int \frac{\kappa}{A_1}\left|Z_{,\alpha'}\left((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c}\right) - \mathfrak{Z}_{,\alpha'}\circ l\left((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta}\circ l + \mathfrak{c}\right)\right|^2\\ + i\partial_{\alpha'}(\Theta - \tilde{\Theta}\circ l)\overline{(\Theta - \tilde{\Theta}\circ l)}\, d\alpha',\end{aligned} \tag{5.2}$$

where $\kappa = \sqrt{\frac{A_1}{\tilde{A}_1\circ l}}l_{\alpha'}$, $\mathfrak{c}$ is a constant, and

$$\mathbf{F}(t) = \int \left|Z_{,\alpha'}\left((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c}\right) - \mathfrak{Z}_{,\alpha'}\circ l\left((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta}\circ l + \mathfrak{c}\right)\right|^2 d\alpha'. \tag{5.3}$$

Then

$$\begin{aligned}\mathfrak{F}'(t) \lesssim \mathbf{F}(t)^{\frac{1}{2}}\left\|\frac{\kappa}{A_1}\right\|_{L^\infty}\left\|Z_{,\alpha'}G - \mathfrak{Z}_{,\alpha'}\circ l\tilde{G}\circ l\right\|_{L^2}\\ + \mathbf{F}(t)^{\frac{1}{2}}\left\|\frac{\kappa}{A_1}\right\|_{L^\infty}\left\|\frac{(\partial_t + b\partial_{\alpha'})\kappa}{\kappa}\right\|_{L^2}\left(\left\|Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})\right\|_{L^\infty}\right.\end{aligned}$$

$$
\begin{aligned}
&+\left\|\mathfrak{Z}_{,\alpha'}((\partial_t+\tilde{b}\partial_{\alpha'})\tilde{\Theta}+\mathfrak{c})\right\|_{L^\infty}\Big)\\
&+\mathbf{F}(t)^{\frac{1}{2}}\left\|\frac{\kappa}{A_1}\right\|_{L^\infty}\left(\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}-\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ h^{-1}\right\|_{L^2}+\left\|D_{\alpha'}Z_t-\tilde{D}_{\alpha'}\mathfrak{Z}_t\circ l\right\|_{L^2}\right)\\
&\quad\left\|\mathfrak{Z}_{,\alpha'}((\partial_t+\tilde{b}\partial_{\alpha'})\tilde{\Theta}+\mathfrak{c})\right\|_{L^\infty}\\
&+\mathbf{F}(t)\left\|\frac{\kappa}{A_1}\right\|_{L^\infty}\left(\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}+\|D_{\alpha'}Z_t\|_{L^\infty}\right)\\
&+\|1-\kappa\|_{L^2}\mathbf{F}(t)^{\frac{1}{2}}\left(\|D_{\alpha'}\Theta\|_{L^\infty}+\left\|\frac{l_{\alpha'}}{\kappa}\right\|_{L^\infty}\left\|\tilde{D}_{\alpha'}\tilde{\Theta}\right\|_{L^\infty}\right)\\
&+2\,\mathrm{Re}\,i\int\overline{\left(\frac{1}{Z_{,\alpha'}}-U_l\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)}\left(\overline{U_l(\mathfrak{Z}_{,\alpha'}((\partial_t+\tilde{b}\partial_{\alpha'})\tilde{\Theta}+\mathfrak{c}))}\Theta_{\alpha'}\right.\\
&\left.-\overline{Z_{,\alpha'}((\partial_t+b\partial_{\alpha'})\Theta+\mathfrak{c})}(\tilde{\Theta}\circ l)_{\alpha'}\right)d\alpha'. \qquad (5.4)
\end{aligned}
$$

Remark 5.2 By definition, $\frac{\kappa}{A_1}=\sqrt{\frac{l_{\alpha'}}{A_1\tilde{A}_1\circ l}}$. And in what follows, $\sqrt{\tilde{a}h_{\alpha'}}\,\kappa\circ h=\frac{\sqrt{A_1}}{\mathfrak{Z}_{,\alpha'}\circ l}\circ h$, $\sqrt{ah_{\alpha'}}=\frac{\sqrt{A_1}}{Z_{,\alpha'}}\circ h$.

Proof Let $\theta=\Theta\circ h$, and $\tilde{\theta}=\tilde{\Theta}\circ\tilde{h}$. We know $\theta,\tilde{\theta}$ satisfy

$$
\begin{aligned}
\partial_t^2\theta+i\mathfrak{a}\partial_\alpha\theta&=G\circ h,\\
\partial_t^2\tilde{\theta}+i\tilde{\mathfrak{a}}\partial_\alpha\tilde{\theta}&=\tilde{G}\circ\tilde{h}.
\end{aligned}
\qquad (5.5)
$$

Changing coordinate by h, we get

$$
\mathfrak{F}(t)=\int\left|\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde{a}k}}\left(\tilde{\theta}_t+\mathfrak{c}\right)\right|^2+i\,\partial_\alpha(\theta-\tilde{\theta})\overline{(\theta-\tilde{\theta})}\,d\alpha, \qquad (5.6)
$$

where $k=\kappa\circ h$, $\sqrt{a}:=\frac{\sqrt{A_1\circ h h_\alpha}}{z_\alpha}$ and $\sqrt{\tilde{a}}:=\frac{\sqrt{\tilde{A}_1\circ\tilde{h}\tilde{h}_\alpha}}{\mathfrak{z}_\alpha}$. Notice that here, $\sqrt{a}$ and $\sqrt{\tilde{a}}$ are complex valued, and $|\sqrt{a}|^2=\mathfrak{a}$, $|\sqrt{\tilde{a}}|^2=\tilde{\mathfrak{a}}$. Differentiating to t, integrating by parts, then applying Eq. (5.5), we get

$$
\begin{aligned}
\mathfrak{F}'(t)=2\,\mathrm{Re}\int&\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde{a}k}}\left(\tilde{\theta}_t+\mathfrak{c}\right)\right\}}\left\{\sqrt{\frac{k}{a}}G\circ h-\frac{1}{\sqrt{\tilde{a}k}}\tilde{G}\circ\tilde{h}\right\}\\
&+\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde{a}k}}\left(\tilde{\theta}_t+\mathfrak{c}\right)\right\}}\left\{\frac{1}{2}\frac{k_t}{k}\left(\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})+\frac{1}{\sqrt{\tilde{a}k}}\left(\tilde{\theta}_t+\mathfrak{c}\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{\tilde\theta_t+\mathfrak{c}}{\sqrt{\tilde a\tilde k}}\right\}}\left\{\left(\frac{1}{2}\frac{\mathfrak{a}_t}{\mathfrak{a}}-i\,\mathrm{Im}\,D_\alpha z_t\right)\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})\right.\\
&\left.-\left(\frac{1}{2}\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}-i\,\mathrm{Im}\,D_\alpha\mathfrak{z}_t\right)\frac{\tilde\theta_t+\mathfrak{c}}{\sqrt{\tilde a\tilde k}}\right\}d\alpha\\
&+2\,\mathrm{Re}\,i\int\overline{(\theta_t-\tilde\theta_t)}(\theta_\alpha-\tilde\theta_a)-\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde a\tilde k}}\left(\tilde\theta_t+\mathfrak{c}\right)\right\}}\\
&\left\{\sqrt{k}\overline{\sqrt{a}}\theta_\alpha-\frac{\overline{\sqrt{\tilde a}}}{\sqrt{k}}\tilde\theta_\alpha\right\}d\alpha\\
&=I+II
\end{aligned}
\tag{5.7}
$$

where I consists of the terms in the first three lines and II is the last line

$$
\begin{aligned}
II:=\,&2\,\mathrm{Re}\,i\int\overline{(\theta_t-\tilde\theta_t)}(\theta_\alpha-\tilde\theta_a)-\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde a\tilde k}}\left(\tilde\theta_t+\mathfrak{c}\right)\right\}}\\
&\left\{\sqrt{k}\overline{\sqrt{a}}\theta_\alpha-\frac{\overline{\sqrt{\tilde a}}}{\sqrt{k}}\tilde\theta_\alpha\right\}d\alpha.
\end{aligned}
$$

Further regrouping terms in II we get

$$
\begin{aligned}
II=\,&2\,\mathrm{Re}\,i\int\frac{(1-k)}{\sqrt{k}}\overline{\left\{\sqrt{\frac{k}{a}}\,(\theta_t+\mathfrak{c})-\frac{1}{\sqrt{\tilde a\tilde k}}\left(\tilde\theta_t+\mathfrak{c}\right)\right\}}\left\{\overline{\sqrt{a}}\theta_\alpha+\overline{\sqrt{\tilde a}}\tilde\theta_\alpha\right\}d\alpha\\
&+2\,\mathrm{Re}\,i\int\overline{(\sqrt{a}-\sqrt{\tilde a\tilde k})}\left(\overline{\frac{1}{\sqrt{\tilde a\tilde k}}(\tilde\theta_t+\mathfrak{c})}\theta_\alpha-\overline{\frac{1}{\sqrt{a}}(\theta_t+\mathfrak{c})}\tilde\theta_\alpha\right)d\alpha.
\end{aligned}
\tag{5.8}
$$

Changing variables by h^{-1} in the integrals in (5.7) and (5.8), and then applying Cauchy–Schwarz and Hölder's inequalities, we obtain (5.4). □

We have the following basic identities and inequalities.

Proposition 5.3 *Let $\mathcal{Q}_l=U_l\mathbb{H}U_{l^{-1}}-\mathbb{H}$, where $l:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism,*[21] *with $l_{\alpha'}-1\in L^2$. For any $f\in H^1(\mathbb{R})$, we have*

$$
\|\mathcal{Q}_l f\|_{\dot H^{\frac12}}\le C\big(\big\|(l^{-1})_{\alpha'}\big\|_{L^\infty},\|l_{\alpha'}\|_{L^\infty}\big)\|l_{\alpha'}-1\|_{L^2}\|\partial_{\alpha'}f\|_{L^2};\tag{5.9}
$$

$$
\|\mathcal{Q}_l f\|_{L^\infty}\le C\big(\big\|(l^{-1})_{\alpha'}\big\|_{L^\infty},\|l_{\alpha'}\|_{L^\infty}\big)\|l_{\alpha'}-1\|_{L^2}\|\partial_{\alpha'}f\|_{L^2};\tag{5.10}
$$

$$
\|\mathcal{Q}_l f\|_{L^2}\le C\big(\big\|(l^{-1})_{\alpha'}\big\|_{L^\infty},\|l_{\alpha'}\|_{L^\infty}\big)\|l_{\alpha'}-1\|_{L^2}\|f\|_{L^\infty};\tag{5.11}
$$

[21] We say $l:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism, if $l:\mathbb{R}\to\mathbb{R}$ is one-to-one and onto, and $l,\ l^{-1}\in C^1(\mathbb{R})$, with $\|l_{\alpha'}\|_{L^\infty}+\|(l^{-1})_{\alpha'}\|_{L^\infty}<\infty$.

$$\|\mathcal{Q}_l f\|_{L^2} \le C\big(\big\|(l^{-1})_{\alpha'}\big\|_{L^\infty}, \|l_{\alpha'}\|_{L^\infty}\big)\|l_{\alpha'} - 1\|_{L^2}\|f\|_{\dot{H}^{1/2}}. \tag{5.12}$$

Proof We know

$$U_l \mathbb{H} U_{l^{-1}} f(\alpha') = \frac{1}{\pi i}\int \frac{f(\beta')l_{\beta'}(\beta')}{l(\alpha') - l(\beta')}\,d\beta' \tag{5.13}$$

so

$$\begin{aligned}\mathcal{Q}_l f &= \frac{1}{\pi i}\int \left(\frac{l_{\beta'}(\beta') - 1}{l(\alpha') - l(\beta')} + \frac{\alpha' - l(\alpha') - \beta' + l(\beta')}{(l(\alpha') - l(\beta'))(\alpha' - \beta')}\right) f(\beta')\,d\beta' \\ &= \frac{1}{\pi i}\int \left(\frac{l_{\beta'}(\beta') - 1}{l(\alpha') - l(\beta')} + \frac{\alpha' - l(\alpha') - \beta' + l(\beta')}{(l(\alpha') - l(\beta'))(\alpha' - \beta')}\right) (f(\beta') - f(\alpha'))\,d\beta',\end{aligned} \tag{5.14}$$

here in the second step we inserted $-f(\alpha')$ because $\mathbb{H}1 = 0$. Apply Cauchy–Schwarz inequality and Hardy's inequality (A.8) on the second equality in (5.14) we obtain (5.10) and (5.12). Using (A.11) and (A.12) on the first equality in (5.14) we get (5.11). We are left with (5.9).

Differentiate with respect to α' and integrate by parts gives

$$\partial_{\alpha'}\mathcal{Q}_l f(\alpha') = \frac{1}{\pi i}\int \left(\frac{l_{\alpha'}(\alpha')}{l(\alpha') - l(\beta')} - \frac{1}{\alpha' - \beta'}\right) f_{\beta'}(\beta')\,d\beta' \tag{5.15}$$

Let $p \in C_0^\infty(\mathbb{R})$. We have, by using the fact $\mathbb{H}1 = 0$ to insert $-p(\beta')$, that

$$\begin{aligned}&\int p(\alpha')\partial_{\alpha'}\mathcal{Q}_l f(\alpha')\,d\alpha' \\ &= \frac{1}{\pi i}\iint \left(\frac{l_{\alpha'}(\alpha')}{l(\alpha') - l(\beta')} - \frac{1}{\alpha' - \beta'}\right) f_{\beta'}(\beta')(p(\alpha') - p(\beta'))\,d\alpha' d\beta' \\ &= \frac{1}{\pi i}\iint \frac{p(\alpha') - p(\beta')}{l(\alpha') - l(\beta')}(l_{\alpha'}(\alpha') - 1) f_{\beta'}(\beta')\,d\alpha' d\beta' \\ &\quad + \frac{1}{\pi i}\iint \frac{\alpha' - l(\alpha') - \beta' + l(\beta')}{(l(\alpha') - l(\beta'))(\alpha' - \beta')} f_{\beta'}(\beta')(p(\alpha') - p(\beta'))\,d\alpha' d\beta'.\end{aligned} \tag{5.16}$$

Applying Cauchy–Schwarz inequality and Hardy's inequality (A.8) to (5.16). We get, for some constant c depending only on $\|l_{\alpha'}\|_{L^\infty}$ and $\|(l^{-1})_{\alpha'}\|_{L^\infty}$,

$$\left|\int p(\alpha')\partial_{\alpha'}\mathcal{Q}_l f(\alpha')\,d\alpha'\right| \le c\|p\|_{\dot{H}^{1/2}}\|l_{\alpha'} - 1\|_{L^2}\|\partial_{\alpha'} f\|_{L^2}. \tag{5.17}$$

This proves inequality (5.9). □

Lemma 5.4 *Assume that* $f,\ g,\ f_1,\ g_1 \in H^1(\mathbb{R})$ *are the boundary values of some holomorphic functions on* $\mathscr{P}_-$. *Then*

$$\begin{aligned}&\int \partial_{\alpha'}\mathbb{P}_A(\overline{f}g)(\alpha')\,f_1(\alpha')\overline{g}_1(\alpha')\,d\alpha'\\&\quad=-\frac{1}{2\pi i}\iint\frac{(\overline{f}(\alpha')-\overline{f}(\beta'))(f_1(\alpha')-f_1(\beta'))}{(\alpha'-\beta')^2}g(\beta')\overline{g}_1(\alpha')\,d\alpha' d\beta'.\end{aligned}\tag{5.18}$$

Proof Let $f,\ g,\ f_1,\ g_1 \in H^1(\mathbb{R})$, and are the boundary values of some holomorphic functions in $\mathscr{P}_-$. We have

$$2\mathbb{P}_A(\overline{f}g)=(I-\mathbb{H})(\overline{f}g)=[\overline{f},\mathbb{H}]g\tag{5.19}$$

and

$$2\partial_{\alpha'}\mathbb{P}_A(\overline{f}g)=\partial_{\alpha'}\overline{f}\,\mathbb{H}g-\frac{1}{\pi i}\int\frac{\overline{f}(\alpha')-\overline{f}(\beta')}{(\alpha'-\beta')^2}g(\beta')\,d\beta'.\tag{5.20}$$

Because $\overline{g}_1\partial_{\alpha'}\mathbb{P}_A(\overline{f}f_1g) \in L^1(\mathbb{R})$ is the boundary value of an anti-holomorphic function in $\mathscr{P}_-$, by Cauchy integral theorem,

$$\begin{aligned}0&=2\int\overline{g}_1\partial_{\alpha'}\mathbb{P}_A(\overline{f}f_1g)\,d\alpha'\\&=\int\partial_{\alpha'}\overline{f}\,f_1g\overline{g}_1\,d\alpha'-\frac{1}{\pi i}\iint\frac{\overline{f}(\alpha')-\overline{f}(\beta')}{(\alpha'-\beta')^2}f_1(\beta')g(\beta')\overline{g}_1(\alpha')\,d\alpha' d\beta',\end{aligned}\tag{5.21}$$

here we applied formula (5.20) to the pair of holomorphic functions f and f_1g, and used the fact that $\mathbb{H}(f_1g)=f_1g$. Now we use (5.20) to compute, because $\mathbb{H}g=g$,

$$\begin{aligned}&2\int\partial_{\alpha'}\mathbb{P}_A(\overline{f}g)\,f_1\overline{g}_1\,d\alpha'\\&\quad=\int\partial_{\alpha'}\overline{f}\,gf_1\overline{g}_1\,d\alpha'-\frac{1}{\pi i}\iint\frac{\overline{f}(\alpha')-\overline{f}(\beta')}{(\alpha'-\beta')^2}g(\beta')f_1(\alpha')\overline{g}_1(\alpha')\,d\alpha' d\beta'.\end{aligned}\tag{5.22}$$

Substituting (5.21) in (5.22), we get (5.18). □

Remark 5.5 By Cauchy integral theorem, we know for $f,\ g,\ f_1,\ g_1 \in H^1(\mathbb{R})$,

$$\begin{aligned}\int \partial_{\alpha'}\mathbb{P}_A(\overline{f}g)(\alpha')f_1(\alpha')\overline{g}_1(\alpha')\,d\alpha' &= \int \partial_{\alpha'}\mathbb{P}_A(\overline{f}g)\mathbb{P}_H(f_1\overline{g}_1)\,d\alpha' \\ &= \int \partial_{\alpha'}\mathbb{P}_A(\overline{f}g)\overline{\mathbb{P}_A(\overline{f}_1 g_1)}\,d\alpha'.\end{aligned}$$

As a corollary of Lemma 5.4 and Remark 5.5 we have

Proposition 5.6 *Assume that* $f,\ g \in H^1(\mathbb{R})$. *We have*

$$\|[f,\mathbb{H}]\,g\|_{\dot{H}^{1/2}} \lesssim \|f\|_{\dot{H}^{1/2}}(\|g\|_{L^\infty} + \|\mathbb{H}g\|_{L^\infty}); \tag{5.23}$$

$$\|[f,\mathbb{H}]\,g\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'}f\|_{L^2}\|g\|_{L^2}; \tag{5.24}$$

$$\|[f,\mathbb{H}]\,\partial_{\alpha'}g\|_{\dot{H}^{1/2}} \lesssim \|g\|_{\dot{H}^{1/2}}(\|\partial_{\alpha'}f\|_{L^\infty} + \|\partial_{\alpha'}\mathbb{H}f\|_{L^\infty}). \tag{5.25}$$

Proof By Proposition A.1 and the decompositions $f = \mathbb{P}_A f + \mathbb{P}_H f$, $g = \mathbb{P}_A g + \mathbb{P}_H g$,

$$[f,\mathbb{H}]\,g = [\mathbb{P}_A f,\mathbb{H}]\,\mathbb{P}_H g + [\mathbb{P}_H f,\mathbb{H}]\,\mathbb{P}_A g. \tag{5.26}$$

So without loss of generality, we assume f is anti-holomorphic and g is holomorphic, i.e. $f = -\mathbb{H}f$, $g = \mathbb{H}g$. (5.23) is straightforward from (5.18), Remark 5.5 and the definition (A.2); and (5.24) can be easily obtained by applying Cauchy–Schwarz inequality and Hardy's inequality (A.8) to (5.18). We are left with (5.25).

By integration by parts, we know

$$[f,\mathbb{H}]\partial_{\alpha'}g + [g,\mathbb{H}]\partial_{\alpha'}f = \frac{1}{\pi i}\int \frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))}{(\alpha'-\beta')^2}\,d\beta' := \mathbf{r}; \tag{5.27}$$

and by (5.23),

$$\|[g,\mathbb{H}]\partial_{\alpha'}f\|_{\dot{H}^{1/2}} \lesssim \|g\|_{\dot{H}^{1/2}}\|\partial_{\alpha'}f\|_{L^\infty}.$$

For the term $\mathbf{r}$ in the right hand side of (5.27), we have

$$\partial_{\alpha'}\mathbf{r} = \frac{-2}{\pi i}\int \frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))}{(\alpha'-\beta')^3}\,d\beta' + f_{\alpha'}\mathbb{H}g_{\alpha'} + g_{\alpha'}\mathbb{H}f_{\alpha'}; \tag{5.28}$$

and using $f = -\mathbb{H}f$, $g = \mathbb{H}g$, we find

$$f_{\alpha'}\mathbb{H}g_{\alpha'} + g_{\alpha'}\mathbb{H}f_{\alpha'} = f_{\alpha'}g_{\alpha'} - g_{\alpha'}f_{\alpha'} = 0.$$

Let $p \in C_0^\infty(\mathbb{R})$. We have, using the symmetry of the integrand,

$$\int p\partial_{\alpha'}\mathbf{r}\, d\alpha' = \frac{-1}{\pi i}\iint \frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))(p(\alpha')-p(\beta'))}{(\alpha'-\beta')^3}\, d\alpha' d\beta'; \tag{5.29}$$

applying Cauchy–Schwarz inequality and the definition (A.2), we get

$$\left|\int p\partial_{\alpha'}\mathbf{r}\, d\alpha'\right| \lesssim \|\partial_{\alpha'} f\|_{L^\infty}\|g\|_{\dot{H}^{1/2}}\|p\|_{\dot{H}^{1/2}}, \tag{5.30}$$

so $\|\mathbf{r}\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'} f\|_{L^\infty}\|g\|_{\dot{H}^{1/2}}$. This finishes the proof for (5.25). □

Proposition 5.7 *Assume $f,\ g,\ f_1,\ g_1 \in H^1(\mathbb{R})$, and $l : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, with $l_{\alpha'} - 1 \in L^2$. Then*

$$\begin{aligned}\|[f,\mathbb{H}]\,\partial_{\alpha'} g - U_l\,[f_1,\mathbb{H}]\,\partial_{\alpha'} g_1\|_{L^2} &\lesssim \|f - f_1\circ l\|_{\dot{H}^{1/2}}\,\|\partial_{\alpha'} g\|_{L^2}\\ &+\|\partial_{\alpha'} f_1\|_{L^2}\|l_{\alpha'}\|_{L^\infty}^{\frac{1}{2}}\,\|g - g_1\circ l\|_{\dot{H}^{1/2}} + \|\partial_{\alpha'} f_1\|_{L^2}\|\partial_{\alpha'} g_1\|_{L^2}\,\|l_{\alpha'} - 1\|_{L^2}.\end{aligned} \tag{5.31}$$

Proof We know

$$\begin{aligned}[f,\mathbb{H}]\,\partial_{\alpha'} g - U_l\,[f_1,\mathbb{H}]\,\partial_{\alpha'} g_1 &= [f,\mathbb{H}]\,\partial_{\alpha'} g - \left[f_1\circ l, U_l\mathbb{H}U_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l)\\ &= [f,\mathbb{H}]\,\partial_{\alpha'} g - [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l) + \left[f_1\circ l, \mathbb{H} - U_l\mathbb{H}U_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l);\end{aligned} \tag{5.32}$$

applying Proposition A.8 to the term

$$\begin{aligned}&[f,\mathbb{H}]\,\partial_{\alpha'} g - [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l)\\ &\quad = [f - f_1\circ l,\mathbb{H}]\,\partial_{\alpha'} g + [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g - g_1\circ l)\end{aligned} \tag{5.33}$$

gives

$$\begin{aligned}&\|[f,\mathbb{H}]\,\partial_{\alpha'} g - [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l)\|_{L^2}\\ &\lesssim \|f - f_1\circ l\|_{\dot{H}^{1/2}}\,\|\partial_{\alpha'} g\|_{L^2} + \|\partial_{\alpha'}(f_1\circ l)\|_{L^2}\,\|g - g_1\circ l\|_{\dot{H}^{1/2}}.\end{aligned} \tag{5.34}$$

Now by (5.13),

$$\begin{aligned}&\left[f_1\circ l, \mathbb{H} - U_l\mathbb{H}U_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l)\\ &\quad = \frac{1}{\pi i}\int \frac{(f_1\circ l(\alpha') - f_1\circ l(\beta'))(l(\alpha')-\alpha'-l(\beta')+\beta')}{(l(\alpha')-l(\beta'))(\alpha'-\beta')}\,\partial_{\beta'}(g_1\circ l)(\beta')\, d\beta';\end{aligned} \tag{5.35}$$

applying Cauchy–Schwarz inequality and Hardy's inequality (A.8) we get

$$\begin{aligned}&\left\|\left[f_1\circ l, \mathbb{H}-U_l\mathbb{H}U_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l)\right\|_{L^2}\\ &\lesssim \|\partial_{\alpha'}f_1\|_{L^2}\|l_{\alpha'}-1\|_{L^2}\|\partial_{\alpha'}g_1\|_{L^2}.\end{aligned}\tag{5.36}$$

This finishes the proof for (5.31). □

Proposition 5.8 *Assume that $f,\ g,\ f_1,\ g_1$ are smooth and decay at infinity, and $l:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism, with $l_{\alpha'}-1\in L^2$. Then there is a constant $c(\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty})$, depending on $\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty}$, such that*

$$\begin{aligned}&\|[f,\mathbb{H}]\,\partial_{\alpha'}g-U_l\,[f_1,\mathbb{H}]\,\partial_{\alpha'}g_1\|_{L^2}\\ &\quad\le c(\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty})\,\|\partial_{\alpha'}f-\partial_{\alpha'}(f_1\circ l)\|_{L^2}\,\|g\|_{L^\infty}\\ &\qquad+c(\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty})(\|\partial_{\alpha'}f_1\|_{L^\infty}\,\|g-g_1\circ l\|_{L^2}\\ &\qquad+\|\partial_{\alpha'}f_1\|_{L^\infty}\|g_1\|_{L^\infty}\,\|l_{\alpha'}-1\|_{L^2}).\end{aligned}\tag{5.37}$$

$$\begin{aligned}&\|[f,\mathbb{H}]\,\partial_{\alpha'}g-U_l\,[f_1,\mathbb{H}]\,\partial_{\alpha'}g_1\|_{L^2}\lesssim\|\partial_{\alpha'}f-\partial_{\alpha'}(f_1\circ l)\|_{L^2}\,\|g\|_{\dot H^{1/2}}\\ &\qquad+c(\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty})(\|\partial_{\alpha'}f_1\|_{L^\infty}\,\|g-g_1\circ l\|_{L^2}\\ &\qquad+\|\partial_{\alpha'}f_1\|_{L^\infty}\|g_1\|_{\dot H^{1/2}}\,\|l_{\alpha'}-1\|_{L^2}).\end{aligned}\tag{5.38}$$

Proof We use the same computation as in the proof for Proposition 5.7, and apply Proposition A.7 to the terms in (5.33) and (5.35) to get (5.37). To obtain (5.38) we apply (A.19) and (A.14) to (5.33); and for the term in (5.35), we first integrate by parts to get

$$\begin{aligned}&\frac{1}{\pi i}\int\frac{(f_1\circ l(\alpha')-f_1\circ l(\beta'))(l(\alpha')-\alpha'-l(\beta')+\beta')}{(l(\alpha')-l(\beta'))(\alpha'-\beta')}\,\partial_{\beta'}(g_1\circ l)(\beta')\,d\beta'\\ &=\frac{-1}{\pi i}\int\partial_{\beta'}\left(\frac{(f_1\circ l(\alpha')-f_1\circ l(\beta'))(l(\alpha')-\alpha'-l(\beta')+\beta')}{(l(\alpha')-l(\beta'))(\alpha'-\beta')}\right)\\ &\quad(g_1\circ l(\beta')-g_1\circ l(\alpha'))\,d\beta'\end{aligned}\tag{5.39}$$

and use the product rule to expand the factor on the right hand side of (5.39), we then apply Cauchy–Schwarz inequality and Hardy's inequality (A.8) to the resulting terms. □

Proposition 5.9 *Assume that $f,\ g,\ f_1,\ g_1$ are smooth and decay at infinity, and $l:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism, with $l_{\alpha'}-1\in L^2$. Then there is a constant $c:=c(\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty})$, depending on $\|l_{\alpha'}\|_{L^\infty},\|(l^{-1})_{\alpha'}\|_{L^\infty}$,*

such that

$$\begin{aligned}
&\|[f,\mathbb{H}]\,\partial_{\alpha'} g - U_l\,[f_1,\mathbb{H}]\,\partial_{\alpha'} g_1\|_{\dot{H}^{1/2}} \\
&\lesssim \|\partial_{\alpha'} f - \partial_{\alpha'}(f_1\circ l)\|_{L^2}\,\|\partial_{\alpha'} g_1\|_{L^2}\|l_{\alpha'}\|_{L^\infty}^{\frac{1}{2}} \\
&\quad + (\|\partial_{\alpha'} f\|_{L^\infty} + \|\partial_{\alpha'}\mathbb{H} f\|_{L^\infty})\,\|g - g_1\circ l\|_{\dot{H}^{1/2}} \\
&\quad + c\|\partial_{\alpha'} f_1\|_{L^\infty}\|\partial_{\alpha'} g_1\|_{L^2}\,\|l_{\alpha'} - 1\|_{L^2}\,.
\end{aligned} \tag{5.40}$$

Proof We begin with (5.32) and write the first two terms on the right hand side as

$$\begin{aligned}
&[f,\mathbb{H}]\,\partial_{\alpha'} g - [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l) \\
&\quad = [f - f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l) + [f,\mathbb{H}]\,\partial_{\alpha'}(g - g_1\circ l);
\end{aligned} \tag{5.41}$$

applying (5.24) and (5.25) to (5.41) we get

$$\begin{aligned}
&\|[f,\mathbb{H}]\,\partial_{\alpha'} g - [f_1\circ l,\mathbb{H}]\,\partial_{\alpha'}(g_1\circ l)\|_{\dot{H}^{1/2}} \\
&\lesssim \|\partial_{\alpha'}(f - f_1\circ l)\|_{L^2}\,\|\partial_{\alpha'}(g_1\circ l)\|_{L^2} \\
&\quad + (\|\partial_{\alpha'} f\|_{L^\infty} + \|\partial_{\alpha'}\mathbb{H} f\|_{L^\infty})\,\|g - g_1\circ l\|_{\dot{H}^{1/2}}\,.
\end{aligned} \tag{5.42}$$

Consider the last term on the right hand side of (5.32). For any $p \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}
&\int \partial_{\alpha'} p\,\left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}U_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l)\,d\alpha' \\
&\quad = \int \partial_{\alpha'}(g_1\circ l)\left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'} p\,d\alpha';
\end{aligned} \tag{5.43}$$

the same argument as in the proof of (5.38), that is, integrating by parts, then applying Cauchy–Schwarz inequality and Hardy's inequality (A.8) gives

$$\|\left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'} p\|_{L^2} \le c\,\|\partial_{\alpha'} f_1\|_{L^\infty}\|l_{\alpha'} - 1\|_{L^2}\|p\|_{\dot{H}^{1/2}},$$

where $c := c(\|l_{\alpha'}\|_{L^\infty}, \|(l^{-1})_{\alpha'}\|_{L^\infty})$ is a constant depending on $\|l_{\alpha'}\|_{L^\infty}$ and $\|(l^{-1})_{\alpha'}\|_{L^\infty}$; so

$$\begin{aligned}
&\left|\int \partial_{\alpha'} p\,\left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}(l_{\alpha'})^{-1}\right]\partial_{\alpha'}(g_1\circ l)\,d\alpha'\right| \\
&\quad \le c\|\partial_{\alpha'}(g_1\circ l)\|_{L^2}\|\partial_{\alpha'} f_1\|_{L^\infty}\|l_{\alpha'} - 1\|_{L^2}\|p\|_{\dot{H}^{1/2}}.
\end{aligned}$$

This finishes the proof for (5.40). □

Proposition 5.10 *Assume that $f,\ g,\ f_1,\ g_1$ are smooth and decay at infinity, and $l : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, with $l_{\alpha'} - 1 \in L^2$. Then there is a*

constant $c := c(\|l_{\alpha'}\|_{L^\infty}, \|(l^{-1})_{\alpha'}\|_{L^\infty})$, *depending on* $\|l_{\alpha'}\|_{L^\infty}, \|(l^{-1})_{\alpha'}\|_{L^\infty}$, *such that*

$$\begin{aligned}\|[f,\mathbb{H}]\,g - U_l\,[f_1,\mathbb{H}]\,g_1\|_{\dot{H}^{1/2}} &\lesssim \|f - f_1\circ l\|_{\dot{H}^{1/2}}\,(\|g\|_{L^\infty} + \|\mathbb{H}g\|_{L^\infty})\\ &+\|\partial_{\alpha'} f_1\|_{L^2}\|l_{\alpha'}\|_{L^\infty}^{\frac{1}{2}}\,\|g - g_1\circ l\|_{L^2} + c\|\partial_{\alpha'} f_1\|_{L^2}\|g_1\|_{L^\infty}\,\|l_{\alpha'} - 1\|_{L^2}\,.\end{aligned} \tag{5.44}$$

Proof Similar to the proof of Proposition 5.7, we have

$$\begin{aligned}&[f,\mathbb{H}]\,g - U_l\,[f_1,\mathbb{H}]\,g_1\\ &\quad= [f,\mathbb{H}]\,g - [f_1\circ l,\mathbb{H}]\,(g_1\circ l) + \left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}\right](g_1\circ l);\end{aligned} \tag{5.45}$$

writing

$$[f,\mathbb{H}]\,g - [f_1\circ l,\mathbb{H}]\,(g_1\circ l) = [f - f_1\circ l,\mathbb{H}]\,g + [f_1\circ l,\mathbb{H}]\,(g - g_1\circ l) \tag{5.46}$$

and applying (5.23) and (5.24) gives,

$$\begin{aligned}&\|[f,\mathbb{H}]\,g - [f_1\circ l,\mathbb{H}]\,(g_1\circ l)\|_{\dot{H}^{1/2}}\\ &\quad\lesssim \|f - f_1\circ l\|_{\dot{H}^{1/2}}\,(\|g\|_{L^\infty} + \|\mathbb{H}g\|_{L^\infty})\\ &\qquad+ \|\partial_{\alpha'}(f_1\circ l)\|_{L^2}\,\|g - g_1\circ l\|_{L^2}\,.\end{aligned} \tag{5.47}$$

Consider the second term on the right hand side of (5.45). We write

$$\begin{aligned}&\left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}\right](g_1\circ l)\\ &\quad= \left[f_1\circ l,\mathbb{H} - U_l\mathbb{H}\mathbb{U}_{l^{-1}}\frac{1}{l_{\alpha'}}\right](g_1\circ l) + \left[f_1\circ l, U_l\mathbb{H}U_{l^{-1}}\right](\frac{1}{l_{\alpha'}} - 1)(g_1\circ l).\end{aligned} \tag{5.48}$$

Now

$$\left[f_1\circ l, U_l\mathbb{H}\mathbb{U}_{l^{-1}}\right](((l_{\alpha'})^{-1} - 1)(g_1\circ l)) = U_l[f_1,\mathbb{H}]\left(((l^{-1})_{\alpha'} - 1)g_1\right). \tag{5.49}$$

Changing variables, and then using (5.24) yields

$$\left\|U_l[f_1,\mathbb{H}]\left(((l^{-1})_{\alpha'} - 1)g_1\right)\right\|_{\dot{H}^{1/2}} \le c\|\partial_{\alpha'} f_1\|_{L^2}\|g_1\|_{L^\infty}\,\|l_{\alpha'} - 1\|_{L^2} \tag{5.50}$$

for some constant c depending on $\|l_{\alpha'}\|_{L^\infty}$, $\|(l^{-1})_{\alpha'}\|_{L^\infty}$.

For the first term on the right hand side of (5.48) we use the duality argument in (5.43). Let $p \in C_0^\infty(\mathbb{R})$,

$$\int \partial_{\alpha'} p \left[f_1 \circ l, \mathbb{H} - U_l \mathbb{H} U_{l^{-1}} (l_{\alpha'})^{-1} \right] (g_1 \circ l)\, d\alpha' = \int g_1 \circ l \left[f_1 \circ l, \mathbb{H} - U_l \mathbb{H} \mathbb{U}_{l^{-1}} (l_{\alpha'})^{-1} \right] \partial_{\alpha'} p\, d\alpha', \tag{5.51}$$

and

$$\left[f_1 \circ l, \mathbb{H} - U_l \mathbb{H} U_{l^{-1}} (l_{\alpha'})^{-1} \right] \partial_{\alpha'} p = \frac{1}{\pi i} \int \frac{(f_1 \circ l(\alpha') - f_1 \circ l(\beta'))(l(\alpha') - \alpha' - l(\beta') + \beta')}{(l(\alpha') - l(\beta'))(\alpha' - \beta')} \partial_{\beta'} p(\beta')\, d\beta'. \tag{5.52}$$

Integrating by parts, then apply Cauchy–Schwarz inequality and Hardy's inequalities (A.8) and (A.9) gives

$$\left\| \left[f_1 \circ l, \mathbb{H} - U_l \mathbb{H} U_{l^{-1}} (l_{\alpha'})^{-1} \right] \partial_{\alpha'} p \right\|_{L^1} \le c \|\partial_{\alpha'} f_1\|_{L^2} \|l_{\alpha'} - 1\|_{L^2} \|p\|_{\dot{H}^{1/2}}, \tag{5.53}$$

for some constant c depending on $\|l_{\alpha'}\|_{L^\infty}$, $\|(l^{-1})_{\alpha'}\|_{L^\infty}$, so

$$\left| \int \partial_{\alpha'} p \left[f_1 \circ l, \mathbb{H} - U_l \mathbb{H} U_{l^{-1}} (l_{\alpha'})^{-1} \right] (g_1 \circ l)\, d\alpha' \right| \le c \|g_1\|_{L^\infty} \|\partial_{\alpha'} f_1\|_{L^2} \|l_{\alpha'} - 1\|_{L^2} \|p\|_{\dot{H}^{1/2}}. \tag{5.54}$$

This finishes the proof for (5.44). □

We define

$$[f, m; \partial_{\alpha'} g]_n := \frac{1}{\pi i} \int \frac{(f(\alpha') - f(\beta'))(m(\alpha') - m(\beta'))^n}{(\alpha' - \beta')^{n+1}} \partial_{\beta'} g(\beta')\, d\beta'.$$

So $[f, m; \partial_{\alpha'} g] = [f, m; \partial_{\alpha'} g]_1$, and $[f, \mathbb{H}] \partial_{\alpha'} g = [f, m; \partial_{\alpha'} g]_0$.

Proposition 5.11 *Assume that f, m, g, f_1, m_1, g_1 are smooth and f, g, f_1, g_1 decay at infinity, and $l : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, with $l_{\alpha'} - 1 \in L^2$. Then there is a constant c, depending on $\|l_{\alpha'}\|_{L^\infty}$, $\|(l^{-1})_{\alpha'}\|_{L^\infty}$, such that*

$$\begin{aligned}\|[f,m;\partial_{\alpha'}g]_n - U_l[f_1,m_1;\partial_{\alpha'}g_1]_n\|_{L^2} &\le c\,\|f - f_1\circ l\|_{\dot H^{1/2}}\,\|\partial_{\alpha'}m\|_{L^\infty}^n\,\|\partial_{\alpha'}g\|_{L^2}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}(\|\partial_{\alpha'}m\|_{L^\infty}+\|\partial_{\alpha'}m_1\|_{L^\infty})^{n-1}\,\|\partial_{\alpha'}(m-m_1\circ l)\|_{L^2}\,\|\partial_{\alpha'}g\|_{L^2}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}\,\|\partial_{\alpha'}m_1\|_{L^\infty}^n\,\|g-g_1\circ l\|_{\dot H^{1/2}}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}\,\|\partial_{\alpha'}m_1\|_{L^\infty}^n\,\|\partial_{\alpha'}g_1\|_{L^2}\,\|l_{\alpha'}-1\|_{L^2}\,. \end{aligned}\tag{5.55}$$

Proposition 5.11 can be proved similarly as for Proposition 5.7, we omit the details.

Proposition 5.12 *Assume that $f,\ m,\ g,\ f_1,\ m_1,\ g_1$ are smooth and decay at infinity, and $l:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism, with $l_{\alpha'}-1\in L^2$. Then there is a constant c, depending on $\|l_{\alpha'}\|_{L^\infty}$, $\|(l^{-1})_{\alpha'}\|_{L^\infty}$, such that*

$$\begin{aligned}\|[f,g;m]-U_l[f_1,g_1;m_1]\|_{L^2} &\le c\,\|f-f_1\circ l\|_{\dot H^{1/2}}\,\|\partial_{\alpha'}g\|_{L^2}\,\|m\|_{L^\infty}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}\,\|g-g_1\circ l\|_{\dot H^{1/2}}\,\|m\|_{L^\infty}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}\,\|\partial_{\alpha'}g_1\|_{L^2}\,\|m-m_1\circ l\|_{L^2}\\ &+c\|\partial_{\alpha'}f_1\|_{L^2}\,\|m_1\|_{L^\infty}\,\|\partial_{\alpha'}g_1\|_{L^2}\,\|l_{\alpha'}-1\|_{L^2}\,. \end{aligned}\tag{5.56}$$

Proposition 5.12 straightforwardly follows from Cauchy–Schwarz inequality, Hardy's inequality and the definition of $\dot H^{1/2}$ norm.

Proposition 5.13 *Assume $f\in\dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$, $g\in\dot H^{1/2}(\mathbb{R})$, and g can be decomposed by*

$$g = g_1 + pq \tag{5.57}$$

with $g_1\in L^\infty(\mathbb{R})$, $q\in L^2(\mathbb{R})$, and $\partial_{\alpha'}p\in L^2(\mathbb{R})$, satisfying $\partial_{\alpha'}(pf)\in L^2(\mathbb{R})$. Then $fg\in\dot H^{1/2}(\mathbb{R})$, and

$$\begin{aligned}\|fg\|_{\dot H^{1/2}} \lesssim\ & \|f\|_{L^\infty}\,\|g\|_{\dot H^{1/2}} + \|g_1\|_{L^\infty}\,\|f\|_{\dot H^{1/2}}\\ &+\|q\|_{L^2}\,\|\partial_{\alpha'}(pf)\|_{L^2} + \|q\|_{L^2}\,\|\partial_{\alpha'}p\|_{L^2}\,\|f\|_{L^\infty}\,. \end{aligned}\tag{5.58}$$

Proof The proof is straightforward by definition. We have

$$\begin{aligned}\|fg\|_{\dot H^{1/2}}^2 &\lesssim \iint \frac{|f(\beta')|^2|g(\alpha')-g(\beta')|^2}{(\alpha'-\beta')^2}\,d\alpha' d\beta'\\ &\quad+\iint \frac{|g(\alpha')|^2|f(\alpha')-f(\beta')|^2}{(\alpha'-\beta')^2}\,d\alpha' d\beta'\\ &\lesssim \|f\|_{L^\infty}^2\,\|g\|_{\dot H^{1/2}}^2 + \|g_1\|_{L^\infty}^2\,\|f\|_{\dot H^{1/2}}^2\\ &\quad+\iint \frac{|q(\alpha')|^2|p(\alpha')f(\alpha')-p(\beta')f(\beta')|^2}{(\alpha'-\beta')^2}\,d\alpha' d\beta' \end{aligned}$$

$$+\iint \frac{|q(\alpha')|^2|p(\alpha')-p(\beta')|^2|f(\beta')|^2}{(\alpha'-\beta')^2}\,d\alpha' d\beta'$$
$$\lesssim \|f\|_{L^\infty}^2\|g\|_{\dot{H}^{1/2}}^2+\|g_1\|_{L^\infty}^2\|f\|_{\dot{H}^{1/2}}^2+\|q\|_{L^2}^2\|\partial_{\alpha'}(pf)\|_{L^2}^2$$
$$+\|q\|_{L^2}^2\|\partial_{\alpha'}p\|_{L^2}^2\|f\|_{L^\infty}^2, \tag{5.59}$$

where in the last step we used Fubini's Theorem and Hardy's inequality (A.8). □

5.2 The proof of Theorem 3.7

In addition to what have already been given, we use the following convention in this section, Sect. 5.2: we write $A \lesssim B$ if there is a constant c, depending only on $\sup_{[0,T]}\mathfrak{E}(t)$ and $\sup_{[0,T]}\tilde{\mathfrak{E}}(t)$, such that $A \le cB$. We assume the reader is familiar with the quantities that are controlled by the functional $\mathfrak{E}(t)$, see Sect. 4 and Appendix C. We don't always give precise references on these estimates.

Let $Z = Z(\alpha', t), \mathfrak{Z} = \mathfrak{Z}(\alpha', t), t \in [0, T]$ be solutions of the system (2.9)–(2.11)–(2.14)–(2.15), satisfying the assumptions of Theorem 3.7. Recall we defined in (3.16)

$$l = l(\alpha', t) = \tilde{h} \circ h^{-1}(\alpha', t) = \tilde{h}(h^{-1}(\alpha', t), t). \tag{5.60}$$

We will apply Lemma 5.1 to $\Theta = \overline{Z}_t$, $\frac{1}{Z_{,\alpha'}} - 1$, $\overline{Z}_{tt}$ and Lemma 4.3 to $l_{\alpha'} - 1$ to construct an energy functional $\mathcal{F}(t)$, and show that the time derivative $\mathcal{F}'(t)$ can be controlled by $\mathcal{F}(t)$ and the initial data. We begin with computing the evolutionary equations for these quantities. We have

$$\partial_t(l_{\alpha'} \circ h) = \partial_t\left(\frac{\tilde{h}_\alpha}{h_\alpha}\right) = \frac{\tilde{h}_\alpha}{h_\alpha}\left(\frac{\tilde{h}_{t\alpha}}{\tilde{h}_\alpha} - \frac{h_{t\alpha}}{h_\alpha}\right) = (l_{\alpha'} \circ h)(\tilde{b}_{\alpha'} \circ \tilde{h} - b_{\alpha'} \circ h); \tag{5.61}$$

precomposing with h^{-1} yields

$$(\partial_t + b\partial_\alpha)l_{\alpha'} = l_{\alpha'}(\tilde{b}_{\alpha'} \circ l - b_{\alpha'}). \tag{5.62}$$

The equation for $\overline{Z}_t$ is given by (2.24)–(2.25). To find the equation for $\overline{Z}_{tt}$ we take a derivative to t to (2.18):

$$
\begin{aligned}
(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\overline{z}_{tt} &= -i\mathfrak{a}_t\overline{z}_{t\alpha} + \partial_t\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)(\overline{z}_{tt} - i) + \frac{\mathfrak{a}_t}{\mathfrak{a}}\overline{z}_{ttt} \\
&= \partial_t\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)(\overline{z}_{tt} - i) + \frac{\mathfrak{a}_t}{\mathfrak{a}}(\overline{z}_{ttt} - i\mathfrak{a}\overline{z}_{t\alpha}) \\
&= (\overline{z}_{tt} - i)\left(\partial_t\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right) + \left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)^2 + 2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\right)\overline{D_\alpha z_t}\right), \qquad (5.63)
\end{aligned}
$$

here we used Eq. (2.18) and substituted by (2.2): $-i\mathfrak{a}\overline{z}_\alpha = \overline{z}_{tt} - i$ in the last step. Precomposing with h^{-1}, and then substituting $\overline{Z}_{tt} - i$ by (2.38), yields, for $\mathcal{P} = (\partial_t + b\partial_{\alpha'})^2 + i\mathcal{A}\partial_{\alpha'}$,

$$
\begin{aligned}
\mathcal{P}\overline{Z}_{tt} &= -i\,\frac{A_1}{Z_{,\alpha'}}\left((\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right. \\
&\quad \left. + \left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)^2 + 2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\overline{D_{\alpha'}Z_t}\right) := G_3. \qquad (5.64)
\end{aligned}
$$

To find the equation for $\frac{1}{Z_{,\alpha'}}$ we begin with (2.35). Precomposing with h, then differentiate with respect to t gives

$$
\partial_t^2\left(\frac{h_\alpha}{z_\alpha}\right) = \frac{h_\alpha}{z_\alpha}\left((b_{\alpha'}\circ h - D_\alpha z_t)^2 + \partial_t(b_{\alpha'}\circ h - 2\,\mathrm{Re}\,D_\alpha z_t) + \partial_t\overline{D_\alpha z_t}\right) \qquad (5.65)
$$

here we subtracted $\partial_t\overline{D_\alpha z_t}$ and then added $\partial_t\overline{D_\alpha z_t}$ in the second factor on the right hand side to take advantage of the formula (2.22). We compute

$$
\partial_t\overline{D_\alpha z_t} = \overline{D_\alpha z_{tt}} - (\overline{D_\alpha z_t})^2, \qquad (5.66)
$$

replacing $\overline{Z}_{tt}$ by (2.38) yields

$$
\overline{D_{\alpha'}Z_{tt}} = \frac{1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\left(-i\,\frac{A_1}{Z_{,\alpha'}}\right) = -i\,\frac{A_1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\right) - i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1; \qquad (5.67)
$$

precomposing Eq. (5.65) with h^{-1} and substitute in by (5.66)–(5.67), we get

$$
\begin{aligned}
\mathcal{P}\frac{1}{Z_{,\alpha'}} &= \frac{1}{Z_{\alpha'}}\left((b_{\alpha'} - D_{\alpha'}Z_t)^2 + (\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t) - \left(\overline{D_{\alpha'}Z_t}\right)^2 - i\,\frac{\partial_{\alpha'}A_1}{|Z_{,\alpha'}|^2}\right) \\
&:= G_2. \qquad (5.68)
\end{aligned}
$$

We record here the equation for $\overline{Z}_t$, which is the first equation in (2.24), in which we substituted in by (2.38),

$$\mathcal{P}\overline{Z}_t = -i\,\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\,\frac{A_1}{Z_{,\alpha'}} := G_1. \tag{5.69}$$

5.2.1 *The energy functional $\mathcal{F}(t)$*

The energy functional $\mathcal{F}(t)$ for the differences of the solutions will consist of $\|l_{\alpha'}(t)-1\|^2_{L^2(\mathbb{R})}$ and the functionals $\mathfrak{F}(t)$ when applied to $\Theta = \overline{Z}_t(t)$, $\frac{1}{Z_{,\alpha'}}-1$ and $\overline{Z}_{tt}$, taking $\mathfrak{c} = -i,\ 0,\ 0$ respectively. Let

$$\mathfrak{F}_0(t) \ = \ \|l_{\alpha'}(t)-1\|^2_{L^2(\mathbb{R})}; \tag{5.70}$$

$$\begin{aligned}\mathfrak{F}_1(t) := &\int \frac{\kappa}{A_1}\left|Z_{,\alpha'}\left(\overline{Z}_{tt}-i\right)-\mathfrak{Z}_{,\alpha'}\circ l\left(\overline{\mathfrak{Z}}_{tt}\circ l-i\right)\right|^2\\ &+i\partial_{\alpha'}(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\overline{(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)}\,d\alpha';\end{aligned} \tag{5.71}$$

$$\begin{aligned}\mathfrak{F}_2(t) := &\int \frac{\kappa}{A_1}\left|Z_{,\alpha'}(\partial_t+b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}\right)-\mathfrak{Z}_{,\alpha'}\circ l(\partial_t+\tilde{b}\partial_{\alpha'})\left(\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)\circ l\right|^2 d\alpha'\\ &+i\int \partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)\overline{\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)}\,d\alpha';\end{aligned} \tag{5.72}$$

and

$$\mathfrak{F}_3(t) := \int \frac{\kappa}{A_1}\left|Z_{,\alpha'}\overline{Z}_{ttt}-\mathfrak{Z}_{,\alpha'}\circ l\overline{\mathfrak{Z}}_{ttt}\circ l\right|^2+i\partial_{\alpha'}(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)\overline{(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)}\,d\alpha'. \tag{5.73}$$

Substituting the evolutionary equations (2.36), (2.35) and (2.37) in the functionals $\mathfrak{F}_i$ we get

$$\mathfrak{F}_1(t) = \int \frac{\kappa}{A_1}\left|A_1-\tilde{A}_1\circ l\right|^2+i\partial_{\alpha'}(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\overline{(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)}\,d\alpha'; \tag{5.74}$$

$$\begin{aligned}\mathfrak{F}_2(t) = &\int \frac{\kappa}{A_1}\left|(b_{\alpha'}-D_{\alpha'}Z_t)-(\tilde{b}_{\alpha'}-\tilde{D}_{\alpha'}\mathfrak{Z}_t)\circ l\right|^2\\ &+i\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)\overline{\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)}\,d\alpha';\end{aligned} \tag{5.75}$$

and

$$\begin{aligned}\mathfrak{F}_3(t) = &\int \frac{\kappa}{A_1}\left|A_1\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}+\overline{D_{\alpha'}Z_t}\right)-\left(\tilde{A}_1\left(\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ\tilde{h}^{-1}+\overline{\tilde{D}_{\alpha'}\mathfrak{Z}_t}\right)\right)\circ l\right|^2 d\alpha'\\ &+i\int \partial_{\alpha'}(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)\overline{(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)}\,d\alpha'.\end{aligned} \tag{5.76}$$

Remark 5.14 Assume that the assumption of Theorem 3.7 holds. Because $h_t = b(h,t)$, and $h(\alpha,0)=\alpha$, where b is given by (2.14), we have $h_{t\alpha} = h_\alpha b_{\alpha'}\circ h$, and

$$h_\alpha(\cdot,t) = e^{\int_0^t b_{\alpha'}\circ h(\cdot,\tau)\,d\tau}. \tag{5.77}$$

So there are constants $c_1>0$, $c_2>0$, depending only on $\sup_{[0,T]}\mathfrak{E}(t)$, such that

$$c_1 \le h_\alpha(\alpha,t)\le c_2, \qquad \text{for all } \alpha\in\mathbb{R}, t\in[0,T]. \tag{5.78}$$

Consequently, because $l_{\alpha'} = \frac{\tilde h_a}{h_\alpha}\circ h^{-1}$, there is a constant $0<c<\infty$, depending only on $\sup_{[0,T]}\mathfrak{E}(t)$ and $\sup_{[0,T]}\tilde{\mathfrak{E}}(t)$, such that

$$c^{-1}\le l_{\alpha'}(\alpha',t)\le c, \qquad \text{for all } \alpha\in\mathbb{R}, t\in[0,T]. \tag{5.79}$$

It is easy to check that for each $t\in[0,T]$, $b_{\alpha'}(t)\in L^2(\mathbb{R})$, so $h_\alpha(t)-1\in L^2(\mathbb{R})$, and hence $l_{\alpha'}(t)-1\in L^2(\mathbb{R})$. It is clear that under the assumption of Theorem 3.7, the functionals $\mathfrak{F}_i(t)$, $i=1,2,3$ are well-defined.

Notice that the functionals $\mathfrak{F}_i(t)$, $i=1,2,3$ are not necessarily positive definite, see Lemma A.2. We prove the following

Lemma 5.15 *There is a constant M_0, depending only on $\sup_{[0,T]}\mathfrak{E}(t)$ and $\sup_{[0,T]}\tilde{\mathfrak{E}}(t)$, such that for all $M\ge M_0$, and $t\in[0,T]$,*

$$\begin{aligned}&\|l_{\alpha'}(t)-1\|_{L^2}+\|(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)(t)\|^2_{\dot H^{1/2}}+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)(t)\right\|^2_{\dot H^{1/2}}\\ &\quad\le M\mathfrak{F}_0(t)+\mathfrak{F}_1(t)+\mathfrak{F}_2(t);\end{aligned} \tag{5.80}$$

$$\begin{aligned}&\left\|(A_1-\tilde A_1\circ l)(t)\right\|^2_{L^2}+\left\|(D_{\alpha'}Z_t-\tilde D_{\alpha'}\mathfrak{Z}_t\circ l)(t)\right\|^2_{L^2}+\left\|b_{\alpha'}-\tilde b_{\alpha'}\circ l(t)\right\|^2_{L^2}\\ &\quad\lesssim M\mathfrak{F}_0(t)+\mathfrak{F}_1(t)+\mathfrak{F}_2(t).\end{aligned} \tag{5.81}$$

Proof By Lemma A.2,

$$\begin{aligned}&\int i\partial_{\alpha'}(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\overline{(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)}\,d\alpha'\\ &\quad=\|\mathbb{P}_H(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\|^2_{\dot H^{1/2}}-\|\mathbb{P}_A(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\|^2_{\dot H^{1/2}};\end{aligned} \tag{5.82}$$

and

$$\begin{aligned}&\|\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l\|^2_{\dot H^{1/2}}\\ &\quad=\int i\partial_{\alpha'}(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\overline{(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)}\,d\alpha'+2\|\mathbb{P}_A(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)\|^2_{\dot H^{1/2}}.\end{aligned} \tag{5.83}$$

Because $\overline{Z}_t = \mathbb{H}\overline{Z}_t$ and $\overline{\mathfrak{Z}}_t = \mathbb{H}\overline{\mathfrak{Z}}_t$,

$$2\mathbb{P}_A(\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l) = -2\mathbb{P}_A(\overline{\mathfrak{Z}}_t \circ l) = -\mathcal{Q}_l(\overline{\mathfrak{Z}}_t \circ l)$$

and by (5.9),

$$\|\mathcal{Q}_l(\overline{\mathfrak{Z}}_t \circ l)\|_{\dot{H}^{1/2}} \le C(\|l_{\alpha'}\|_{L^\infty}, \|(l^{-1})_{\alpha'}\|_{L^\infty})\|\partial_{\alpha'}\overline{\mathfrak{Z}}_t\|_{L^2}\|l_{\alpha'} - 1\|_{L^2} \lesssim \|l_{\alpha'} - 1\|_{L^2}.$$

So there is a constant M_0, depending only on $\sup_{[0,T]} \mathfrak{E}(t)$ and $\sup_{[0,T]} \tilde{\mathfrak{E}}(t)$, such that for all $t \in [0, T]$ and $M \ge M_0$,

$$\begin{aligned}\|(\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l)(t)\|^2_{\dot{H}^{1/2}} \le \int i\,\partial_{\alpha'}(\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l)\overline{(\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l)}\,d\alpha' \\ + M\|l_{\alpha'} - 1\|^2_{L^2} \le \mathfrak{F}_1(t) + M\mathfrak{F}_0(t) \quad (5.84)\end{aligned}$$

A similar argument holds for $\left\|\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)(t)\right\|^2_{\dot{H}^{1/2}}$. This proves (5.80).

Now by $\frac{\kappa}{A_1} = \sqrt{\frac{l_{\alpha'}}{A_1 \tilde{A}_1 \circ l}}$, Remark 5.14 and the estimate (4.20), there is a constant $0 < c < \infty$, depending on $\sup_{[0,T]} \mathfrak{E}(t)$ and $\sup_{[0,T]} \tilde{\mathfrak{E}}(t)$, such that

$$\frac{1}{c} \le \frac{\kappa}{A_1} \le c, \quad (5.85)$$

so

$$\begin{aligned}&\left\|(A_1 - \tilde{A}_1 \circ l)(t)\right\|^2_{L^2} + \left\|(b_{\alpha'} - D_{\alpha'}Z_t - (\tilde{b}_{\alpha'} - \tilde{D}_{\alpha'}\mathfrak{Z}_t \circ l))(t)\right\|^2_{L^2} \\ &\lesssim M\mathfrak{F}_0(t) + \mathfrak{F}_1(t) + \mathfrak{F}_2(t), \quad (5.86)\end{aligned}$$

for large enough M, depending only on $\sup_{[0,T]} \mathfrak{E}(t)$ and $\sup_{[0,T]} \tilde{\mathfrak{E}}(t)$. Using (2.22) we have, from Proposition 5.7,

$$\begin{aligned}&\left\|b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t - (\tilde{b}_{\alpha'} - 2\,\mathrm{Re}\,\tilde{D}_{\alpha'}\mathfrak{Z}_t \circ l)\right\|_{L^2} \\ &\lesssim \|\overline{Z}_t - \overline{\mathfrak{Z}}_t \circ l\|_{\dot{H}^{1/2}} + \left\|\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right\|_{\dot{H}^{1/2}} + \|l_{\alpha'} - 1\|_{L^2}; \quad (5.87)\end{aligned}$$

combining with (5.86) yields

$$\begin{aligned}&\left\|(D_{\alpha'}Z_t - \tilde{D}_{\alpha'}\mathfrak{Z}_t \circ l)(t)\right\|^2_{L^2} + \left\|(b_{\alpha'} - \tilde{b}_{\alpha'} \circ l)(t)\right\|^2_{L^2} \\ &\lesssim M\mathfrak{F}_0(t) + \mathfrak{F}_1(t) + \mathfrak{F}_2(t),\end{aligned}$$

for large enough M, depending only on $\sup_{[0,T]} \mathfrak{E}(t)$ and $\sup_{[0,T]} \tilde{\mathfrak{E}}(t)$. This proves (5.81). □

Lemma 5.16 *Let M_0 be the constant in Lemma 5.15. Then for all $M \geq M_0$, and $t \in [0, T]$,*

$$\|\mathbb{P}_A(\overline{Z}_{tt} - \overline{\mathfrak{Z}}_{tt} \circ l)(t)\|^2_{\dot{H}^{1/2}} \lesssim M\mathfrak{F}_0(t) + \mathfrak{F}_1(t) + \mathfrak{F}_2(t) \tag{5.88}$$

Proof We have

$$2\mathbb{P}_A(\overline{Z}_{tt} - \overline{\mathfrak{Z}}_{tt} \circ l) = 2\mathbb{P}_A(\overline{Z}_{tt}) - 2U_l\mathbb{P}_A(\overline{\mathfrak{Z}}_{tt}) - \mathcal{Q}_l(\overline{\mathfrak{Z}}_{tt} \circ l); \tag{5.89}$$

and by (5.9),

$$\left\| \mathcal{Q}_l(\overline{\mathfrak{Z}}_{tt} \circ l) \right\|_{\dot{H}^{1/2}} \lesssim \|l_{\alpha'} - 1\|_{L^2}.$$

Consider the first two terms on the right hand side of (5.89). We use (B.17) and the fact that $\overline{Z}_t = \mathbb{H}\overline{Z}_t$ to rewrite

$$2\mathbb{P}_A(\overline{Z}_{tt}) = [\partial_t + b\partial_{\alpha'}, \mathbb{H}]\overline{Z}_t = [b, \mathbb{H}]\partial_{\alpha'}\overline{Z}_t. \tag{5.90}$$

We would like to use (5.40) to estimate $\left\|2\mathbb{P}_A(\overline{Z}_{tt}) - 2U_l\mathbb{P}_A(\overline{\mathfrak{Z}}_{tt})\right\|_{\dot{H}^{1/2}}$, observe that we have controlled all the quantities on the right hand side of (5.40), except for $\|\mathbb{H}b_{\alpha'}\|_{L^\infty}$.

By (2.22),

$$b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'}Z_t = \mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} + [Z_t, \mathbb{H}]\, \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right), \tag{5.91}$$

and by the fact $Z_{t,\alpha'} = -\mathbb{H}Z_{t,\alpha'}$,

$$\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} = -(I + \mathbb{H})D_{\alpha'}Z_t, \tag{5.92}$$

so

$$\mathbb{H}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} = \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'}. \tag{5.93}$$

This gives, by (A.18),

$$\left\|\mathbb{H}\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'}\right\|_{L^\infty} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \left\|Z_{t,\alpha'}\right\|_{L^2} \lesssim C(\mathfrak{E}(t)); \tag{5.94}$$

similarly $\left\|\mathbb{H}\,[Z_t,\mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \lesssim C(\mathfrak{E}(t))$, therefore $\left\|\mathbb{H}(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t)\right\|_{L^\infty} \lesssim C(\mathfrak{E}(t))$. Observe that the argument from (5.92)–(5.94) also shows that

$$\|(I+\mathbb{H})D_{\alpha'}Z_t\|_{L^\infty} \lesssim C(\mathfrak{E}(t)),$$

therefore

$$\|\mathbb{H}D_{\alpha'}Z_t\|_{L^\infty} \le \|(I+\mathbb{H})D_{\alpha'}Z_t\|_{L^\infty} + \|D_{\alpha'}Z_t\|_{L^\infty} \lesssim C(\mathfrak{E}(t)), \tag{5.95}$$

and hence $\|\mathbb{H}b_{\alpha'}\|_{L^\infty} \lesssim C(\mathfrak{E}(t))$. Notice that

$$\left\|\partial_{\alpha'}(b-\tilde{b}\circ l)\right\|_{L^2} \lesssim \left\|b_{\alpha'}-\tilde{b}_{\alpha'}\circ l\right\|_{L^2} + \|l_{\alpha'}-1\|_{L^2}.$$

Applying (5.40) to (5.90) we get

$$\begin{aligned}&\left\|2\mathbb{P}_A(\overline{Z}_{tt}) - 2U_l\mathbb{P}_A(\overline{\mathfrak{Z}}_{tt})\right\|_{\dot{H}^{1/2}}\\ &\lesssim \left\|b_{\alpha'}-\tilde{b}_{\alpha'}\circ l\right\|_{L^2} + \|l_{\alpha'}-1\|_{L^2} + \left\|\overline{Z}_t - \overline{\mathfrak{Z}}_t\circ l\right\|_{\dot{H}^{1/2}}.\end{aligned}$$

A further application of Lemma 5.15 yields Lemma 5.16. □

As a consequence of Lemmas 5.15, and 5.16 we have

Proposition 5.17 *There is a constant $M_0 > 0$, depending only on $\sup_{[0,T]}\mathfrak{E}(t)$ and $\sup_{[0,T]}\tilde{\mathfrak{E}}(t)$, such that for all $M \ge M_0$, and $t \in [0,T]$,*

$$\begin{aligned}&\|(\overline{Z}_t - \overline{\mathfrak{Z}}_t\circ l)(t)\|^2_{\dot{H}^{1/2}} + \left\|\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)(t)\right\|^2_{\dot{H}^{1/2}} + \|(\overline{Z}_{tt} - \overline{\mathfrak{Z}}_{tt}\circ l)(t)\|^2_{\dot{H}^{1/2}}\\ &+ \left\|(A_1 - \tilde{A}_1\circ l)(t)\right\|^2_{L^2} + \left\|(D_{\alpha'}Z_t - \tilde{D}_{\alpha'}\mathfrak{Z}_t\circ l)(t)\right\|^2_{L^2}\\ &+ \left\|(b_{\alpha'}-\tilde{b}_{\alpha'}\circ l)(t)\right\|^2_{L^2} + \|l_{\alpha'}(t)-1\|^2_{L^2}\\ &+ \left\|\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1} - \frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ h^{-1}\right)(t)\right\|^2_{L^2} \lesssim M\mathfrak{F}_0(t) + \mathfrak{F}_1(t) + \mathfrak{F}_2(t) + M^{-1}\mathfrak{F}_3(t).\end{aligned} \tag{5.96}$$

Remark 5.18 It is clear that the reverse of inequality (5.96) also holds. Observe that by (2.15) and Proposition 5.7, we have for any $t \in [0,T]$,

$$\left\|(A_1 - \tilde{A}_1\circ l)(t)\right\|_{L^2} \lesssim \|(\overline{Z}_t - \overline{\mathfrak{Z}}_t\circ l)(t)\|_{\dot{H}^{1/2}} + \|l_{\alpha'}(t)-1\|_{L^2}. \tag{5.97}$$

and by (2.21)–(2.22)–(2.23) and Propositions 5.7, 5.11,

$$\begin{aligned}
&\left\|\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}-\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ h^{-1}\right)(t)\right\|_{L^2}=\left\|\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}-\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ \tilde{h}^{-1}\circ l\right)(t)\right\|_{L^2}\\
&\lesssim \|(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)(t)\|_{\dot{H}^{1/2}}+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)(t)\right\|_{\dot{H}^{1/2}}\\
&\quad+\|(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)(t)\|_{\dot{H}^{1/2}}\\
&\quad+\left\|(b_{\alpha'}-\tilde{b}_{\alpha'}\circ l)(t)\right\|_{L^2}+\|l_{\alpha'}(t)-1\|_{L^2}. \qquad (5.98)
\end{aligned}$$

This, together with (5.87) shows that for all $t\in[0,T]$,

$$\begin{aligned}
&M\mathfrak{F}_0(t)+\mathfrak{F}_1(t)+\mathfrak{F}_2(t)+M^{-1}\mathfrak{F}_3(t)\\
&\lesssim \|(\overline{Z}_t-\overline{\mathfrak{Z}}_t\circ l)(t)\|^2_{\dot{H}^{1/2}}+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\circ l\right)(t)\right\|^2_{\dot{H}^{1/2}}\\
&\quad+\|(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\circ l)(t)\|^2_{\dot{H}^{1/2}}+\left\|(D_{\alpha'}Z_t-\tilde{D}_{\alpha'}\mathfrak{Z}_t\circ l)(t)\right\|^2_{L^2}+\|l_{\alpha'}(t)-1\|^2_{L^2}. \qquad (5.99)
\end{aligned}$$

Now fix a constant M, with $M\geq M_0>0$, so that (5.96) holds. We define

$$\mathcal{F}(t):=M\mathfrak{F}_0(t)+\mathfrak{F}_1(t)+\mathfrak{F}_2(t)+M^{-1}\mathfrak{F}_3(t). \qquad (5.100)$$

We have

Proposition 5.19 *Assume that $Z=Z(\alpha',t)$, $\mathfrak{Z}=\mathfrak{Z}(\alpha',t)$ are solutions of the system (2.9)–(2.11)–(2.15)–(2.14), satisfying the assumption of Theorem 3.7. Then there is a constant C, depending only on T, $\sup_{[0,T]}\mathfrak{E}(t)$ and $\sup_{[0,T]}\tilde{\mathfrak{E}}(t)$, such that for $t\in[0,T]$,*

$$\frac{d}{dt}\mathcal{F}(t)\leq C\left(\mathcal{F}(t)+\int_0^t\mathcal{F}(\tau)\,d\tau+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(0)\right\|_{L^\infty}\mathcal{F}(t)^{\frac{1}{2}}\right). \qquad (5.101)$$

Assuming Proposition 5.19 holds, we have, by (5.101),

$$\begin{aligned}
&\frac{d}{dt}\left(\mathcal{F}(t)+\int_0^t\mathcal{F}(\tau)\,d\tau\right)\\
&\leq C\left(\mathcal{F}(t)+\int_0^t\mathcal{F}(\tau)\,d\tau+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(0)\right\|^2_{L^\infty}\right); \qquad (5.102)
\end{aligned}$$

and by Gronwall's inequality,

$$\mathcal{F}(t) + \int_0^t \mathcal{F}(\tau)\, d\tau \le C\left(\mathcal{F}(0) + \left\| \left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(0)\right\|_{L^\infty}^2\right), \qquad \text{for } t \in [0,T], \tag{5.103}$$

for some constant C depending on T, $\sup_{[0,T]} \mathfrak{E}(t)$ and $\sup_{[0,T]} \tilde{\mathfrak{E}}(t)$. This together with (5.96) and (5.99) gives (3.17).

We now give the proof for Proposition 5.19.

Proof To prove Proposition 5.19 we apply Lemma 4.3 to $\Theta = l_{\alpha'} - 1$, and Lemma 5.1 to $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$. We have, by Lemma 4.3 and (5.62),

$$\mathfrak{F}_0'(t) \le 2\left\| l_{\alpha'}(b_{\alpha'} - \tilde{b}_{\alpha'} \circ l)\right\|_{L^2} \mathfrak{F}_0(t)^{1/2} + \|b_{\alpha'}\|_{L^\infty} \mathfrak{F}_0(t) \lesssim \mathcal{F}(t), \tag{5.104}$$

here we used (5.79), (5.96), and Sect. 4.1.1, (4.41).

Now we apply Lemma 5.1 to $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$ to get the estimates for $\mathfrak{F}_1'(t), \mathfrak{F}_2'(t)$ and $\mathfrak{F}_3'(t)$. Checking through the right hand sides of the inequalities (5.4) for $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$, we find that we have controlled almost all of the quantities, respectively by $\mathcal{F}(t)$ or $\mathfrak{E}(t)$, $\tilde{\mathfrak{E}}(t)$, except for the following:

- 1. $\left\| \frac{(\partial_t + b\partial_{\alpha'})\kappa}{\kappa} \right\|_{L^2}$;
- 2. $\|1 - \kappa\|_{L^2}$;
- 3. $2\,\mathrm{Re}\, i \int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)}\Big(\mathfrak{Z}_{,\alpha'} \circ l((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta} \circ l + \mathfrak{c})\Theta_{\alpha'} - \overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}(\tilde{\Theta} \circ l)_{\alpha'}\Big)\, d\alpha'$, for $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$; with $\mathfrak{c} = -i$ for $\Theta = \overline{Z}_t$, and $\mathfrak{c} = 0$ for $\Theta = \frac{1}{Z_{,\alpha'}} - 1$ and $\overline{Z}_{tt}$;
- 4. $\left\| Z_{,\alpha'} G_i - \mathfrak{Z}_{,\alpha'} \circ l \tilde{G}_i \circ l \right\|_{L^2}$, for $i = 1, 2, 3$.

We begin with items 1. and 2. By definition $\kappa = \sqrt{\frac{A_1}{\tilde{A}_1 \circ l} l_{\alpha'}}$, so

$$2\frac{(\partial_t + b\partial_{\alpha'})\kappa}{\kappa} = \frac{(\partial_t + b\partial_{\alpha'})A_1}{A_1} - \frac{(\partial_t + \tilde{b}\partial_{\alpha'})\tilde{A}_1}{\tilde{A}_1} \circ l + (\tilde{b}_{\alpha'} \circ l - b_{\alpha'}); \tag{5.105}$$

and by (2.21),

$$\frac{(\partial_t + b\partial_{\alpha'})A_1}{A_1} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} - (b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t); \tag{5.106}$$

therefore

$$\left\| \frac{(\partial_t + b\partial_{\alpha'})\kappa}{\kappa} \right\|_{L^2} \lesssim \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} - \frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}} \circ h^{-1} \right\|_{L^2} + \left\| b_{\alpha'} - \tilde{b}_{\alpha'} \circ l \right\|_{L^2} + \left\| D_{\alpha'} Z_t - \tilde{D}_{\alpha'} \mathfrak{Z}_t \circ l \right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.107}$$

And it is clear that by the definition of κ,

$$\|1 - \kappa\|_{L^2} \lesssim \left\| A_1 - \tilde{A}_1 \circ l \right\|_{L^2} + \|l_{\alpha'} - 1\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.108}$$

What remains to be controlled are the quantities in items 3. and 4. We first consider item 4. We have, by (5.69), $Z_{,\alpha'} G_1 = -i\, \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} A_1$, so

$$\left\| Z_{,\alpha'} G_1 - \mathfrak{Z}_{,\alpha'} \circ l \tilde{G}_1 \circ l \right\|_{L^2} \lesssim \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} - \frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}} \circ h^{-1} \right\|_{L^2} + \left\| A_1 - \tilde{A}_1 \circ l \right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}; \tag{5.109}$$

and by (5.68), $Z_{\alpha'} G_2 = (b_{\alpha'} - D_{\alpha'} Z_t)^2 + (\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t) - \left(\overline{D_{\alpha'} Z_t}\right)^2 - i\, \frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2}$, so

$$\begin{aligned} \left\| Z_{,\alpha'} G_2 - \mathfrak{Z}_{,\alpha'} \circ l \tilde{G}_2 \circ l \right\|_{L^2} &\lesssim \left\| b_{\alpha'} - \tilde{b}_{\alpha'} \circ l \right\|_{L^2} + \left\| D_{\alpha'} Z_t - \tilde{D}_{\alpha'} \mathfrak{Z}_t \circ l \right\|_{L^2} \\ &+ \left\| (\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t) - (\partial_t + \tilde{b}\partial_{\alpha'})(\tilde{b}_{\alpha'} - 2\,\mathrm{Re}\, \tilde{D}_{\alpha'} \mathfrak{Z}_t) \circ l \right\|_{L^2} \\ &+ \left\| \frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2} - \frac{\partial_{\alpha'} \tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2} \circ l \right\|_{L^2}; \end{aligned} \tag{5.110}$$

observe that we have controlled all but the last two quantities on the right hand side of (5.110) by $\mathcal{F}(t)^{1/2}$. By (5.64), $Z_{,\alpha'}G_3 = -i\,A_1\left((\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) + \left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)^2 + 2\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\overline{D_{\alpha'}Z_t}\right)$, so

$$\begin{aligned}\left\|Z_{,\alpha'}G_3 - \mathfrak{Z}_{,\alpha'}\circ l\tilde{G}_3\circ l\right\|_{L^2} &\lesssim \left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1} - \frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ h^{-1}\right\|_{L^2} + \left\|D_{\alpha'}Z_t - \tilde{D}_{\alpha'}\mathfrak{Z}_t\circ l\right\|_{L^2}\\ &+ \left\|A_1 - \tilde{A}_1\circ l\right\|_{L^2}\left(\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^\infty} + 1\right)\\ &+ \left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) - (\partial_t + \tilde{b}\partial_{\alpha'})\left(\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ\tilde{h}^{-1}\right)\circ l\right\|_{L^2}.\end{aligned} \tag{5.111}$$

We have controlled all but the factor in the third quantity and the very last quantity on the right hand side of (5.111).

In the remaining part of the proof for Proposition 5.19, we will show the following inequalities

- $\left\|\frac{\partial_{\alpha'}A_1}{|Z_{,\alpha'}|^2} - \frac{\partial_{\alpha'}\tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2}\circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}$;
- $\left\|(\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t) - (\partial_t + \tilde{b}\partial_{\alpha'})(\tilde{b}_{\alpha'} - 2\,\mathrm{Re}\,\tilde{D}_{\alpha'}\mathfrak{Z}_t)\circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}$;
- $\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^\infty} \le C(\mathfrak{E}(t))$;
- $\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) - (\partial_t + \tilde{b}\partial_{\alpha'})\left(\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ\tilde{h}^{-1}\right)\circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}$;
- and control the quantities in item 3.

Our main strategy is the same as always, that is, to rewrite the quantities in forms to which the results in Sect. 5.1 can be applied.

5.2.2 *Some additional quantities controlled by $\mathfrak{E}(t)$ and by $\mathcal{F}(t)$*

We begin with deriving some additional estimates that will be used in the proof. First we record the conclusions from the computations of (5.91)–(5.95),

$$\|\mathbb{H}D_{\alpha'}Z_t\|_{L^\infty} \le C(\mathfrak{E}(t)), \qquad \|\mathbb{H}b_{\alpha'}\|_{L^\infty} \le C(\mathfrak{E}(t)). \tag{5.112}$$

Because $\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = \mathbb{H}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$,

$$2\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = [Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}; \tag{5.113}$$

and we have, by (A.18) and (5.31),

$$\left\|\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{L^\infty}\lesssim\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\leq C(\mathfrak{E}(t));\tag{5.114}$$

$$\left\|\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)-U_l\mathbb{P}_A\left(\mathfrak{Z}_t\partial_{\alpha'}\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)\right\|_{L^2}\lesssim\mathcal{F}(t)^{1/2}.\tag{5.115}$$

Similarly we have

$$\left\|\mathbb{P}_A\left(Z_t\partial_{\alpha'}\overline{Z}_t\right)-U_l\mathbb{P}_A\left(\mathfrak{Z}_t\partial_{\alpha'}\overline{\mathfrak{Z}}_t\right)\right\|_{L^2}\lesssim\mathcal{F}(t)^{1/2}.\tag{5.116}$$

By (2.15), $iA_1=i-\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})+\mathbb{P}_H(\overline{Z}_tZ_{t,\alpha'})$, and by (2.38),

$$\overline{Z}_{tt}-i=-\frac{i}{Z_{,\alpha'}}+\frac{\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})}{Z_{,\alpha'}}-\frac{\mathbb{P}_H(\overline{Z}_tZ_{t,\alpha'})}{Z_{,\alpha'}};\tag{5.117}$$

applying $\mathbb{P}_H$ to both sides of (5.117) and rewriting the second term on the right hand side as a commutator gives

$$\frac{\mathbb{P}_H(\overline{Z}_tZ_{t,\alpha'})}{Z_{,\alpha'}}=-i\left(\frac{1}{Z_{,\alpha'}}-1\right)-\frac{1}{2}\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})-\mathbb{P}_H(\overline{Z}_{tt}).\tag{5.118}$$

Now we apply (5.44), (5.9), (5.96) and (5.116) to get

$$\left\|\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})-U_l\left[\frac{1}{\mathfrak{Z}_{,\alpha'}},\mathbb{H}\right]\mathbb{P}_A(\mathfrak{Z}_t\overline{\mathfrak{Z}}_{t,\alpha'})\right\|_{\dot{H}^{1/2}}\lesssim\mathcal{F}(t)^{1/2};\tag{5.119}$$

$$\left\|\frac{\mathbb{P}_H(\overline{Z}_tZ_{t,\alpha'})}{Z_{,\alpha'}}-U_l\frac{\mathbb{P}_H(\overline{\mathfrak{Z}}_t\mathfrak{Z}_{t,\alpha'})}{\mathfrak{Z}_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\lesssim\mathcal{F}(t)^{1/2};\tag{5.120}$$

consequently by (5.117) and (5.120), (5.96),

$$\left\|\frac{\mathbb{P}_A(\overline{Z}_tZ_{t,\alpha'})}{Z_{,\alpha'}}-U_l\frac{\mathbb{P}_A(\overline{\mathfrak{Z}}_t\mathfrak{Z}_{t,\alpha'})}{\mathfrak{Z}_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\lesssim\mathcal{F}(t)^{1/2}.\tag{5.121}$$

Similarly we have

$$\left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - U_l \left[\frac{1}{\mathfrak{Z}_{,\alpha'}}, \mathbb{H} \right] \mathbb{P}_A \left(\mathfrak{Z}_t \partial_{\alpha'} \frac{1}{\mathfrak{Z}_{,\alpha'}} \right) \right\|_{\dot{H}^{1/2}} \lesssim \mathcal{F}(t)^{1/2}; \tag{5.122}$$

$$\left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \left(A_1 \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right) - U_l \left[\frac{1}{\mathfrak{Z}_{,\alpha'}}, \mathbb{H} \right] \left(\tilde{A}_1 \left(\overline{\tilde{D}_{\alpha'} \mathfrak{Z}_t} + \frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}} \circ \tilde{h}^{-1} \right) \right) \right\|_{\dot{H}^{1/2}} \lesssim \mathcal{F}(t)^{1/2}; \tag{5.123}$$

provided we can show that

$$\left\| \mathbb{H} \left(A_1 \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right) \right\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{5.124}$$

We now prove (5.124). It suffices to show $\left\| \mathbb{P}_A \left(A_1 \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right) \right\|_{L^\infty} \lesssim C(\mathfrak{E})$, since we have $\left\| A_1 \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^\infty} \lesssim C(\mathfrak{E})$. We know

$$2 \mathbb{P}_A (A_1 \overline{D_{\alpha'} Z_t}) = \left[\frac{A_1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \overline{Z}_{t,\alpha'} = -i [Z_{tt}, \mathbb{H}] \overline{Z}_{t,\alpha'},$$

hence by Cauchy–Schwarz inequality and Hardy's inequality (A.8),

$$\left\| 2 \mathbb{P}_A (A_1 \overline{D_{\alpha'} Z_t}) \right\|_{L^\infty} \lesssim \| Z_{tt,\alpha'} \|_{L^2} \| Z_{t,\alpha'} \|_{L^2} \lesssim C(\mathfrak{E}). \tag{5.125}$$

For the second term we use the formula (2.23) of [43],[22]

$$A_1 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = -\operatorname{Im}(2[Z_t, \mathbb{H}] \overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \partial_{\alpha'} \overline{Z}_t - [Z_t, Z_t; D_{\alpha'} \overline{Z}_t]), \tag{5.126}$$

observe that the quantities $[Z_t, \mathbb{H}] \overline{Z}_{tt,\alpha'}$, $[Z_{tt}, \mathbb{H}] \partial_{\alpha'} \overline{Z}_t$ are anti-holomorphic by (A.1), and $[Z_t, Z_t; D_{\alpha'} \overline{Z}_t]$ is anti-holomorphic by integration by parts and (A.1), so

$$\mathbb{P}_A \left(A_1 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) = i \left([Z_t, \mathbb{H}] \overline{Z}_{tt,\alpha'} + [Z_{tt}, \mathbb{H}] \partial_{\alpha'} \overline{Z}_t - \frac{1}{2} [Z_t, Z_t; D_{\alpha'} \overline{Z}_t] \right); \tag{5.127}$$

therefore

$$\left\| \mathbb{P}_A \left(A_1 \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^\infty} \lesssim C(\mathfrak{E}) \tag{5.128}$$

[22] This formula can be checked directly from (2.21)–(2.22)–(2.23) via similar manipulations as in (5.181)–(5.193).

by Cauchy–Schwarz inequality and Hardy's inequality (A.8). This proves (5.124).

In what follows we will need the bound for $\left\| Z_{ttt,\alpha'} \right\|_{L^2}$. We begin with (2.37) and calculate $\overline{Z}_{ttt,\alpha'}$. We have

$$\overline{Z}_{ttt,\alpha'} = \overline{Z}_{tt,\alpha'}(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}) - i A_1 D_{\alpha'} \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \tag{5.129}$$

where we substituted the factor $\overline{Z}_{tt} - i$ in the second term by $-\frac{i A_1}{Z_{,\alpha'}}$, see (2.36). We know from Sect. 4 that all the quantities in (5.129) are controlled and we have

$$\left\| Z_{ttt,\alpha'} \right\|_{L^2} \leq C(\mathfrak{E}(t)). \tag{5.130}$$

5.2.3 *Controlling the $\dot{H}^{1/2}$ norms of $Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})$ for $\Theta = \overline{Z}_t, \frac{1}{Z_{,\alpha'}} - 1, \overline{Z}_{tt}$, with $\mathfrak{c} = -i, 0, 0$ respectively*

We will use Proposition 5.13 to control the item 3 above. To do so we need to check that the assumptions of the proposition hold. One of them is $Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c}) \in \dot{H}^{1/2} \cap L^\infty$, for $\Theta = \overline{Z}_t, \frac{1}{Z_{,\alpha'}} - 1, \overline{Z}_{tt}; \mathfrak{c} = -i, 0, 0$ respectively; with the norms bounded by $C(\mathfrak{E}(t))$. By (2.36), (2.37) and (2.35),

$$\begin{aligned} Z_{,\alpha'}(\overline{Z}_{tt} - i) &= -i A_1, \quad Z_{,\alpha'}(\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{,\alpha'}} \\ &= b_{\alpha'} - D_{\alpha'} Z_t, \quad Z_{,\alpha'} \overline{Z}_{ttt} = -i A_1 \left(\overline{D_{\alpha'} Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right). \end{aligned} \tag{5.131}$$

In Sect. 4, we have shown that these quantities are in L^∞, with their L^∞ norms controlled by $C(\mathfrak{E}(t))$. So we only need to estimate their $\dot{H}^{1/2}$ norms.

Applying Proposition 5.6, (5.24) to (2.15) and (2.22), we get $A_1, b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t \in \dot{H}^{1/2}$, with

$$\| A_1 \|_{\dot{H}^{1/2}} \lesssim \| \partial_{\alpha'} Z_t \|_{L^2}^2 \lesssim C(\mathfrak{E}(t)); \tag{5.132}$$

$$\| b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t \|_{\dot{H}^{1/2}} \lesssim \| \partial_{\alpha'} Z_t \|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim C(\mathfrak{E}(t)). \tag{5.133}$$

We next compute $\|D_{\alpha'} Z_t(t)\|_{\dot{H}^{1/2}}$. By definition,

$$\begin{aligned}\|D_{\alpha'} Z_t(t)\|_{\dot{H}^{1/2}}^2 &= \int (i\partial_{\alpha'} \mathbb{H} D_{\alpha'} Z_t)\, \overline{D_{\alpha'} Z_t}\, d\alpha' \\ &= \int i\partial_{\alpha'} \left[\mathbb{H}, \frac{1}{Z_{,\alpha'}}\right] Z_{t,\alpha'}\, \overline{D_{\alpha'} Z_t}\, d\alpha' \\ &\quad + \int i\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \mathbb{H} Z_{t,\alpha'}\right) \overline{D_{\alpha'} Z_t}\, d\alpha' \\ &= \int i\, \overline{D_{\alpha'}} \left[\mathbb{H}, \frac{1}{Z_{,\alpha'}}\right] Z_{t,\alpha'}\, \partial_{\alpha'} \overline{Z_t}\, d\alpha' \\ &\quad + \int i Z_{t,\alpha'} (D_{\alpha'} \overline{D_{\alpha'} Z_t})\, d\alpha' \end{aligned} \tag{5.134}$$

where in the last step we used integration by parts and the fact $\mathbb{H} Z_{t,\alpha'} = -Z_{t,\alpha'}$. Recall in (4.67), we have shown $\left\| D_{\alpha'} \left[\mathbb{H}, \frac{1}{Z_{,\alpha'}}\right] Z_{t,\alpha'} \right\|_{L^2} \le C(\mathfrak{E}(t))$. So by Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|D_{\alpha'} Z_t(t)\|_{\dot{H}^{1/2}}^2 &\lesssim \left\| D_{\alpha'} \left[\mathbb{H}, \frac{1}{Z_{,\alpha'}}\right] Z_{t,\alpha'} \right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2} \\ &\quad + \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'}^2 Z_t\|_{L^2} \le C(\mathfrak{E}(t)). \end{aligned} \tag{5.135}$$

Now we consider $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{\dot{H}^{1/2}}$. By (2.21)–(2.22)–(2.23), we know Proposition 5.6, (5.24) can be used to handle all terms, except for $[Z_t, b; \overline{Z}_{t,\alpha'}]$.

Let $p \in C_0^\infty(\mathbb{R})$, we have, by duality,

$$\begin{aligned}\left| \int \partial_{\alpha'} p[Z_t, b; \overline{Z}_{t,\alpha'}]\, d\alpha' \right| &= \left| \int [Z_t, b; \partial_{\alpha'} p] \overline{Z}_{t,\alpha'}\, d\alpha' \right| \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \|b_{\alpha'}\|_{L^\infty} \|p\|_{\dot{H}^{1/2}}, \end{aligned} \tag{5.136}$$

where in the last step we used Cauchy–Schwarz inequality and (A.20). Therefore $\left\| [Z_t, b; \overline{Z}_{t,\alpha'}] \right\|_{\dot{H}^{1/2}} \le C(\mathfrak{E}(t))$. Applying Proposition 5.6, (5.24) to the remaining terms and using (A.5) yields

$$\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{\dot{H}^{1/2}} \le C(\mathfrak{E}(t)). \tag{5.137}$$

We can now conclude that for $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$, with $\mathfrak{c} = i, 0, 0$ respectively,

$$\left\| Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c}) \right\|_{L^\infty} + \left\| Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c}) \right\|_{\dot{H}^{1/2}} \leq C(\mathfrak{E}(t)). \tag{5.138}$$

5.2.4 Controlling $\int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)} \Big(\overline{\mathfrak{Z}_{,\alpha'} \circ l((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta} \circ l + \mathfrak{c})}\Theta_{\alpha'} - \overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}(\tilde{\Theta} \circ l)_{\alpha'} \Big)\, d\alpha'$

We begin with studying $\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l$. By (4.118), $\frac{1}{Z_{,\alpha'}}(h(\alpha, t), t) = \frac{1}{Z_{,\alpha'}}(\alpha, 0) e^{\int_0^t (b_{\alpha'} \circ h(\alpha,\tau) - D_\alpha z_t(\alpha,\tau))\, d\tau}$, so

$$\begin{aligned}\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right) \circ h = &\left(\frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right) e^{\int_0^t (\tilde{b}_{\alpha'} - \tilde{D}_{\alpha'}\mathfrak{Z}_t) \circ \tilde{h}(\tau)\, d\tau} \\ &+ \frac{1}{Z_{,\alpha'}} \circ h \left(1 - e^{\int_0^t ((\tilde{b}_{\alpha'} - \tilde{D}_{\alpha'}\mathfrak{Z}_t) \circ \tilde{h} - (b_{\alpha'} - D_{\alpha'} Z_t) \circ h)(\tau)\, d\tau}\right).\end{aligned} \tag{5.139}$$

We know for $t \in [0, T]$,

$$\begin{aligned}&\left\| \left(\frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right) e^{\int_0^t (\tilde{b}_{\alpha'} - \tilde{D}_{\alpha'}\mathfrak{Z}_t) \circ \tilde{h}(\tau)\, d\tau} \right\|_{L^\infty} \\ &\leq C(\sup_{[0,T]} \tilde{\mathfrak{E}}(t)) \left\| \frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0) \right\|_{L^\infty};\end{aligned} \tag{5.140}$$

and

$$\left\| 1 - e^{\int_0^t ((\tilde{b}_{\alpha'} - \tilde{D}_{\alpha'}\mathfrak{Z}_t) \circ \tilde{h} - (b_{\alpha'} - D_{\alpha'} Z_t) \circ h)(\tau)\, d\tau} \right\|_{L^2} \lesssim \int_0^t \mathcal{F}(\tau)^{1/2}\, d\tau. \tag{5.141}$$

Now we rewrite

$$\begin{aligned}&\int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)} \Big(\overline{\mathfrak{Z}_{,\alpha'} \circ l((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta} \circ l + \mathfrak{c})}\Theta_{\alpha'} \\ &\quad - \overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}(\tilde{\Theta} \circ l)_{\alpha'} \Big)\, d\alpha' \\ &= \int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)}\Theta_{\alpha'} \Big(\overline{\mathfrak{Z}_{,\alpha'} \circ l((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta} \circ l + \mathfrak{c})}\end{aligned}$$

$$-\overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}\Big)\, d\alpha'$$
$$+\int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)}\,\overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}\,(\Theta - \tilde{\Theta} \circ l)_{\alpha'}\, d\alpha' = I + II. \quad (5.142)$$

We apply Proposition 5.13 to II, with $g = \frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l$, and $f = Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})$, where $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$, with $\mathfrak{c} = i, 0, 0$ respectively. We know

$$\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}} f\right) = \overline{Z}_{tt,\alpha'}, \quad \partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}, \quad \overline{Z}_{ttt,\alpha'}, \qquad \text{for } \Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt},$$

so $\left\|\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}} f\right)\right\|_{L^2} \le C(\mathfrak{E}(t))$, by Sect. 4 and (5.130). Applying Proposition 5.13 to the g and f given above yields

$$\left\|\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right) Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})\right\|_{\dot{H}^{1/2}}$$
$$\lesssim \left\|\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right\|_{\dot{H}^{1/2}} + \left\|\frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right\|_{L^\infty} + \int_0^t \mathcal{F}(\tau)^{1/2}\, d\tau; \quad (5.143)$$

consequently

$$|II| \lesssim \mathcal{F}(t) + T\int_0^t \mathcal{F}(\tau)\, d\tau + \left\|\frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right\|_{L^\infty} \mathcal{F}(t)^{1/2}. \quad (5.144)$$

We apply the decomposition (5.139) and Cauchy–Schwarz inequality to I, notice that $\|\Theta_{\alpha'}\|_{L^2} \le C(\mathfrak{E}(t))$, and $\|D_{\alpha'}\Theta\|_{L^\infty} \le C(\mathfrak{E}(t))$, for $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$. We have

$$|I| \lesssim \left\|\frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right\|_{L^\infty} \mathcal{F}(t)^{1/2} + \mathcal{F}(t)^{1/2}\int_0^t \mathcal{F}(\tau)^{1/2}\, d\tau. \quad (5.145)$$

This shows that for $\Theta = \overline{Z}_t,\ \frac{1}{Z_{,\alpha'}} - 1,\ \overline{Z}_{tt}$, with $\mathfrak{c} = i, 0, 0$ respectively,

$$\Bigg|\int \overline{\left(\frac{1}{Z_{,\alpha'}} - \frac{1}{\mathfrak{Z}_{,\alpha'}} \circ l\right)}\Big(\overline{\mathfrak{Z}_{,\alpha'} \circ l((\partial_t + \tilde{b}\partial_{\alpha'})\tilde{\Theta} \circ l + \mathfrak{c})}\Theta_{\alpha'}$$
$$-\overline{Z_{,\alpha'}((\partial_t + b\partial_{\alpha'})\Theta + \mathfrak{c})}(\tilde{\Theta} \circ l)_{\alpha'}\Big)\, d\alpha'\Bigg|$$

$$\lesssim \mathcal{F}(t) + T\int_0^t \mathcal{F}(\tau)\,d\tau + \left\| \frac{1}{Z_{,\alpha'}}(0) - \frac{1}{\mathfrak{Z}_{,\alpha'}}(0)\right\|_{L^\infty} \mathcal{F}(t)^{1/2}. \tag{5.146}$$

5.2.5 *Controlling* $\left\| \frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2} - \frac{\partial_{\alpha'} \tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2} \circ l \right\|_{L^2}$

We will take advantage of the fact that $\frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2}$ is purely real to use $(I + \mathbb{H})$ to convert it to some commutator forms to which the Propositions in Sect. 5.1 can be applied.

Observe that

$$i\,\frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2} = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{iA_1}{\overline{Z}_{,\alpha'}} - \frac{iA_1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'} Z_{tt} + (\overline{Z}_{tt} - i)\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}; \tag{5.147}$$

we apply $(I + \mathbb{H})$ to (5.147), and use the fact $\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}} = -\mathbb{H}\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)$ to write the second term in a commutator form. We have

$$i\,(I + \mathbb{H})\frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2} = (I + \mathbb{H})\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'} Z_{tt}\right) - \left[\overline{Z}_{tt}, \mathbb{H}\right]\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}. \tag{5.148}$$

For the first term on the right hand side, we commute out $\frac{1}{Z_{,\alpha'}}$, then use the fact $Z_t = -\mathbb{H} Z_t$ to write $(I + \mathbb{H})Z_{tt}$ as a commutator [see (5.90)],

$$(I + \mathbb{H})\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'} Z_{tt}\right) = \left[\mathbb{H}, \frac{1}{Z_{,\alpha'}}\right]\partial_{\alpha'} Z_{tt} - \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}[b, \mathbb{H}]Z_{t,\alpha'}; \tag{5.149}$$

we compute

$$\begin{aligned}&\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}[b, \mathbb{H}]Z_{t,\alpha'}\\ &= \frac{1}{Z_{,\alpha'}} b_{\alpha'}\mathbb{H} Z_{t,\alpha'} - \frac{1}{\pi i\, Z_{,\alpha'}}\int \frac{b(\alpha', t) - b(\beta', t)}{(\alpha' - \beta')^2} Z_{t,\beta'}\,d\beta'\end{aligned}$$

$$
\begin{aligned}
&= -b_{\alpha'} D_{\alpha'} Z_t - \left[\frac{1}{Z_{,\alpha'}}, b;\, Z_{t,\alpha'}\right] - \frac{1}{\pi i}\int \frac{b(\alpha',t)-b(\beta',t)}{(\alpha'-\beta')^2} D_{\beta'} Z_t \, d\beta' \\
&= -b_{\alpha'} D_{\alpha'} Z_t - \left[\frac{1}{Z_{,\alpha'}}, b;\, Z_{t,\alpha'}\right] + [b, \mathbb{H}]\partial_{\alpha'} D_{\alpha'} Z_t - \mathbb{H}(b_{\alpha'} D_{\alpha'} Z_t),
\end{aligned}
\tag{5.150}
$$

in the last step we performed integration by parts. We have converted the right hand side of (5.148) in the desired forms. Applying (5.31), (5.37), (5.55), (5.11) and (5.96), then take the imaginary parts gives

$$
\left\| \frac{\partial_{\alpha'} A_1}{|Z_{,\alpha'}|^2} - \frac{\partial_{\alpha'} \tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2} \circ l \right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.151}
$$

In what follows we will use the following identities in the calculations: for $f,\ g,\ p$, satisfying $g = \mathbb{H}g$ and $p = \mathbb{H}p$,

$$
[f, \mathbb{H}](gp) = [fg, \mathbb{H}]p = [\mathbb{P}_A(fg), \mathbb{H}]p; \tag{5.152}
$$

$$
\begin{aligned}
[f, \mathbb{H}]\partial_{\alpha'}(gp) &= [f\partial_{\alpha'} g, \mathbb{H}]p + [fg, \mathbb{H}]\partial_{\alpha'} p \\
&= [\mathbb{P}_A(f\partial_{\alpha'} g), \mathbb{H}]p + [\mathbb{P}_A(fg), \mathbb{H}]\partial_{\alpha'} p
\end{aligned}
\tag{5.153}
$$

(5.152) is obtained by using the fact that the product of holomorphic functions is holomorphic, and (A.1); (5.153) is a consequence of (5.152) and the product rules.

5.2.6 *Controlling* $\left\|(\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t) - (\partial_t + \tilde{b}\partial_{\alpha'})(\tilde{b}_{\alpha'} - 2\,\mathrm{Re}\, \tilde{D}_{\alpha'} \mathfrak{Z}_t) \circ l\right\|_{L^2}$

We begin with (4.74),

$$
\begin{aligned}
&(\partial_t + b\partial_{\alpha'})\,(b_{\alpha'} - 2\,\mathrm{Re}\, D_{\alpha'} Z_t) \\
&= \mathrm{Re}\left(\left[(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}\right) \\
&\quad + \mathrm{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{tt,\alpha'} + [Z_{tt}, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right. \\
&\quad \left. - \left[\frac{1}{Z_{,\alpha'}}, b;\, Z_{t,\alpha'}\right] - \left[Z_t, b;\, \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right]\right);
\end{aligned}
\tag{5.154}
$$

observe that using Propositions 5.7, 5.11 and 5.17 we are able to get the desired estimates for the last four terms on the right hand side of (5.154). We

need to rewrite the first two terms in order to apply the results in Sect. 5.1. First, by (2.35) we have

$$(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}\right) = \frac{1}{Z_{,\alpha'}}\left(b_{\alpha'} - D_{\alpha'}Z_t\right); \tag{5.155}$$

and by $\mathbb{H}\mathbb{Z}_{t,\alpha'} = -Z_{t,\alpha'}$,

$$\left[(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} = -(I+\mathbb{H})\left((D_{\alpha'}Z_t)(b_{\alpha'} - D_{\alpha'}Z_t)\right); \tag{5.156}$$

so we can conclude from (5.11) and (5.96) that

$$\left\| \left[(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} - U_l \left[(\partial_t + \tilde{b}\partial_{\alpha'})\frac{1}{\mathfrak{Z}_{,\alpha'}}, \mathbb{H}\right] \mathfrak{Z}_{t,\alpha'} \right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.157}$$

For the second term on the right hand side of (5.154), we use (2.14) to further rewrite (5.155),

$$\begin{aligned}
(\partial_t + b\partial_{\alpha'})\left(\frac{1}{Z_{,\alpha'}}\right) &= \frac{1}{Z_{,\alpha'}}\left(\partial_{\alpha'}\mathrm{Re}(I-\mathbb{H})\frac{Z_t}{Z_{,\alpha'}} - D_{\alpha'}Z_t\right) \\
&= \frac{1}{Z_{,\alpha'}}\left(\mathbb{P}_A\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} + \mathbb{P}_H\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} + \mathrm{Re}(I-\mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - D_{\alpha'}Z_t\right) \\
&= \frac{1}{Z_{,\alpha'}}\left(\mathbb{P}_H\left(\overline{D_{\alpha'}Z_t} - D_{\alpha'}Z_t\right) + \mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + \overline{\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)}\right).
\end{aligned} \tag{5.158}$$

We substitute the right hand side of (5.158) in the second term, $[Z_t, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})\frac{1}{Z_{,\alpha'}}$ of (5.154), term by term. For the first term we have, by (5.153),

$$\begin{aligned}
[Z_t, \mathbb{H}]\,\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\mathbb{P}_H\left(\mathrm{Im}\,\overline{D_{\alpha'}Z_t}\right)\right) &= \left[\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right), \mathbb{H}\right]\mathbb{P}_H\left(\mathrm{Im}\,\overline{D_{\alpha'}Z_t}\right) \\
&\quad + \left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}\mathbb{P}_H\left(\mathrm{Im}\,\overline{D_{\alpha'}Z_t}\right) \\
&= (I-\mathbb{H})\left(\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\mathbb{P}_H\left(\mathrm{Im}\,\overline{D_{\alpha'}Z_t}\right)\right) + [b, \mathbb{H}]\,\partial_{\alpha'}\mathbb{P}_H\left(\mathrm{Im}\,\overline{D_{\alpha'}Z_t}\right);
\end{aligned} \tag{5.159}$$

in the last step we used (4.50) and (A.1). Therefore by (5.114)–(5.115), (5.96), (5.11) and (5.37),

$$\left\| [Z_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \mathbb{P}_H \left(\mathrm{Im}\, \overline{D_{\alpha'} Z_t} \right) \right) - U_l\, [\mathfrak{Z}_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{\mathfrak{Z}_{,\alpha'}} \mathbb{P}_H \left(\mathrm{Im}\, \overline{\tilde{D}_{\alpha'} \mathfrak{Z}_t} \right) \right) \right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.160}$$

We substitute in the second term and rewrite further by (A.1),

$$\begin{aligned} &[Z_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \\ &= [Z_t, \mathbb{H}]\, \partial_{\alpha'} \mathbb{P}_H \left(\frac{1}{Z_{,\alpha'}} \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \\ &= -\frac{1}{2} [Z_t, \mathbb{H}]\, \partial_{\alpha'} \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \end{aligned} \tag{5.161}$$

This allows us to conclude, by (5.31), and (5.122), (5.96),[23]

$$\left\| [Z_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) - U_l\, [\mathfrak{Z}_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{\mathfrak{Z}_{,\alpha'}} \mathbb{P}_A \left(\mathfrak{Z}_t \partial_{\alpha'} \frac{1}{\mathfrak{Z}_{,\alpha'}} \right) \right) \right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.162}$$

Now we substitute in the last term and rewrite further by (5.153),

$$\begin{aligned} &[Z_t, \mathbb{H}]\, \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \overline{\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)} \right) \\ &= \left[\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right), \mathbb{H} \right] \overline{\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)} \\ &\quad + \left[\mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} \overline{\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)} \\ &= (I - \mathbb{H}) \left(\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \overline{\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)} \right) \\ &\quad + [b, \mathbb{H}]\, \partial_{\alpha'} \overline{\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)}. \end{aligned} \tag{5.163}$$

[23] For the estimate $\left\| \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right\|_{L^2} \leq C(\mathfrak{E}(t))$, see (4.68)–(4.69).

Again, this puts it in the right form to allow us to conclude, from (5.114)–(5.115), (5.11), and (5.37), that

$$\left\| [Z_t, \mathbb{H}]\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}\overline{\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)}\right) - U_l\,[\mathfrak{Z}_t, \mathbb{H}]\partial_{\alpha'}\left(\frac{1}{\mathfrak{Z}_{,\alpha'}}\overline{\mathbb{P}_A\left(\mathfrak{Z}_t\partial_{\alpha'}\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)}\right)\right\|_{L^2} \lesssim \mathcal{F}(t)^{\frac{1}{2}}. \tag{5.164}$$

This finishes the proof of

$$\left\|(\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t) - (\partial_t + \tilde{b}\partial_{\alpha'})(\tilde{b}_{\alpha'} - 2\,\mathrm{Re}\,\tilde{D}_{\alpha'}\mathfrak{Z}_t)\circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.165}$$

5.2.7 *Controlling* $\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^\infty}$

We begin with (2.21) and take a $\partial_t + b\partial_{\alpha'}$ derivative. We get

$$(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) = \frac{(\partial_t + b\partial_{\alpha'})^2 A_1}{A_1} - \left(\frac{(\partial_t + b\partial_{\alpha'})A_1}{A_1}\right)^2 + (\partial_t + b\partial_{\alpha'})(b_{\alpha'} - 2\,\mathrm{Re}\,D_{\alpha'}Z_t). \tag{5.166}$$

We have controlled all the quantities on the right hand side of (5.166) in Sect. 4, except for $\|(\partial_t + b\partial_{\alpha'})^2 A_1\|_{L^\infty}$. We proceed from (2.23) and use (B.19) to compute,

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})^2 A_1 &= -\,\mathrm{Im}\left(\left[2\,[Z_{tt}, \mathbb{H}]\,\overline{Z}_{tt,\alpha'} - [Z_{tt}, b; \overline{Z}_{t,\alpha'}] - [Z_t, b; \overline{Z}_{tt,\alpha'}]\right]\right)\\ &\quad - \mathrm{Im}\left([Z_{ttt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{ttt} - (\partial_t + b\partial_{\alpha'})[Z_t, b; \overline{Z}_{t,\alpha'}]\right),\end{aligned} \tag{5.167}$$

and we expand similarly

$$\begin{aligned}&(\partial_t + b\partial_{\alpha'})[Z_t, b; \overline{Z}_{t,\alpha'}]\\ &\quad = [Z_{tt}, b; \overline{Z}_{t,\alpha'}] + [Z_t, (\partial_t + b\partial_{\alpha'})b; \overline{Z}_{t,\alpha'}] + [Z_t, b; \overline{Z}_{tt,\alpha'}]\\ &\qquad -\frac{2}{\pi i}\int \frac{(b(\alpha', t) - b(\beta', t))^2 (Z_t(\alpha', t) - Z_t(\beta', t))}{(\alpha' - \beta')^3}\overline{Z}_{t,\beta'}\,d\beta'\end{aligned} \tag{5.168}$$

Applying Cauchy–Schwarz inequality and Hardy's inequality, we get

$$\begin{aligned}&\left\|(\partial_t + b\partial_{\alpha'})^2 A_1\right\|_{L^\infty}\\ &\lesssim \left\|Z_{tt,\alpha'}\right\|_{L^2}^2 + \left\|Z_{tt,\alpha'}\right\|_{L^2}\|b_{\alpha'}\|_{L^\infty}\left\|Z_{t,\alpha'}\right\|_{L^2} + \left\|Z_{ttt,\alpha'}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}\\ &\quad + \|\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b\|_{L^\infty}\left\|Z_{t,\alpha'}\right\|_{L^2}^2 + \|b_{\alpha'}\|_{L^\infty}^2\left\|Z_{t,\alpha'}\right\|_{L^2}^2 \end{aligned} \tag{5.169}$$

Observe that all quantities on the right hand side of (5.169) are controlled in Sect. 4 and in (5.130). This shows that

$$\left\|(\partial_t + b\partial_{\alpha'})^2 A_1\right\|_{L^\infty} \le C(\mathfrak{E}(t)), \qquad \left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right\|_{L^\infty} \le C(\mathfrak{E}(t)). \tag{5.170}$$

5.2.8 *Controlling* $\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right) - (\partial_t + \tilde{b}\partial_{\alpha'})\left(\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}}\circ \tilde{h}^{-1}\right)\circ l\right\|_{L^2}$

By the expansions (2.23), (5.166), and (5.167), (5.168), we see that by the results in Sect. 5.1 and by (5.165), we can directly conclude the desired estimates for all but the following three

- $\left\|[Z_{ttt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} - ([\mathfrak{Z}_{ttt}, \mathbb{H}]\,\overline{\mathfrak{Z}}_{t,\alpha'})\circ l\right\|_{L^2}$;
- $\left\|[Z_t, \mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{ttt} - ([\mathfrak{Z}_t, \mathbb{H}]\,\partial_{\alpha'}\overline{\mathfrak{Z}}_{ttt})\circ l\right\|_{L^2}$;
- $\left\|[Z_t, (\partial_t + b\partial_{\alpha'})b; \overline{Z}_{t,\alpha'}] - U_l[\mathfrak{Z}_t, (\partial_t + b\partial_{\alpha'})b; \overline{\mathfrak{Z}}_{t,\alpha'}]\right\|_{L^2}$.

The first two items can be analyzed similarly as in Sect. 5.2.6. We begin with $[Z_{ttt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'}$ and rewrite it using $\mathbb{H}\overline{Z}_{t,\alpha'} = \overline{Z}_{t,\alpha'}$, and substitute in by (2.37), (2.36),

$$[Z_{ttt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} = (I - \mathbb{H})(Z_{ttt}\overline{Z}_{t,\alpha'}) = (I - \mathbb{H})\left(iA_1\overline{D_{\alpha'}Z_t}\left(D_{\alpha'}Z_t + \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right). \tag{5.171}$$

From here we are ready to conclude from (5.11), (5.96) that

$$\left\|[Z_{ttt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} - ([\mathfrak{Z}_{ttt}, \mathbb{H}]\,\overline{\mathfrak{Z}}_{t,\alpha'})\circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.172}$$

Now substitute in by (2.37), (2.36), and use the identity $\mathbb{P}_H + \mathbb{P}_A = I$, then use (5.153) and (A.1),

$$\begin{aligned}&[Z_t, \mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{ttt}\\ &= -i\,[Z_t, \mathbb{H}]\,\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}}(\mathbb{P}_H + \mathbb{P}_A)\left(A_1\left(\overline{D_{\alpha'}Z_t} + \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= -i\left[\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right),\mathbb{H}\right]\mathbb{P}_H\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\quad - i\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}\mathbb{P}_H\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\quad - i\,[Z_t,\mathbb{H}]\,\partial_{\alpha'}\mathbb{P}_H\left(\frac{1}{Z_{,\alpha'}}\mathbb{P}_A\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right)\\
&= -i(I-\mathbb{H})\left(\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\mathbb{P}_H\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\right)\\
&\quad - i\,[b,\mathbb{H}]\,\partial_{\alpha'}\mathbb{P}_H\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right)\\
&\quad - i\,[Z_t,\mathbb{H}]\,\partial_{\alpha'}\left[\mathbb{P}_H,\frac{1}{Z_{,\alpha'}}\right]\mathbb{P}_A\left(A_1\left(\overline{D_{\alpha'}Z_t}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right)\right). \qquad (5.173)
\end{aligned}
$$

From here we can apply the Propositions in Sect. 5.1 and (5.114)–(5.115), (5.123) to conclude

$$
\left\|[Z_t,\mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{ttt}-([\mathfrak{Z}_t,\mathbb{H}]\,\partial_{\alpha'}\overline{\mathfrak{Z}}_{ttt})\circ l\right\|_{L^2}\lesssim\mathcal{F}(t)^{1/2}. \qquad (5.174)
$$

Now we consider the last term, $[Z_t,(\partial_t+b\partial_{\alpha'})b;\overline{Z}_{,\alpha'}]$. The problem with this term is that we don't yet have the estimate, $\left\|\partial_{\alpha'}(\partial_t+b\partial_{\alpha'})b-(\partial_{\alpha'}(\partial_t+\tilde{b}\partial_{\alpha'})\tilde{b})\circ l\right\|_{L^2}\lesssim\mathcal{F}(t)^{1/2}$, to apply Proposition 5.11. We will not prove this estimate. Instead, we will identify the trouble term in $\partial_{\alpha'}(\partial_t+b\partial_{\alpha'})b$, and handle it differently. We compute, by (B.12), (B.6),

$$
\begin{aligned}
&\partial_{\alpha'}(\partial_t+b\partial_{\alpha'})b=b_{\alpha'}^2+(\partial_t+b\partial_{\alpha'})b_{\alpha'}\\
&\quad=b_{\alpha'}^2+(\partial_t+b\partial_{\alpha'})(b_{\alpha'}-2\,\mathrm{Re}\,D_{\alpha'}Z_t)-2\,\mathrm{Re}\{(D_{\alpha'}Z_t)^2\}+2\,\mathrm{Re}\,D_{\alpha'}Z_{tt};
\end{aligned} \qquad (5.175)
$$

observe that we have the estimate for the first three terms. We expand the last term by substituting in (2.38),

$$
\begin{aligned}
2\,\mathrm{Re}\,D_{\alpha'}Z_{tt}&=2\,\mathrm{Re}\,\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{iA_1}{\overline{Z}_{,\alpha'}}=\partial_{\alpha'}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right)\\
&\quad-\frac{\partial_{\alpha'}(iA_1)}{|Z_{,\alpha'}|^2}-2\frac{iA_1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}. \qquad (5.176)
\end{aligned}
$$

Substitute (5.176) in (5.175), and then apply $\mathbb{P}_A$, writing the last term as a commutator; we get

$$\begin{aligned}
&\mathbb{P}_A\partial_{\alpha'}\left((\partial_t+b\partial_{\alpha'})b-\frac{iA_1}{|Z_{,\alpha'}|^2}\right)\\
&\quad=\mathbb{P}_A\left(b_{\alpha'}^2+(\partial_t+b\partial_{\alpha'})(b_{\alpha'}-2\,\mathrm{Re}\,D_{\alpha'}Z_t)-2\,\mathrm{Re}\{(D_{\alpha'}Z_t)^2\}-i\frac{\partial_{\alpha'}A_1}{|Z_{,\alpha'}|^2}\right)\\
&\qquad-\left[\frac{iA_1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}};
\end{aligned}\tag{5.177}$$

a direct application of the results in Sect. 5.1, 5.2.6 and 5.2.5 to the right hand side of (5.177) yields

$$\begin{aligned}
&\left\|\mathbb{P}_A\partial_{\alpha'}\left((\partial_t+b\partial_{\alpha'})b-\frac{iA_1}{|Z_{,\alpha'}|^2}\right)-U_l\mathbb{P}_A\partial_{\alpha'}\left((\partial_t+\tilde{b}\partial_{\alpha'})\tilde{b}-\frac{i\tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2}\right)\right\|_{L^2}\\
&\lesssim\mathcal{F}(t)^{1/2},
\end{aligned}\tag{5.178}$$

which of course holds also for its real part. We know the real part

$$\mathrm{Re}\,\mathbb{P}_A\partial_{\alpha'}\left((\partial_t+b\partial_{\alpha'})b-\frac{iA_1}{|Z_{,\alpha'}|^2}\right)=\frac{1}{2}\partial_{\alpha'}\left((\partial_t+b\partial_{\alpha'})b+\mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right)\right).$$

We split $[Z_t,(\partial_t+b\partial_{\alpha'})b;\overline{Z}_{t,\alpha'}]$ in two:

$$\begin{aligned}
[Z_t,(\partial_t+b\partial_{\alpha'})b;\overline{Z}_{t,\alpha'}]=[Z_t,(\partial_t+b\partial_{\alpha'})b&\\
+\mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right);\overline{Z}_{t,\alpha'}]-[Z_t,\mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right);\overline{Z}_{t,\alpha'}]&
\end{aligned}\tag{5.179}$$

and we can conclude from Proposition 5.11 for the first term that,[24]

$$\begin{aligned}
&\left\|\left[Z_t,(\partial_t+b\partial_{\alpha'})b+\mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right);\overline{Z}_{t,\alpha'}\right]\right.\\
&\qquad\left.-U_l\left[\mathfrak{Z}_t,(\partial_t+\tilde{b}\partial_{\alpha'})\tilde{b}+\mathbb{H}\left(\frac{i\tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2}\right);\overline{\mathfrak{Z}}_{t,\alpha'}\right]\right\|_{L^2}\\
&\lesssim\mathcal{F}(t)^{1/2}.
\end{aligned}\tag{5.180}$$

[24] The fact that $\left\|\partial_{\alpha'}\left((\partial_t+b\partial_{\alpha'})b+\mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right)\right)\right\|_{L^\infty}\le C(\mathfrak{E}(t))$ follows from (4.59) and (4.79).

We are left with the term $[Z_t, \mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'}]$. We will convert it to a form so that on which we can directly apply known results to conclude the desired estimate,

$$\left\| \left[Z_t, \mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'} \right] - U_l \left[\mathfrak{Z}_t, \mathbb{H}\left(\frac{i\tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2}\right); \overline{\mathfrak{Z}}_{t,\alpha'} \right] \right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}.$$

We need the following basic identities: 1. for f, g satisfying $f = \mathbb{H}f$, $g = \mathbb{H}g$,

$$[f, g; 1] = 0; \tag{5.181}$$

2. for f, p, g, satisfying $g = \mathbb{H}g$ and $p = \mathbb{H}p$,

$$[\overline{p}, \mathbb{P}_H f; g] = [\mathbb{P}_H f, \overline{p}g; 1] = [f, \mathbb{P}_A(\overline{p}g); 1] \tag{5.182}$$

(5.181) can be verified by (A.1) and integration by parts. (5.182) can be verified by (5.181).

We split

$$\begin{aligned}\left[Z_t, \mathbb{H}\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'} \right] &= \left[Z_t, 2\mathbb{P}_H\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'} \right] \\ &\quad - \left[Z_t, \frac{iA_1}{|Z_{,\alpha'}|^2}; \overline{Z}_{t,\alpha'} \right] = 2I - II.\end{aligned} \tag{5.183}$$

Applying (5.182) to I yields

$$I := \left[Z_t, \mathbb{P}_H\left(\frac{iA_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'} \right] = \left[\frac{iA_1}{|Z_{,\alpha'}|^2}, \mathbb{P}_A(Z_t \overline{Z}_{t,\alpha'}); 1 \right]; \tag{5.184}$$

substituting in (5.184) the identity

$$\frac{iA_1(\alpha')}{|Z_{,\alpha'}|^2} - \frac{iA_1(\beta')}{|Z_{,\beta'}|^2} = \left(\frac{iA_1(\alpha')}{Z_{,\alpha'}} - \frac{iA_1(\beta')}{Z_{,\beta'}}\right)\frac{1}{\overline{Z}_{,\beta'}} + \frac{iA_1(\alpha')}{Z_{,\alpha'}}\left(\frac{1}{\overline{Z}_{,\alpha'}} - \frac{1}{\overline{Z}_{,\beta'}}\right); \tag{5.185}$$

gives

$$I=\frac{1}{\pi i}\int \frac{\left(\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})(\alpha')-\mathbb{P}_A(Z_t\overline{Z}_{t,\beta'})(\beta')\right)\left(\frac{iA_1(\alpha')}{Z_{,\alpha'}}-\frac{iA_1(\beta')}{Z_{,\beta'}}\right)\frac{1}{\overline{Z}_{,\beta'}}}{(\alpha'-\beta')^2}d\beta'; \tag{5.186}$$

here the second term disappears because of the fact (5.181). Using the identity

$$\begin{aligned}\frac{\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})-\mathbb{P}_A(Z_t\overline{Z}_{t,\beta'})}{\overline{Z}_{,\beta'}}&=\frac{\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})}{\overline{Z}_{,\alpha'}}-\frac{\mathbb{P}_A(Z_t\overline{Z}_{t,\beta'})}{\overline{Z}_{,\beta'}}\\&\quad-\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})\left(\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{\overline{Z}_{,\beta'}}\right)\end{aligned} \tag{5.187}$$

we get

$$I=\left[\frac{\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})}{\overline{Z}_{,\alpha'}},\frac{iA_1}{Z_{,\alpha'}};1\right]-\mathbb{P}_A(Z_t\overline{Z}_{t,\alpha'})\left[\frac{1}{\overline{Z}_{,\alpha'}},\frac{iA_1}{Z_{,\alpha'}};1\right] \tag{5.188}$$

from here we are readily to conclude from Proposition 5.12. We now work on II. By (5.185),

$$II:=\left[Z_t,\frac{iA_1}{|Z_{,\alpha'}|^2};\overline{Z}_{t,\alpha'}\right]=\left[Z_t,\frac{iA_1}{Z_{,\alpha'}};\overline{D_{\alpha'}Z_t}\right]+\frac{iA_1}{Z_{,\alpha'}}\left[Z_t,\frac{1}{\overline{Z}_{,\alpha'}};\overline{Z}_{t,\alpha'}\right]; \tag{5.189}$$

the first term can be handled by Proposition 5.12. We focus on the second term. By a (5.187) type identity, we have

$$\begin{aligned}&\frac{1}{Z_{,\alpha'}}\left[Z_t,\frac{1}{\overline{Z}_{,\alpha'}};\overline{Z}_{t,\alpha'}\right]=\left[\frac{Z_t}{Z_{,\alpha'}},\frac{1}{\overline{Z}_{,\alpha'}};\overline{Z}_{t,\alpha'}\right]-\left[\frac{1}{Z_{,\alpha'}},\frac{1}{\overline{Z}_{,\alpha'}};Z_t\overline{Z}_{t,\alpha'}\right]\\&=\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\frac{1}{\overline{Z}_{,\alpha'}};\overline{Z}_{t,\alpha'}\right]-\left[\frac{1}{Z_{,\alpha'}},\frac{1}{\overline{Z}_{,\alpha'}};\mathbb{P}_A\left(Z_t\overline{Z}_{t,\alpha'}\right)\right]\\&\quad+\left[\mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right),\frac{1}{\overline{Z}_{,\alpha'}};\overline{Z}_{t,\alpha'}\right]-\left[\frac{1}{Z_{,\alpha'}},\frac{1}{\overline{Z}_{,\alpha'}};\mathbb{P}_H\left(Z_t\overline{Z}_{t,\alpha'}\right)\right]\\&=I_1-I_2+I_3-I_4\end{aligned} \tag{5.190}$$

The first two terms I_1, I_2 in (5.190) can be handed by Propositions 5.11 and 5.12, because $\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)=\mathbb{P}_A b$. We need to manipulate further the last two

terms. We begin with I_4, and use the first equality in (5.182), then use the identity $\mathbb{P}_H = -\mathbb{P}_A + I$,

$$\begin{aligned} I_4 &:= \left[\frac{1}{\overline{Z}_{,\alpha'}}, \frac{1}{Z_{,\alpha'}}; \mathbb{P}_H\left(Z_t \overline{Z}_{t,\alpha'}\right)\right] = \left[\frac{1}{Z_{,\alpha'}}, \frac{\mathbb{P}_H\left(Z_t \overline{Z}_{t,\alpha'}\right)}{\overline{Z}_{,\alpha'}}; 1\right] \\ &= -\left[\frac{1}{Z_{,\alpha'}}, \frac{\mathbb{P}_A\left(Z_t \overline{Z}_{t,\alpha'}\right)}{\overline{Z}_{,\alpha'}}; 1\right] + \left[\frac{1}{Z_{,\alpha'}}, \mathbb{P}_A\left(Z_t \mathbb{P}_H \overline{D_{\alpha'} Z_t}\right); 1\right] \\ &\quad + \left[\frac{1}{Z_{,\alpha'}}, Z_t \mathbb{P}_A \overline{D_{\alpha'} Z_t}; 1\right] \end{aligned} \tag{5.191}$$

because of the fact (5.181), the $\mathbb{P}_A$ can be inserted in the second term. Now the first two terms on the right hand side of (5.191) can be handled by Propositions 5.12 and 5.10, we need to work further on the last term,

$$I_{43} := \left[\frac{1}{Z_{,\alpha'}}, Z_t \mathbb{P}_A \overline{D_{\alpha'} Z_t}; 1\right].$$

We consider it together with I_3. By (5.182),

$$I_3 := \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{P}_H\left(\frac{Z_t}{Z_{,\alpha'}}\right); \overline{Z}_{t,\alpha'}\right] = \left[\frac{Z_t}{Z_{,\alpha'}}, \mathbb{P}_A(\overline{D_{\alpha'} Z_t}); 1\right]. \tag{5.192}$$

Sum up I_3 and $-I_{43}$ gives

$$\begin{aligned} I_3 - I_{43} &= \frac{1}{\pi i} \int \frac{\left(\frac{\mathbb{P}_A(\overline{D_{\alpha'} Z_t})}{Z_{,\beta'}} - \frac{\mathbb{P}_A(\overline{D_{\beta'} Z_t})}{Z_{,\alpha'}}\right)(Z_t(\alpha', t) - Z_t(\beta', t))}{(\alpha' - \beta')^2} d\beta' \\ &= -\mathbb{P}_A(\overline{D_{\alpha'} Z_t}) \left[Z_t, \frac{1}{Z_{,\alpha'}}; 1\right] \end{aligned} \tag{5.193}$$

here we used (5.181) in the second step.

Through the steps in (5.183)–(5.193), we have converted $\left[Z_t, \mathbb{H}\left(\frac{i A_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'}\right]$ into a sum of terms that can be handled with known results in Sect. 5.1–5.2.2. We can conclude now that

$$\left\| \left[Z_t, \mathbb{H}\left(\frac{i A_1}{|Z_{,\alpha'}|^2}\right); \overline{Z}_{t,\alpha'}\right] - U_l \left[\mathfrak{Z}_t, \mathbb{H}\left(\frac{i \tilde{A}_1}{|\mathfrak{Z}_{,\alpha'}|^2}\right); \overline{\mathfrak{Z}}_{t,\alpha'}\right] \right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.194}$$

Combine with (5.180), we obtain

$$\left\|\left[Z_t, (\partial_t + b\partial_{\alpha'})b; \overline{Z}_{t,\alpha'}\right] - U_l\left[\mathfrak{Z}_t, (\partial_t + \tilde{b}\partial_{\alpha'})\tilde{b}; \overline{\mathfrak{Z}}_{t,\alpha'}\right]\right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.195}$$

Now combine all the steps in Sect. 5.2.8, we get

$$\left\|(\partial_t + b\partial_{\alpha'})\left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}\right) - (\partial_t + \tilde{b}\partial_{\alpha'})\left(\frac{\tilde{\mathfrak{a}}_t}{\tilde{\mathfrak{a}}} \circ \tilde{h}^{-1}\right) \circ l\right\|_{L^2} \lesssim \mathcal{F}(t)^{1/2}. \tag{5.196}$$

Combine all the steps above we have (5.101). This finishes the proof for Proposition 5.19, and Theorem 3.7. □

6 The proof of Theorem 3.9

For the data given in Sect. 3.3.1, we construct the solution of the Cauchy problem in the class where $\mathcal{E} < \infty$ via a sequence of approximating solutions obtained by mollifying the initial data by the Poisson kernel, where we use Theorem 3.7 and a compactness argument to prove the convergence of the sequence. To prove the uniqueness of the solutions we use Theorem 3.7.

In what follows, we denote $z' = x' + iy'$, where x', $y' \in \mathbb{R}$. K is the Poisson kernel as defined by (3.12), $f * g$ is the convolution in the spatial variable. For any function φ, $\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon})$ for $x \in \mathbb{R}$.

6.1 Some basic preparations

Observe that in inequality (3.17), the stability is proved for the difference of the solutions in Lagrangian coordinates. We begin with some inequalities that will allow us to control the difference in Riemann mapping coordinates.

We have

Lemma 6.1 *Let $l : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism with $l - \alpha' \in H^1(\mathbb{R})$. Then*
1. for any $f \in \dot{H}^1(\mathbb{R})$,

$$\begin{aligned}\|f \circ l - f\|_{\dot{H}^{1/2}} \lesssim\ & \|\partial_{\alpha'} f\|_{L^2}\|l - \alpha'\|_{L^2}^{1/4}\|l_{\alpha'} \\ & -1\|_{L^2}^{1/4} + C(\left\|(l^{-1})_{\alpha'}\right\|_{L^\infty}, \|l_{\alpha'}\|_{L^\infty})\|l_{\alpha'} - 1\|_{L^2}\|\partial_{\alpha'} f\|_{L^2}. \end{aligned} \tag{6.1}$$

2. for any function $b : \mathbb{R} \to \mathbb{R}$, with $b_{\alpha'} \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\|b_{\alpha'} \circ l - b_{\alpha'}\|_{L^2}^2 \lesssim \|b_{\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^\infty}\|l_{\alpha'}\|_{L^\infty}^{1/2}\|l_{\alpha'} - 1\|_{L^2} + \|b_{\alpha'}\|_{\dot{H}^{1/2}}\|b \circ l - b\|_{\dot{H}^{1/2}} + \|b_{\alpha'}\|_{L^\infty}^2\|l_{\alpha'} - 1\|_{L^2}^2. \tag{6.2}$$

Proof We know

$$i\int \partial_{\alpha'}(f \circ l - f)\overline{(f \circ l - f)}\, d\alpha' = 2\,\mathrm{Re}\, i\int \partial_{\alpha'} f\overline{(f - f \circ l)}\, d\alpha' \tag{6.3}$$

so

$$\left| i\int \partial_{\alpha'}(f \circ l - f)\overline{(f \circ l - f)}\, d\alpha' \right| \le 2\|\partial_{\alpha'} f\|_{L^2}\|f - f \circ l\|_{L^2}. \tag{6.4}$$

Now

$$\begin{aligned}\int |f(\alpha') - f(l(\alpha'))|^2\, d\alpha' &\le \|l - \alpha'\|_{L^\infty}^2 \int |\mathcal{M}(\partial_{\alpha'} f))(\alpha')|^2\, d\alpha' \\ &\lesssim \|l - \alpha'\|_{L^\infty}^2\|\partial_{\alpha'} f\|_{L^2}^2,\end{aligned} \tag{6.5}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Therefore by Sobolev embedding (A.7) and Lemma A.2,

$$\|f \circ l - f\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'} f\|_{L^2}\|l - \alpha'\|_{L^2}^{1/4}\|l_{\alpha'} - 1\|_{L^2}^{1/4} + \|\mathbb{P}_A(f \circ l - f)\|_{\dot{H}^{1/2}}, \tag{6.6}$$

$$\|f \circ l - f\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'} f\|_{L^2}\|l - \alpha'\|_{L^2}^{1/4}\|l_{\alpha'} - 1\|_{L^2}^{1/4} + \|\mathbb{P}_H(f \circ l - f)\|_{\dot{H}^{1/2}}. \tag{6.7}$$

Now

$$2\mathbb{P}_A(f \circ l - f) = (2\mathbb{P}_A f) \circ l - 2\mathbb{P}_A f + \mathcal{Q}_l(f \circ l).$$

Applying (6.7) to $(\mathbb{P}_A f) \circ l - \mathbb{P}_A f$ and using (5.9) gives

$$\|\mathbb{P}_A(f \circ l - f)\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'} f\|_{L^2}\|l - \alpha'\|_{L^2}^{1/4}\|l_{\alpha'} - 1\|_{L^2}^{1/4} + C\Big(\Big\|(l^{-1})_{\alpha'}\Big\|_{L^\infty}, \|l_{\alpha'}\|_{L^\infty}\Big)\|l_{\alpha'} - 1\|_{L^2}\|\partial_{\alpha'} f\|_{L^2}.$$

This proves (6.1).

To prove (6.2), we begin with

$$b_{\alpha'} \circ l - b_{\alpha'} = \partial_{\alpha'}(b \circ l - b) + b_{\alpha'} \circ l(1 - l_{\alpha'}); \tag{6.8}$$

and by expanding the integral, we have

$$\|\partial_{\alpha'}(b \circ l - b)\|_{L^2}^2 = \int (b_{\alpha'} \circ l)^2 (l_{\alpha'}^2 - l_{\alpha'})\, d\alpha' + 2 \int b_{\alpha'} \partial_{\alpha'}(b - b \circ l)\, d\alpha'. \tag{6.9}$$

(6.2) follows directly from the Triangular, Cauchy–Schwarz and Hölder's inequalities. □

Lemma 6.2 *For any $\varphi \in C^\infty(\mathbb{R})$, with $\int \varphi(x)\, dx = 1$ and $\int |x\varphi(x)|^2\, dx < \infty$, and for any $f \in \dot{H}^1(\mathbb{R})$,*

$$\|\varphi_\epsilon * f - f\|_{L^\infty} \lesssim \epsilon^{1/2} \|\partial_x f\|_{L^2} \|x\varphi\|_{L^2}. \tag{6.10}$$

The proof is straightforward by Cauchy–Schwarz inequality and Hardy's inequality (A.8). We omit the details.

Let $Z, \mathfrak{Z}$ be solutions of the system (2.9)–(2.11)–(2.15)–(2.14), satisfying the assumptions of Theorem 3.7, and let l be given by (3.16). We know

$$(\partial_t + b\partial_{\alpha'})(l - \alpha') = U_{h^{-1}}(\tilde{h}_t - h_t) = \tilde{b} \circ l - b,$$

and $l(\alpha', 0) = \alpha'$ for $\alpha' \in \mathbb{R}$. By Lemma 4.3,

$$\begin{aligned} \frac{d}{dt}\|l(t) - \alpha'\|_{L^2}^2 &\le 2\|\tilde{b} \circ l(t) - b(t)\|_{L^2} \|l(t) - \alpha'\|_{L^2} \\ &\quad + \|b_{\alpha'}(t)\|_{L^\infty} \|l(t) - \alpha'\|_{L^2}^2, \end{aligned} \tag{6.11}$$

and from (2.14) and Sobolev embedding,

$$\|b(t)\|_{H^1(\mathbb{R})} \lesssim \|Z_t(t)\|_{H^1(\mathbb{R})} \left(\left\| \frac{1}{Z_{,\alpha'}}(t) - 1 \right\|_{H^1(\mathbb{R})} + 1 \right). \tag{6.12}$$

Therefore by Gronwall's inequality, we have

$$\sup_{[0,T]} \|l(t) - \alpha'\|_{L^2(\mathbb{R})} \le C, \tag{6.13}$$

where C is a constant depending on $\sup_{[0,T]} \Big(\|Z_t(t)\|_{L^2} + \|\mathfrak{Z}_t(t)\|_{L^2} + \left\| \frac{1}{Z_{,\alpha'}}(t) - 1 \right\|_{L^2} + \left\| \frac{1}{\mathfrak{Z}_{,\alpha'}}(t) - 1 \right\|_{L^2} \Big)$ and $\sup_{[0,T]}(\mathcal{E}(t) + \tilde{\mathcal{E}}(t))$.

Let

$$\begin{aligned}\|(Z-\mathfrak{Z})(0)\| :=& \left\|\left(\overline{Z}_t-\overline{\mathfrak{Z}}_t\right)(0)\right\|_{\dot{H}^{1/2}}+\left\|\left(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\right)(0)\right\|_{\dot{H}^{1/2}}\\ &+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(0)\right\|_{\dot{H}^{1/2}}\\ &+\left\|\left(D_{\alpha'}Z_t-(\tilde{D}_{\alpha'}\mathfrak{Z}_t)\right)(0)\right\|_{L^2}+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(0)\right\|_{L^\infty}.\end{aligned} \tag{6.14}$$

Applying (6.1) to $f=\overline{\mathfrak{Z}}_t$, $\frac{1}{\mathfrak{Z}_{,\alpha'}}-1$ and $\overline{\mathfrak{Z}}_{tt}$ and use (3.17) gives

$$\begin{aligned}&\sup_{[0,T]}\left(\left\|\left(\overline{Z}_t-\overline{\mathfrak{Z}}_t\right)(t)\right\|_{\dot{H}^{1/2}(\mathbb{R})}+\left\|\left(\frac{1}{Z_{,\alpha'}}-\frac{1}{\mathfrak{Z}_{,\alpha'}}\right)(t)\right\|_{\dot{H}^{1/2}(\mathbb{R})}\right.\\ &\quad\left.+\left\|\left(\overline{Z}_{tt}-\overline{\mathfrak{Z}}_{tt}\right)(t)\right\|_{\dot{H}^{1/2}(\mathbb{R})}\right)\\ &\quad\le C(\|(Z-\mathfrak{Z})(0)\|+\|(Z-\mathfrak{Z})(0)\|^{1/4});\end{aligned} \tag{6.15}$$

and applying (6.2), (6.1) to $\tilde{b}$ and use (6.12), (5.133), (5.135), Appendix C and (3.17) yields

$$\sup_{[0,T]}\|(b_{\alpha'}-\tilde{b}_{\alpha'})(t)\|_{L^2(\mathbb{R})}\le C(\|(Z-\mathfrak{Z})(0)\|+\|(Z-\mathfrak{Z})(0)\|^{1/8}), \tag{6.16}$$

where C is a constant depending on $\sup_{[0,T]}\left(\|Z_t(t)\|_{L^2}+\|\mathfrak{Z}_t(t)\|_{L^2}+\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^2}+\left\|\frac{1}{\mathfrak{Z}_{,\alpha'}}(t)-1\right\|_{L^2}\right)$ and $\sup_{[0,T]}(\mathcal{E}(t)+\tilde{\mathcal{E}}(t))$. By Sobolev embedding (A.7),

$$\|l(\cdot,t)-\alpha'\|^2_{L^\infty(\mathbb{R})}\lesssim\|l(\cdot,t)-\alpha'\|_{L^2(\mathbb{R})}\|(l_{\alpha'}-1)(t)\|_{L^2(\mathbb{R})}, \tag{6.17}$$

therefore by (6.13), (5.77)–(5.78), and (3.17),

$$\sup_{[0,T]}(\|h(t)-\tilde{h}(t)\|^2_{L^\infty(\mathbb{R})}+\|h^{-1}(t)-\tilde{h}^{-1}(t)\|^2_{L^\infty(\mathbb{R})})\le C\|(Z-\mathfrak{Z})(0)\|, \tag{6.18}$$

where C is a constant depending on $\sup_{[0,T]}\left(\|Z_t(t)\|_{L^2}+\|\mathfrak{Z}_t(t)\|_{L^2}+\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^2}+\left\|\frac{1}{\mathfrak{Z}_{,\alpha'}}(t)-1\right\|_{L^2}\right)$ and $\sup_{[0,T]}(\mathcal{E}(t)+\tilde{\mathcal{E}}(t))$.

We also have, from Sobolev embedding (A.7), (6.12), (6.13), (6.17) and (3.17) that for $t \in [0, T]$,

$$\begin{aligned}\left\|(b-\tilde{b})(t)\right\|^2_{L^\infty(\mathbb{R})} &\lesssim \left\|(b-\tilde{b}\circ l)(t)\right\|^2_{L^\infty(\mathbb{R})} + \left\|(\tilde{b}\circ l-\tilde{b})(t)\right\|^2_{L^\infty(\mathbb{R})}\\ &\lesssim \left\|(b-\tilde{b}\circ l)(t)\right\|_{L^2(\mathbb{R})}\left\|\partial_{\alpha'}(b-\tilde{b}\circ l)(t)\right\|_{L^2(\mathbb{R})} + \|l(t)-\alpha'\|^2_{L^\infty(\mathbb{R})}\left\|\tilde{b}_{\alpha'}(t)\right\|^2_{L^\infty(\mathbb{R})}\\ &\le C(\|(Z-\mathfrak{Z})(0)\| + \|(Z-\mathfrak{Z})(0)\|^2), \end{aligned} \tag{6.19}$$

where C is a constant depending on $\sup_{[0,T]}\Big(\|Z_t(t)\|_{L^2} + \|\mathfrak{Z}_t(t)\|_{L^2} + \left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^2} + \left\|\frac{1}{\mathfrak{Z}_{,\alpha'}}(t)-1\right\|_{L^2}\Big)$ and $\sup_{[0,T]}(\mathcal{E}(t)+\tilde{\mathcal{E}}(t))$.

We have

Lemma 6.3 *1. Assume that $f \in H^{1/2}(\mathbb{R})$. Then*

$$\|f\|^2_{L^4(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}\|f\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{6.20}$$

2. Let $\varphi \in C^\infty(\mathbb{R}) \cap L^q(\mathbb{R})$, and $f \in L^p(\mathbb{R})$, where $1 \le p \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. For any $y' < 0$, $x' \in \mathbb{R}$,

$$|\varphi_{y'} * f(x')| \le (-y')^{-1/p}\|\varphi\|_{L^q(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}. \tag{6.21}$$

Proof By the Theorem 1 on page 119 of [32], Plancherel's Theorem and Cauchy–Schwarz inequality, we have, for any $f \in H^{1/2}(\mathbb{R})$,

$$\|f\|_{L^4(\mathbb{R})} \lesssim \|\partial_x^{1/4} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}\|f\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

(6.21) is a direct consequence of Hölder's inequality. □

We need in addition the following compactness results in the proof of the existence of solutions.

Lemma 6.4 *Let $\{f_n\}$ be a sequence of smooth functions on $\mathbb{R}\times[0,T]$. Let $1 < p \le \infty$. Assume that there is a constant C, independent of n, such that*

$$\sup_{[0,T]}\|f_n(t)\|_{L^\infty} + \sup_{[0,T]}\|\partial_x f_n(t)\|_{L^p} + \sup_{[0,T]}\|\partial_t f_n(t)\|_{L^\infty} \le C. \tag{6.22}$$

Then there is a function f, continuous and bounded on $\mathbb{R}\times[0,T]$, and a subsequence $\{f_{n_j}\}$, such that $f_{n_j} \to f$ uniformly on compact subsets of $\mathbb{R}\times[0,T]$.

Lemma 6.4 is an easy consequence of Arzela-Ascoli Theorem, we omit the proof.

Lemma 6.5 *Assume that $f_n \to f$ uniformly on compact subsets of $\mathbb{R}\times[0,T]$, and assume there is a constant C, such that $\sup_n \|f_n\|_{L^\infty(\mathbb{R}\times[0,T])} \le C$. Then $K_{y'} * f_n$ converges uniformly to $K_{y'} * f$ on compact subsets of $\overline{\mathscr{P}}_- \times [0,T]$.*

The proof follows easily by considering the convolution on the sets $|x'| < N$, and $|x'| \ge N$ separately. We omit the proof.

Definition 6.6 We write

$$f_n \Rightarrow f \qquad \text{on } E \tag{6.23}$$

if f_n converge uniformly to f on compact subsets of E.

6.2 The proof of Theorem 3.9

The uniqueness of the solution to the Cauchy problem is a direct consequence of (6.15) and Definition 3.8. In what follows we prove the existence of solutions to the Cauchy problem.

6.2.1 The initial data

Let $U(z',0)$ be the initial fluid velocity in the Riemann mapping coordinate, $\Psi(z',0) : \mathscr{P}_- \to \Omega(0)$ be the Riemann mapping as given in Sect. 3.3.1 with $Z(\alpha',0) = \Psi(\alpha',0)$ the initial interface. We note that by the assumption

$$\begin{aligned}
&\sup_{y'<0} \left\| \partial_{z'} \left(\frac{1}{\Psi_{z'}(z',0)} \right) \right\|_{L^2(\mathbb{R},dx')} \le \mathcal{E}_1(0) < \infty,\\
&\sup_{y'<0} \left\| \frac{1}{\Psi_{z'}(z',0)} - 1 \right\|_{L^2(\mathbb{R},dx')} \le c_0 < \infty,\\
&\sup_{y'<0} \|U_{z'}(z',0)\|_{L^2(\mathbb{R},dx')} \le \mathcal{E}_1(0) < \infty \quad \text{and} \quad \sup_{y'<0} \|U(z',0)\|_{L^2(\mathbb{R},dx')} \le c_0 < \infty,
\end{aligned}$$

$\frac{1}{\Psi_{z'}}(\cdot,0), U(\cdot,0)$ can be extended continuously onto $\overline{\mathscr{P}}_-$. So $Z(\cdot,0) := \Psi(\cdot + i0, 0)$ is continuous differentiable on the open set where $\frac{1}{\Psi_{z'}}(\alpha',0) \ne 0$, and $\frac{1}{\Psi_{z'}}(\alpha',0) = \frac{1}{Z_{,\alpha'}(\alpha',0)}$ where $\frac{1}{\Psi_{z'}}(\alpha',0) \ne 0$. By $\frac{1}{\Psi_{z'}}(\cdot,0) - 1 \in H^1(\mathbb{R})$ and Sobolev embedding, there is $N > 0$ sufficiently large, such that for $|\alpha'| \ge N$, $|\frac{1}{\Psi_{z'}}(\alpha',0) - 1| \le 1/2$, so $Z = Z(\cdot,0)$ is continuous differentiable on $(-\infty,-N)\cup(N,\infty)$, with $|Z_{,\alpha'}(\alpha',0)| \le 2$, for all $|\alpha'| \ge N$. Moreover, $Z_{,\alpha'}(\cdot,0) - 1 \in H^1\{(-\infty,-N)\cup(N,\infty)\}$.

6.2.2 *The mollified data and the approximate solutions*

Let $\epsilon > 0$. We take

$$\begin{aligned} &Z^\epsilon(\alpha', 0) = \Psi(\alpha' - \epsilon i, 0), \quad \overline{Z}^\epsilon_t(\alpha', 0) = U(\alpha' - \epsilon i, 0), \quad h^\epsilon(\alpha, 0) = \alpha, \\ &U^\epsilon(z', 0) = U(z' - \epsilon i, 0), \quad \Psi^\epsilon(z', 0) = \Psi(z' - \epsilon i, 0). \end{aligned} \tag{6.24}$$

Notice that $U^\epsilon(\cdot, 0)$, $\Psi^\epsilon(\cdot, 0)$ are holomorphic on $\mathscr{P}_-$, $Z^\epsilon(0)$ satisfies (2.11) and $\overline{Z}^\epsilon_t(0) = \mathbb{H}\overline{Z}^\epsilon_t(0)$. Let $Z^\epsilon_{tt}(0)$ be given by (2.17). It is clear that $Z^\epsilon(0)$, $Z^\epsilon_t(0)$ and $Z^\epsilon_{tt}(0)$ satisfy the assumption of Theorem 3.6. Let $Z^\epsilon(t) := Z^\epsilon(\cdot, t)$, $t \in [0, T^*_\epsilon)$, be the solution given by Theorem 3.6, with the maximal time of existence T^*_ϵ, the diffeomorphism $h^\epsilon(t) = h^\epsilon(\cdot, t) : \mathbb{R} \to \mathbb{R}$, the quantity $b^\epsilon := h^\epsilon_t \circ (h^\epsilon)^{-1}$, and $z^\epsilon(\alpha, t) = Z^\epsilon(h^\epsilon(\alpha, t), t)$. We know $z^\epsilon_t(\alpha, t) = Z^\epsilon_t(h^\epsilon(\alpha, t), t)$. Let

$$U^\epsilon(x' + iy', t) = K_{y'} * \overline{Z}^\epsilon_t(x', t), \quad \Psi^\epsilon_{z'}(x' + iy', t) = K_{y'} * Z^\epsilon_{,\alpha'}(x', t), \quad \Psi^\epsilon(\cdot, t)$$

be the holomorphic functions on $\mathscr{P}_-$ with boundary values $\overline{Z}^\epsilon_t(t)$, $Z^\epsilon_{,\alpha'}(t)$ and $Z^\epsilon(t)$; we have

$$\frac{1}{\Psi^\epsilon_{z'}}(x' + iy', t) = K_{y'} * \frac{1}{Z^\epsilon_{,\alpha'}}(x', t)$$

by uniqueness.[25] We denote the energy functional $\mathcal{E}$ for $\left(Z^\epsilon(t), \overline{Z}^\epsilon_t(t)\right)$ by $\mathcal{E}^\epsilon(t)$ and the energy functional $\mathcal{E}_1$ for $(U^\epsilon(t), \Psi^\epsilon(t))$ by $\mathcal{E}^\epsilon_1(t)$. It is clear that $\mathcal{E}^\epsilon_1(0) \le \mathcal{E}_1(0)$, and $\|Z^\epsilon_t(0)\|_{L^2} + \left\| \frac{1}{Z^\epsilon_{,\alpha'}(0)} - 1 \right\|_{L^2} \le c_0$ for all $\epsilon > 0$; and by the continuity of $\frac{1}{\Psi_{z'}}(\cdot, 0)$ on $\overline{\mathscr{P}}_-$, there is a $\epsilon_0 > 0$, such that for all $0 < \epsilon \le \epsilon_0$, $|\frac{1}{Z^\epsilon_{,\alpha'}}(0, 0)|^2 \le |\frac{1}{Z_{,\alpha'}}(0, 0)|^2 + 1$. By Theorem 3.6, Theorem 3.1 and Proposition 3.3, there exists $T_0 > 0$, T_0 depends only on $\mathcal{E}(0) = \mathcal{E}_1(0) + |\frac{1}{Z_{,\alpha'}}(0, 0)|^2$,[26] such that for all $0 < \epsilon \le \epsilon_0$, $T^*_\epsilon > T_0$ and

$$\sup_{[0,T_0]} \left(\mathcal{E}^\epsilon_1(t) + \left| \frac{1}{Z^\epsilon_{,\alpha'}}(0, t) \right|^2 \right) = \sup_{[0,T_0]} \mathcal{E}^\epsilon(t) \le M\left(\mathcal{E}(0)\right) < \infty; \tag{6.25}$$

[25] By the maximum principle, $\left(K_{y'} * \frac{1}{Z^\epsilon_{,\alpha'}}\right)\left(K_{y'} * Z^\epsilon_{,\alpha'}\right) \equiv 1$ on $\mathscr{P}_-$.

[26] By (3.13).

and by (2.17), (4.129) and (4.130),

$$\sup_{[0,T_0]}\left(\|Z_t^\epsilon(t)\|_{L^2}+\|Z_{tt}^\epsilon(t)\|_{L^2}+\left\|\frac{1}{Z_{,\alpha'}^\epsilon(t)}-1\right\|_{L^2}\right)\le c\,(c_0,\mathcal{E}(0))\,, \tag{6.26}$$

so there is a constant $C_0:=C(c_0,\mathcal{E}(0))>0$, such that

$$\sup_{[0,T_0]}\{\sup_{y'<0}\|U^\epsilon(\cdot+iy',t)\|_{L^2(\mathbb{R})}+\sup_{y'<0}\left\|\frac{1}{\Psi_{z'}^\epsilon(\cdot+iy',t)}-1\right\|_{L^2(\mathbb{R})}\}<C_0<\infty. \tag{6.27}$$

6.2.3 *Uniformly bounded quantities*

Besides (6.15), (6.18) and (6.19), we would like to apply the compactness results Lemma 6.4, Lemma 6.5 to pass to limits of some of the quantities. To this end we discuss the boundedness properties of these quantities. We begin with two inequalities.

We have, from (2.35),

$$\left\|(\partial_t+b^\epsilon\partial_{\alpha'})\frac{1}{Z_{,\alpha'}^\epsilon}(t)\right\|_{L^\infty}\le\left\|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\right\|_{L^\infty}(\|b_{\alpha'}^\epsilon(t)\|_{L^\infty}+\|D_{\alpha'}Z_t^\epsilon(t)\|_{L^\infty}) \tag{6.28}$$

and by (2.37),

$$\begin{aligned}\|(\partial_t+b^\epsilon\partial_{\alpha'})Z_{tt}^\epsilon(t)\|_{L^\infty}\le{}&\|Z_{tt}^\epsilon(t)\\&+i\|_{L^\infty}\left(\|D_{\alpha'}Z_t^\epsilon(t)\|_{L^\infty}+\left\|\frac{\mathfrak{a}_t^\epsilon}{\mathfrak{a}^\epsilon}\circ(h^\epsilon)^{-1}(t)\right\|_{L^\infty}\right).\end{aligned} \tag{6.29}$$

Let $0<\epsilon\le\epsilon_0$, and $M(\mathcal{E}(0))$, $c(c_0,\mathcal{E}(0))$, C_0 be the bounds in (6.25), (6.26) and (6.27). By Proposition 3.3, Sobolev embedding, Appendix C and (6.28), (6.12), the following quantities are uniformly bounded with bounds depending only on $M(\mathcal{E}(0))$, $c(c_0,\mathcal{E}(0))$, C_0:

$$\begin{aligned}
&\sup_{[0,T_0]}\|Z_t^\epsilon(t)\|_{L^\infty},\quad \sup_{[0,T_0]}\|Z_{t,\alpha'}^\epsilon(t)\|_{L^2},\quad \sup_{[0,T_0]}\|Z_{tt}^\epsilon(t)\|_{L^\infty},\quad \sup_{[0,T_0]}\|Z_{tt,\alpha'}^\epsilon(t)\|_{L^2},\\
&\sup_{[0,T_0]}\left\|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\right\|_{L^\infty},\quad \sup_{[0,T_0]}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}^\epsilon}(t)\right\|_{L^2},\\
&\sup_{[0,T_0]}\left\|(\partial_t+b^\epsilon\partial_{\alpha'})\frac{1}{Z_{,\alpha'}^\epsilon}(t)\right\|_{L^\infty},\quad \sup_{[0,T_0]}\|b^\epsilon\|_{L^\infty};
\end{aligned}\tag{6.30}$$

and with a change of the variables and (5.78), (6.29) and Appendix C,

$$\begin{aligned}
&\sup_{[0,T_0]}\|z_t^\epsilon(t)\|_{L^\infty}+\sup_{[0,T_0]}\|z_{t\alpha}^\epsilon(t)\|_{L^2}+\sup_{[0,T_0]}\|z_{tt}^\epsilon(t)\|_{L^\infty}\le C(c_0,\mathcal{E}(0)),\\
&\sup_{[0,T_0]}\left\|\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(t)\right\|_{L^\infty}+\sup_{[0,T_0]}\left\|\partial_\alpha(\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon})(t)\right\|_{L^2}+\sup_{[0,T_0]}\left\|\partial_t\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(t)\right\|_{L^\infty}\le C(c_0,\mathcal{E}(0)),\\
&\sup_{[0,T_0]}\|z_{tt}^\epsilon(t)\|_{L^\infty}+\sup_{[0,T_0]}\|z_{tt\alpha}^\epsilon(t)\|_{L^2}+\sup_{[0,T_0]}\|z_{ttt}^\epsilon(t)\|_{L^\infty}\le C(c_0,\mathcal{E}(0)).
\end{aligned}\tag{6.31}$$

Observe that $h^\epsilon(\alpha,t)-\alpha=\int_0^t h_t^\epsilon(\alpha,s)\,ds$, so

$$\sup_{\mathbb{R}\times[0,T_0]}|h^\epsilon(\alpha,t)-\alpha|\le T_0\sup_{[0,T_0]}\|h_t^\epsilon(t)\|_{L^\infty}\le T_0C(c_0,\mathcal{E}(0))<\infty.\tag{6.32}$$

Furthermore by (5.78) and Appendix C, there are $c_1,c_2>0$, depending only on $\mathcal{E}(0)$, such that

$$0<c_1\le\frac{h^\epsilon(\alpha,t)-h^\epsilon(\beta,t)}{\alpha-\beta}\le c_2<\infty,\qquad\forall\alpha,\beta\in\mathbb{R},\ t\in[0,T_0].\tag{6.33}$$

6.2.4 Passing to the limit

It is easy to check by Lemma 6.2 and (A.5), (5.24) that the sequence $(Z^\epsilon(0),\overline{Z}_t^\epsilon(0))$ converges in the norm $\|\cdot\|$ defined by (6.14), so by (6.16), (6.18) and (6.19), there are functions b and $h-\alpha$, continuous and bounded on $\mathbb{R}\times[0,T_0]$, with $h(\cdot,t):\mathbb{R}\to\mathbb{R}$ a homeomorphism for $t\in[0,T_0]$, $b_{\alpha'}\in L^\infty([0,T_0],L^2(\mathbb{R}))$, such that

$$\lim_{\epsilon\to0}\left(b^\epsilon,\ h^\epsilon,\ (h^\epsilon)^{-1}\right)=\left(b,\ h,\ h^{-1}\right),\qquad\text{uniformly on }\mathbb{R}\times[0,T_0];\tag{6.34}$$

$$\lim_{\epsilon\to0}b_{\alpha'}^\epsilon=b_{\alpha'}\qquad\text{in}\quad L^\infty([0,T_0],L^2(\mathbb{R}));\tag{6.35}$$

and (6.33) yields

$$0 < c_1 \le \frac{h(\alpha,t)-h(\beta,t)}{\alpha-\beta} \le c_2 < \infty, \qquad \forall \alpha,\beta \in \mathbb{R},\ t \in [0,T_0]. \tag{6.36}$$

By Lemma 6.4, (6.31) and (6.15), there are functions w, u, $q := w_t$, continuous and bounded on $\mathbb{R}\times[0,T_0]$, such that

$$z^\epsilon_t \Rightarrow w, \quad \frac{h^\epsilon_\alpha}{z^\epsilon_\alpha} \Rightarrow u, \quad z^\epsilon_{tt} \Rightarrow q, \qquad \text{on } \mathbb{R}\times[0,T_0], \tag{6.37}$$

as $\epsilon \to 0$; this gives

$$\overline{Z}^\epsilon_t \Rightarrow w\circ h^{-1}, \qquad \frac{1}{Z^\epsilon_{,\alpha'}} \Rightarrow u\circ h^{-1}, \quad \overline{Z}^\epsilon_{tt} \Rightarrow w_t\circ h^{-1}, \qquad \text{on } \mathbb{R}\times[0,T_0] \tag{6.38}$$

as $\epsilon\to 0$. (6.15) also gives that

$$\lim_{\epsilon\to 0}\left(\overline{Z}^\epsilon_t,\ \frac{1}{Z^\epsilon_{,\alpha'}},\ \overline{Z}^\epsilon_{tt}\right) = \left(w\circ h^{-1},\ u\circ h^{-1},\ w_t\circ h^{-1}\right), \qquad \text{in } L^\infty([0,T_0],\dot{H}^{1/2}(\mathbb{R})). \tag{6.39}$$

Now

$$U^\epsilon(z',t) = K_{y'} * \overline{Z}^\epsilon_t, \qquad \frac{1}{\Psi^\epsilon_{z'}}(z',t) = K_{y'} * \frac{1}{Z^\epsilon_{,\alpha'}}. \tag{6.40}$$

Let $U(z',t) = K_{y'} * (w\circ h^{-1})(x',t)$, $\Lambda(z',t) = K_{y'} * (u\circ h^{-1})(x',t)$. By Lemma 6.5,

$$U^\epsilon(z',t) \Rightarrow U(z',t), \qquad \frac{1}{\Psi^\epsilon_{z'}}(z',t) \Rightarrow \Lambda(z',t) \qquad \text{on } \overline{\mathscr{P}}_-\times[0,T_0]; \tag{6.41}$$

as $\epsilon\to 0$. Moreover $U(\cdot,t)$, $\Lambda(\cdot,t)$ are holomorphic on $\mathscr{P}_-$ for each $t\in[0,T_0]$, and continuous on $\overline{\mathscr{P}}_-\times[0,T]$. Applying Cauchy integral formula to the first limit in (6.41) yields, as $\epsilon\to 0$,

$$U^\epsilon_{z'}(z',t) \Rightarrow U_{z'}(z',t) \qquad \text{on } \mathscr{P}_-\times[0,T_0]. \tag{6.42}$$

Step 1. The limit of Ψ^ϵ

We consider the limit of Ψ^ϵ, as $\epsilon \to 0$. Let $0 < \epsilon \le \epsilon_0$. We know

$$\begin{aligned} z^\epsilon(\alpha, t) &= z^\epsilon(\alpha, 0) + \int_0^t z_t^\epsilon(\alpha, s)\, ds \\ &= \Psi(\alpha - \epsilon i, 0) + \int_0^t z_t^\epsilon(\alpha, s)\, ds, \end{aligned} \tag{6.43}$$

therefore

$$\begin{aligned} Z^\epsilon(\alpha', t) - Z^\epsilon(\alpha', 0) = \Psi((h^\epsilon)^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0) \\ + \int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha', t), s)\, ds. \end{aligned} \tag{6.44}$$

Let

$$\begin{aligned} W^\epsilon(\alpha', t) \ &:= \Psi((h^\epsilon)^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0) \\ &+ \int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha', t), s)\, ds. \end{aligned} \tag{6.45}$$

Observe $Z^\epsilon(\alpha', t) - Z^\epsilon(\alpha', 0)$ is the boundary value of the holomorphic function $\Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0)$. By (6.37) and (6.34), $\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha', t), s)\, ds \to \int_0^t w(h^{-1}(\alpha', t), s)\, ds$ uniformly on compact subsets of $\mathbb{R} \times [0, T_0]$, and by (6.31), $\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha', t), s)\, ds$ is continuous and uniformly bounded in $L^\infty(\mathbb{R} \times [0, T_0])$. By the assumptions $\lim_{z' \to 0} \Psi_{z'}(z', 0) = 1$ and $\Psi(\cdot, 0)$ is continuous on $\overline{\mathscr{P}}_-$, and by (6.32), (6.34),

$$\Psi((h^\epsilon)^{-1}(\alpha', t) - \epsilon i, 0) - \Psi(\alpha' - \epsilon i, 0)$$

is continuous and uniformly bounded in $L^\infty(\mathbb{R} \times [0, T_0])$ for $0 < \epsilon < 1$, and converges uniformly on compact subsets of $\mathbb{R} \times [0, T_0]$ as $\epsilon \to 0$. This gives[27]

$$\Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0) = K_{y'} * W^\epsilon(x', t) \tag{6.46}$$

and by Lemma 6.5, $\Psi^\epsilon(z', t) - \Psi^\epsilon(z', 0)$ converges uniformly on compact subsets of $\overline{\mathscr{P}}_- \times [0, T_0]$ to a function that is holomorphic on $\mathscr{P}_-$ for every $t \in [0, T_0]$ and continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$. Therefore there is a function $\Psi(\cdot, t)$,

[27] Because $W^\epsilon(\cdot, t)$ and $\partial_{\alpha'} W^\epsilon(\cdot, t) := Z^\epsilon_{,\alpha'}(\alpha', t) - Z^\epsilon_{,\alpha'}(\alpha', 0)$ are continuous and bounded on $\mathbb{R}$, $\Psi^\epsilon_{z'}(z', t) - \Psi^\epsilon_{z'}(z', 0) = K_{y'} * (\partial_{\alpha'} W^\epsilon)(x', t) = \partial_{z'} K_{y'} * W^\epsilon(x', t)$. (6.46) holds because both sides of (6.46) have the same value on $\partial \mathscr{P}_-$.

holomorphic on $\mathscr{P}_-$ for every $t \in [0, T_0]$ and continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$, such that

$$\Psi^\epsilon(z', t) \Rightarrow \Psi(z', t) \qquad \text{on } \overline{\mathscr{P}}_- \times [0, T_0] \tag{6.47}$$

as $\epsilon \to 0$; as a consequence of the Cauchy integral formula,

$$\Psi^\epsilon_{z'}(z', t) \Rightarrow \Psi_{z'}(z', t) \qquad \text{on } \mathscr{P}_- \times [0, T_0] \tag{6.48}$$

as $\epsilon \to 0$. Combining with (6.41), we have $\Lambda(z', t) = \frac{1}{\Psi_{z'}(z', t)}$, so $\Psi_{z'}(z', t) \neq 0$ for all $(z', t) \in \overline{\mathscr{P}}_- \times [0, T_0]$ and

$$\frac{1}{\Psi^\epsilon_{z'}(z', t)} \Rightarrow \frac{1}{\Psi_{z'}(z', t)} \qquad \text{on } \overline{\mathscr{P}}_- \times [0, T_0] \tag{6.49}$$

as $\epsilon \to 0$.

Denote $Z(\alpha', t) := \Psi(\alpha', t)$, $\alpha' \in \mathbb{R}$, and $z(\alpha, t) = Z(h(\alpha, t), t)$. (6.47) yields $Z^\epsilon(\alpha', t) \Rightarrow Z(\alpha', t)$, together with (6.34) it implies $z^\epsilon(\alpha, t) \Rightarrow z(\alpha, t)$ on $\mathbb{R} \times [0, T_0]$, as $\epsilon \to 0$. Furthermore by (6.43),

$$z(\alpha', t) = z(\alpha', 0) + \int_0^t w(\alpha, s)\, ds,$$

so $w = z_t$. We denote $Z_t = z_t \circ h^{-1}$.

Step 2. The limits of Ψ^ϵ_t and U^ϵ_t

Observe that by (6.45), for fixed $\epsilon > 0$, $\partial_t W^\epsilon(\cdot, t)$ is a bounded function on $\mathbb{R} \times [0, T_0]$, so by (6.46) and the dominated convergence Theorem, $\Psi^\epsilon_t = K_{y'} * \partial_t W^\epsilon$, hence Ψ^ϵ_t is bounded on $\mathscr{P}_- \times [0, T_0]$.

Since for given $t \in [0, T_0]$ and $\epsilon > 0$, $\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}}$ is bounded and holomorphic on $\mathscr{P}_-$, by (2.29),

$$\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}} = K_{y'} * \left(\frac{Z^\epsilon_t}{Z^\epsilon_{,\alpha'}} - b^\epsilon \right). \tag{6.50}$$

Therefore by (6.34), (6.38) and Lemma 6.5, as $\epsilon \to 0$, $\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}}$ converges uniformly on compact subsets of $\overline{\mathscr{P}}_- \times [0, T_0]$ to a function that is holomorphic on $\mathscr{P}_-$ for each $t \in [0, T_0]$ and continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$. Hence we can conclude

from (6.47) and (6.48) that Ψ is continuously differentiable and

$$\Psi_t^\epsilon \Rightarrow \Psi_t \qquad \text{on } \mathscr{P}_- \times [0, T_0] \tag{6.51}$$

as $\epsilon \to 0$.

Now we consider the limit of U_t^ϵ as $\epsilon \to 0$. Since for fixed $\epsilon > 0$, $\partial_t Z_t^\epsilon = Z_{tt}^\epsilon - b^\epsilon Z_{t,\alpha'}^\epsilon$ is in $L^\infty(\mathbb{R} \times [0, T_0])$, by (6.40) and the dominated convergence Theorem,

$$U_t^\epsilon(z', t) = K_{y'} * \partial_t \overline{Z}_t^\epsilon = K_{y'} * (\overline{Z}_{tt}^\epsilon - b^\epsilon \overline{Z}_{t,\alpha'}^\epsilon). \tag{6.52}$$

We rewrite

$$K_{y'} * (\overline{Z}_{tt}^\epsilon - b^\epsilon \overline{Z}_{t,\alpha'}^\epsilon) = K_{y'} * \overline{Z}_{tt}^\epsilon - (\partial_{x'} K_{y'}) * (b^\epsilon \overline{Z}_t^\epsilon) + K_{y'} * (b_{\alpha'}^\epsilon \overline{Z}_t^\epsilon). \tag{6.53}$$

Now we apply (6.34), (6.35), (6.39) and Lemma 6.3 to each term on the right hand side of (6.53). We can conclude that U is continuously differentiable with respect to t, and

$$U_t^\epsilon \Rightarrow U_t \qquad \text{on } \mathscr{P}_- \times [0, T_0] \tag{6.54}$$

as $\epsilon \to 0$.

Step 3. The limit of $\mathfrak{P}^\epsilon$

By the calculation in Sect. 2.5, we know there is a real valued function $\mathfrak{P}^\epsilon$, such that

$$\Psi_{z'}^\epsilon U_t^\epsilon - \Psi_t^\epsilon U_{z'}^\epsilon + \overline{U}^\epsilon U_{z'}^\epsilon - i\Psi_{z'}^\epsilon = -(\partial_{x'} - i\partial_{y'})\mathfrak{P}^\epsilon, \qquad \text{in } \mathscr{P}_-; \tag{6.55}$$

and

$$\mathfrak{P}^\epsilon = constant, \qquad \text{on } \partial\mathscr{P}_-. \tag{6.56}$$

Without loss of generality we take the $constant = 0$. We now explore a few other properties of $\mathfrak{P}^\epsilon$. Moving $\overline{U}^\epsilon U_{z'}^\epsilon = \partial_{z'}(\overline{U}^\epsilon U^\epsilon)$ to the right of (6.55) gives

$$\Psi_{z'}^\epsilon U_t^\epsilon - \Psi_t^\epsilon U_{z'}^\epsilon - i\Psi_{z'}^\epsilon = -(\partial_{x'} - i\partial_{y'})(\mathfrak{P}^\epsilon + \frac{1}{2}|U^\epsilon|^2), \qquad \text{in } \mathscr{P}_-; \tag{6.57}$$

Applying $(\partial_{x'} + i\partial_{y'}) = 2\overline{\partial}_{z'}$ to (6.57) yields

$$-\Delta(\mathfrak{P}^\epsilon + \frac{1}{2}|U^\epsilon|^2) = 0, \qquad \text{in } \mathscr{P}_-. \tag{6.58}$$

So $\mathfrak{P}^\epsilon + \frac{1}{2}|U^\epsilon|^2$ is a harmonic function on $\mathscr{P}_-$ with boundary value $\frac{1}{2}|\overline{Z}_t^\epsilon|^2$. On the other hand, it is easy to check that $\lim_{y'\to-\infty}(\Psi_{z'}^\epsilon U_t^\epsilon - \Psi_t^\epsilon U_{z'}^\epsilon - i\Psi_{z'}^\epsilon) = -i$. Therefore

$$\mathfrak{P}^\epsilon(z', t) = -\frac{1}{2}|U^\epsilon(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}_t^\epsilon|^2)(x', t). \tag{6.59}$$

By (6.41), (6.38) and Lemma 6.5,

$$\mathfrak{P}^\epsilon(z', t) \Rightarrow -\frac{1}{2}|U(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}_t|^2)(x', t), \qquad \text{on } \overline{\mathscr{P}}_- \times [0, T_0] \tag{6.60}$$

as $\epsilon \to 0$. We write

$$\mathfrak{P} := -\frac{1}{2}|U(z', t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}_t|^2)(x', t).$$

We have $\mathfrak{P}$ is continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$ with $\mathfrak{P} \in C([0, T_0], C^\infty(\mathscr{P}_-))$, and

$$\mathfrak{P} = 0, \qquad \text{on } \partial\mathscr{P}_-. \tag{6.61}$$

Moreover, since $K_{y'} * (|\overline{Z}_t^\epsilon|^2)(x', t)$ is harmonic on $\mathscr{P}_-$, by interior derivative estimate for harmonic functions and by (6.41),

$$(\partial_{x'} - i\partial_{y'})\mathfrak{P}^\epsilon \Rightarrow (\partial_{x'} - i\partial_{y'})\mathfrak{P} \qquad \text{on } \mathscr{P}_- \times [0, T_0] \tag{6.62}$$

as $\epsilon \to 0$. And by (6.59) and a similar argument as that in (6.52)–(6.54), we have that $\mathfrak{P}$ is continuously differentiable with respect to t and

$$\partial_t\mathfrak{P}^\epsilon \Rightarrow \partial_t\mathfrak{P} \qquad \text{on } \mathscr{P}_- \times [0, T_0] \tag{6.63}$$

as $\epsilon \to 0$.

Step 4. Conclusion

We now sum up Steps 1–3. We have shown that there are functions $\Psi(\cdot, t)$ and $U(\cdot, t)$, holomorphic on $\mathscr{P}_-$ for each fixed $t \in [0, T_0]$, continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$, and continuous differentiable on $\mathscr{P}_- \times [0, T_0]$, with $\frac{1}{\Psi_{z'}}$ continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$, such that $\Psi^\epsilon \to \Psi$, $\frac{1}{\Psi_{z'}^\epsilon} \to \frac{1}{\Psi_{z'}}$, $U^\epsilon \to U$ uniform on compact subsets of $\overline{\mathscr{P}}_- \times [0, T_0]$, $\Psi_t^\epsilon \to \Psi_t$, $\Psi_{z'}^\epsilon \to \Psi_{z'}$, $U_{z'}^\epsilon \to U_{z'}$ and

$U_t^\epsilon \to U_t$ uniform on compact subsets of $\mathscr{P}_- \times [0, T_0]$, as $\epsilon \to 0$. We have also shown that there is $\mathfrak{P}$, continuous on $\overline{\mathscr{P}}_- \times [0, T_0]$ with $\mathfrak{P} = 0$ on $\partial\mathscr{P}_-$, and continuous differentiable on $\mathscr{P}_- \times [0, T_0]$, such that $\mathfrak{P}^\epsilon \to \mathfrak{P}$ uniform on compact subsets of $\bar{\mathscr{P}}_- \times [0, T_0]$ and, $(\partial_{x'} - i\partial_{y'})\mathfrak{P}^\epsilon \to (\partial_{x'} - i\partial_{y'})\mathfrak{P}$ and $\partial_t\mathfrak{P}^\epsilon \to \partial_t\mathfrak{P}$ uniformly on compact subsets of $\mathscr{P}_- \times [0, T_0]$, as $\epsilon \to 0$. Let $\epsilon \to 0$ in Eq. (6.55), we have

$$\Psi_{z'}U_t - \Psi_t U_{z'} + \overline{U}U_{z'} - i\Psi_{z'} = -(\partial_{x'} - i\partial_{y'})\mathfrak{P}, \qquad \text{on } \mathscr{P}_- \times [0, T_0]. \tag{6.64}$$

This shows that $(U, \Psi, \mathfrak{P})$ is a solution of the Cauchy problem for the system (2.31)–(2.32)–(2.33) in the sense of Definition 3.8. Furthermore because of (6.25), (6.27), letting $\epsilon \to 0$ gives

$$\sup_{[0,T_0]} \mathcal{E}(t) \le M(\mathcal{E}(0)) < \infty. \tag{6.65}$$

and

$$\sup_{[0,T_0]} \{\sup_{y'<0} \|U(x' + iy', t)\|_{L^2(\mathbb{R},dx')} + \sup_{y'<0} \|\frac{1}{\Psi_{z'}(x' + iy', t)} - 1\|_{L^2(\mathbb{R},dx')}\} < C_0 < \infty. \tag{6.66}$$

By the argument at the end of Sect. 2.5, if $\Sigma(t) := \{Z = \Psi(\alpha', t) \,|\, \alpha' \in \mathbb{R}\}$ is a Jordan curve, then $\Psi(\cdot, t) : \mathscr{P}_- \to \Omega(t)$, where $\Omega(t)$ is the domain bounded from the above by $\Sigma(t)$, is invertible; and the solution $(U, \Psi, \mathfrak{P})$ gives rise to a solution $(\overline{\mathbf{v}}, P) := (U \circ \Psi^{-1}, \mathfrak{P} \circ \Psi^{-1})$ of the water wave equation (1.1). This finishes the proof for part 1 of Theorem 3.9.

6.3 The chord–arc interfaces

Now assume at time $t = 0$, the interface $Z = \Psi(\alpha', 0) := Z(\alpha', 0), \alpha' \in \mathbb{R}$ is chord–arc, that is, there is $0 < \delta < 1$, such that

$$\delta \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma, 0)|\, d\gamma \le |Z(\alpha', 0) - Z(\beta', 0)| \le \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma, 0)|\, d\gamma, \quad \forall -\infty < \alpha' < \beta' < \infty.$$

We want to show there is $T_1 > 0$, depending only on $\mathcal{E}(0)$, such that for $t \in [0, \min\{T_0, \frac{\delta}{T_1}\}]$, the interface $Z = Z(\alpha', t) := \Psi(\alpha', t)$ remains chord–arc. We begin with

$$-z^\epsilon(\alpha,t)+z^\epsilon(\beta,t)+z^\epsilon(\alpha,0)-z^\epsilon(\beta,0)=\int_0^t\int_\alpha^\beta z^\epsilon_{t\alpha}(\gamma,s)\,d\gamma\,ds \tag{6.67}$$

for $\alpha<\beta$. Because

$$\frac{d}{dt}|z^\epsilon_\alpha|^2=2|z^\epsilon_\alpha|^2\,\mathrm{Re}\,D_\alpha z^\epsilon_t, \tag{6.68}$$

by Gronwall's inequality, for $t\in[0,T_0]$,

$$|z^\epsilon_\alpha(\alpha,t)|^2\le|z^\epsilon_\alpha(\alpha,0)|^2e^{2\int_0^t|D_\alpha z^\epsilon_t(\alpha,\tau)|\,d\tau}; \tag{6.69}$$

so

$$|z^\epsilon_{t\alpha}(\alpha,t)|\le|z^\epsilon_\alpha(\alpha,0)||D_\alpha z^\epsilon_t(\alpha,t)|e^{\int_0^t|D_\alpha z^\epsilon_t(\alpha,\tau)|\,d\tau}; \tag{6.70}$$

by Appendix C, (6.25) and Proposition 3.3,

$$\sup_{[0,T_0]}|z^\epsilon_{t\alpha}(\alpha,t)|\le|z^\epsilon_\alpha(\alpha,0)|C(\mathcal{E}(0)); \tag{6.71}$$

therefore for $t\in[0,T_0]$,

$$\int_0^t\int_\alpha^\beta|z^\epsilon_{t\alpha}(\gamma,s)|\,d\gamma\,ds\le tC(\mathcal{E}(0))\int_\alpha^\beta|z^\epsilon_\alpha(\gamma,0)|\,d\gamma. \tag{6.72}$$

Now $z^\epsilon(\alpha,0)=Z^\epsilon(\alpha,0)=\Psi(\alpha-\epsilon i,0)$. Because $Z_{,\alpha'}(\cdot,0)\in L^1_{loc}(\mathbb{R})$, and $Z_{,\alpha'}(\cdot,0)-1\in H^1(\mathbb{R}\backslash[-N,N])$ for some large N,

$$\overline{\lim_{\epsilon\to0}}\int_\alpha^\beta|\Psi_{z'}(\gamma-\epsilon i,0)|\,d\gamma\le\int_\alpha^\beta|Z_{,\alpha'}(\gamma,0)|\,d\gamma. \tag{6.73}$$

Let $\epsilon\to0$ in (6.67). We get, for $t\in[0,T_0]$,

$$||z(\alpha,t)-z(\beta,t)|-|Z(\alpha,0)-Z(\beta,0)||\le tC(\mathcal{E}_1(0))\int_\alpha^\beta|Z_{,\alpha'}(\gamma,0)|\,d\gamma, \tag{6.74}$$

hence for all $\alpha<\beta$ and $0\le t\le\min\{T_0,\frac{\delta}{2C(\mathcal{E}(0))}\}$,

$$\frac{1}{2}\delta\int_\alpha^\beta|Z_{,\alpha'}(\gamma,0)|\,d\gamma\le|z(\alpha,t)-z(\beta,t)|\le2\int_\alpha^\beta|Z_{,\alpha'}(\gamma,0)|\,d\gamma. \tag{6.75}$$

This show that for $\leq t \leq \min\{T_0, \frac{\delta}{2C(\mathcal{E}(0))}\}$, $z = z(\cdot, t)$ is absolute continuous on compact intervals of $\mathbb{R}$, with $z_\alpha(\cdot, t) \in L^1_{loc}(\mathbb{R})$, and is chord–arc. So $\Sigma(t) = \{z(\alpha, t) \mid \alpha \in \mathbb{R}\}$ is Jordan. This finishes the proof of Theorem 3.9.

Appendix A. Basic analysis preparations

We present in this section some basic analysis results that will be used in this paper. First we have, as a consequence of the fact that product of holomorphic functions is holomorphic, the following identity.

Proposition A.1 *Assume that $f,\ g \in L^2(\mathbb{R})$. Assume either both f, g are holomorphic: $f = \mathbb{H}f$, $g = \mathbb{H}g$, or both are anti-holomorphic: $f = -\mathbb{H}f$, $g = -\mathbb{H}g$. Then*

$$[f, \mathbb{H}]g = 0. \tag{A.1}$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a function in $\dot{H}^{1/2}(\mathbb{R})$, we define

$$\begin{aligned} \|f\|^2_{\dot{H}^{1/2}} = \|f\|^2_{\dot{H}^{1/2}(\mathbb{R})} &:= \int i\mathbb{H}\partial_x f(x)\overline{f}(x)\,dx \\ &= \frac{1}{2\pi}\iint \frac{|f(x) - f(y)|^2}{(x-y)^2}\,dx\,dy. \end{aligned} \tag{A.2}$$

We have the following results on $\dot{H}^{1/2}$ norms and $\dot{H}^{1/2}$ functions.

Lemma A.2 *For any function $f \in \dot{H}^{1/2}(\mathbb{R})$,*

$$\|f\|^2_{\dot{H}^{1/2}} = \|\mathbb{P}_H f\|^2_{\dot{H}^{1/2}} + \|\mathbb{P}_A f\|^2_{\dot{H}^{1/2}}; \tag{A.3}$$

$$\int i\,\partial_{\alpha'} f\ \overline{f}\,d\alpha' = \|\mathbb{P}_H f\|^2_{\dot{H}^{1/2}} - \|\mathbb{P}_A f\|^2_{\dot{H}^{1/2}}. \tag{A.4}$$

Proof Lemma A.2 is an easy consequence of the decomposition $f = \mathbb{P}_H f + \mathbb{P}_A f$, the definition (A.2) and the Cauchy integral Theorem. We omit the details. □

Proposition A.3 *Let $f,\ g \in C^1(\mathbb{R})$. Then*

$$\|fg\|_{\dot{H}^{1/2}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{H}^{1/2}} + \|g\|_{L^\infty}\|f\|_{\dot{H}^{1/2}}; \tag{A.5}$$

$$\|g\|_{\dot{H}^{1/2}} \lesssim \|f^{-1}\|_{L^\infty}(\|fg\|_{\dot{H}^{1/2}} + \|f'\|_{L^2}\|g\|_{L^2}). \tag{A.6}$$

The proof of (A.5) is straightforward from the definition of $\dot{H}^{1/2}$, and the proof of (A.6) is straightforward from the definition of $\dot{H}^{1/2}$ and the Hardy's inequality[28]. We omit the details.

We next present the basic estimates we will rely on for this paper. We start with the Sobolev inequality.

Proposition A.4 (Sobolev inequality) *Let $f \in C_0^1(\mathbb{R})$. Then*

$$\|f\|_{L^\infty}^2 \le 2\|f\|_{L^2}\|f'\|_{L^2}. \tag{A.7}$$

Proposition A.5 (Hardy's inequalities) *Let $f \in C^1(\mathbb{R})$, with $f' \in L^2(\mathbb{R})$. Then there exists $C > 0$ independent of f such that for any $x \in \mathbb{R}$,*

$$\left| \int \frac{(f(x) - f(y))^2}{(x-y)^2} dy \right| \le C \left\| f' \right\|_{L^2}^2 ; \tag{A.8}$$

and

$$\iint \frac{|f(x) - f(y)|^4}{|x-y|^4} dxdy \le C \left\| f' \right\|_{L^2}^4 . \tag{A.9}$$

Let $H \in C^1(\mathbb{R}; \mathbb{R}^d)$, $A_i \in C^1(\mathbb{R})$, $i = 1, \dots m$, and $F \in C^\infty(\mathbb{R})$. Define

$$C_1(A_1, \dots, A_m, f)(x) = \text{pv.} \int F\left(\frac{H(x) - H(y)}{x-y}\right) \frac{\Pi_{i=1}^m (A_i(x) - A_i(y))}{(x-y)^{m+1}} f(y)\, dy. \tag{A.10}$$

Proposition A.6 *There exist constants $c_1 = c_1(F, \|H'\|_{L^\infty})$, $c_2 = c_2(F, \|H'\|_{L^\infty})$, such that*

1. For any $f \in L^2$, $A_i' \in L^\infty$, $1 \le i \le m$,

$$\|C_1(A_1, \dots, A_m, f)\|_{L^2} \le c_1 \|A_1'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^2}. \tag{A.11}$$

2. For any $f \in L^\infty$, $A_i' \in L^\infty$, $2 \le i \le m$, $A_1' \in L^2$,

$$\|C_1(A_1, \dots, A_m, f)\|_{L^2} \le c_2 \|A_1'\|_{L^2} \|A_2'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^\infty}. \tag{A.12}$$

(A.11) is a result of Coifman, McIntosh and Meyer [14]. (A.12) is a consequence of the Tb Theorem, a proof is given in [40].

[28] First write $g(x) - g(y) = f(x)^{-1}(f(x)g(x) - f(y)g(y)) + g(y)f(x)^{-1}(f(y) - f(x))$.

Let H, A_i F satisfy the same assumptions as in (A.10). Define

$$C_2(A, f)(x) = \int F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{\Pi_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^m} \partial_y f(y)\, dy. \tag{A.13}$$

We have the following inequalities.

Proposition A.7 *There exist constants c_3, c_4 and c_5, depending on F and $\|H'\|_{L^\infty}$, such that*

1. For any $f \in L^2$, $A_i' \in L^\infty$, $1 \le i \le m$,

$$\|C_2(A, f)\|_{L^2} \le c_3 \|A_1'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^2}. \tag{A.14}$$

2. For any $f \in L^\infty$, $A_i' \in L^\infty$, $2 \le i \le m$, $A_1' \in L^2$,

$$\|C_2(A, f)\|_{L^2} \le c_4 \|A_1'\|_{L^2} \|A_2'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^\infty}. \tag{A.15}$$

3. For any $f' \in L^2$, $A_1 \in L^\infty$, $A_i' \in L^\infty$, $2 \le i \le m$,

$$\|C_2(A, f)\|_{L^2} \le c_5 \|A_1\|_{L^\infty} \|A_2'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f'\|_{L^2}. \tag{A.16}$$

Using integration by parts, the operator $C_2(A, f)$ can be easily converted into a sum of operators of the form $C_1(A, f)$. (A.14) and (A.15) follow from (A.11) and (A.12). To get (A.16), we rewrite $C_2(A, f)$ as the difference of the two terms $A_1 C_1(A_2, \dots, A_m, f')$ and $C_1(A_2, \dots, A_m, A_1 f')$ and apply (A.11) to each term.

Proposition A.8 *There exists a constant $C > 0$ such that for any f, g, m smooth and decays fast at infinity,*

$$\|[f, \mathbb{H}]g\|_{L^2} \le C \|f\|_{\dot{H}^{1/2}} \|g\|_{L^2}; \tag{A.17}$$

$$\|[f, \mathbb{H}]g\|_{L^\infty} \le C \left\|f'\right\|_{L^2} \|g\|_{L^2}; \tag{A.18}$$

$$\|[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \le C \left\|f'\right\|_{L^2} \|g\|_{\dot{H}^{1/2}}; \tag{A.19}$$

$$\|[f, m; \partial_{\alpha'} g]\|_{L^2} \le C \left\|f'\right\|_{L^2} \left\|m'\right\|_{L^\infty} \|g\|_{\dot{H}^{1/2}}. \tag{A.20}$$

Here $[f, g; h]$ is as given in (1.5). (A.17) is straightforward by Cauchy–Schwarz inequality and the definition of $\dot{H}^{1/2}$. (A.18) is straightforward from Cauchy–Schwarz inequality and Hardy's inequality (A.8). (A.19) and (A.20) follow from integration by parts, then Cauchy–Schwarz inequality, Hardy's inequality (A.8), and the definition of $\dot{H}^{1/2}$.

Proposition A.9 *There exists a constant $C > 0$ such that for any f, g, h, smooth and decay fast at spatial infinity,*

$$\|[f, g; h]\|_{L^2} \le C \left\| f' \right\|_{L^2} \left\| g' \right\|_{L^2} \|h\|_{L^2}; \tag{A.21}$$

$$\|[f, g; h]\|_{L^\infty} \le C \left\| f' \right\|_{L^2} \left\| g' \right\|_{L^\infty} \|h\|_{L^2}; \tag{A.22}$$

$$\|[f, g; h]\|_{L^\infty} \le C \left\| f' \right\|_{L^2} \left\| g' \right\|_{L^2} \|h\|_{L^\infty}. \tag{A.23}$$

(A.21) follows directly from Cauchy–Schwarz inequality, Hardy's inequality (A.8) and Fubini Theorem; (A.22) follows from Cauchy–Schwarz inequality, Hardy's inequality (A.8) and the mean value Theorem; (A.23) follows from Cauchy–Schwarz inequality and Hardy's inequality (A.8).

Appendix B. Identities

B.1. Commutator identities

We include here various commutator identities that are necessary for the proofs. The first set: (B.1)–(B.5) has already appeared in [24].

$$[\partial_t, D_\alpha] = -(D_\alpha z_t) D_\alpha; \tag{B.1}$$

$$\left[\partial_t, D_\alpha^2\right] = -2(D_\alpha z_t) D_\alpha^2 - (D_\alpha^2 z_t) D_\alpha; \tag{B.2}$$

$$\left[\partial_t^2, D_\alpha\right] = (-D_\alpha z_{tt}) D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t) D_\alpha \partial_t; \tag{B.3}$$

$$\left[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha\right] = (-2D_\alpha z_{tt}) D_\alpha - 2(D_\alpha z_t)\partial_t D_\alpha; \tag{B.4}$$

and

$$\begin{aligned}\left[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), D_\alpha^2\right] = {} & (-4D_\alpha z_{tt}) D_\alpha^2 + 6(D_\alpha z_t)^2 D_\alpha^2 - (2D_\alpha^2 z_{tt}) D_\alpha \\ & + 6(D_\alpha z_t)(D_\alpha^2 z_t) D_\alpha - 2(D_\alpha^2 z_t) D_\alpha \partial_t - 4(D_\alpha z_t) D_\alpha^2 \partial_t.\end{aligned} \tag{B.5}$$

Let

$$\mathcal{P} := (\partial_t + b\partial_{\alpha'})^2 + i\mathcal{A}\partial_{\alpha'}.$$

Notice that $U_h^{-1}\partial_t U_h = \partial_t + b\partial_{\alpha'}$, $U_h^{-1} D_\alpha U_h = D_{\alpha'}$ and $\mathcal{P} = U_h^{-1}(\partial_t^2 + i\mathfrak{a}\partial_\alpha)U_h$, we precompose with h^{-1} to Eqs. (B.1)–(B.5), and get

$$[\partial_t + b\partial_{\alpha'}, D_{\alpha'}] = -(D_{\alpha'} Z_t) D_{\alpha'}; \tag{B.6}$$

$$\left[\partial_t + b\partial_{\alpha'}, D_{\alpha'}^2\right] = -2(D_{\alpha'} Z_t) D_{\alpha'}^2 - (D_{\alpha'}^2 Z_t) D_{\alpha'}; \tag{B.7}$$

$$\left[(\partial_t + b\partial_{\alpha'})^2, D_{\alpha'}\right] = (-D_{\alpha'}Z_{tt})D_{\alpha'} + 2(D_{\alpha'}Z_t)^2 D_{\alpha'} - 2(D_{\alpha'}Z_t)D_{\alpha'}(\partial_t + b\partial_{\alpha'}); \tag{B.8}$$

$$[\mathcal{P}, D_{\alpha'}] = (-2D_{\alpha'}Z_{tt})D_{\alpha'} - 2(D_{\alpha'}Z_t)(\partial_t + b\partial_{\alpha'})D_{\alpha'}; \tag{B.9}$$

and

$$\begin{aligned}\left[\mathcal{P}, D_{\alpha'}^2\right] = {} & (-4D_{\alpha'}Z_{tt})D_{\alpha'}^2 + 6(D_{\alpha'}Z_t)^2 D_{\alpha'}^2 - (2D_{\alpha'}^2 Z_{tt})D_{\alpha'} \\ & + 6(D_{\alpha'}Z_t)(D_{\alpha'}^2 Z_t)D_{\alpha'} - 2(D_{\alpha'}^2 Z_t)D_{\alpha'}(\partial_t + b\partial_{\alpha'}) \\ & - 4(D_{\alpha'}Z_t)D_{\alpha'}^2(\partial_t + b\partial_{\alpha'}).\end{aligned} \tag{B.10}$$

We need some additional commutator identities. In general, for operators A, B and C,

$$[A, BC^k] = [A, B]C^k + B[A, C^k] = [A, B]C^k + \sum_{i=1}^{k} BC^{i-1}[A, C]C^{k-i}. \tag{B.11}$$

We have

$$[\partial_t + b\partial_{\alpha'}, \partial_{\alpha'}]f = -b_{\alpha'}\partial_{\alpha'}f; \tag{B.12}$$

$$[(\partial_t + b\partial_{\alpha'})^2, \partial_{\alpha'}]f = -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}f) - b_{\alpha'}\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})f; \tag{B.13}$$

$$[i\mathcal{A}\partial_{\alpha'}, \partial_{\alpha'}]f = -i\mathcal{A}_{\alpha'}\partial_{\alpha'}f; \tag{B.14}$$

$$[\mathcal{P}, \partial_{\alpha'}]f = -(\partial_t + b\partial_{\alpha'})(b_{\alpha'}\partial_{\alpha'}f) - b_{\alpha'}\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})f - i\mathcal{A}_{\alpha'}\partial_{\alpha'}f; \tag{B.15}$$

$$[\partial_t + b\partial_{\alpha'}, \partial_{\alpha'}^2]f = -\partial_{\alpha'}(b_{\alpha'}\partial_{\alpha'}f) - b_{\alpha'}\partial_{\alpha'}^2 f. \tag{B.16}$$

Here (B.13), (B.16) are obtained by (B.11) and (B.12). We also have

$$[\partial_t + b\partial_{\alpha'}, \mathbb{H}] = [b, \mathbb{H}]\partial_{\alpha'} \tag{B.17}$$

We compute

$$\begin{aligned}&(\partial_t + b\partial_{\alpha'})[f, \mathbb{H}]g \\ &\quad = [(\partial_t + b\partial_{\alpha'})f, \mathbb{H}]g + [f, [\partial_t + b\partial_{\alpha'}, \mathbb{H}]]\, g + [f, \mathbb{H}](\partial_t + b\partial_{\alpha'})g \\ &\quad = [(\partial_t + b\partial_{\alpha'})f, \mathbb{H}]g + [f, [b, \mathbb{H}]\, \partial_{\alpha'}]\, g + [f, \mathbb{H}](\partial_t + b\partial_{\alpha'})g \\ &\quad = [(\partial_t + b\partial_{\alpha'})f, \mathbb{H}]g + [f, \mathbb{H}]\,((\partial_t + b\partial_{\alpha'})g + b_{\alpha'}g) \\ &\qquad + [f, [b, \mathbb{H}]]\, \partial_{\alpha'}g - [b, \mathbb{H}](f_{\alpha'}g) - [f, \mathbb{H}](b_{\alpha'}g).\end{aligned}$$

It can be checked easily, by integration by parts, that

$$[f, [b, \mathbb{H}]]\, \partial_{\alpha'}g - [b, \mathbb{H}](f_{\alpha'}g) - [f, \mathbb{H}](b_{\alpha'}g) = -[f, b; g].$$

So

$$(\partial_t + b\partial_{\alpha'})[f, \mathbb{H}]g = [(\partial_t + b\partial_{\alpha'})f, \mathbb{H}]g + [f, \mathbb{H}]((\partial_t + b\partial_{\alpha'})g + b_{\alpha'}g) - [f, b; g]; \tag{B.18}$$

with an application of (B.12) yields

$$(\partial_t + b\partial_{\alpha'})[f, \mathbb{H}]\partial_{\alpha'}g = [(\partial_t + b\partial_{\alpha'})f, \mathbb{H}]\partial_{\alpha'}g + [f, \mathbb{H}]\partial_{\alpha'}(\partial_t + b\partial_{\alpha'})g - [f, b; \partial_{\alpha'}g]. \tag{B.19}$$

We compute, by (B.17), (B.11) and (B.19) that

$$\begin{aligned}\left[(\partial_t + b\partial_{\alpha'})^2, \mathbb{H}\right] f &= (\partial_t + b\partial_{\alpha'})\,[b, \mathbb{H}]\,\partial_{\alpha'} f + [b, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'}) f \\ &= [(\partial_t + b\partial_{\alpha'})b, \mathbb{H}]\,\partial_{\alpha'} f \\ &\quad + 2\,[b, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'}) f - [b, b; \partial_{\alpha'} f].\end{aligned} \tag{B.20}$$

We also have

$$[i\mathcal{A}\partial_{\alpha'}, \mathbb{H}]\, f = [i\mathcal{A}, \mathbb{H}]\,\partial_{\alpha'} f. \tag{B.21}$$

Sum up (B.20) and (B.21) yields

$$\begin{aligned}[\mathcal{P}, \mathbb{H}]\, f &= [(\partial_t + b\partial_{\alpha'})b, \mathbb{H}]\,\partial_{\alpha'} f \\ &\quad + 2\,[b, \mathbb{H}]\,\partial_{\alpha'}(\partial_t + b\partial_{\alpha'}) f - [b, b; \partial_{\alpha'} f] + [i\mathcal{A}, \mathbb{H}]\,\partial_{\alpha'} f.\end{aligned} \tag{B.22}$$

We have, by product rules, that

$$\begin{aligned}\left[(\partial_t + b\partial_{\alpha'})^2, \frac{1}{Z_{,\alpha'}}\right] f &= (\partial_t + b\partial_{\alpha'})^2 \left(\frac{1}{Z_{,\alpha'}}\right) f \\ &\quad + 2(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}}\right) (\partial_t + b\partial_{\alpha'}) f;\end{aligned} \tag{B.23}$$

and

$$\left[i\mathcal{A}\partial_{\alpha'}, \frac{1}{Z_{,\alpha'}}\right] f = i\mathcal{A}\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}}\right) f; \tag{B.24}$$

so

$$\left[\mathcal{P}, \frac{1}{Z_{,\alpha'}}\right] f = (\partial_t + b\partial_{\alpha'})^2 \left(\frac{1}{Z_{,\alpha'}}\right) f + 2(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}}\right) (\partial_t + b\partial_{\alpha'}) f + i\mathcal{A}\partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}}\right) f. \tag{B.25}$$

And we compute, by (2.35),

$$(\partial_t + b\partial_{\alpha'}) \left(\frac{1}{Z_{,\alpha'}}\right) = \frac{1}{Z_{,\alpha'}} (b_{\alpha'} - D_{\alpha'} Z_t); \tag{B.26}$$

$$(\partial_t + b\partial_{\alpha'})^2 \left(\frac{1}{Z_{,\alpha'}}\right) = \frac{1}{Z_{,\alpha'}} (b_{\alpha'} - D_{\alpha'} Z_t)^2 + \frac{1}{Z_{,\alpha'}} (\partial_t + b\partial_{\alpha'})(b_{\alpha'} - D_{\alpha'} Z_t). \tag{B.27}$$

Appendix C. Main quantities controlled by $\mathfrak{E}$

We have shown in Sect. 4.1.1 that the following quantities are controlled by a polynomial of $\mathfrak{E}$ (or equivalently by $\mathcal{E}$):

$$\begin{aligned} &\|D_{\alpha'} \overline{Z}_t\|_{\dot{H}^{1/2}}, \quad \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t \right\|_{\dot{H}^{1/2}}, \quad \|\overline{Z}_{t,\alpha'}\|_{L^2}, \quad \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \\ &\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \quad \left| \frac{1}{Z_{,\alpha'}}(0,t) \right|, \\ &\|A_1\|_{L^\infty}, \quad \|b_{\alpha'}\|_{L^\infty}, \quad \|D_{\alpha'} Z_t\|_{L^\infty}, \quad \|D_{\alpha'} Z_{tt}\|_{L^\infty}, \quad \|(\partial_t + b\partial_{\alpha'}) \overline{Z}_{t,\alpha'}\|_{L^2} \\ &\|Z_{tt,\alpha'}\|_{L^2}, \quad \|D_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2}, \quad \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \quad \|(\partial_t + b\partial_{\alpha'}) D_{\alpha'}^2 \overline{Z}_t\|_{L^2}, \quad \|D_{\alpha'}^2 Z_t\|_{L^2}. \end{aligned} \tag{C.1}$$

In the remainder of Sect. 4.1 we have controlled the following quantities by a polynomila of $\mathfrak{E}$ (or equivalently by $\mathcal{E}$):

$$\begin{aligned} &\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right\|_{L^\infty}, \quad \|(\partial_t + b\partial_{\alpha'}) A_1\|_{L^\infty}, \quad \|\mathcal{A}_{\alpha'}\|_{L^\infty}, \quad \left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}, \\ &\left\| \partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \quad \left\| (\partial_t + b\partial_{\alpha'}) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \\ &\|\partial_{\alpha'} (\partial_t + b\partial_{\alpha'}) b\|_{L^\infty}, \quad \|(\partial_t + b\partial_{\alpha'}) b_{\alpha'}\|_{L^\infty}, \\ &\|(\partial_t + b\partial_{\alpha'}) D_{\alpha'} Z_t\|_{L^\infty}, \quad \left\| D_{\alpha'} \left(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} \right) \right\|_{L^2}. \end{aligned} \tag{C.2}$$

As a consequence of (4.70) and (C.1) we have

$$\|D_{\alpha'} b_{\alpha'}\|_{L^2} \lesssim C(\mathfrak{E}).$$

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