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Almost global wellposedness of the 2-D full water wave problem

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Abstract We consider the problem of global in time existence and uniqueness of solutions of the 2-D infinite depth full water wave equation. It is known that this equation has a solution for a time period $[0, T/\epsilon]$ for initial data of the form $\epsilon\Psi$, where T depends only on Ψ. In this paper, we show that for such data there exists a unique solution for a time period $[0, e^{T/\epsilon}]$. This is achieved by better understandings of the nature of the nonlinearity of the full water wave equation.

1 Introduction

The mathematical problem of n-dimensional water wave concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{k} - \nabla P \quad \text{on } \Omega(t),\ t \geq 0, \tag{1.1}$$

$$\operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, \quad \text{on } \Omega(t),\ t \geq 0, \tag{1.2}$$

$$P = 0, \quad \text{on } \Sigma(t) \tag{1.3}$$

$$(1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \tag{1.4}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. It is well-known that when surface tension is neglected, the motion of the interface between an inviscid fluid and vacuum can

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be subject to Taylor instability [2, 3, 36]. Assume that the free interface $\Sigma(t)$ is described by $z = z(\alpha, t)$, where $\alpha \in R^{n-1}$ is the Lagrangian coordinate, i.e. $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the fluid velocity on the interface, $z_{tt}(\alpha, t) = (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v})(z(\alpha, t), t)$ is the acceleration. Let $\mathbf{n}$ be the unit normal pointing out of $\Omega(t)$. The Taylor sign condition relating to Taylor instability is

$$-\frac{\partial P}{\partial \mathbf{n}} = (z_{tt} + \mathbf{k}) \cdot \mathbf{n} \geq c_0 > 0, \tag{1.5}$$

point-wisely on the interface for some positive constant c_0. In previous works [38, 39], we showed that the Taylor sign condition (1.5) always holds for the n-dimensional infinite depth water wave problem (1.1)–(1.4), $n \geq 2$, as long as the interface is nonself-intersecting. In other words, the motion of infinite depth water wave is not subject to Taylor instability. Furthermore we showed that the initial value problem of the full nonlinear water wave system (1.1)–(1.4) is uniquely solvable *locally* in time in Sobolev spaces, for any initially nonself-intersecting interfaces and incompressible irrotational velocities. Earlier Nalimov [27], and Yosihara [40] obtained the existence and uniqueness of solutions *locally* in time in Sobolev classes, for two dimensional water wave of infinite and finite depths, under the assumption that the initial interface and velocity is a small perturbation of the still water, and the bottom is a small perturbation of the flat horizontal one. There has been much work recently concerning water waves with additional effects such as a nonzero surface tension, a bottom, and a bounded nonzero vorticity in the fluid region [1, 6, 7, 14, 24, 26, 28, 31, 41]. Energy estimates were established and *local* in time existence in Sobolev spaces were proved for water wave motion with these additional effects. However the question concerning the solution behaviors globally in time remains open.

We study the global in time behavior of solutions of the infinite depth water wave system (1.1)–(1.4), assuming that the initial velocity and acceleration are small.

Notice that for given initial velocity and acceleration of the form $\epsilon\Psi(\cdot)$, where ϵ is small, Ψ is a smooth vector-valued function decay fast at spatial infinity, the same energy estimates and analysis in [38, 39] gives a long time existence of classical solutions for a time period $[0, T/\epsilon]$, with T depending only on Ψ. Indeed this is also one of the key elements used in justifying asymptotic models of the water wave equation. In [11, 29], Craig [11] and Schneider and Wayne [29] proved the existence of classical solutions up to time T/ϵ^3 for rescaled initial data $\epsilon^2\Psi(\epsilon\cdot)$ in the case of 2D finite depth water wave, but most importantly they gave a rigorous justification of the KdV equation as a long wave limit of the full 2D water wave equation. Justification of some other asymptotic models can be found in [15, 17, 18, 25, etc.].

Therefore for initial velocity and acceleration of the form $\epsilon\Psi(\cdot)$, where ϵ is small, the question remains to be answered is what happens to the classical solution during the time period $[T/\epsilon, \infty)$? Notice that in previous works [38, 39] we used the Lagrangian coordinates. We approach the question of global well-posedness with two novel ideas. One idea is to explore the possibility of a better coordinate system. Another idea is to use the dispersive aspect of the water wave equation that hasn't been used so far.[1]

We focus on the infinite depth 2D water wave in this paper. We will study the 3D case in an upcoming paper, since it requires some different treatments in certain parts of the calculations.

[1] A preprint [5] on the local smoothing effect of surface tension appeared recently after the submission of this manuscript.

The main results of this paper are that we find some quantities Θ, that are the traces on the interface $\Sigma(t)$ of some holomorphic or almost holomorphic functions in the air region $\Omega(t)^c$, and a new coordinate system, such that in this coordinate system, these quantities satisfy equations of the form

$$\partial_t^2\Theta - i\partial_\alpha\Theta = G,$$

where G consists of nonlinear terms of only cubic and higher orders. Using these equations we are able to show that for data of the form $\epsilon\Psi(\cdot)$, there exists a unique classical solution for a time period $[0, e^{T/\epsilon}]$ for the 2D water wave system (1.1)–(1.4), with T depending only on Ψ. We give some more detailed explanations in the following.

We first recall some of the main ideas in proving the local in time well-posedness of the water wave system (1.1)–(1.4) [38, 39], while doing so we give a brief derivation of a quasilinear system that is equivalent to (1.1)–(1.4). This derivation is somewhat different from that in [38] but is the same as that in [39], while in [39] we only derived in the context of the 3D water wave.

Let $z = z(\alpha, t) = (x(\alpha, t), y(\alpha, t))$, $\alpha \in R$, be the equation of the free interface $\Sigma(t)$ at time t in Lagrangian coordinates α. In what follows, we regard the 2D space as a complex plane and use the same notation for a complex form $z = x + iy$ and a point $z = (x, y)$. So $\bar{z} = x - iy$. We write $[A, B] = AB - BA$.

As we know it is difficult to solve (1.1)–(1.4) directly since it is defined on moving domains. The first step used in [38, 39] in solving (1.1)–(1.4) was to reduce it to an equivalent equation defined on the moving interface $\Sigma(t)$. This was done based on the following observations. First notice that (1.2) implies that the complex conjugate of the velocity field $\bar{\mathbf{v}}$ is holomorphic in $\Omega(t)$. Therefore (1.2) is equivalent to $\bar{z}_t = \mathfrak{H}\bar{z}_t$, where

$$\mathfrak{H}f(\alpha, t) = \frac{1}{\pi i}\text{p.v.}\int \frac{f(\beta, t)z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)}\,d\beta \tag{1.6}$$

is the Hilbert transform on $\Sigma(t)$ associated with the parameterization $z = z(\alpha, t)$. Now (1.3) implies that on $\Sigma(t)$, ∇P points in the direction normal to $\Sigma(t)$, therefore $-\nabla P = i\mathfrak{a}z_\alpha$, with $\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}}\frac{1}{|z_\alpha|}$. So (1.1), (1.3) is equivalent to $z_{tt} + i = i\mathfrak{a}z_\alpha$, and the system (1.1)–(1.4) is equivalent to the following system on the interface $\Sigma(t)$:

$$\begin{cases} z_{tt} + i = i\mathfrak{a}z_\alpha, \\ \bar{z}_t = \mathfrak{H}\bar{z}_t. \end{cases} \tag{1.7}$$

However (1.7) is fully nonlinear. To solve (1.7), we further reduced it into a quasilinear equation [38, 39], on which the classical energy method could be applied. This was done by taking one derivative to t to the first equation in (1.7). Taking derivative to t to the first equation in (1.7), we got

$$\begin{cases} z_{ttt} - i\mathfrak{a}z_{t\alpha} = i\mathfrak{a}_t z_\alpha, \\ \bar{z}_t = \mathfrak{H}\bar{z}_t. \end{cases} \tag{1.8}$$

Using the fact that $\bar{z}_t = \mathfrak{H}\bar{z}_t$, we deduced

$$\begin{aligned} &(I - \mathfrak{H})(-i\mathfrak{a}_t\bar{z}_\alpha) \\ &\quad= (I - \mathfrak{H})(\bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha}) \\ &\quad= [\partial_t^2 + i\mathfrak{a}\partial_\alpha, \mathfrak{H}]\bar{z}_t \\ &\quad= 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 \bar{z}_{t\beta}\,d\beta. \end{aligned} \tag{1.9}$$

Further using the fact that $\mathfrak{a}$ and $\mathfrak{a}_t$ are real valued gave

$$(I+\mathfrak{K}^*)(\mathfrak{a}_t|z_\alpha|) \\ = \operatorname{Re}\left(\frac{iz_\alpha}{|z_\alpha|}\left\{2[z_t,\mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha}+2[z_{tt},\mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha}-\frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2\bar{z}_{t\beta}\,d\beta\right\}\right) \tag{1.10}$$

here $\operatorname{Re} z$ indicates the real part of z,

$$\mathfrak{K}^* f(\alpha,t) = \text{p.v.}\int \operatorname{Re}\left\{\frac{-1}{\pi i}\frac{z_\alpha}{|z_\alpha|}\frac{|z_\beta(\beta,t)|}{(z(\alpha,t)-z(\beta,t))}\right\} f(\beta,t)\,d\beta \tag{1.11}$$

is the adjoint of the double layer potential $\mathfrak{K}$ in $L^2(\Sigma(t),dS)$. Notice that $I+\mathfrak{K}^*$ is invertible on $L^2(\Sigma(t),dS)$. From (1.10) we see that $\mathfrak{a}_t|z_\alpha|$ has the same regularity as that of $\bar{z}_{tt}$ and $\bar{z}_t$.

After proving the Taylor sign condition (1.5) holds, i.e. $\mathfrak{a}>0$ [38, 39], we rewrote (1.8) as

$$\begin{cases} z_{ttt} - i\mathfrak{a}z_{t\alpha} = \dfrac{z_{tt}+i}{|z_{tt}+i|}\mathfrak{a}_t|z_\alpha|, \\ \bar{z}_t = \mathfrak{H}\bar{z}_t. \end{cases} \tag{1.12}$$

Using (1.10) for $\mathfrak{a}_t|z_\alpha|$, we see that (1.12) is a weakly hyperbolic quasi-linear system with the right hand side of the first equation in (1.12) consisting of terms of lower order derivatives of z_t.

Let $\mathfrak{u}=z_t$. The system (1.12)–(1.10)–(1.11) is the 2D version of the quasi-linear system (5.21)–(5.22) derived in [39]. The local in time wellposedness of (1.12)–(1.10)–(1.11) in Sobolev spaces (with $(\mathfrak{u},\mathfrak{u}_t)\in C([0,T],H^{s+1/2}\times H^s)$) was proved by energy estimates and iterative scheme. Through establishing the equivalence of (1.7) with (1.12)–(1.10)–(1.11), we obtained the local in time well-posedness in Sobolev spaces of the full water wave equation (1.7) [38, 39].

Let's now take a look at a possible change of variables. For $f=f(\alpha,t)$, $g=g(\alpha,t)$, we use the notation $U_g f(\alpha,t)=f\circ g(\alpha,t)=f(g(\alpha,t),t)$. For fixed t, let $k=k(\alpha,t):R\to R$ be a diffeomorphism and $k_\alpha>0$. Let k^{-1} be such that $k\circ k^{-1}(\alpha,t)=\alpha$. Define

$$\zeta = z\circ k^{-1}, \qquad u = z_t\circ k^{-1} = \mathfrak{u}\circ k^{-1}, \quad \text{and} \quad w = z_{tt}\circ k^{-1} = \mathfrak{u}_t\circ k^{-1}. \tag{1.13}$$

Let

$$b = k_t\circ k^{-1} \quad \text{and} \quad A\circ k = \mathfrak{a}k_\alpha. \tag{1.14}$$

By a simple application of the chain rule, we find

$$w = u_t + bu_\alpha \quad \text{and} \quad U_k^{-1}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)U_k = (\partial_t + b\partial_\alpha)^2 - iA\partial_\alpha. \tag{1.15}$$

In this new coordinate system, (1.12) becomes

$$\begin{cases} (\partial_t + b\partial_\alpha)^2 u - iA\partial_\alpha u = (\mathfrak{a}_t|z_\alpha|)\circ k^{-1}\frac{w+i}{|w+i|}, \\ \bar{u} = \mathcal{H}\bar{u} \end{cases} \tag{1.16}$$

with

$$(I+\mathcal{K}^*)((\mathfrak{a}_t|z_\alpha|)\circ k^{-1})$$
$$=\operatorname{Re}\left(\frac{i\zeta_\alpha}{|\zeta_\alpha|}\left\{2[w,\mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha}+2[u,\mathcal{H}]\frac{\bar{w}_\alpha}{\zeta_\alpha}-\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2\bar{u}_\beta\,d\beta\right\}\right) \quad (1.17)$$

and

$$\mathcal{H}f(\alpha,t)=\frac{1}{\pi i}\text{p.v.}\int\frac{f(\beta,t)\zeta_\beta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\,d\beta, \quad (1.18)$$
$$\mathcal{K}^*f(\alpha,t)=\text{p.v.}\int\operatorname{Re}\left\{\frac{-1}{\pi i}\frac{\zeta_\alpha}{|\zeta_\alpha|}\frac{|\zeta_\beta(\beta,t)|}{(\zeta(\alpha,t)-\zeta(\beta,t))}\right\}f(\beta,t)\,d\beta. \quad (1.19)$$

Notice the remarkable similarities between the terms in (1.12)–(1.10)–(1.11) and in (1.16)–(1.17)–(1.19). In particular, the structures of the terms in (1.6), (1.10), (1.11) do not change under the change of variables. This makes it convenient for us to work in another coordinate system and to choose a different coordinate system when there is advantage to do so. In fact, using (1.16)–(1.17) one can prove the local in time well-posedness of the water wave problem in this arbitrarily chosen coordinate system using the same analysis as in [38, 39].

We now discuss the dispersive aspect of (1.12)–(1.10)–(1.11). Let $\mathfrak{u}=z_t$. Linearize (1.12)–(1.10)–(1.11) about the zero solution (the still water), we obtain the free equation

$$\mathfrak{u}_{tt}+|D_\alpha|\mathfrak{u}=0, \quad (1.20)$$

where $|D_\alpha|=\sqrt{-\partial_\alpha^2}$. The system (1.12)–(1.10)–(1.11) can be rewritten as

$$\mathfrak{u}_{tt}+|D_\alpha|\mathfrak{u}=F(z,\mathfrak{u},\mathfrak{u}_t,\mathfrak{u}_{tt},\mathfrak{u}_\alpha) \quad (1.21)$$

with F consisting of the nonlinear terms. Notice that F contains both quadratic and higher order nonlinear terms. As we know, for small solutions, it is in general the most difficult to handle the lowest order nonlinearity. In (1.21), the most difficult terms are the quadratic nonlinear terms in F. Using the method of stationary phase [34], we find that the free solution $\mathfrak{u}$ of the linear equation (1.20) for 2D water wave has a L^1-L^∞ time decay rate[2] $1/t^{1/2}$. Such slow decay rate coupled with the quadratic nonlinearity in (1.21) doesn't give us a much longer existence time period than already obtained by the energy estimates and the standard Sobolev embedding. This motivates us to explore possible cancellations in the quadratic nonlinearity in F. One possible method is the method of normal form. The idea of the method of normal form is to make a nonlinear change of unknowns (if it exists)

$$\mathfrak{v}=\mathfrak{u}+K(\mathfrak{u})$$

[2]Let $|D|=\sqrt{-\Delta}$, Δ the Laplacian on R^n. If $\mathfrak{u}\in C^\infty((x,t)\in R^{n+1})$ is a solution of

$$\mathfrak{u}_{tt}+|D|\mathfrak{u}=0,\qquad \mathfrak{u}(\cdot,0)=u_0,\qquad \mathfrak{u}_t(\cdot,0)=u_1,$$

and vanishes rapidly as $|x|\to\infty$, we have by an application of the method of stationary phase,

$$|\mathfrak{u}(\cdot,t)|_{L^\infty(R)}\le\frac{c}{(t+1)^{n/2}}\sum_{j=0}^{1}\sum_{|k|\le n+1}\int|\partial_x^k u_j(x)|\,dx.$$

We do not give detailed derivations of this estimate, since we will not use it in this article. Instead we will use the decay estimate in Proposition 3.1.

so that $\mathfrak{v}$ satisfies an equation

$$\mathfrak{v}_{tt} + |D_\alpha|\mathfrak{v} = F_1(z, \mathfrak{u}, \mathfrak{u}_t, \mathfrak{u}_{tt}, \mathfrak{u}_\alpha)$$

with F_1 consisting of only cubic and higher order nonlinear terms, and $\|\mathfrak{v}\| \approx \|\mathfrak{u}\|$ in various norms $\|\cdot\|$ involved in the estimates. However there are difficulties in carrying out this method on our system (1.12)–(1.10)–(1.11), since we do not know a priori whether such a transformation exists; and we do not know the form of the transformation if it indeed exists. In [30], Shatah extended Poincaré's method of normal forms for the ordinary differential equations to the study of the nonlinear Klein-Gordon equations. Using a transformation of the type $v = u + B(u, u)$, where B is a bounded bilinear form, he successfully reduced a quadratic nonlinear Klein-Gordon equation $\partial_t^2 u - \Delta u + u = f(u, Du, D^2 u)$ for u into a cubic nonlinear Klein-Gordon equation for the new function $v = u + B(u, u)$. He then proved the global in time existence of classical solutions for small initial data in dimensions $n > 2$. The bounded-ness of the bilinear form $B(u, u)$ is crucial for his method to work, as it implies that $\|v\| \approx \|u\|$ when $\|u\|$ is small. (We note that the bilinear form $B(u, u)$ in Shatah's work is only explicit in its Fourier symbol, and the time decay rate[3] for the Klein-Gordon equation in n-D is $1/t^{n/2}$. See also Simon [32] and Simon and Taflin [33] where they gave a different transformation for the nonlinear Klein-Gordon equation cancelling out quadratic terms.)

We therefore tried a transformation of the type $\mathfrak{v} = \mathfrak{u} + B(\mathfrak{u}, \mathfrak{u})$, where $B(\mathfrak{u}, \mathfrak{u})$ is bilinear. However, this type of transformation works only partially for our system (1.12)–(1.10)–(1.11). There are three difficulties: firstly, not all of the quadratic nonlinear terms in (1.21) (or equivalently (1.12)) can be transformed into cubic and higher orders through a transformation $\mathfrak{v} = \mathfrak{u} + B(\mathfrak{u}, \mathfrak{u})$ with a bounded bilinear form $B(\mathfrak{u}, \mathfrak{u})$. Secondly, even with the partial transformation that transforms only those quadratic nonlinear terms that can be transformed into cubic and higher orders with a bounded bilinear form $B(\mathfrak{u}, \mathfrak{u})$, the result is still unsatisfying: the partial transformation destroys the structure of (1.12) and the resulting terms are not in favorable forms for us to have good decay estimates. For example, the estimates for the new nonlinearity depend now on the estimates of the function $\mathfrak{u}$ itself. While one may use L^p–L^q estimates to get a time decay estimate for $\mathfrak{u}$, it would be slower than the rates (directly obtainable from the equation) for $\mathfrak{u}_t$ and $|D_\alpha|^{1/2}\mathfrak{u}$. Therefore the existence time period of classical solutions one may obtain for the new nonlinearity depending on the estimates of $\mathfrak{u}$ would be shorter than the case when the estimates don't depend on the estimates of $\mathfrak{u}$. Thirdly, the structures of the partial transformation and the resulting equation are no longer coordinate invariant and are too complicated to be understood.[4]

Nevertheless the idea of the method of normal form is the one we employ in this paper. Starting from the partial transformation, and with much further analysis and efforts in understanding the nature, relations and possible cancellations of the terms in the equation, we find that the quantities

$$\Pi = (I - \mathfrak{H})(z - \bar{z}), \quad \text{and} \quad \mathfrak{v} = \partial_t (I - \mathfrak{H})(z - \bar{z}) = (I - \mathfrak{H})\mathfrak{u} + [\mathfrak{u}, \mathfrak{H}]\frac{\bar{z}_\alpha - z_\alpha}{z_\alpha} \tag{1.22}$$

[3]See [13, p. 151, Corollary 7.2.4].

[4]The Fourier symbol of the partial bilinear transform is not of the types [8] for which satisfactory estimates of their corresponding bilinear forms are known. It becomes necessary for us to find its structures in the physical space. In 2D it is relatively easy to find the structures of the partial bilinear transform in the physical space, while in 3D it is quite difficult to find it. A more natural and understandable transformation for the 2D water wave will be helpful for finding the one for the 3D water wave, and will be useful for studying other questions on water waves.

satisfy the equations

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi = -2\left[z_t, \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha} + \mathfrak{H}\frac{1}{z_\alpha}\right]z_{t\alpha} + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2(z_\beta - \bar{z}_\beta)\,d\beta \tag{1.23}$$

and

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} = \partial_t\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(I-\mathfrak{H})(z-\bar{z})\} + i\mathfrak{a}_t\partial_\alpha(I-\mathfrak{H})(z-\bar{z}). \tag{1.24}$$

Notice that both the right hand sides of (1.23) and (1.24) are consisting of only cubic and higher order nonlinear terms, while the left hand sides of (1.23) and (1.24) still involve quadratic nonlinear terms. We resolve this difficulty by looking for a better coordinate system. (Notice that both the structures of Π, $\mathfrak{v}$ and (1.23), (1.24) are coordinate invariant.[5])

Let $\Phi(\cdot,t):\Omega(t)\to P_-$ be the Riemann mapping from the fluid domain $\Omega(t)$ to the lower half plane P_-, satisfying $\lim_{z\to\infty}\Phi_z(z,t)=1$ and $\Phi(z(0,t),t)=2x(0,t)$. Let $h(\alpha,t)=\Phi(z(\alpha,t),t)$ and

$$k(\alpha,t) = 2x(\alpha,t) - h(\alpha,t). \tag{1.25}$$

We find that both $b = k_t\circ k^{-1}$ and $A-1=(ak_\alpha)\circ k^{-1}-1$ are consisting of only quadratic and higher order nonlinear terms in $w=u_t+bu_\alpha$ and u, here u, w are as defined in (1.13). Therefore the quantities

$$v = \mathfrak{v}\circ k^{-1} \quad \text{and} \quad \chi = U_k^{-1}(I-\mathfrak{H})(z-\bar{z}) \tag{1.26}$$

satisfy equations

$$(\partial_t + b\partial_\alpha)^2 v - iA\partial_\alpha v = U_k^{-1}(\partial_t\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(I-\mathfrak{H})(z-\bar{z})\} + i\mathfrak{a}_t\partial_\alpha(I-\mathfrak{H})(z-\bar{z})) \tag{1.27}$$

and

$$(\partial_t + b\partial_\alpha)^2\chi - iA\partial_\alpha\chi = -2\left[u, \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha} + \mathcal{H}\frac{1}{\zeta_\alpha}\right]u_\alpha + \frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(\zeta_\beta - \bar{\zeta}_\beta)\,d\beta \tag{1.28}$$

and they are of the form

$$\partial_t^2 v - i\partial_\alpha v = G_1(w,u), \qquad \partial_t^2\chi - i\partial_\alpha\chi = G_2(w,u) \tag{1.29}$$

with G_1, G_2 consisting of only cubic and higher order nonlinear terms in w and u. Furthermore, since u and $v=(\partial_t+b\partial_\alpha)\chi$ are equivalent in the various norms involved in our estimates, we can directly obtain estimates for u from (1.28). The finding of this change of

[5] We will see in later sections, in particular in Propositions 3.2, 3.3, 3.4 and (3.12), (3.14) that the structures of Π, $\mathfrak{v}$ and the terms in (1.23), (1.24) and (1.27), (1.28) are of the types for which we can obtain satisfactory estimates.

variables k is inspired by the formula for $h_t \circ h^{-1}$ in our previous work (see [38, the formula for b at the bottom of p. 50]).

Notice that the quantity Π or χ is the boundary value of a holomorphic function in the air region $\Omega(t)^c$ and $\mathfrak{v}$ or v is almost holomorphic in $\Omega(t)^c$ (from (1.22), we see that it is a holomorphic term plus a quadratic or higher order term). We remark that the initial value problem for the equation

$$\partial_t^2 \Theta - i\partial_\alpha \Theta = G$$

is ill-posed for a general complex valued unknown function Θ. However it is well-posed if the unknown function Θ is the boundary value of a holomorphic function in $\{\Omega(t)\}^c$ and in this case $\partial_t^2\Theta - i\partial_\alpha\Theta = \partial_t^2\Theta + |\zeta_\alpha|\nabla_{\mathbf{n}}\Theta$, where $\nabla_{\mathbf{n}}$ is the Dirichlet-Neumann operator (see [39] for a definition). Therefore when viewed in terms of its real or imaginary part separately, the left hand sides of the equations in (1.29) are in fact non-linear and contains quadratic nonlinearity. However when viewed as a complex valued term, $\partial_t^2\Theta - i\partial_\alpha\Theta$ is linear.

We now see that the nonlinearity of the infinite depth 2D water wave system (1.1)–(1.4) is essentially consisting of only nonlinear terms of cubic and higher orders when we work with the right quantities in the right coordinate system. To study the global wellposedness of the 2D water wave system (1.1)–(1.4), we use the method of invariant vector fields to (1.27) and (1.28). The advantage of this method is that it gives a better L^2–L^∞ type time decay estimate (with decay rate $1/t^{1/2}$ for 2D water wave). Klainerman developed the method of invariant vector fields for the nonlinear wave equations [20–23]. The invariant vector fields for $\partial_t^2 - i\partial_\alpha$ are

$$\partial_t, \qquad \partial_\alpha, \qquad L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha, \quad \text{and} \quad \Omega_0 = \alpha\partial_t + \frac{1}{2}ti. \tag{1.30}$$

However the inhomogeneity of Ω_0 makes it difficult to use it to get estimates for terms such as $\Omega_0 v$, $\Omega_0\chi$ directly from (1.27), (1.28). We notice that

$$\Omega_0\partial_\alpha = L_0\partial_t - \frac{1}{2}t(\partial_t^2 - i\partial_\alpha). \tag{1.31}$$

Therefore we resort to use only ∂_t, ∂_α, and L_0, and resolve the problems from the absence of Ω_0 by finding one more quantity of one time derivative less than χ satisfying an equation of the form (1.29). We find the quantity

$$\lambda = U_k^{-1}(I - \mathfrak{H})\psi, \tag{1.32}$$

where $\psi(\alpha, t) = \phi(z(\alpha, t), t)$, ϕ the velocity potential, satisfies an equation of the form (1.29). Using (1.27), (1.28), and the equation for λ, and the method of invariant vector fields, we are able to derive a priori estimates for solutions of the 2D water wave equation that hold for a time period $[0, e^{T/\epsilon}]$, as long as the initial energy is no more than $H^2\epsilon^2$. Using the local existence result and a continuity argument, we arrive at the following almost global existence result for the 2D water wave.

Let the initial interface be a graph $z(\alpha, 0) = \alpha + i\epsilon f(\alpha)$, the initial velocity $z_t(\alpha, 0) = \epsilon g(\alpha)$, $\alpha \in R$, f and g are smooth and decay fast at infinity, and $\bar{g} = \mathfrak{H}^0\bar{g}$, here $\mathfrak{H}^0$ is the Hilbert transform associated with the initial interface $z(\cdot, 0)$.

Theorem *There exist* $\epsilon_0 > 0$, $T > 0$, *depending only on* f *and* g, *such that for* $0 < \epsilon < \epsilon_0$, *the initial value problem of the* 2*-D water wave system* (1.7) (*equivalently* (1.1)–(1.4)) *has a unique classical solution for a time period* $[0, e^{T/\epsilon}]$. *During this time period, the solution has the same regularity as the initial data and remains small, and the interface is a graph.*

For precise assumptions on f and g and the precise statement of the above Theorem, see Theorem 5.5. See also the discussion on initial data following the proof of Theorem 5.5.

For the rest of this paper if not otherwise mentioned, $z = z(\alpha, t) = x(\alpha, t) + i\, y(\alpha, t)$ indicates a solution of the water wave system (1.7), $\mathfrak{H}$ is as defined in (1.6), k indicates the change of variable given in (1.25). ζ, u, w, b, A and $\mathcal{H}, \mathcal{K}^*$ are as defined in (1.13)–(1.19) for this particular k. Π, $\mathfrak{v}$ and v, χ, λ are as defined by (1.22), (1.26), (1.32). $\Lambda = (I - \mathfrak{H})\psi = \lambda \circ k$.

The equations for the quantities v, χ and λ (equivalently $\mathfrak{v}$, Π and Λ), as well as the relations between the transformed quantities v, χ and λ and untransformed quantities w, u and ζ will be derived in details in Sect. 2. In Sect. 3 we present some basic analytic tools that will be used in deriving a priori estimates. In Sect. 4 we construct the energy from the equations for v, χ and λ and derive the energy estimates. In Sect. 5 we present and prove the almost global wellposedness result of the 2D water wave system (1.7).

2 The derivation of the main equations

In this section, we give a derivation of the equations for v, χ and λ.[6] As noted in the introduction, to find possible cancellations in the quadratic nonlinearity of the water wave system, we started from a partial transformation of the type $\mathfrak{u} + B(\mathfrak{u}, \mathfrak{u})$ with a bounded bilinear form $B(\mathfrak{u}, \mathfrak{u})$, and finally arrived at the quantity v through much further analysis and understandings of possible cancellations involved, and through choosing a better coordinate system. Because of the impossibility of getting estimates for such quantities u, $\Omega_0 u$, v, $\Omega_0 v$ from the equation and because of (1.31), we find further the quantities χ and λ. However for the sake of clarity we do not go over this process. Instead we give a derivation of the equations assuming we have already found the quantities v, χ, λ and the change of variables k as introduced in the introduction.

We know $\mathfrak{H}^2 = I$ in L^2. We use the convention $\mathfrak{H}1 = 0$.

We start from the equivalent water wave system (1.7):

$$z_{tt} + i = i\mathfrak{a} z_\alpha, \tag{2.1}$$

$$\bar{z}_t = \mathfrak{H}\bar{z}_t, \tag{2.2}$$

where $\mathfrak{a}$ is a real valued function determined by the system (2.1)–(2.2). Let $z = z(\alpha, t)$ be a solution of (2.1)–(2.2), with $z_t \in C([0, T], H^{s+1/2}) \cap C^1([0, T], H^s)$, $s \ge 4$ as obtained in [38]. We have the following identities.

Lemma 2.1 *Assume that* $f \in C^1(R \times (0, T))$ *satisfies* $f_\alpha(\alpha, t) \to 0$, *as* $|\alpha| \to \infty$. *We have*

$$[\partial_t, \mathfrak{H}]f = [z_t, \mathfrak{H}]\frac{f_\alpha}{z_\alpha}, \tag{2.3}$$

$$[\partial^2, \mathfrak{H}]f = [z_{tt}, \mathfrak{H}]\frac{f_\alpha}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 f_\beta \, d\beta, \tag{2.4}$$

[6]The physical meanings of v, χ, and λ are unknown at the time of writing. It is desirable to have good understandings of these quantities from the physical point of view.

$$[\mathfrak{a}\partial_\alpha, \mathfrak{H}]f = [\mathfrak{a}z_\alpha, \mathfrak{H}]\frac{f_\alpha}{z_\alpha}, \qquad \partial_\alpha \mathfrak{H} f = z_\alpha \mathfrak{H}\frac{f_\alpha}{z_\alpha}, \tag{2.5}$$

$$[\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}]f = 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2 f_\beta \, d\beta, \tag{2.6}$$

$$(I-\mathfrak{H})(-i\mathfrak{a}_t\bar{z}_\alpha) = 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2 \bar{z}_{t\beta}\, d\beta. \tag{2.7}$$

Remark In fact, (2.3) to (2.5) hold for a generally given $z = z(\alpha, t)$ that defines a non-selfintersecting curve for each fixed time t, with $z_t, z_\alpha - 1 \in C^1([0,T], H^1(R))$, and its associated Hilbert transform $\mathfrak{H}$.

The proof of (2.3) to (2.5) is straightforwardly integration by parts, we omit the proof. (2.6) is a straightforward consequence of (2.4), (2.5) and (2.1). (2.7) is basically (1.9), in which the last step is explained by (2.4), (2.5) and (2.1).

We record in the following some basic facts on holomorphic functions that will be used in this paper. Let Ω be a C^1 domain in the complex plane. Assume that the boundary of Ω is parametrized by $z = z(\alpha)$, $\alpha \in R$, oriented in the clockwise sense, and there exist constants $\mu_1, \ \mu_2 > 0$, such that $\mu_1|\alpha - \beta| \le |z(\alpha) - z(\beta)| \le \mu_2|\alpha-\beta|$ for all $\alpha, \ \beta \in R$. Let

$$\mathbf{H} f(\alpha) = \frac{1}{\pi i}\text{p.v.}\int \frac{z_\beta(\beta)}{z(\alpha)-z(\beta)} f(\beta)\, d\beta$$

be the associated Hilbert transform. We have

Lemma 2.2 1. $f(\cdot) = F(z(\cdot)) \in L^2(\partial\Omega)$ *is the boundary value of a holomorphic function* F *in* Ω *if and only if* $f = \mathbf{H} f$.
2. *If* $f = \mathbf{H} f$, $g = \mathbf{H} g$, *then* $[f, \mathbf{H}]g = 0$.
3. *For any* $f, g \in L^2(\partial\Omega)$, *we have* $[f, \mathbf{H}]\mathbf{H} g = -[\mathbf{H} f, \mathbf{H}]g$.

The first statement above is a consequence of the Cauchy integral formula. The second statement follows from the fact that the product of holomorphic functions is holomorphic. Applying the second statement to $f + \mathbf{H} f$ and $g + \mathbf{H} g$, we get

$$[f + \mathbf{H} f, \mathbf{H}](\mathbf{H} g) = -[f + \mathbf{H} f, \mathbf{H}](g). \tag{2.8}$$

Notice that $-\mathbf{H}$ is the Hilbert transform associated with the domain Ω^c. Applying the second statement to $f - \mathbf{H} f$ and $g - \mathbf{H} g$ gives

$$[f - \mathbf{H} f, \mathbf{H}](\mathbf{H} g) = [f - \mathbf{H} f, \mathbf{H}](g). \tag{2.9}$$

Summing up (2.8), (2.9) we arrive at the third statement.

We now derive the three main equations that will be used to obtain a priori estimates and prove the almost global wellposedness of the water wave equation (2.1)–(2.2). Since we are concerned with small solutions in this paper, the order of smallness in the nonlinearity is important. We understand that z_t, z_{tt} and $z_\alpha - 1$ each is of order one in smallness.

Let ϕ be the velocity potential, i.e. $\mathbf{v} = \nabla\phi$, and let $\psi(\alpha, t) = \phi(z(\alpha, t), t)$.[7] The main point of the following proposition is to show that the quantities $(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})(z - \bar{z})\}$, $(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\partial_t\{(I - \mathfrak{H})(z - \bar{z})\}$ and $(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})\psi\}$ are each consisting of only cubic and higher order nonlinear terms. Furthermore, efforts are made to ensure these nonlinear terms are in formats on which we are able to carry out the (optimal) estimates.

Proposition 2.3 *Let $z = z(\alpha, t)$, with $z_t \in C([0, T], H^{s+1/2}) \cap C^1([0, T], H^s)$, $s \geq 4$ be a solution of* (2.1)–(2.2). *Let $\Pi = (I - \mathfrak{H})(z - \bar{z})$, $\mathfrak{v} = \partial_t \Pi$, and $\Lambda = (I - \mathfrak{H})\psi$. We have*

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi = -2\left[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}\right]z_{t\alpha} + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 (z_\beta - \bar{z}_\beta)\, d\beta,^{8} \tag{2.10}$$

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} = \partial_t\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi\} + i\mathfrak{a}_t\partial_\alpha(I - \mathfrak{H})(z - \bar{z}), \tag{2.11}$$

$$\begin{aligned}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Lambda = &-\left[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}\right](\bar{z}_\alpha z_{tt}) + [z_t, \bar{\mathfrak{H}}]\left(\bar{z}_t\frac{z_{t\alpha}}{\bar{z}_\alpha}\right) + z_t[z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} \\ &- 2[z_t, \mathfrak{H}]\frac{z_t \cdot z_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 z_t \cdot z_\beta\, d\beta.\end{aligned} \tag{2.12}$$

Proof Let's first prove (2.10). We have

$$\begin{aligned}&(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})(z - \bar{z})\} \\ &= (I - \mathfrak{H})\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(z - \bar{z})\} - [\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}](z - \bar{z}) \\ &= (I - \mathfrak{H})(-2\bar{z}_{tt}) - 2[z_t, \mathfrak{H}]\frac{z_{t\alpha} - \bar{z}_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 (z - \bar{z})_\beta\, d\beta \\ &= -2[z_t, \mathfrak{H}]\frac{z_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 (z - \bar{z})_\beta\, d\beta \\ &= -2\left[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}\right]z_{t\alpha} + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 (z_\beta - \bar{z}_\beta)\, d\beta.\end{aligned} \tag{2.13}$$

Here in the second step we used (2.1), (2.6), the third step we used (2.2) and (2.3), and in the fourth step we inserted the term $[z_t, \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]z_{t\alpha}$ to make it an at least cubic nonlinear term. $[z_t, \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]z_{t\alpha} = 0$ because both $\bar{z}_t$ and $\frac{\bar{z}_{t\alpha}}{z_\alpha}$ are boundary values of some holomorphic functions in $\Omega(t)$ and because of the part 2 of Lemma 2.2. (2.11) is obtained by taking a derivative to t to (2.10). We see from (2.7) that $\mathfrak{a}_t$ is consisting of terms of quadratic and higher orders.

[7] Notice that in general, the velocity potential ψ does not decay at the spatial infinity. Therefore here we do not assume ψ decay at the spatial infinity. The case we consider is that the spatial L^2 norms of $\partial_t\Lambda$, $\partial_\alpha\Lambda$ etc., where $\Lambda = (I - \mathfrak{H})\psi$, are finite, this is roughly equivalent to asking the amplitude of the wave and the velocity etc. being in L^2.

[8] The right hand side of (2.10) is consisting of cubic and higher order terms, since it has an expansion as that in (2.27).

Now let's derive (2.12). We know the velocity potential ϕ satisfies the Bernoulli equation: $\phi_t + \frac{1}{2}|\nabla\phi|^2 + P + y = \text{constant}$ in $\Omega(t)$, therefore we may choose the ϕ so that

$$\phi_t + \frac{1}{2}|z_t|^2 + y = 0 \quad \text{on } \Sigma(t), \tag{2.14}$$

and we have

$$\begin{aligned} \psi_t &= \phi_t + \nabla\phi\cdot z_t = \phi_t + |z_t|^2 = -y + \frac{1}{2}|z_t|^2, \\ \psi_{tt} &= -y_t + z_t\cdot z_{tt}, \\ \psi_\alpha &= \nabla\phi\cdot z_\alpha = z_t\cdot z_\alpha = \mathrm{Re}\{\bar z_t z_\alpha\}. \end{aligned} \tag{2.15}$$

Using (2.1) we get

$$\psi_{tt} - i\mathfrak{a}\partial_\alpha\psi = z_t\bar z_{tt} - i\bar z_t. \tag{2.16}$$

Now from (2.16), (2.2), (2.6), and (2.15),

$$\begin{aligned} &(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I-\mathfrak{H})\psi\} \\ &= (I-\mathfrak{H})(\psi_{tt} - i\mathfrak{a}\psi_\alpha) - [\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}]\psi \\ &= (I-\mathfrak{H})(z_t\bar z_{tt}) - 2[z_t,\mathfrak{H}]\frac{\psi_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2\psi_\beta\,d\beta \\ &= (I-\mathfrak{H})(z_t\bar z_{tt}) - 2[z_t,\mathfrak{H}]\frac{-y_\alpha + z_t\cdot z_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2 z_t\cdot z_\beta\,d\beta. \end{aligned} \tag{2.17}$$

In order to show that the term $(I-\mathfrak{H})(z_t\bar z_{tt}) + 2[z_t,\mathfrak{H}]\frac{y_\alpha}{z_\alpha}$ is in fact consisting of only cubic and higher order terms, we manipulate further

$$(I-\mathfrak{H})(z_t\bar z_{tt}) = [z_t,\mathfrak{H}]\bar z_{tt} + z_t[\partial_t,\mathfrak{H}]\bar z_t = [z_t,\mathfrak{H}]\bar z_{tt} + z_t[z_t,\mathfrak{H}]\frac{\bar z_{t\alpha}}{z_\alpha}$$

and

$$\begin{aligned} [z_t,\mathfrak{H}]\bar z_{tt} + 2[z_t,\mathfrak{H}]\frac{y_\alpha}{z_\alpha} &= [z_t,\mathfrak{H}]\left(\bar z_{tt} + \frac{2y_\alpha}{z_\alpha}\right) = [z_t,\mathfrak{H}]\left(\frac{-\bar z_\alpha z_{tt}}{z_\alpha}\right) \\ &= -\left[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}\right](\bar z_\alpha z_{tt}) + \frac{1}{2}[z_t,\bar{\mathfrak{H}}](z_{tt} - \bar{\mathfrak{H}} z_{tt}) \\ &= -\left[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}\right](\bar z_\alpha z_{tt}) + \frac{1}{2}[z_t,\bar{\mathfrak{H}}][\bar z_t,\bar{\mathfrak{H}}]\frac{z_{t\alpha}}{\bar z_\alpha}. \end{aligned}$$

In the second step above we used (2.1), the third step we used the fact that both z_t and $z_{tt} + \bar{\mathfrak{H}} z_{tt}$ are boundary values of anti-holomorphic functions in $\Omega(t)$ and the part 2 of Lemma 2.2. In the last step we used (2.3). Further using the fact that $\mathfrak{H}\frac{\bar z_{t\alpha}}{z_\alpha} = \frac{\bar z_{t\alpha}}{z_\alpha}$ (see (2.5) and (2.2)), (2.2) and the third statement of Lemma 2.2, we rewrite

$$\frac{1}{2}[z_t,\bar{\mathfrak{H}}][\bar z_t,\bar{\mathfrak{H}}]\frac{z_{t\alpha}}{\bar z_\alpha} = \frac{1}{2}[z_t,\bar{\mathfrak{H}}]\left(\bar z_t\frac{z_{t\alpha}}{\bar z_\alpha} - \bar{\mathfrak{H}}\left(\bar z_t\frac{z_{t\alpha}}{\bar z_\alpha}\right)\right) = [z_t,\bar{\mathfrak{H}}]\left(\bar z_t\frac{z_{t\alpha}}{\bar z_\alpha}\right).$$

This gives us (2.12). □

We now show that after applying the change of variables k as given in (1.25),[9] the nonlinearities in the left hand sides of (2.10)–(2.12) will be consisting of only cubic and higher order terms.

Let $\Phi(\cdot,t):\Omega(t)\to P_-$ be the Riemann mapping from the fluid domain $\Omega(t)$ to the lower half plane P_-, satisfying $\lim_{z\to\infty}\Phi_z(z,t)=1$ and $\Phi(z(0,t),t)=2x(0,t)$. Let $h(\alpha,t)=\Phi(z(\alpha,t),t)$,

$$k(\alpha,t)=2x(\alpha,t)-h(\alpha,t). \tag{2.18}$$

We know $k(0,t)=0$. Assume that $k(\cdot,t):R\to R$ is a diffeomorphism and $k_\alpha(\cdot,t)>0$. (We consider the question of when the function $k(\cdot,t)$ is a diffeomorphism in Sect. 5. As shown in Lemma 5.3 part 2, a smallness in $z_\alpha(\cdot,t)-1$ is sufficient for $k(\cdot,t)$ to be a diffeomorphism from R to R and $k_\alpha(\cdot,t)>0$.)

Let the notations introduced in (1.13)–(1.19) relating to change of variables be for this particular k. In particular, $b=k_t\circ k^{-1}$, and $A=(\mathfrak{a}k_\alpha)\circ k^{-1}$. We know

$$U_k^{-1}\mathfrak{H}U_k=\mathcal{H},\qquad U_k^{-1}\partial_t U_k=\partial_t+b\partial_\alpha,\qquad U_k^{-1}\mathfrak{a}\partial_\alpha U_k=A\partial_\alpha. \tag{2.19}$$

In the following proposition we show that b and $A-1$ are consisting of only quadratic and higher order terms.

Proposition 2.4 *Let $b=k_t\circ k^{-1}$ and $A=(\mathfrak{a}k_\alpha)\circ k^{-1}$. We have*

$$(I-\mathcal{H})b=-[z_t\circ k^{-1},\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}, \tag{2.20}$$

$$(I-\mathcal{H})A=1+i[z_t\circ k^{-1},\mathcal{H}]\frac{(\bar z_t\circ k^{-1})_\alpha}{\zeta_\alpha}+i[z_{tt}\circ k^{-1},\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}. \tag{2.21}$$

Proof We know $h_t=\Phi_t+\Phi_z z_t$, $h_\alpha=\Phi_z z_\alpha$, Φ_t, Φ_z are holomorphic in $\Omega(t)$, and $\lim_{z\to\infty}\Phi_z-1=0$. We have

$$\bar z_t-k_t=h_t-z_t=\Phi_t+(\Phi_z-1)z_t,\qquad \bar z_\alpha-k_\alpha=h_\alpha-z_\alpha=(\Phi_z-1)z_\alpha. \tag{2.22}$$

Therefore

$$\begin{aligned}-(I-\mathfrak{H})k_t&=(I-\mathfrak{H})(\bar z_t-k_t)=(I-\mathfrak{H})\{(\Phi_z-1)z_t\}=[z_t,\mathfrak{H}](\Phi_z-1)\\&=[z_t,\mathfrak{H}]\frac{\bar z_\alpha-k_\alpha}{z_\alpha}.\end{aligned} \tag{2.23}$$

Applying the change of variable U_k^{-1} to (2.23) gives (2.20). Now using (2.22), (2.1), and the fact that Φ_z-1 is holomorphic, we have

$$(I-\mathfrak{H})(i\mathfrak{a}\bar z_\alpha-i\mathfrak{a}k_\alpha)=(I-\mathfrak{H})(i\mathfrak{a}z_\alpha(\Phi_z-1))=[z_{tt},\mathfrak{H}](\Phi_z-1)$$

[9] The basic requirements on k are that $b=k_t\circ k^{-1}$ and $A-1=(\mathfrak{a}k_\alpha)\circ k^{-1}-1$ must be consisting of quadratic and higher order terms. However we also need them to have "good" structures for optimal estimates. For this article, being "good" includes being coordinate invariant, being invariant under the application of the invariant vector fields, see (3.14). Also see Propositions 3.2, 3.3, 3.4 for some examples of structures that posses satisfactory estimates. The one given by (2.18) is a good one, but certainly it is not necessarily the unique one.

therefore from (2.1), (2.2), (2.3), (2.22),

$$-(I-\mathfrak{H})(i\mathfrak{a}k_\alpha)=-(I-\mathfrak{H})(i\mathfrak{a}\bar z_\alpha)+[z_{tt},\mathfrak{H}](\Phi_z-1)=-i+[z_t,\mathfrak{H}]\frac{\bar z_{t\alpha}}{z_\alpha}+[z_{tt},\mathfrak{H}]\frac{\bar z_\alpha-k_\alpha}{z_\alpha}.$$

Again with the change of variable U_k^{-1} we arrive at (2.21). □

Now let

$$\mathcal{P}=(\partial_t+b\partial_\alpha)^2-iA\partial_\alpha, \tag{2.24}$$

$$v=\mathfrak{v}\circ k^{-1}, \qquad \chi=\Pi\circ k^{-1}, \quad \text{and} \quad \lambda=\Lambda\circ k^{-1} \tag{2.25}$$

and as defined in (1.13),

$$\zeta=z\circ k^{-1}=\mathfrak{x}+i\mathfrak{y}, \qquad u=z_t\circ k^{-1} \quad \text{and} \quad w=z_{tt}\circ k^{-1}. \tag{2.26}$$

We have from (2.10) to (2.12) (by the chain rule)

$$\begin{aligned}\mathcal{P}\chi&=-2\left[u,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right]u_\alpha+\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(\zeta_\beta-\bar\zeta_\beta)\,d\beta\\&=\frac{4}{\pi}\int\frac{(u(\alpha,t)-u(\beta,t))(\mathfrak{y}(\alpha,t)-\mathfrak{y}(\beta,t))}{|\zeta(\alpha,t)-\zeta(\beta,t)|^2}u_\beta\,d\beta\\&\quad+\frac{2}{\pi}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2\mathfrak{y}_\beta\,d\beta,\end{aligned} \tag{2.27}$$

$$\mathcal{P}v=(\partial_t+b\partial_\alpha)(\mathcal{P}\chi)+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}iA\partial_\alpha\chi, \tag{2.28}$$

$$\begin{aligned}\mathcal{P}\lambda&=-\left[u,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right](\bar\zeta_\alpha w)+[u,\bar{\mathcal{H}}]\left(\bar u\frac{u_\alpha}{\bar\zeta_\alpha}\right)+u[u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}\\&\quad-2[u,\mathcal{H}]\frac{u\cdot u_\alpha}{\zeta_\alpha}+\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2u\cdot\zeta_\beta\,d\beta.\end{aligned} \tag{2.29}$$

(2.20) and (2.21) give

$$(I-\mathcal{H})b=-[u,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}, \qquad (I-\mathcal{H})A=1+i[u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}+i[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha} \tag{2.30}$$

and from (2.1) and (2.7), we have

$$w+i=iA\zeta_\alpha, \quad \text{and} \tag{2.31}$$

$$\begin{aligned}&(I-\mathcal{H})\left(-i\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\bar\zeta_\alpha\right)\\&\quad=2[w,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}+2[u,\mathcal{H}]\frac{\bar w_\alpha}{\zeta_\alpha}-\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2\bar u_\beta\,d\beta.\end{aligned} \tag{2.32}$$

Notice that (2.27) to (2.29) are of the form $\partial_t^2\Theta-i\partial_\alpha\Theta=G$ with G consisting of only cubic and higher order terms. These equations together with (2.30) to (2.32) and the relations among the quantities u, ζ, v, χ and λ:

$$\chi = (I-\mathcal{H})(\zeta-\bar\zeta), \qquad v = (\partial_t + b\partial_\alpha)\chi, \quad \text{and} \quad \lambda = (I-\mathcal{H})(\psi\circ k^{-1}) \tag{2.33}$$

will be the main equations we use to derive our estimates on the solution z of the water wave system (2.1)–(2.2). In what follows we record some further relations consequent to (2.33).

Proposition 2.5 *Let χ, λ, v and ζ, u be as defined in* (2.25), (2.26). *We have*

$$\partial_\alpha\chi = \left(I - \zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}\right)(\zeta_\alpha - \bar\zeta_\alpha), \tag{2.34}$$

$$v = (\partial_t + b\partial_\alpha)\chi = 2u - (\mathcal{H}+\bar{\mathcal{H}})u - [u,\mathcal{H}]\frac{\zeta_\alpha-\bar\zeta_\alpha}{\zeta_\alpha}, \tag{2.35}$$

$$\bar u\zeta_\alpha = \partial_\alpha\bar\lambda + \frac{1}{2}\left(\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha} + \bar\zeta_\alpha\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u\zeta_\alpha). \tag{2.36}$$

(2.34) is straightforward by applying (2.5) to the definition $\chi = (I-\mathcal{H})(\zeta-\bar\zeta)$. (2.35) is a straightforward consequence of the definition of $\mathfrak{v}$, by applying (2.3):

$$\begin{aligned}\mathfrak{v} &= \partial_t(I-\mathfrak{H})(z-\bar z) = (I-\mathfrak{H})(z_t-\bar z_t) - [z_t,\mathfrak{H}]\frac{z_\alpha-\bar z_\alpha}{z_\alpha}\\ &= 2z_t - (\bar{\mathfrak{H}}+\mathfrak{H})z_t - [z_t,\mathfrak{H}]\frac{z_\alpha-\bar z_\alpha}{z_\alpha}\end{aligned} \tag{2.37}$$

we used (2.2) in the last step above. Applying the change of variable U_k^{-1} gives us (2.35). Now taking derivative to α to the definition of λ, we get

$$\begin{aligned}\partial_\alpha\lambda &= \left(I-\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}\right)\partial_\alpha(\psi\circ k^{-1}) = \left(I-\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}\right)\mathrm{Re}\{\bar z_t\circ k^{-1}\zeta_\alpha\}\\ &= -i\left(I-\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}\right)\mathrm{Im}\{\bar z_t\circ k^{-1}\zeta_\alpha\}.\end{aligned} \tag{2.38}$$

Here we used (2.5), (2.15) and again (2.2). Im z indicates the imaginary part of z. Therefore from the fact that $\mathrm{Re}\{\bar z_t\circ k^{-1}\zeta_\alpha\}$ is real valued, we have

$$\begin{aligned}\mathrm{Re}\{\bar z_t\circ k^{-1}\zeta_\alpha\} &= \partial_\alpha\lambda + \zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}\mathrm{Re}\{\bar z_t\circ k^{-1}\zeta_\alpha\}\\ &= \mathrm{Re}\{\partial_\alpha\lambda\} + \frac{1}{2}\left(\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha} + \bar\zeta_\alpha\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)\mathrm{Re}\{\bar z_t\circ k^{-1}\zeta_\alpha\}\end{aligned} \tag{2.39}$$

and similarly from the last equality of (2.38)

$$\mathrm{Im}\{\bar z_t\circ k^{-1}\zeta_\alpha\} = \mathrm{Re}\{i\partial_\alpha\lambda\} + \frac{1}{2}\left(\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha} + \bar\zeta_\alpha\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)\mathrm{Im}\{\bar z_t\circ k^{-1}\zeta_\alpha\}. \tag{2.40}$$

(2.39), (2.40) together gives us (2.36).

Notice that $\zeta_\alpha - \bar\zeta_\alpha$ is pure imaginary. Similar to (2.39), we get from (2.34) that

$$\begin{aligned}2i\mathfrak{y}_\alpha &= \zeta_\alpha - \bar\zeta_\alpha = \partial_\alpha\chi + \zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha}(\zeta_\alpha-\bar\zeta_\alpha)\\ &= i\,\mathrm{Im}\{\partial_\alpha\chi\} + \frac{1}{2}\left(\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha} + \bar\zeta_\alpha\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\zeta_\alpha-\bar\zeta_\alpha).\end{aligned} \tag{2.41}$$

To get the relations of w vs. $\partial_\alpha \chi$ and w vs. $(\partial_t + b\partial_\alpha)v$, we take derivative to t to (2.37) and apply $\partial_t k_\alpha U_k$ to (2.36).

We first take derivative to t to (2.37). We get

$$\begin{aligned}\partial_t \mathfrak{v} &= 2z_{tt} - [\bar z_t, \bar{\mathfrak{H}}]\frac{z_{t\alpha}}{\bar z_\alpha} - [z_t, \mathfrak{H}]\frac{z_{t\alpha}}{z_\alpha} - (\bar{\mathfrak{H}} + \mathfrak{H})z_{tt} \\ &\quad - \frac{1}{\pi i}\int \frac{z_{tt}(\alpha,t) - z_{tt}(\beta,t)}{z(\alpha,t) - z(\beta,t)}(z_\beta - \bar z_\beta)\,d\beta - \frac{1}{\pi i}\int \frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}(z_{t\beta} - \bar z_{t\beta})\,d\beta \\ &\quad + \frac{1}{\pi i}\int \left(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\right)^2 (z_\beta - \bar z_\beta)\,d\beta. \end{aligned} \tag{2.42}$$

Here we used (2.3) in the first step. Applying the coordinate change U_k^{-1} gives us

$$\begin{aligned}(\partial_t + b\partial_\alpha)v &= 2w - [\bar u, \bar{\mathcal{H}}]\frac{u_\alpha}{\bar\zeta_\alpha} - [u, \mathcal{H}]\frac{u_\alpha}{\zeta_\alpha} - (\bar{\mathcal{H}} + \mathcal{H})w \\ &\quad - [w, \mathcal{H}]\frac{\zeta_\alpha - \bar\zeta_\alpha}{\zeta_\alpha} - [u, \mathcal{H}]\frac{u_\alpha - \bar u_\alpha}{\zeta_\alpha} \\ &\quad + \frac{1}{\pi i}\int \left(\frac{u(\alpha,t) - u(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\right)^2 (\zeta_\beta - \bar\zeta_\beta)\,d\beta. \end{aligned} \tag{2.43}$$

To get the relation between w and $\partial_\alpha \chi$ we first rewrite (2.36) as

$$\begin{aligned}\bar u \zeta_\alpha &= \partial_\alpha \bar\lambda + \frac{1}{2}(\zeta_\alpha - \bar\zeta_\alpha)\mathcal{H}\bar u + \frac{1}{2}\bar\zeta_\alpha\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u \zeta_\alpha) \\ &= \partial_\alpha \bar\lambda + \frac{1}{2}(\zeta_\alpha - \bar\zeta_\alpha)\bar u + \frac{1}{2}\bar\zeta_\alpha\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u \zeta_\alpha). \end{aligned} \tag{2.44}$$

Applying $k_\alpha U_k$ to (2.44) gives us

$$\bar z_t z_\alpha = \partial_\alpha \bar\Lambda + \frac{1}{2}(z_\alpha - \bar z_\alpha)\bar z_t + \frac{1}{2}\bar z_\alpha\left(\mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}\right)(\bar z_t z_\alpha).$$

Now taking derivative to t to both sides we get

$$\begin{aligned}\bar z_{tt} z_\alpha + \bar z_t z_{t\alpha} &= \partial_\alpha \partial_t \bar\Lambda + \frac{1}{2}(z_{t\alpha} - \bar z_{t\alpha})\bar z_t + \frac{1}{2}(z_\alpha - \bar z_\alpha)\bar z_{tt} + \frac{1}{2}\bar z_{t\alpha}\left(\mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}\right)(\bar z_t z_\alpha) \\ &\quad + \frac{1}{2}\bar z_\alpha\left(\mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}\right)(\bar z_{tt} z_\alpha + \bar z_t z_{t\alpha}) \\ &\quad - \frac{\bar z_\alpha}{\pi}\int \operatorname{Im}\left(\frac{z_t(\alpha,t) - z_t(\beta,t)}{(z(\alpha,t) - z(\beta,t))^2}\right)\bar z_t z_\beta\,d\beta. \end{aligned} \tag{2.45}$$

For convenience in obtaining estimates in the later part of the paper we further rewrite $\partial_\alpha \partial_t \bar\Lambda$. First we know from the definition and (2.3) that

$$\begin{aligned}\partial_t \Lambda &= (I - \mathfrak{H})\psi_t - [z_t, \mathfrak{H}]\frac{\psi_\alpha}{z_\alpha} = (I - \mathfrak{H})\left(-y + \frac{1}{2}|z_t|^2\right) - [z_t, \mathfrak{H}]\frac{\operatorname{Re}\{\bar z_t z_\alpha\}}{z_\alpha} \\ &= \frac{i}{2}\Pi + \frac{1}{2}[z_t, \mathfrak{H}]\left(\bar z_t - 2\frac{\operatorname{Re}\{\bar z_t z_\alpha\}}{z_\alpha}\right) = \frac{i}{2}\Pi - \frac{1}{2}[z_t, \mathfrak{H}]\frac{z_t \bar z_\alpha}{z_\alpha}. \end{aligned} \tag{2.46}$$

Here we used (2.15) in the second step and (2.2) in the third step. Applying the change of variable U_k^{-1} then take derivative to α to both sides of (2.46), we obtain

$$\partial_\alpha(\partial_t + b\partial_\alpha)\lambda = \frac{i}{2}\partial_\alpha\chi - \frac{1}{2}[u_\alpha, \mathcal{H}]\frac{u\bar{\zeta}_\alpha}{\zeta_\alpha} - \frac{1}{2}[u, \mathcal{H}]\frac{\partial_\alpha(u\bar{\zeta}_\alpha)}{\zeta_\alpha} + \frac{1}{2\pi i}\int \frac{(u(\alpha,t)-u(\beta,t))(\zeta_\alpha-\zeta_\beta)}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}u\bar{\zeta}_\beta\, d\beta. \tag{2.47}$$

Now applying the change of variable $U_k^{-1}\frac{1}{k_\alpha}$ to (2.45), move the second term on the left to the right, replace $\partial_\alpha(\partial_t + b\partial_\alpha)\bar{\lambda}$ with (2.47) and canceling out cancelable terms using (2.2), we arrive at a relation between w and $\partial_\alpha\chi$. We record this relation and (2.41), (2.43), (2.46) (with the change of variables k^{-1}) in the following proposition.

Proposition 2.6 *We have*

$$2i\mathfrak{y}_\alpha = \zeta_\alpha - \bar{\zeta}_\alpha = i\mathrm{Im}\{\partial_\alpha\chi\} + \frac{1}{2}\left(\zeta_\alpha\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\zeta}_\alpha\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)(\zeta_\alpha - \bar{\zeta}_\alpha), \tag{2.48}$$

$$(\partial_t + b\partial_\alpha)v = 2w - [\bar{u}, \bar{\mathcal{H}}]\frac{u_\alpha}{\bar{\zeta}_\alpha} - [u, \mathcal{H}]\frac{2u_\alpha - \bar{u}_\alpha}{\zeta_\alpha} - (\bar{\mathcal{H}} + \mathcal{H})w - [w, \mathcal{H}]\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha} + \frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(\zeta_\beta - \bar{\zeta}_\beta)\, d\beta, \tag{2.49}$$

$$\begin{aligned}\bar{w}\zeta_\alpha = &-\frac{i}{2}\partial_\alpha\bar{\chi} + \frac{1}{2}\bar{\mathcal{H}}\left(\bar{u}_\alpha\frac{\bar{u}\zeta_\alpha}{\bar{\zeta}_\alpha}\right) - \frac{1}{2}[\bar{u}, \bar{\mathcal{H}}]\frac{\partial_\alpha(\bar{u}\zeta_\alpha)}{\bar{\zeta}_\alpha}\\ &- \frac{1}{2\pi i}\int\frac{(\bar{u}(\alpha,t)-\bar{u}(\beta,t))(\bar{\zeta}_\alpha-\bar{\zeta}_\beta)}{(\bar{\zeta}(\alpha,t)-\bar{\zeta}(\beta,t))^2}\bar{u}\zeta_\beta\, d\beta + \frac{1}{2}(\zeta_\alpha - \bar{\zeta}_\alpha)\bar{w} - \frac{1}{2}u_\alpha\bar{u}\\ &+ \frac{1}{2}\bar{\zeta}_\alpha\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)(\bar{w}\zeta_\alpha + \bar{u}u_\alpha)\\ &- \frac{\bar{\zeta}_\alpha}{\pi}\int \mathrm{Im}\left(\frac{u(\alpha,t)-u(\beta,t)}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}\right)\bar{u}\zeta_\beta\, d\beta,\end{aligned} \tag{2.50}$$

$$(\partial_t + b\partial_\alpha)\lambda = \frac{i}{2}\chi - \frac{1}{2}[u, \mathcal{H}]\frac{u\bar{\zeta}_\alpha}{\zeta_\alpha}. \tag{2.51}$$

From (2.35), (2.36), (2.48) to (2.50), we see that the differences:

$$v - 2u, \qquad u - \partial_\alpha\lambda, \qquad 2\mathfrak{y}_\alpha - \mathrm{Im}\,\partial_\alpha\chi, \qquad 2w - (\partial_t + b\partial_\alpha)v \quad \text{and} \quad w - \frac{i}{2}\partial_\alpha\chi$$

are each at least of quadratic order. In the next section, after developing some basic analytic tools, we will use the relations (2.35), (2.49), (2.31) to derive estimates of the L^2 norms of derivatives of the quantities u, w and $\zeta_\alpha - 1$ via the L^2 norms of derivatives of $(\partial_t + b\partial_\alpha)\chi$ and $(\partial_t + b\partial_\alpha)v$; and we will use (2.36), (2.48), (2.50) to derive estimates of the L^∞ norms of derivatives of u, w and $\zeta_\alpha - 1$ via the L^∞ norms of derivatives of $\partial_\alpha\lambda$ and $\partial_\alpha\chi$.

We record in the following one more identity.

Proposition 2.7 *We have*

$$(I-\mathcal{H})(\partial_t+b\partial_\alpha)b=[u,\mathcal{H}]\frac{\partial_\alpha(2b-\bar{u})}{\zeta_\alpha}-[w,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}$$
$$+\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(\bar{\zeta}_\beta-1)\,d\beta. \tag{2.52}$$

We prove (2.52) from (2.23). Taking derivative to t to (2.23) and using (2.3), we have

$$-\partial_t(I-\mathfrak{H})k_t=-(I-\mathfrak{H})\partial_t k_t+[z_t,\mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}k_t=\partial_t[z_t,\mathfrak{H}]\frac{\bar{z}_\alpha-k_\alpha}{z_\alpha}$$
$$=[z_{tt},\mathfrak{H}]\frac{\bar{z}_\alpha-k_\alpha}{z_\alpha}+[z_t,\mathfrak{H}]\frac{\bar{z}_{t\alpha}-k_{t\alpha}}{z_\alpha}$$
$$-\frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2(\bar{z}_\beta-k_\beta)\,d\beta. \tag{2.53}$$

Recall $b=k_t\circ k^{-1}$, so $(\partial_t+b\partial_\alpha)b=k_{tt}\circ k^{-1}$. Making the change of variables U_k^{-1} to (2.53) we get (2.52).

3 Basic analysis preparations

In this section we first present and develop some necessary analytic tools. We then give the estimates of the L^2 and L^∞ norms of the derivatives of the quantities u, w and $\zeta_\alpha-1$ respectively via the L^2 norms of the derivatives of $(\partial_t+b\partial_\alpha)\chi$ and $(\partial_t+b\partial_\alpha)v$ and the L^∞ norms of derivatives of $\partial_\alpha\lambda$ and $\partial_\alpha\chi$.

For a function $f=f(\alpha,t)$, we use the notation

$$\|f(t)\|_2=\|f(t)\|_{L^2}=\|f(\cdot,t)\|_{L^2(R)},\qquad \|f(t)\|_\infty=\|f(t)\|_{L^\infty}=\|f(\cdot,t)\|_{L^\infty(R)}.$$

We use C, c to indicate universal constants, while $c(F,\|H'\|_{L^\infty})$, $c(L,\mu)$, $c(M)$ etc. indicate constants depending on F, $\|H'\|_{L^\infty}$, or L, μ, or M respectively. Constants $c(F,\|H'\|_{L^\infty})$, $c(L,\mu)$, $c(M)$ and C, c etc. appearing in different contexts need not be the same. $[a]$ indicates the largest integer $\le a$.

We know the invariant vector fields for $\partial_t^2-i\partial_\alpha$[10] are consisting of (see [4, 35])

$$\partial_t,\qquad \partial_\alpha,\qquad L_0=\frac{1}{2}t\partial_t+\alpha\partial_\alpha\quad\text{and}\quad \Omega_0=\alpha\partial_t+\frac{1}{2}ti \tag{3.1}$$

with the commutativity relations:

$$[\partial_t,L_0]=\frac{1}{2}\partial_t,\qquad [\partial_\alpha,L_0]=\partial_\alpha,\qquad [\partial_t,\partial_\alpha]=0\quad\text{and} \tag{3.2}$$
$$[\partial_t,\Omega_0]=\frac{i}{2}I,\qquad [\partial_\alpha,\Omega_0]=\partial_t,\qquad [L_0,\Omega_0]=\frac{1}{2}\Omega_0. \tag{3.3}$$

[10]Notice that $\partial_t^2-i\partial_\alpha$ is the Schrödinger operator with t, α interchanged.

In what follows, we let

$$\partial_t, \qquad \partial_\alpha \quad \text{and} \quad L_0$$

be denoted respectively by Γ_i, $i = 1, 2, 3$. At times we shall suppress the subscript. If $k = (k_1, k_2, k_3)$ is a multi-index, we shall write

$$\Gamma^k = \Gamma_1^{k_1}\Gamma_2^{k_2}\Gamma_3^{k_3}.$$

For a nonnegative integer k, we shall also use Γ^k to indicate a k-product of Γ_i, $i = 1, 2, 3$. We have the following generalized Sobolev inequality.

Proposition 3.1 *Let $f \in C^\infty(R^{1+1})$ vanish as $|\alpha| \to \infty$. We have for $t > 0$,*

$$(1 + t + |\alpha|)^{1/2}|f(\alpha, t)| \le C\left(\sum_{|k|\le 2} \|\Gamma^k f(t)\|_2 + \sum_{|k|\le 1} \|\Omega_0\Gamma^k f(t)\|_2\right), \tag{3.4}$$

where C is a constant independent of f.

Proof Notice that

$$\alpha L_0 - \frac{t}{2}\Omega_0 = \alpha^2\partial_\alpha - \frac{t^2}{4}i. \tag{3.5}$$

Therefore

$$\alpha^2\partial_\alpha(e^{i\frac{t^2}{4\alpha}} f) = e^{i\frac{t^2}{4\alpha}}\left(\alpha L_0 f - \frac{t}{2}\Omega_0 f\right)$$

and we have

$$\begin{aligned} |f(\alpha, t)| &\le \int_{|\alpha|}^{\infty} |\partial_\beta(e^{i\frac{t^2}{4\beta}} f)|\, d\beta + \int_{-\infty}^{-|\alpha|} |\partial_\beta(e^{i\frac{t^2}{4\beta}} f)|\, d\beta \\ &\le \int_{|\alpha|}^{\infty} \frac{1}{\beta^2}\left|\beta L_0 f - \frac{t}{2}\Omega_0 f\right| d\beta + \int_{-\infty}^{-|\alpha|} \frac{1}{\beta^2}\left|\beta L_0 f - \frac{t}{2}\Omega_0 f\right| d\beta \\ &\le \frac{2}{|\alpha|^{1/2}}\|L_0 f(t)\|_2 + \frac{t}{|\alpha|^{3/2}}\|\Omega_0 f(t)\|_2. \end{aligned} \tag{3.6}$$

On the other hand, (3.5) implies that

$$\frac{t^2 i}{4} f = -\alpha L_0 f + \frac{t}{2}\Omega_0 f + \alpha^2\partial_\alpha f.$$

So from the standard Sobolev embedding and (3.6), we have

$$\begin{aligned} \frac{t^2}{4}|f(\alpha, t)| &\le |\alpha||L_0 f| + \frac{t}{2}|\Omega_0 f| + \alpha^2|\partial_\alpha f| \\ &\le |\alpha|(\|L_0 f(t)\|_2 + \|\partial_\alpha L_0 f(t)\|_2) + t(\|\Omega_0 f(t)\|_2 + \|\partial_\alpha\Omega_0 f(t)\|_2) \\ &\quad + 2|\alpha|^{3/2}\|L_0\partial_\alpha f(t)\|_2 + t|\alpha|^{1/2}\|\Omega_0\partial_\alpha f(t)\|_2. \end{aligned} \tag{3.7}$$

We obtain (3.4) by applying the standard Sobolev embedding to the case that $t+|\alpha|\le 2$, applying (3.6) to the case that $|\alpha|\ge t$ and $|\alpha|\ge 1$, and (3.7) to the case that $|\alpha|\le t$ and $t\ge 1$. □

We now record some commutativity relations. We note that these relations hold for operators $\mathcal{P}$ with general coefficients b, A, and for general surfaces ζ and its associated Hilbert transform $\mathcal{H}$.

We have the following commutativity relations between $\partial_t+b\partial_\alpha$ and the vector fields Γ_i, $i=1,2,3$,

$$\begin{aligned} &[\partial_t,\partial_t+b\partial_\alpha]=b_t\partial_\alpha, \qquad [\partial_\alpha,\partial_t+b\partial_\alpha]=b_\alpha\partial_\alpha,\\ &[L_0,\partial_t+b\partial_\alpha]=\left(L_0b-\frac{1}{2}b\right)\partial_\alpha-\frac{1}{2}(\partial_t+b\partial_\alpha). \end{aligned} \tag{3.8}$$

For any positive integer k, and any operator P,

$$[\Gamma^j,P]=\sum_{k=1}^{j}\Gamma^{j-k}[\Gamma,P]\Gamma^{k-1}. \tag{3.9}$$

Between $\mathcal{P}=(\partial_t+b\partial_\alpha)^2-iA\partial_\alpha$, and Γ_i, $i=1,2,3$, we have

$$\begin{aligned} [\partial_t,\mathcal{P}]&=(\partial_t+b\partial_\alpha)b_t\partial_\alpha+b_t\partial_\alpha(\partial_t+b\partial_\alpha)-iA_t\partial_\alpha\\ &=\{\partial_t(\partial_t+b\partial_\alpha)b-b_tb_\alpha\}\partial_\alpha+b_t\{(\partial_t+b\partial_\alpha)\partial_\alpha+\partial_\alpha(\partial_t+b\partial_\alpha)\}-iA_t\partial_\alpha,\\ [\partial_\alpha,\mathcal{P}]&=(\partial_t+b\partial_\alpha)b_\alpha\partial_\alpha+b_\alpha\partial_\alpha(\partial_t+b\partial_\alpha)-iA_\alpha\partial_\alpha\\ &=\{\partial_\alpha(\partial_t+b\partial_\alpha)b-b_\alpha^2\}\partial_\alpha+b_\alpha\{(\partial_t+b\partial_\alpha)\partial_\alpha+\partial_\alpha(\partial_t+b\partial_\alpha)\}-iA_\alpha\partial_\alpha,\\ [L_0,\mathcal{P}]&=-\mathcal{P}+(\partial_t+b\partial_\alpha)\left(L_0b-\frac{1}{2}b\right)\partial_\alpha+\left(L_0b-\frac{1}{2}b\right)\partial_\alpha(\partial_t+b\partial_\alpha)-i(L_0A)\partial_\alpha\\ &=-\mathcal{P}+\left\{L_0(\partial_t+b\partial_\alpha)b-\left(L_0b-\frac{1}{2}b\right)b_\alpha\right\}\partial_\alpha\\ &\quad+\left(L_0b-\frac{1}{2}b\right)\{(\partial_t+b\partial_\alpha)\partial_\alpha+\partial_\alpha(\partial_t+b\partial_\alpha)\}-i(L_0A)\partial_\alpha. \end{aligned} \tag{3.10}$$

Between $\mathcal{H}$ and Γ_i, $i=1,2,3$, we have

$$\begin{aligned} &[\partial_t,\mathcal{H}]f=[\zeta_t,\mathcal{H}]\frac{f_\alpha}{\zeta_\alpha}, \qquad [\partial_\alpha,\mathcal{H}]f=[\zeta_\alpha,\mathcal{H}]\frac{f_\alpha}{\zeta_\alpha}, \quad \text{and}\\ &[L_0,\mathcal{H}]f=[L_0\zeta-\zeta,\mathcal{H}]\frac{f_\alpha}{\zeta_\alpha} \end{aligned} \tag{3.11}$$

for $f\in C^1$, f, f_α vanish as $|\alpha|\to\infty$. Define Γ', such that

$$\Gamma'=\Gamma, \quad \text{for } \Gamma=\partial_t,\ \partial_\alpha; \quad \text{and} \quad \Gamma'=\Gamma-I, \quad \text{for } \Gamma=L_0.$$

We can rewrite (3.11) as

$$[\Gamma,\mathcal{H}]f=[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha f}{\zeta_\alpha}. \tag{3.12}$$

Notice that

$$\partial_\alpha \Gamma' = \Gamma \partial_\alpha. \tag{3.13}$$

Let

$$\mathbf{K} f(\alpha, t) = \text{p.v.} \int K(\alpha, \beta; t) f(\beta, t)\, d\beta,$$

where either K or $(\alpha - \beta)K(\alpha, \beta; t)$ is continuous and bounded, and K is smooth away from the diagonal $\Delta = \{(\alpha, \beta) \mid \alpha = \beta\}$. We have

$$\begin{aligned}
[\partial_t, \mathbf{K}] f(\alpha, t) &= \int \partial_t K(\alpha, \beta; t) f(\beta, t)\, d\beta, \\
[\partial_\alpha, \mathbf{K}] f(\alpha, t) &= \int (\partial_\alpha + \partial_\beta) K(\alpha, \beta; t) f(\beta, t)\, d\beta, \\
[L_0, \mathbf{K}] f(\alpha, t) &= \int \left(\alpha \partial_\alpha + \beta \partial_\beta + \frac{1}{2} t \partial_t \right) K(\alpha, \beta; t) f(\beta, t)\, d\beta + \mathbf{K} f(\alpha, t)
\end{aligned} \tag{3.14}$$

for $f \in C^1(R^{1+1})$ vanish as $|\alpha| \to \infty$.

The proof of (3.8) to (3.14) are straightforward. We omit them.

Let $H \in C^1(R; R^d),\ A_i \in C^1(R), i = 1, \dots, m,\ F \in C^\infty(R^d)$. Define

$$C_1(H, A, f)(x) = \text{p.v.} \int F\left(\frac{H(x) - H(y)}{x - y} \right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^{m+1}} f(y)\, dy. \tag{3.15}$$

Proposition 3.2 *There exist constants $c_1 = c_1(F, \|H'\|_{L^\infty}), c_2 = c_2(F, \|H'\|_{L^\infty})$, such that*

1. *For any $f \in L^2,\ A_i' \in L^\infty,\ 1 \le i \le m$,*

$$\|C_1(H, A, f)\|_{L^2} \le c_1 \|A_1'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^2}. \tag{3.16}$$

2. *For any $f \in L^\infty,\ A_i' \in L^\infty,\ 2 \le i \le m,\ A_1' \in L^2$,*

$$\|C_1(H, A, f)\|_{L^2} \le c_2 \|A_1'\|_{L^2} \|A_2'\|_{L^\infty} \dots \|A_m'\|_{L^\infty} \|f\|_{L^\infty}. \tag{3.17}$$

(3.16) is a result of Coifman, McIntosh and Meyer [9, 10].

We prove (3.17) by the Tb Theorem [12]. Without loss of generality, we assume $f \ge \frac{1}{3}\|f\|_{L^\infty}$, for otherwise we consider $f + 2\|f\|_{L^\infty}$ and $2\|f\|_{L^\infty}$ respectively. Furthermore for simplicity, we assume $m = 1$, and we let $a = A_1'$. The general case can be proved similarly. We know

$$C_1(H, A, f)(x) = \int\int_y^x F\left(\frac{H(x) - H(y)}{x - y} \right) \frac{f(y)}{(x - y)^2} a(z)\, dz\, dy = \int k(x, z) a(z)\, dz,$$

where

$$k(x, z) = \begin{cases} \int_{-\infty}^z F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{f(y)}{(x - y)^2}\, dy, & \text{for } z < x, \\ -\int_z^\infty F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{f(y)}{(x - y)^2}\, dy, & \text{for } z > x. \end{cases} \tag{3.18}$$

Let

$$T(a)=\int k(x,z)a(z)\,dz=C_1(H,A,f).$$

It is easy to check that the kernel function k in (3.18) satisfies the standard estimates:

$$|k(x,z)|\le c(F,\|H'\|_{L^\infty})\|f\|_{L^\infty}\frac{1}{|x-z|},$$

$$|\partial_z k(x,z)|+|\partial_x k(x,z)|\le c(F,\|H'\|_{L^\infty})\|f\|_{L^\infty}\frac{1}{|x-z|^2}$$

for some constant $c(F,\|H'\|_{L^\infty})$ depending on F and $\|H'\|_{L^\infty}$. Now

$$T(1)=\int F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{f(y)}{x-y}\,dy\in BMO$$

with $\|T(1)\|_{BMO}\le c(F,\|H'\|_{L^\infty})\|f\|_{L^\infty}$ (this is a consequence of (3.16) and the Theorem in [16, p. 49]), and

$$\begin{aligned}\langle T^*(f),\ a\rangle&=\langle f,\ T(a)\rangle\\&=\iint f(x)F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{A_1(x)-A_1(y)}{(x-y)^2}f(y)\,dy\,dx=0,\quad\forall\, a.\end{aligned}$$

This implies that $T^*(f)=0$. What remains to be checked is that fT satisfies the weak bounded-ness property.

Take $\eta,\varsigma\in C_0^\infty(R)$, with supp η, supp $\varsigma\subset Q$ for some interval Q. Let $\delta=\int\varsigma$. Take $\rho\in C_0^\infty(R)$, such that $\rho(x)=1$ for $x\in 3Q$, $\rho(x)=0$ for $x\notin 4Q$, and $0\le\rho(x)\le 1$. Here mQ is the interval with the same center as Q, of length $|mQ|=m|Q|$. We have

$$\begin{aligned}\langle fT(\varsigma),\eta\rangle&=\iint f(x)\eta(x)F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\delta(x)-\delta(y)}{(x-y)^2}f(y)\,dx\,dy\\&=\iint f(x)\eta(x)F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\delta(x)-\delta(y)}{(x-y)^2}f(y)\rho(y)\,dx\,dy\\&\quad+\iint f(x)\eta(x)F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\delta(x)-\delta(y)}{(x-y)^2}f(y)(1-\rho(y))\,dx\,dy\\&=I+II.\end{aligned}$$

Now from (3.16) we have

$$|I|\le c_1\|f\eta\|_{L^2}\|\varsigma\|_{L^\infty}\|f\rho\|_{L^2}\le 2c_1\|f\|_{L^\infty}^2|Q|\|\eta\|_{L^\infty}\|\varsigma\|_{L^\infty}.$$

On the other hand,

$$\begin{aligned}|II|&\le\iint_{|x-y|\ge|Q|}\left|f(x)\eta(x)F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\delta(x)-\delta(y)}{(x-y)^2}f(y)\right|dx\,dy\\&\le c(F,\|H'\|_{L^\infty})\|f\|_{L^\infty}^2\|\varsigma\|_{L^1}\iint_{|x-y|>|Q|}\frac{|\eta(x)|}{(x-y)^2}\,dx\,dy\\&\le c(F,\|H'\|_{L^\infty})\|f\|_{L^\infty}^2|Q|\|\varsigma\|_{L^\infty}\|\eta\|_{L^\infty},\end{aligned}$$

where $c(F, \|H'\|_{L^\infty})$ is a constant depending on F and $\|H'\|_{L^\infty}$. Therefore $|\langle fT(\varsigma), \eta\rangle| \le c\|f\|_{L^\infty}^2|Q|\|\varsigma\|_{L^\infty}\|\eta\|_{L^\infty}$ for some constant c. This proves that fT is weakly bounded. Therefore from the Tb Theorem [12], we have (3.17).

Now let H, A_i, F satisfy the same assumptions as in (3.15). Define

$$C_2(H, A, f)(x) = \int F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^m} \partial_y f(y)\, dy. \tag{3.19}$$

We have the following inequalities.

Proposition 3.3 *There exist constants $c_3 = c_3(F, \|H'\|_{L^\infty})$, $c_4 = c_4(F, \|H'\|_{L^\infty})$, such that*

1. *For any $f \in L^2,\ A_i' \in L^\infty,\ 1 \le i \le m$,*

$$\|C_2(H, A, f)\|_{L^2} \le c_3\|A_1'\|_{L^\infty} \dots \|A_m'\|_{L^\infty}\|f\|_{L^2}. \tag{3.20}$$

2. *For any $f \in L^\infty,\ A_i' \in L^\infty,\ 2 \le i \le m,\ A_1' \in L^2$,*

$$\|C_2(H, A, f)\|_{L^2} \le c_4\|A_1'\|_{L^2}\|A_2'\|_{L^\infty} \dots \|A_m'\|_{L^\infty}\|f\|_{L^\infty}. \tag{3.21}$$

Remark (3.16), (3.20) hold for $m = 0$ as well. In this case, $\|A_1'\|_{L^\infty} \dots \|A_m'\|_{L^\infty}$ is replaced by 1.

Using integration by parts, one can easily convert the operator $C_2(H, A, f)$ into a sum of operators of the form $C_1(H, A, f)$. (3.20) and (3.21) then follow from (3.16) and (3.17). We omit the details.

Let $C_1(H, A, f)$ be as defined in (3.15). We have the following L^∞ estimate.

Proposition 3.4 *There exists a constant $c = c(F, \|H'\|_{L^\infty}, \|H''\|_{L^\infty})$, such that for any real number $r > 0$,*

$$\|C_1(H, A, f)\|_{L^\infty} \le c\left(\prod_{i=1}^m (\|A_i'\|_{L^\infty} + \|A_i''\|_{L^\infty})(\|f\|_{L^\infty} + \|f'\|_{L^\infty}) + \prod_{i=1}^m \|A_i'\|_{L^\infty}\|f\|_{L^\infty} \ln r + \prod_{i=1}^m \|A_i'\|_{L^\infty}\|f\|_{L^2}\frac{1}{r^{1/2}}\right). \tag{3.22}$$

Remark (3.22) holds for $m = 0$ as well. In this case, $\prod_{i=1}^m(\|A_i'\|_{L^\infty} + \|A_i''\|_{L^\infty})$ and $\prod_{i=1}^m \|A_i'\|_{L^\infty}$ are replaced by 1.

Proof For simplicity, we assume $m = 1$. The general case can be proved similarly. We have

$$\begin{aligned} C_1(H, A, f) &= \text{p.v.}\int_{|x-y|\le 1} + \int_{1<|x-y|<r} \\ &\quad + \int_{|x-y|\ge r} F\left(\frac{H(x) - H(y)}{x - y}\right)\frac{A_1(x) - A_1(y)}{(x - y)^2} f(y)\, dy \\ &= I + II + III. \end{aligned}$$

Now

$$
\begin{aligned}
|I| &= \bigg| \int_{|x-y|\le 1} \left(F\left(\frac{H(x)-H(y)}{x-y}\right) - F(H'(x)) \right) \frac{A_1(x)-A_1(y)}{(x-y)^2} f(y)\,dy \\
&\quad + F(H'(x)) \int_{|x-y|\le 1} \frac{A_1(x)-A_1(y)-A_1'(x)(x-y)}{(x-y)^2} f(y)\,dy \\
&\quad + F(H'(x)) \int_{|x-y|\le 1} \frac{A_1'(x)}{x-y}(f(y)-f(x))\,dy \bigg| \\
&\le 2\|F'\|_{L^\infty}\|H''\|_{L^\infty}\|A_1'\|_{L^\infty}\|f\|_{L^\infty} + 2\|F\|_{L^\infty}\|A_1''\|_{L^\infty}\|f\|_{L^\infty} \\
&\quad + 2\|F\|_{L^\infty}\|A_1'\|_{L^\infty}\|f'\|_{L^\infty}, \\
|II| &\le \|F\|_{L^\infty}\|A_1'\|_{L^\infty}\|f\|_{L^\infty} \ln r,
\end{aligned}
$$

and

$$
|III| \le \|F\|_{L^\infty}\|A_1'\|_{L^\infty} \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|}\,dy \le \|F\|_{L^\infty}\|A_1'\|_{L^\infty}\|f\|_{L^2}\frac{1}{r^{1/2}}.
$$

This proves (3.22). □

For referencing convenience, we record the following standard Sobolev inequality.

Proposition 3.5 *There exists a constant* $c>0$, *such that for any* $f\in C^\infty(R)$,

$$
\|f\|_{L^\infty} \le c(\|f\|_{L^2} + \|f'\|_{L^2}). \tag{3.23}
$$

3.1 Estimates between the quantities u, w, $\zeta_\alpha - 1$ and v, χ, λ

Before moving onto proving a priori estimates of the solutions of the water wave system (2.1)–(2.2), we establish estimates of the L^2 and L^∞ norms of the Γ-derivatives of the quantities $u,\ w$ and $\zeta_\alpha - 1$ respectively via the L^2 norms of the Γ-derivatives of $(\partial_t + b\partial_\alpha)\chi$ and $(\partial_t + b\partial_\alpha)v$ and the L^∞ norms of the Γ-derivatives of $\partial_\alpha\lambda$ and $\partial_\alpha\chi$.

We make the following assumptions on the solution for the rest of this section.

Let $l\ge 2$, $z=z(\alpha,t)$, $t\in[0,T]$ be a solution of the water wave system (2.1)–(2.2). Assume that the function $k(\cdot,t): R\to R$ defined by (2.18) is a diffeomorphism and $k_\alpha>0$ for $t\in[0,T]$. Assume

$$
\Gamma^j(\zeta_\alpha - 1),\ \Gamma^j u,\ \Gamma^j w \in C([0,T], L^2(R)), \quad \text{for all } |j|\le l.
$$

Let $t\in[0,T]$ be fixed. Assume that at this time t,

$$
\sum_{|j|\le l}(\|\Gamma^j(\zeta_\alpha-1)(t)\|_2 + \|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le M < \infty \quad \text{and} \tag{3.24}
$$

$$
|\zeta(\alpha,t)-\zeta(\beta,t)| \ge \frac{1}{2}|\alpha-\beta| \quad \text{for } \alpha,\ \beta\in R. \tag{3.25}
$$

We have the following Propositions and Lemmas.

Proposition 3.6 *There exists a constant $M_0 > 0$, independent of t, such that*

1. *For $M \le M_0$, $0 \le j \le l-1$,*

$$\sum_{|k|\le j} \|\Gamma^k(\zeta_\alpha - \bar{\zeta}_\alpha)(t)\|_\infty \le c(M) \sum_{|k|\le j} \|\Gamma^k \mathrm{Im}\partial_\alpha \chi(t)\|_\infty \tag{3.26}$$

$$\sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty \le c(M)\Bigg(\sum_{|k|\le j} \|\Gamma^k \partial_\alpha \lambda(t)\|_\infty + \|\partial_\alpha \chi(t)\|_\infty\Bigg). \tag{3.27}$$

2. *For $M \le M_0$, and $0 \le j \le l-2$,*

$$\sum_{|k|\le j} \|\Gamma^k w(t)\|_\infty \le c(M)\Bigg(\sum_{|k|\le j} \|\Gamma^k \partial_\alpha \chi(t)\|_\infty + \sum_{|k|\le [\frac{j}{2}]+1} \|\Gamma^k \partial_\alpha \lambda(t)\|_\infty\Bigg) \tag{3.28}$$

here $c(M)$ is a constant depending only on M and j.

To prove (3.26), we rewrite (2.48) as

$$\zeta_\alpha - \bar{\zeta}_\alpha = i\mathrm{Im}\{\partial_\alpha \chi\} + \frac{1}{2}(\zeta_\alpha - \bar{\zeta}_\alpha)\mathcal{H}\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha} + \frac{1}{2}\bar{\zeta}_\alpha\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)(\zeta_\alpha - \bar{\zeta}_\alpha).$$

We have for any $0 \le j \le l-1$,

$$\begin{aligned}\|\Gamma^j(\zeta_\alpha - \bar{\zeta}_\alpha)(t)\|_\infty &\le \|\Gamma^j \mathrm{Im}\{\partial_\alpha \chi\}(t)\|_\infty \\ &\quad + c\sum_{|k|\le j} \|\Gamma^k(\zeta_\alpha - \bar{\zeta}_\alpha)(t)\|_\infty \sum_{|k|\le j} \left\|\Gamma^k \mathcal{H}\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_\infty \\ &\quad + c(M)\sum_{|k|\le j}\left\|\Gamma^k\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)(\zeta_\alpha - \bar{\zeta}_\alpha)(t)\right\|_\infty.\end{aligned}$$

Now using Proposition 3.5, (3.9), (3.12), (3.14) and Proposition 3.3, we get

$$\begin{aligned}&\sum_{|k|\le j}\left\|\Gamma^k \mathcal{H}\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_\infty \\ &\le c\sum_{|k|\le j+1}\left\|\Gamma^k \mathcal{H}\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_2 \\ &\le c\sum_{|k|\le j+1}\left(\sum_{m=1}^{k}\left\|\Gamma^{k-m}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{m-1}\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_2 + \left\|\mathcal{H}\Gamma^k\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_2\right) \\ &\le c(M)\sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2.\end{aligned}$$

While

$$\left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)f = -\frac{2}{\pi}\int \frac{\mathfrak{y}(\alpha,t) - \mathfrak{y}(\beta,t)}{|\zeta(\alpha,t) - \zeta(\beta,t)|^2} f(\beta)\,d\beta,$$

using Proposition 3.5, (3.14), Proposition 3.2, we have

$$\begin{aligned}
&\sum_{|k|\le j}\left\|\Gamma^k\left(\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\zeta_\alpha-\bar\zeta_\alpha)(t)\right\|_\infty\\
&\quad\le c\sum_{|k|\le j+1}\left\|\Gamma^k\left(\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\zeta_\alpha-\bar\zeta_\alpha)(t)\right\|_2\\
&\quad\le c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-\bar\zeta_\alpha)(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2.
\end{aligned}$$

Therefore

$$\|\Gamma^j(\zeta_\alpha-\bar\zeta_\alpha)(t)\|_\infty\le\|\Gamma^j\mathrm{Im}\{\partial_\alpha\chi\}\|_\infty+c(M)M\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-\bar\zeta_\alpha)(t)\|_\infty.$$

Taking M_0 sufficiently small and $M\le M_0$, we obtain (3.26).

The proof of (3.27) and (3.28) follows similar argument. To prove (3.27), we first rewrite (2.44) as

$$\bar u=\partial_\alpha\bar\lambda+\bar u(1-\zeta_\alpha)+\frac{1}{2}(\zeta_\alpha-\bar\zeta_\alpha)\bar u+\frac{1}{2}\bar\zeta_\alpha\left(\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u\zeta_\alpha).$$

Using Proposition 3.5 to estimate $\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty$; and using Proposition 3.5, (3.14), and Proposition 3.2 to estimate

$$\begin{aligned}
&\sum_{|k|\le j}\left\|\Gamma^k\left(\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u\zeta_\alpha)(t)\right\|_\infty\\
&\quad\le c\sum_{|k|\le j+1}\left\|\Gamma^k\left(\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}\right)(\bar u\zeta_\alpha)(t)\right\|_2\\
&\quad\le c(M)\sum_{|k|\le j+1}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_2\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty+c(M)\sum_{|k|\le j+1}\|\Gamma^k(u\bar\zeta_\alpha)(t)\|_2\|\mathfrak{y}_\alpha(t)\|_{L^\infty},
\end{aligned}$$

we have for $0\le j\le l-1$,

$$\|\Gamma^j u(t)\|_\infty\le\|\Gamma^j\partial_\alpha\lambda(t)\|_\infty+c(M)M\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty.$$

Using (3.26) to estimate $\|\mathfrak{y}_\alpha(t)\|_\infty$. So for sufficiently small M we have (3.27).

To prove (3.28) we use (2.50) and assume $0\le j\le l-2$. The argument is similar as above. We expand

$$\begin{aligned}
&\frac{1}{2\pi i}\int\frac{(\bar u(\alpha,t)-\bar u(\beta,t))(\bar\zeta_\alpha-\bar\zeta_\beta)}{(\bar\zeta(\alpha,t)-\bar\zeta(\beta,t))^2}\bar u\zeta_\beta\,d\beta\\
&\quad=\frac{\bar\zeta_\alpha}{2\pi i}\int\frac{(\bar u(\alpha,t)-\bar u(\beta,t))}{(\bar\zeta(\alpha,t)-\bar\zeta(\beta,t))^2}\bar u\zeta_\beta\,d\beta-\frac{1}{2\pi i}\int\frac{(\bar u(\alpha,t)-\bar u(\beta,t))\bar\zeta_\beta}{(\bar\zeta(\alpha,t)-\bar\zeta(\beta,t))^2}\bar u\zeta_\beta\,d\beta
\end{aligned}$$

and estimate it term by term. Using Propositions 3.2, 3.3 and 3.5, we arrive at

$$\|\Gamma^j w(t)\|_\infty \le \|\Gamma^j \partial_\alpha \chi(t)\|_\infty + c(M) \sum_{|k|\le j+1} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 \sum_{|k|\le j} \|\Gamma^k w(t)\|_\infty + c(M) \sum_{|k|\le j+2} \|\Gamma^k u(t)\|_2 \sum_{|k|\le [\frac{j}{2}]+1} \|\Gamma^k u(t)\|_\infty + c(M)\|\mathfrak{y}_\alpha(t)\|_\infty.$$

Using (3.26), (3.27) to estimate $\sum_{|k|\le[\frac{j}{2}]+1} \|\Gamma^k u(t)\|_\infty$ and $\|\mathfrak{y}_\alpha(t)\|_\infty$. Therefore for sufficiently small $M > 0$ we have (3.28).

Proposition 3.7 1. *There exists a constant $M_0 > 0$, independent of t, such that for $M \le M_0$, $1 \le j \le l$,*

$$\sum_{|k|\le j} \|\Gamma^k u(t)\|_2 + \sum_{|k|\le j} \|\Gamma^k w(t)\|_2 \le c(M)\Bigg(\sum_{|k|\le j} (\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k v(t)\|_2)\Bigg). \tag{3.29}$$

2. *For $1 \le j \le l$,*

$$\sum_{|k|\le j} (\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k v(t)\|_2) \le c(M)\Bigg(\sum_{|k|\le j} \|\Gamma^k u(t)\|_2 + \sum_{|k|\le j} \|\Gamma^k w(t)\|_2\Bigg). \tag{3.30}$$

Here $c(M)$ is a constant depending only on M and j.

The proof of Proposition 3.7 uses (2.35) and (2.49).
From (2.35), rewriting

$$(\mathcal{H} + \bar{\mathcal{H}})u = \mathcal{H}\frac{(\zeta_\alpha - \bar{\zeta}_\alpha)u}{\zeta_\alpha} + \left(\mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right)(\bar{\zeta}_\alpha u) \tag{3.31}$$

and using (3.9), (3.12), (3.14) and Propositions 3.5, 3.2, we have for $1 \le j \le l$,

$$2\|\Gamma^j u(t)\|_2 \le \|\Gamma^j v(t)\|_2 + c(M) \sum_{|k|\le j} \|\Gamma^k u(t)\|_2 \sum_{|k|\le j} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2.$$

On the other hand, from (2.49), rewriting $(\mathcal{H} + \bar{\mathcal{H}})w$ as in (3.31), then applying Propositions 3.5, 3.3, 3.2. In the process using (3.9), (3.12), (3.14). We have for $1 \le j \le l$,

$$2\|\Gamma^j w(t)\|_2 \le \|\Gamma^j(\partial_t + b\partial_\alpha)v(t)\|_2 + c(M) \sum_{|k|\le j} \|\Gamma^k w(t)\|_2 \sum_{|k|\le j} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 + c(M) \sum_{|k|\le j} \|\Gamma^k u(t)\|_2 \sum_{|k|\le \max\{j,2\}} \|\Gamma^k u(t)\|_2.$$

Sum up these two inequalities and using (3.24), we have

$$2(\|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le \|\Gamma^j v(t)\|_2 + \|\Gamma^j(\partial_t + b\partial_\alpha)v(t)\|_2 + c(M)M(\|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2).$$

Therefore for sufficiently small M we have (3.29).

A small modification of the above argument gives us (3.30).

We derive the L^2 and L^∞ estimates of the Γ-derivatives of $\zeta_\alpha - 1$ in turns of that of v, χ and λ as a consequence of (2.31) and Propositions 3.6, 3.7. This requires us to obtain the L^2 and L^∞ estimates of the Γ-derivatives of $A-1$.

We first present the following Lemma.

Lemma 3.8 *Let f be real valued such that*

$$(I - \mathcal{H})f = g.$$

Then

1. *For all $0 \le j \le l$, we have*

$$\sum_{|k|\le j} \|\Gamma^k f(t)\|_2 \le c(M) \sum_{|k|\le j} \|\Gamma^k g(t)\|_2. \tag{3.32}$$

2. *There exists $M_0 > 0$, such that for $M \le M_0$, and $0 \le j \le l-1$,*

$$\sum_{|k|\le j} \|\Gamma^k f(t)\|_\infty \le c(M)\left(\sum_{|k|\le j} \|\Gamma^k g(t)\|_\infty + \|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k g(t)\|_2\right). \tag{3.33}$$

Here $c(M)$ is a constant depending only on M and j.

Remark We remark that Lemma 3.8 in fact holds for a general family of curves $\zeta = \zeta(\cdot, \tau)$, $\tau \in [0, T]$, satisfying $\Gamma^j(\zeta_\alpha - 1) \in C([0,T], L^2(R))$, and at the time t, $\sum_{|j|\le l} \|\Gamma^j(\zeta_\alpha - 1)(t)\|_2 \le M$ and

$$|\zeta(\alpha,t) - \zeta(\beta,t)| \ge \nu|\alpha - \beta| \quad \text{for all } \alpha,\ \beta \in R$$

for some constant $\nu > 0$.

Proof Using (3.12) and (3.9), we have

$$(I - \mathcal{H})\Gamma^j f = \sum_{k=1}^{j} \Gamma^{j-k}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1} f + \Gamma^j g. \tag{3.34}$$

From the fact that f is real valued, we obtain

$$(I - \mathcal{K})\Gamma^j f = \operatorname{Re}\left\{\sum_{k=1}^{j} \Gamma^{j-k}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1} f + \Gamma^j g\right\}, \tag{3.35}$$

where $\mathcal{K} = \frac{1}{2}(\mathcal{H} + \bar{\mathcal{H}})$ is the double layered potential operator. We know the operator $(I - \mathcal{K})^{-1}$ is L^2-bounded, with its $L^2 \to L^2$ operator norm $\|(I-\mathcal{K})^{-1}\|_{2,2} \le c(M)$ for some constant $c(M)$ (see [19], [37, Theorem 2.1.5]).[11] From an inductive argument, (3.14) and Propositions 3.3, 3.5, we obtain (3.32).

[11] Theorem 2.1.5 in [37] in fact states that for the double layered potential operator K defined by the arc-length parametrization, $\|(I-K)^{-1}\|_{2,2}$ only depends on the Lipschitz constant of the domain. This certainly implies that $\|(I-\mathcal{K})^{-1}\|_{2,2} \le c(M)$.

Now we prove (3.33). Let $0 \le j \le l-1$. We know from (3.32) that $\sum_{|k|\le j} \|\Gamma^k f(t)\|_\infty < \infty$ provided $\sum_{|k|\le j+1} \|\Gamma^k g(t)\|_2 < \infty$. Rewriting (3.35)

$$\Gamma^j f = \frac{1}{2}(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f + \operatorname{Re}\left\{\sum_{k=1}^{j} \Gamma^{j-k}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1} f + \Gamma^j g\right\}$$

and using Propositions 3.5, 3.3, we have

$$\begin{aligned}\|\Gamma^j f(t)\|_\infty &\le \frac{1}{2}\|(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f(t)\|_\infty \\ &\quad + c(M)\sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 \sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty + \|\Gamma^j g(t)\|_\infty.\end{aligned}$$

Therefore there exists $M_0 > 0$, such that for $M \le M_0$,

$$\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty \le c(M)\sum_{|k|\le j}\left(\|\Gamma^k g(t)\|_\infty + \frac{1}{2}\|(\mathcal{H}+\bar{\mathcal{H}})\Gamma^k f(t)\|_\infty\right).$$

To estimate $\|(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f(t)\|_\infty$, we use Propositions 3.5, 3.2, and rewriting $\mathcal{H}+\bar{\mathcal{H}}$ as in (3.31), we get

$$\begin{aligned}\|(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f(t)\|_\infty &\le c\|(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f(t)\|_2 + c\|\partial_\alpha(\mathcal{H}+\bar{\mathcal{H}})\Gamma^j f(t)\|_2 \\ &\le c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k f(t)\|_2 \\ &\le c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k g(t)\|_2.\end{aligned}$$

In the last step we used (3.32). This proves (3.33). □

We know from (2.30) that

$$(I-\mathcal{H})(A-1) = i[u,\mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha} + i[w,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}, \tag{3.36}$$

since $\mathcal{H}1 = 0$. Notice that A is real valued. Using Lemma 3.8 (3.32), and using (3.14) and Propositions 3.5, 3.2, 3.3, we have for $1 \le j \le l$,

$$\begin{aligned}\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_2 &\le c(M)\sum_{|k|\le j}\left\|\Gamma^k\left([u,\mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha} + [w,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}\right)(t)\right\|_2 \\ &\le c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2 \sum_{|k|\le \max\{j,2\}}\|\Gamma^k u(t)\|_2 \\ &\quad + c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_2 \sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2.\end{aligned}$$

Therefore for M sufficiently small,

$$\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_2\le c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_2+c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$\text{and}\quad \|(A-1)(t)\|_\infty\le\frac{1}{2}. \tag{3.37}$$

Now (2.31) gives

$$\zeta_\alpha-1=\frac{w}{iA}-\frac{A-1}{A}. \tag{3.38}$$

From (3.38), (3.37) and Proposition 3.7 (3.29), we get

Proposition 3.9 *There exists a constant $M_0>0$, independent of t, such that for $M\le M_0$, $1\le j\le l$,*

$$\sum_{|k|\le j}(\|\Gamma^k w(t)\|_2+\|\Gamma^k u(t)\|_2+\|\Gamma^k(\zeta_\alpha-1)(t)\|_2)$$
$$\le c(M)\sum_{|k|\le j}(\|\Gamma^k(\partial_t+b\partial_\alpha)v(t)\|_2+\|\Gamma^k(\partial_t+b\partial_\alpha)\chi(t)\|_2), \tag{3.39}$$

where $c(M)$ is a constant depending on M and j.

Before deriving the L^∞ estimates of the Γ-derivatives of $\zeta_\alpha-1$, we give the following

Lemma 3.10 *Assume that f, g are smooth and decay fast at infinity. Then for $0\le j\le l-1$,*

$$\left\|\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}(t)\right\|_\infty\le c(M)\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k g(t)\|_2$$
$$+c(M)\|g(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k f(t)\|_2. \tag{3.40}$$

Proof Lemma 3.10 is a consequence of (3.14) and Propositions 3.5, 3.2. Precisely it is obtained by rewriting

$$\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}=\left(\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}-[\Gamma^j f,\mathcal{H}]\frac{g}{\zeta_\alpha}\right)+\Gamma^j f\mathcal{H}\frac{g}{\zeta_\alpha}-\mathcal{H}\frac{g\Gamma^j f}{\zeta_\alpha}$$

and estimating each term using Propositions 3.5, 3.2:

$$\left\|\left(\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}-[\Gamma^j f,\mathcal{H}]\frac{g}{\zeta_\alpha}\right)(t)\right\|_\infty$$
$$\le c\sum_{|k|\le 1}\left\|\Gamma^k\left(\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}-[\Gamma^j f,\mathcal{H}]\frac{g}{\zeta_\alpha}\right)(t)\right\|_2$$
$$\le c(M)\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k g(t)\|_2,$$

$$\left\|\Gamma^j f\mathcal{H}\frac{g}{\zeta_\alpha}(t)\right\|_\infty\le c(M)\|\Gamma^j f(t)\|_\infty\sum_{|k|\le 1}\|\Gamma^k g(t)\|_2$$

and

$$\left\|\mathcal{H}\frac{g\Gamma^j f}{\zeta_\alpha}(t)\right\|_\infty \le c(M)\sum_{|k|\le 1}\|\Gamma^k(g\Gamma^j f)(t)\|_2$$
$$\le c(M)\Big(\|\Gamma^j f(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k g(t)\|_2 + \|g(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k f(t)\|_2\Big). \qquad \square$$

We now derive the L^∞ estimates of the Γ-derivatives of $\zeta_\alpha - 1$. We first have from (3.36), Lemma 3.8 (3.33) and then Propositions 3.5, 3.2 and 3.3, Lemma 3.10 that for M sufficiently small, $0 \le j \le l-2$,

$$\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_\infty$$
$$\le c(M)\Big(\sum_{|k|\le j+1}\left\|\Gamma^k[u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}(t)\right\|_2 + \sum_{|k|\le j}\left\|\Gamma^k[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}(t)\right\|_\infty$$
$$+\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\left\|\Gamma^k[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}(t)\right\|_2\Big)$$
$$\le c(M)\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+2}\|\Gamma^k u(t)\|_2$$
$$+c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2$$
$$+c(M)\|(\zeta_\alpha-1)(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k w(t)\|_2.$$

Therefore

$$\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_\infty \le c(M)\Big\{\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty + \sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty\Big\}$$
$$+c(M)\|(\zeta_\alpha-1)(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k w(t)\|_2.$$

Now (3.38), (3.37) give that for M sufficiently small, $0\le j\le l-1$,

$$\|\Gamma^j(\zeta_\alpha-1)(t)\|_\infty \le c(M)\sum_{|k|\le j}(\|\Gamma^k w(t)\|_\infty + \|\Gamma^k(A-1)(t)\|_\infty).$$

Putting the above two inequalities together we have that for sufficiently small $M>0$, and $0\le j\le l-2$,

$$\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_\infty \le c(M)\Big\{\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty + \sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty\Big\}$$

therefore

$$\|\Gamma^j(\zeta_\alpha - 1)(t)\|_\infty \le c(M)\left\{\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty + \sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty\right\}. \tag{3.41}$$

Estimating the right hand side of (3.41) by Proposition 3.6, we record the result in the following

Proposition 3.11 *There exists a constant $M_0 > 0$, independent of t, such that for $M \le M_0$, $0 \le j \le l-2$,*

$$\|\Gamma^j(\zeta_\alpha - 1)(t)\|_\infty \le c(M)\left(\sum_{|k|\le j}\|\Gamma^k \partial_\alpha \chi(t)\|_\infty + \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k \partial_\alpha \lambda(t)\|_\infty\right) \tag{3.42}$$

for some constant $c(M)$ depending on M and j.

In next section we construct the energy of the water wave system from (2.27) to (2.29). The energy we construct contain terms such as $\|(\partial_t + b\partial_\alpha)\Gamma^j\chi(t)\|_2^2$ and $\|(\partial_t + b\partial_\alpha)\Gamma^j v(t)\|_2^2$. This gives us control of such terms. However the right hand side of (3.39) is in terms of $\|\Gamma^j(\partial_t + b\partial_\alpha)\chi(t)\|_2$ and $\|\Gamma^j(\partial_t + b\partial_\alpha)v(t)\|_2$. So we derive in the following the estimate of $\|\Gamma^j(\partial_t + b\partial_\alpha)\chi(t)\|_2$ and $\|\Gamma^j(\partial_t + b\partial_\alpha)v(t)\|_2$ in terms of $\|(\partial_t + b\partial_\alpha)\Gamma^j\chi(t)\|_2$ and $\|(\partial_t + b\partial_\alpha)\Gamma^j v(t)\|_2$. For this purpose we use (3.8), (3.9). But before we do that we need the estimates of the L^2 norms of the Γ-derivatives of b.

We have from (2.30) and Lemma 3.8 (3.32) that for $1 \le j \le l$,

$$\sum_{|k|\le j}\|\Gamma^k b(t)\|_2 \le c(M)\sum_{|k|\le j}\left\|\Gamma^k[u,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}(t)\right\|_2.$$

Using Propositions 3.2, 3.5, we know for $1 \le j \le l$,

$$\sum_{|k|\le j}\left\|\Gamma^k\left([u,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}\right)(t)\right\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2\sum_{|k|\le j}\|\Gamma^k u(t)\|_2.$$

Therefore for $1 \le j \le l$,

$$\sum_{|k|\le j}\|\Gamma^k b(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2. \tag{3.43}$$

Notice that (3.43) holds without any smallness assumptions on M. We state the following Lemma in terms of a generally given function b.

Lemma 3.12 *Assume that $\Gamma^k b \in C([0,T], L^2(R))$ for $|k| \le l$. We have for $1 \le j \le l$,*

$$\begin{aligned}\|[\partial_t + b\partial_\alpha, \Gamma^j]f(t)\|_2 &\le c\sum_{|k|\le j}\|\Gamma^k b(t)\|_2\sum_{|k|\le\max\{j-1,1\}}\|\Gamma^k\partial_\alpha f(t)\|_2\\ &\quad + c\sum_{|k|\le j-1}\|(\partial_t + b\partial_\alpha)\Gamma^k f(t)\|_2,\end{aligned} \tag{3.44}$$

$$\|[\partial_t + b\partial_\alpha, \Gamma^j]f(t)\|_2 \le c\sum_{|k|\le j}\|\Gamma^k b(t)\|_2 \sum_{|k|\le \max\{j-1,1\}}\|\Gamma^k\partial_\alpha f(t)\|_2 + c\sum_{|k|\le j-1}\|\Gamma^k(\partial_t + b\partial_\alpha)f(t)\|_2, \tag{3.45}$$

where c is a universal constant.

Proof We have from (3.9) that

$$[\Gamma^j, \partial_t + b\partial_\alpha]f = \sum_{k=1}^{j}\Gamma^{j-k}[\Gamma, \partial_t + b\partial_\alpha]\Gamma^{k-1}f.$$

(3.44) and (3.45) follow from (3.8) and Proposition 3.5. □

We now want to use (3.8), (3.9) and (3.43), (3.44) to show that $\|\Gamma^j(\partial_t + b\partial_\alpha)f(t)\|_2$ can be dominated by $\|(\partial_t + b\partial_\alpha)\Gamma^j f(t)\|_2$ for $f = \chi$ and v. To estimate the term $\|\Gamma^k\partial_\alpha\chi(t)\|_2$ in (3.44) for $f = \chi$, we apply Propositions 3.2, 3.3 to (2.34), we have for $1 \le j \le l$,

$$\sum_{|k|\le j}\|\Gamma^k\partial_\alpha\chi(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2. \tag{3.46}$$

Applying (3.39) to the right hand side of (3.46), inserting the estimates (3.43) and (3.46) to (3.44), we get for $2 \le j \le l$,

$$\begin{aligned}\sum_{|k|\le j}&(\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_2)\\ &\le c(M)M\sum_{|k|\le j}(\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_2)\\ &\quad + c\sum_{|k|\le j}(\|(\partial_t + b\partial_\alpha)\Gamma^k v(t)\|_2 + \|(\partial_t + b\partial_\alpha)\Gamma^k\chi(t)\|_2).\end{aligned}$$

From (3.30) we know $\sum_{|k|\le j}(\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_2) < \infty$. Therefore for sufficiently small M, and $2 \le j \le l$, we have

$$\begin{aligned}\sum_{|k|\le j}&(\|\Gamma^k(\partial_t + b\partial_\alpha)v(t)\|_2 + \|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_2)\\ &\le c(M)\sum_{|k|\le j}(\|(\partial_t + b\partial_\alpha)\Gamma^k v(t)\|_2 + \|(\partial_t + b\partial_\alpha)\Gamma^k\chi(t)\|_2).\end{aligned} \tag{3.47}$$

Apply (3.47) to Proposition 3.9, we arrive at the following

Proposition 3.13 *There exists a constant* $M_0 > 0$, *independent of* t, *such that for* $M \le M_0$, $2 \le j \le l$,

$$\sum_{|k|\le j}(\|\Gamma^k w(t)\|_2+\|\Gamma^k u(t)\|_2+\|\Gamma^k(\zeta_\alpha-1)(t)\|_2)$$
$$\le c(M)\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2) \tag{3.48}$$

for some constant $c(M)$ depending on M and j.

Remark A more careful argument with a further estimate of the term $\|\Gamma^k\partial_\alpha v(t)\|_2$ in (3.44) would show that (3.48) holds for $1\le j\le l$. However since we are only concerned with solutions of higher regularities, we are not making efforts to lower the range of j.

We present in the following the estimates of the L^2 norms of the Γ-derivatives of $\partial_\alpha\chi$, $\partial_\alpha\lambda$, v, and v_1, where $v_1=(I-\mathcal{H})v$, in turns of the type of L^2 norms as in the right hand side of (3.48), and the estimates of the L^∞ norms of the Γ-derivatives of $\partial_t\chi$, v, and v_1 in turns of the L^∞ norms of the Γ-derivatives of $\partial_\alpha\chi$ and $\partial_\alpha\lambda$. We derive the estimates of v_1 here, because we will use v_1 instead of v to construct part of the energy functional in the next section.

Let

$$\mathfrak{v}_1=(I-\mathfrak{H})\mathfrak{v},\qquad v_1=\mathfrak{v}_1\circ k^{-1}=(I-\mathcal{H})v. \tag{3.49}$$

We know

$$\mathfrak{v}_1=(I-\mathfrak{H})\mathfrak{v}=(I-\mathfrak{H})\partial_t(I-\mathfrak{H})(z-\bar z).$$

From (2.3) and the fact that $(I-\mathfrak{H})^2=2(I-\mathfrak{H})$, we have

$$\mathfrak{v}_1=2\partial_t(I-\mathfrak{H})(z-\bar z)+[z_t,\mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}(I-\mathfrak{H})(z-\bar z)$$
$$=2\mathfrak{v}+[z_t,\mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\Pi,$$

and with the change of variable U_k^{-1},

$$v_1=2v+[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi. \tag{3.50}$$

Proposition 3.14 *There exists a constant $M_0>0$, independent of t, such that*

1. *for $M\le M_0$, $2\le j\le l$,*

$$\sum_{|k|\le j}(\|\Gamma^k\partial_\alpha\lambda(t)\|_2+\|\Gamma^k\partial_\alpha\chi(t)\|_2)$$
$$\le c(M)\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2)\quad\textit{and} \tag{3.51}$$

$$\sum_{|k|\le j}(\|\Gamma^k v(t)\|_2+\|\Gamma^k v_1(t)\|_2)$$
$$\le c(M)\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2); \tag{3.52}$$

2. *for* $M \le M_0$, $0 \le j \le l-1$,

$$\sum_{|k|\le j}(\|\Gamma^k v(t)\|_\infty + \|\Gamma^k v_1(t)\|_\infty + \|\Gamma^k \partial_t \chi(t)\|_\infty)$$
$$\le c(M)\sum_{|k|\le j}(\|\Gamma^k \partial_\alpha \chi(t)\|_\infty + \|\Gamma^k \partial_\alpha \lambda(t)\|_\infty). \tag{3.53}$$

Here $c(M)$ *is a constant depending on* M *and* j.

Proof From (2.38), we know

$$\partial_\alpha \lambda = -i\left(I - \zeta_\alpha \mathcal{H}\frac{1}{\zeta_\alpha}\right)\mathrm{Im}\{\bar{u}\zeta_\alpha\}.$$

Using Proposition 3.2 and (3.9), (3.12), we get for $1 \le j \le l$,

$$\|\Gamma^j \partial_\alpha \lambda(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k(u\zeta_\alpha)(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2. \tag{3.54}$$

(3.46) and (3.54) with an application of Proposition 3.13 gives us (3.51).

Using Proposition 3.2 and (3.9), (3.12), we have for $1 \le j \le l$,

$$\sum_{|k|\le j}\|\Gamma^k v_1(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k v(t)\|_2.$$

From (3.47), we get (3.52).

To prove (3.53), we use (2.35) and (3.50). First we have by using (3.43) and Proposition 3.5 that for $0 \le j \le l-1$,

$$\|\Gamma^j \partial_t \chi(t)\|_\infty \le \|\Gamma^j(\partial_t + b\partial_\alpha)\chi(t)\|_\infty + c(M)\sum_{|k|\le j}\|\Gamma^k \partial_\alpha \chi(t)\|_\infty. \tag{3.55}$$

Now from (2.35), using Proposition 3.5, rewriting $\mathcal{H} + \bar{\mathcal{H}}$ as in (3.31) and using Proposition 3.2 and Lemma 3.10, we have

$$\begin{aligned}
&\sum_{|k|\le j}\|\Gamma^k v(t)\|_\infty \\
&\quad = \sum_{|k|\le j}\|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_\infty \\
&\quad \le 2\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty + c\sum_{|k|\le j+1}\|\Gamma^k(\mathcal{H}+\bar{\mathcal{H}})u(t)\|_2 + \sum_{|k|\le j}\left\|\Gamma^k[u,\mathcal{H}]\frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}(t)\right\|_\infty \\
&\quad \le 2\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty + c(M)\Bigg(\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2 \\
&\qquad + \sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_2\Bigg).
\end{aligned} \tag{3.56}$$

From (3.50), using Lemma 3.10, we have for $0 \le j \le l-1$,

$$\begin{aligned}\|\Gamma^j v_1(t)\|_\infty &\le 2\|\Gamma^j v(t)\|_\infty + \left\|\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_\infty \\ &\le 2\|\Gamma^j v(t)\|_\infty + c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k \partial_\alpha \chi(t)\|_2 \\ &\quad + c(M)\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2 \|\partial_\alpha \chi(t)\|_\infty. \end{aligned} \tag{3.57}$$

Sum up (3.55)–(3.57), further using (3.46) and Proposition 3.6, we get (3.53). □

The following Lemma will be used to obtain estimates concerning the quantity $\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1}$.

Lemma 3.15 *Let f be a real valued function satisfying*

$$(I - \mathcal{H})(f A\bar{\zeta}_\alpha) = g.$$

There exists a constant $M_0 > 0$, such that

1. *for $M \le M_0$ and $0 \le j \le l$,*

$$\sum_{|k|\le j}\|\Gamma^k f(t)\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k g(t)\|_2. \tag{3.58}$$

2. *for $M \le M_0$ and $0 \le j \le l-1$,*

$$\begin{aligned}\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty &\le c(M)\sum_{|k|\le j}\|\Gamma^k g(t)\|_\infty + c(M)(\|w(t)\|_\infty \\ &\quad + \|\mathfrak{y}_\alpha(t)\|_\infty)\sum_{|k|\le j+1}\|\Gamma^k g(t)\|_2. \end{aligned} \tag{3.59}$$

Here $c(M)$ is a constant depending on M and j.

Proof The proof of (3.58) is similar to that of Lemma 3.8 (3.32). Let

$$\mathcal{K}^* = -\mathrm{Re}\left\{\frac{\zeta_\alpha}{|\zeta_\alpha|}\mathcal{H}\frac{|\zeta_\alpha|}{\zeta_\alpha}\right\}$$

be the adjoint of the double layered potential operator as defined in (1.19). We know the operator $(I+\mathcal{K}^*)^{-1}$ is L^2-bounded, with its $L^2 \to L^2$ operator norm $\|(I+\mathcal{K}^*)^{-1}\|_{2,2} \le c(M)$ for some constant $c(M)$ depending on M (see [19], [37, Theorem 2.1.5]). Using (3.12), (3.9), we have

$$\begin{aligned}&(I-\mathcal{H})(A\bar{\zeta}_\alpha \Gamma^j f) \\ &\quad = \sum_{k=1}^{j}\Gamma^{j-k}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}(fA\bar{\zeta}_\alpha) + \Gamma^j g - (I-\mathcal{H})(\Gamma^j(fA\bar{\zeta}_\alpha) - A\bar{\zeta}_\alpha\Gamma^j f). \end{aligned} \tag{3.60}$$

Multiplying (3.60) by $\frac{\zeta_\alpha}{|\zeta_\alpha|}$ and using the fact that f, A are real valued, we obtain

$$(I+\mathcal{K}^*)(A|\zeta_\alpha|\Gamma^j f) = \mathrm{Re}\left\{\frac{\zeta_\alpha}{|\zeta_\alpha|}\sum_{k=1}^{j}\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}(fA\bar{\zeta}_\alpha)\right\}$$
$$+\mathrm{Re}\left\{\frac{\zeta_\alpha}{|\zeta_\alpha|}\Gamma^j g - \frac{\zeta_\alpha}{|\zeta_\alpha|}(I-\mathcal{H})(\Gamma^j(fA\bar{\zeta}_\alpha) - A\bar{\zeta}_\alpha\Gamma^j f)\right\}.$$

Using an induction argument and (3.24), (3.37), using Propositions 3.3, 3.5, we obtain (3.58).

We now prove (3.59). Let $0\le j\le l-1$. From (2.31) we have

$$A\bar{\zeta}_\alpha = i\bar{w}+1.$$

Therefore

$$(I-\mathcal{H})f = -i(I-\mathcal{H})(\bar{w}f)+g.$$

We know from (3.58) that $\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty<\infty$ provided $\sum_{|k|\le j+1}\|\Gamma^k g(t)\|_2<\infty$. Now using Lemma 3.8 (3.33), we get

$$\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty \le c(M)\sum_{|k|\le j}(\|\Gamma^k g(t)\|_\infty + \|\Gamma^k(I-\mathcal{H})(\bar{w}f)(t)\|_\infty)$$
$$+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le j+1}(\|\Gamma^k g(t)\|_2+\|\Gamma^k(I-\mathcal{H})(\bar{w}f)(t)\|_2).$$

Further using Propositions 3.5, 3.2, we have

$$\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty \le c(M)\sum_{|k|\le j}\|\Gamma^k g(t)\|_\infty + c(M)\sum_{|k|\le j+1}\|\Gamma^k(\bar{w}f)(t)\|_2$$
$$+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le j+1}(\|\Gamma^k g(t)\|_2+\|\Gamma^k(\bar{w}f)(t)\|_2)$$
$$\le c(M)\sum_{|k|\le j}\|\Gamma^k g(t)\|_\infty + c(M)\|w(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k f(t)\|_2$$
$$+c(M)\sum_{|k|\le j+1}\|\Gamma^k w(t)\|_2\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty$$
$$+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le j+1}(\|\Gamma^k g(t)\|_2+\|\Gamma^k f(t)\|_2).$$

Using (3.58) to estimate $\sum_{|k|\le j+1}\|\Gamma^k f(t)\|_2$. We conclude that (3.59) holds for sufficiently small M and $0\le j\le l-1$. □

Finally, we give two technical Lemmas that will be used in Sect. 4.

Lemma 3.16 *Assume that f, g are smooth and decay fast at infinity. Then*

1. *for* $0 \le j \le l-2$,

$$\left\|\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}(t)\right\|_\infty \le c(M)\sum_{|k|\le j+1}\|\Gamma^k f(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k g(t)\|_\infty \ln(t+e) + c(M)\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k g(t)\|_2\frac{1}{t+1}. \tag{3.61}$$

2. *for* $0 \le j \le l$,

$$\left\|\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}(t)\right\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k g(t)\|_2. \tag{3.62}$$

3. *for* $0 \le j \le l$,

$$\left\|\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}(t)\right\|_2 \le c(M)\sum_{|k|\le j}\|\Gamma^k f(t)\|_2\sum_{|k|\le \max(j,1)}\|\Gamma^k g(t)\|_2. \tag{3.63}$$

Proof (3.61) is a consequence of (3.14) and Proposition 3.4, by taking $r=(t+1)^2$. In particular, such terms $[\Gamma^p f,\mathcal{H}]\frac{\Gamma^q g}{\zeta_\alpha}$ are expanded

$$[\Gamma^p f,\mathcal{H}]\frac{\Gamma^q g}{\zeta_\alpha} = (\Gamma^p f)\left(\mathcal{H}\frac{\Gamma^q g}{\zeta_\alpha}\right) - \mathcal{H}\left(\Gamma^p f\frac{\Gamma^q g}{\zeta_\alpha}\right)$$

and estimated term by term using Proposition 3.4.

(3.62), (3.63) are direct consequences of (3.14) and Propositions 3.2, 3.5. □

Lemma 3.17 *Assume that* f, g *are smooth and decay fast at infinity. We have for* $0 \le j \le l$,

$$\begin{aligned}\left\|\Gamma^j[f,\mathcal{H}]\frac{g}{\zeta_\alpha}\right\|_2 &\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k g(t)\|_2 \\ &\quad + \sum_{|k|\le j}\|\Gamma^k f(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k g(t)\|_\infty\ln(t+e) + \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k g(t)\|_2\frac{1}{t+1}\Bigg)\Bigg],\end{aligned} \tag{3.64}$$

$$\begin{aligned}\left\|\Gamma^j[f,\mathcal{H}]\frac{\partial_\alpha g}{\zeta_\alpha}\right\|_2 &\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k f(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k g(t)\|_2 \\ &\quad + \sum_{|k|\le j}\|\Gamma^k f(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k g(t)\|_\infty\ln(t+e) + \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k g(t)\|_2\frac{1}{t+1}\Bigg)\Bigg].\end{aligned} \tag{3.65}$$

Proof (3.64) is obtained by using (3.14) and Propositions 3.2 and 3.4, taking $r=(t+1)^2$. (3.65) is by using (3.14) and Propositions 3.3, 3.4. We omit the details. □

4 A priori estimates

In this section, we construct the energy functional using the expanded set of vector fields $\Gamma = \{\partial_\alpha, \partial_t, L_0\}$ and derive the energy estimates for the water wave system (2.1)–(2.2). We use all three equations (2.27) to (2.29), where the part of energy connecting to (2.28) for v gives estimates for the highest order derivatives and the part connecting to (2.29) for λ gives estimates for the lowest order derivatives.

Let $\dot{H}^{1/2}(R) = \{f \mid |D|^{1/2} f \in L^2(R)\}$, where $|D| = \sqrt{-\partial_\alpha^2}$. In this section we make the following assumptions on the solution.

Let $l \geq 11$, $z = z(\alpha, t)$, $t \in [0, T]$ be a solution of the water wave system (2.1)–(2.2). Assume that the function $k(\cdot, t) : R \to R$ defined by (2.18) is a diffeomorphism and $k_\alpha(\cdot, t) > 0$ for $t \in [0, T]$. Assume

$$\Gamma^j(\zeta_\alpha - 1),\ \Gamma^j u,\ \Gamma^j w,\ \Gamma^j(\partial_t + b\partial_\alpha)w,\ \Gamma^j\partial_\alpha u,\ \Gamma^j\partial_\alpha w \in C([0, T], L^2(R)) \tag{4.1}$$

for all $|j| \leq l$, and

$$\Gamma^j(\partial_t + b\partial_\alpha)\lambda \in C([0, T], L^2(R)), \qquad \Gamma^j\lambda \in C([0, T], \dot{H}^{1/2}(R)) \tag{4.2}$$

for all $|j| \leq l - 2$; Assume

$$|\zeta(\alpha, t) - \zeta(\beta, t)| \geq \frac{1}{2}|\alpha - \beta|, \quad \text{for all } \alpha,\ \beta \in R,\ t \in [0, T], \tag{4.3}$$

and

$$\sup_{[0,T]} \sum_{|k| \leq l} (\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^k u(t)\|_2 + \|\Gamma^k w(t)\|_2) \leq M, \tag{4.4}$$

where $M \leq M_0$, M_0 is the constant such that the inequalities (3.37), (3.47) and Propositions 3.6, 3.7, 3.9, 3.11, 3.13, 3.14, Lemmas 3.8, 3.15 hold. The assumptions (4.1)–(4.3) ensure that the energy functional $\mathfrak{E}$ defined later in this section is in $C^1([0, T])$.

We first present the following basic energy equality.

Lemma 4.1 (Basic energy equality) *Assume that Θ satisfies the equation*

$$((\partial_t + b\partial_\alpha)^2 - iA\partial_\alpha)\Theta = G \tag{4.5}$$

and Θ is smooth and decay fast at spatial infinity. Let

$$E_0(t) = \int \frac{1}{A(\alpha, t)} |(\partial_t + b(\alpha, t)\partial_\alpha)\Theta(\alpha, t)|^2 + i\Theta(\alpha, t)\partial_\alpha\bar{\Theta}(\alpha, t)\, d\alpha. \tag{4.6}$$

Then

$$\frac{dE_0}{dt} = \int \frac{2}{A}\mathrm{Re}((\partial_t + b\partial_\alpha)\Theta\, \bar{G}) - \frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1}|(\partial_t + b\partial_\alpha)\Theta|^2\, d\alpha. \tag{4.7}$$

Moreover if Θ is the boundary value of a holomorphic function in $\Omega(t)^c$, that is if

$$\Theta = \frac{1}{2}(I - \mathcal{H})\Theta, \tag{4.8}$$

then

$$\int i\Theta\partial_\alpha\bar{\Theta}\,d\alpha = -\int i\bar{\Theta}\partial_\alpha\Theta\,d\alpha \geq 0. \tag{4.9}$$

Proof Making the change of variable U_k to (4.5), we get

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(\Theta\circ k) = G\circ k.$$

Now we also make the change of variables in the basic energy E_0, we get

$$E_0 = \int \frac{1}{\mathfrak{a}}|\partial_t(\Theta\circ k)|^2 + i\Theta\circ k\partial_\alpha(\bar{\Theta}\circ k)\,d\alpha.$$

So

$$\begin{aligned}\frac{dE_0}{dt} &= \int \frac{2}{\mathfrak{a}}\mathrm{Re}\{\partial_t(\bar{\Theta}\circ k)\partial_t^2(\Theta\circ k)\} + |\partial_t(\Theta\circ k)|^2\partial_t\left(\frac{1}{\mathfrak{a}}\right) - 2\mathrm{Re}\{i\partial_t(\bar{\Theta}\circ k)\partial_\alpha(\Theta\circ k)\}\,d\alpha\\ &= \int \frac{2}{\mathfrak{a}}\mathrm{Re}\{\partial_t(\bar{\Theta}\circ k)G\circ k\} - \frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\partial_t(\Theta\circ k)|^2\,d\alpha.\end{aligned}$$

Change the variables back by U_k^{-1}, we get (4.7). Now we prove (4.9). First the equality in (4.9) is obtained by integration by parts, therefore $\int i\bar{\Theta}\partial_\alpha\Theta\,d\alpha$ is real valued. Notice that $\zeta = \zeta(\alpha, t)$ for increasing α traverse the boundary of $\Omega(t)^c$ counter clock-wisely; for Θ the boundary value of a holomorphic function θ in $\Omega(t)^c$, we have that

$$-i\partial_\alpha\Theta = |\zeta_\alpha|\frac{\partial\theta}{\partial\mathbf{n}}, \tag{4.10}$$

where $\mathbf{n}$ is the unit normal pointing out of $\Omega(t)^c$. So by the Green's identity we have

$$-\int i\bar{\Theta}\partial_\alpha\Theta\,d\alpha = \mathrm{Re}\left\{-\int i\bar{\Theta}\partial_\alpha\Theta\,d\alpha\right\} = \int_{\partial\Omega(t)^c}\theta\cdot\frac{\partial\theta}{\partial\mathbf{n}}\,ds = \int_{\Omega(t)^c}|\nabla\theta|^2\,dxdy \geq 0.$$

Here the integration in the third integral above is with respect to the arc-length s. □

Remark 1 A more detailed proof of (4.10) can be done using the Cauchy-Riemann equations for θ. We omit the details.

Remark 2 The Green's identity on the unbounded domain $\Omega(t)^c$ can be verified as the following: first verify the Green's identity on the upper half plane P_+ by the Fourier transform and the Plancherel's Theorem. Then use a Riemann Mapping map $\Omega(t)^c$ onto P_+.

We now construct the higher order energy for a Θ that satisfies (4.8) and (4.5). We know for any integer $j \geq 1$,

$$((\partial_t + b\partial_\alpha)^2 - iA\partial_\alpha)\Gamma^j\Theta = G_j^\Theta, \tag{4.11}$$

where

$$G_j^\Theta = \Gamma^j\mathcal{P}\Theta + \sum_{k=1}^{j}\Gamma^{j-k}[\mathcal{P},\Gamma]\Gamma^{k-1}\Theta = \Gamma^j G + \sum_{k=1}^{j}\Gamma^{j-k}[\mathcal{P},\Gamma]\Gamma^{k-1}\Theta \tag{4.12}$$

and $\mathcal{P} = (\partial_t + b\partial_\alpha)^2 - iA\partial_\alpha$. Notice that for $j \geq 1$, $\Gamma^j\Theta$ is in general not the boundary value of a holomorphic function on $\Omega(t)^c$, we decompose $\Gamma^j\Theta$ as

$$\Gamma^j\Theta = \frac{I-\mathcal{H}}{2}\Gamma^j\Theta + \frac{I+\mathcal{H}}{2}\Gamma^j\Theta = \eta_j^\Theta + R_j^\Theta,$$

where

$$\eta_j^\Theta = \frac{I-\mathcal{H}}{2}\Gamma^j\Theta, \qquad R_j^\Theta = \frac{I+\mathcal{H}}{2}\Gamma^j\Theta.$$

Now from the assumption (4.8) and the fact that $\mathcal{H}^2 = I$, (3.9) and (3.12), we have

$$R_j^\Theta = \frac{1}{2}[I+\mathcal{H}, \Gamma^j]\Theta = -\frac{1}{2}\sum_{k=1}^{j}\Gamma^{j-k}[\Gamma'\zeta, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta. \tag{4.13}$$

We also know that

$$\eta_j^\Theta = \frac{I-\mathcal{H}}{2}\eta_j^\Theta = \frac{I-\mathcal{H}}{2}\Gamma^j\Theta, \qquad R_j^\Theta = \frac{I+\mathcal{H}}{2}R_j^\Theta. \tag{4.14}$$

Let

$$E_j^\Theta(t) = \int \frac{1}{A(\alpha,t)}|(\partial_t + b(\alpha,t)\partial_\alpha)\Gamma^j\Theta(\alpha,t)|^2 + i\eta_j^\Theta(\alpha,t)\partial_\alpha\bar{\eta}_j^\Theta(\alpha,t)\, d\alpha. \tag{4.15}$$

Notice that both terms in the definition of E_j^Θ are nonnegative. We know

$$E_j^\Theta = \int \frac{1}{A}|(\partial_t + b\partial_\alpha)\Gamma^j\Theta|^2 + i\Gamma^j\Theta\partial_\alpha\bar{\Gamma}^j\Theta\, d\alpha - i\int(\eta_j^\Theta\partial_\alpha\bar{R}_j^\Theta + R_j^\Theta\partial_\alpha\bar{\eta}_j^\Theta + R_j^\Theta\partial_\alpha\bar{R}_j^\Theta)\, d\alpha. \tag{4.16}$$

Therefore

$$\begin{aligned}\frac{dE_j^\Theta}{dt} &= \int \frac{2}{A}\mathrm{Re}((\partial_t + b\partial_\alpha)\Gamma^j\Theta\, \bar{G}_j^\Theta) - \frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}|(\partial_t + b\partial_\alpha)\Gamma^j\Theta|^2\, d\alpha \\ &\quad - 2\mathrm{Re}\int(i\partial_t\eta_j^\Theta\partial_\alpha\bar{R}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{\eta}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{R}_j^\Theta)\, d\alpha. \end{aligned} \tag{4.17}$$

Here in (4.17) we used (4.7) and integration by parts. If G_j^Θ is at least cubic, then the term

$$\int \frac{2}{A}\mathrm{Re}((\partial_t + b\partial_\alpha)\Gamma^j\Theta\, \bar{G}_j^\Theta) - \frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}|(\partial_t + b\partial_\alpha)\Gamma^j\Theta|^2\, d\alpha$$

is at least of the 4th order. We next show that the term

$$\int(i\partial_t\eta_j^\Theta\partial_\alpha\bar{R}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{\eta}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{R}_j^\Theta)\, d\alpha$$

is at least of the 4th order. We see from (4.13) that R_j^Θ is at least quadratic. Now using (3.12), we have

$$\partial_t\eta_j^\Theta = \partial_t\frac{I-\mathcal{H}}{2}\Gamma^j\Theta = \frac{I-\mathcal{H}}{2}\partial_t\Gamma^j\Theta - \frac{1}{2}[\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^j\Theta \tag{4.18}$$

and

$$\partial_\alpha R_j^\Theta = \partial_\alpha \frac{1}{2}(I+\mathcal{H})R_j^\Theta = \frac{1}{2}(I-\mathcal{H}^*)\partial_\alpha R_j^\Theta, \tag{4.19}$$

where $\mathcal{H}^*$ defined by

$$\mathcal{H}^* f = -\zeta_\alpha \mathcal{H}\frac{1}{\zeta_\alpha} f \tag{4.20}$$

is the adjoint operator of $\mathcal{H}$, i.e.

$$\int g\mathcal{H}^* f\, d\alpha = \int f\mathcal{H}g\, d\alpha.$$

Therefore

$$\int \partial_t \eta_j^\Theta \partial_\alpha \bar{R}_j^\Theta\, d\alpha = \int \left(\frac{I-\mathcal{H}}{2}\partial_t \Gamma^j \Theta\right)\partial_\alpha \bar{R}_j^\Theta\, d\alpha - \int \frac{1}{2}\left([\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^j\Theta\right)\partial_\alpha \bar{R}_j^\Theta\, d\alpha. \tag{4.21}$$

We see the second term on the right hand side of (4.21) is at least of the 4th order. We manipulate further the first term on the right of (4.21) by using (4.19). We have

$$\begin{aligned}\int \left(\frac{I-\mathcal{H}}{2}\partial_t \Gamma^j \Theta\right)\partial_\alpha \bar{R}_j^\Theta\, d\alpha &= \int \left(\frac{I-\mathcal{H}}{2}\partial_t \Gamma^j \Theta\right)\left(\frac{1}{2}(I-\bar{\mathcal{H}}^*)\partial_\alpha \bar{R}_j^\Theta\right) d\alpha \\ &= \int \left(\frac{-\bar{\mathcal{H}}-\mathcal{H}}{2}\partial_t \Gamma^j \Theta\right)\left(\frac{1}{2}(I-\bar{\mathcal{H}}^*)\partial_\alpha \bar{R}_j^\Theta\right) d\alpha \\ &= -\frac{1}{2}\int ((\bar{\mathcal{H}}+\mathcal{H})\partial_t \Gamma^j\Theta)\partial_\alpha \bar{R}_j^\Theta\, d\alpha. \end{aligned} \tag{4.22}$$

Here in the second step in (4.22) we used the fact that $(I-\mathcal{H})(I+\mathcal{H}) = 0$. Therefore

$$\int \partial_t \eta_j^\Theta \partial_\alpha \bar{R}_j^\Theta\, d\alpha = -\frac{1}{2}\int ((\bar{\mathcal{H}}+\mathcal{H})\partial_t \Gamma^j\Theta)\partial_\alpha \bar{R}_j^\Theta\, d\alpha - \int \frac{1}{2}\left([\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^j\Theta\right)\partial_\alpha \bar{R}_j^\Theta\, d\alpha. \tag{4.23}$$

This show that $\int \partial_t \eta_j^\Theta \partial_\alpha \bar{R}_j^\Theta\, d\alpha$ is at least of the 4th order. The discussion for $\int \partial_t R_j^\Theta \partial_\alpha \bar{\eta}_j^\Theta\, d\alpha$ is similar. We know

$$\partial_t R_j^\Theta = \partial_t \frac{I+\mathcal{H}}{2} R_j^\Theta = \frac{I+\mathcal{H}}{2}\partial_t R_j^\Theta + \frac{1}{2}[\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha} R_j^\Theta \tag{4.24}$$

and

$$\partial_\alpha \eta_j^\Theta = \partial_\alpha \frac{1}{2}(I-\mathcal{H})\Gamma^j\Theta = \frac{1}{2}(I+\mathcal{H}^*)\partial_\alpha \Gamma^j\Theta. \tag{4.25}$$

Therefore

$$\begin{aligned}\int \partial_t R_j^\Theta \partial_\alpha \bar{\eta}_j^\Theta\, d\alpha &= \int \left(\frac{1}{2}[\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha} R_j^\Theta\right)\partial_\alpha \bar{\eta}_j^\Theta\, d\alpha \\ &\quad + \int \left(\frac{I+\mathcal{H}}{2}\partial_t R_j^\Theta\right)\left(\frac{1}{2}(I+\bar{\mathcal{H}}^*)\partial_\alpha \Gamma^j\bar{\Theta}\right) d\alpha \\ &= \int \left(\frac{1}{2}[\zeta_t, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha} R_j^\Theta\right)\partial_\alpha \bar{\eta}_j^\Theta\, d\alpha + \int \left(\frac{\bar{\mathcal{H}}+\mathcal{H}}{2}\partial_t R_j^\Theta\right)\partial_\alpha \bar{\eta}_j^\Theta\, d\alpha \end{aligned} \tag{4.26}$$

and it is of at least the 4th order. It is evident that $\int \partial_t R_j^\Theta \partial_\alpha \bar{R}_j^\Theta\, d\alpha$ is of at least the 4th order.

While the energy defined by (4.15) and the subsequent calculations work well for $\Theta = \chi$ and λ, it may lead to lose of derivatives if applied to v.[12] Therefore for v we construct an energy different from (4.15). We first introduce a new quantity

$$\mathfrak{v}_1 = (I - \mathfrak{H})\mathfrak{v}, \qquad v_1 = \mathfrak{v}_1 \circ k^{-1} = (I - \mathcal{H})v \tag{4.27}$$

and show that v_1 satisfy an equation of the type (4.5) with G consisting of cubic and higher order terms. Notice that v_1 defined by (4.27) is the same as that in (3.49). We know from (2.6) that

$$\begin{aligned}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v}_1 &= (I - \mathfrak{H})(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} - [\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}]\mathfrak{v} \\ &= (I - \mathfrak{H})(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} - 2[z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\partial_t\mathfrak{v} \\ &\quad + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\right)^2 \partial_\beta\mathfrak{v}\, d\beta.\end{aligned} \tag{4.28}$$

Recall $\mathfrak{v} = \partial_t \Pi$. To show that the right hand side of (4.28) is of at least cubic order, we rewrite

$$[z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\partial_t\mathfrak{v} = [z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi + [z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}(i\mathfrak{a}\partial_\alpha\Pi). \tag{4.29}$$

Notice that the first term in the right hand side of (4.29) is at least of cubic order. From (2.1) we know

$$i\mathfrak{a}\partial_\alpha\Pi = (z_{tt} + i)\frac{\partial_\alpha}{z_\alpha}\Pi.$$

Now (2.5) implies that for any function f,

$$\frac{\partial_\alpha}{z_\alpha}(I - \mathfrak{H})f = (I - \mathfrak{H})\frac{\partial_\alpha}{z_\alpha}f = \frac{(I - \mathfrak{H})}{2}\frac{\partial_\alpha}{z_\alpha}(I - \mathfrak{H})f$$

here in the last step we used the fact that $(I - \mathfrak{H})^2 f = 2(I - \mathfrak{H})f$. So

$$\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi = \frac{(I - \mathfrak{H})}{2}\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi^{13} \tag{4.30}$$

and using the third statement in Lemma 2.2 and (2.2) we have

$$[z_t, \mathfrak{H}]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi = \left[\frac{(I + \mathfrak{H})}{2}z_t, \mathfrak{H}\right]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi = \frac{1}{2}[(\bar{\mathfrak{H}} + \mathfrak{H})z_t, \mathfrak{H}]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi.$$

Therefore

$$[z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}(i\mathfrak{a}\partial_\alpha\Pi) = [z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\left(z_{tt}\frac{\partial_\alpha}{z_\alpha}\Pi\right) + \frac{i}{2}[(\bar{\mathfrak{H}} + \mathfrak{H})z_t, \mathfrak{H}]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi.$$

[12] We need $\|\partial_\alpha\Gamma^j\Theta(t)\|_2$ to bound $|\int(i\partial_t\eta_j^\Theta\partial_\alpha\bar{R}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{\eta}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{R}_j^\Theta)\,d\alpha|$, see (4.54). This requires a half order more spatial derivative than that provided by the energy $E_j^\Theta(t)$. See also (4.50) for the definition of the total energy. For this total energy, (4.15) can be used for $\Theta = \chi$, λ, but not for v.

[13] (4.30) is equivalent to $\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi$ being the boundary value of a holomorphic function in $\Omega(t)^c$.

Going back to (4.28), we get

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v}_1 = (I-\mathfrak{H})(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} - 2[z_t,\mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi \\ - 2[z_t,\mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\left(z_{tt}\frac{\partial_\alpha}{z_\alpha}\Pi\right) - i[(\bar{\mathfrak{H}}+\mathfrak{H})z_t,\mathfrak{H}]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\Pi \\ + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2\partial_\beta\mathfrak{v}\,d\beta. \tag{4.31}$$

Making the change of variables U_k^{-1} we arrive at the equation for v_1:

$$\mathcal{P}v_1 = (I-\mathcal{H})\mathcal{P}v - 2[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\mathcal{P}\chi - 2[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\left(w\frac{\partial_\alpha}{\zeta_\alpha}\chi\right) \\ - i[(\bar{\mathcal{H}}+\mathcal{H})u,\mathcal{H}]\left(\frac{\partial_\alpha}{\zeta_\alpha}\right)^2\chi + \frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2\partial_\beta v\,d\beta. \tag{4.32}$$

Notice that in these expressions the right hand sides of (4.31), (4.32) are clearly consisting of terms of cubic and higher orders. We have in (2.27) a formula for $\mathcal{P}\chi$. We calculate here an explicit expression for $\mathcal{P}v$. We have from (2.11), (2.10) that

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\mathfrak{v} = \partial_t\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi\} + i\mathfrak{a}_t\partial_\alpha\Pi \\ = -2\left[z_{tt},\mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}\right]z_{t\alpha} - 2\left[z_t,\mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}\right]z_{tt\alpha} \\ + \frac{2}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2 z_{t\beta}\,d\beta - \frac{2}{\pi i}\int\frac{|z_t(\alpha,t)-z_t(\beta,t)|^2}{(\bar{z}(\alpha,t)-\bar{z}(\beta,t))^2}z_{t\beta}\,d\beta \\ + \frac{2}{\pi i}\int\frac{(z_t(\alpha,t)-z_t(\beta,t))(z_{tt}(\alpha,t)-z_{tt}(\beta,t))}{(z(\alpha,t)-z(\beta,t))^2}(z_\beta-\bar{z}_\beta)\,d\beta \\ + \frac{1}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2(z_{t\beta}-\bar{z}_{t\beta})\,d\beta \\ - \frac{2}{\pi i}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^3(z_\beta-\bar{z}_\beta)\,d\beta + i\mathfrak{a}_t\partial_\alpha\Pi.$$

Therefore with the change of variable U_k^{-1}, we get

$$\mathcal{P}v = -2\left[w,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right]u_\alpha - 2\left[u,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right]w_\alpha \\ + \frac{2}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2 u_\beta\,d\beta - \frac{2}{\pi i}\int\frac{|u(\alpha,t)-u(\beta,t)|^2}{(\bar{\zeta}(\alpha,t)-\bar{\zeta}(\beta,t))^2}u_\beta\,d\beta \\ + \frac{2}{\pi i}\int\frac{(u(\alpha,t)-u(\beta,t))(w(\alpha,t)-w(\beta,t))}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}(\zeta_\beta-\bar{\zeta}_\beta)\,d\beta$$

$$+\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(u_\beta-\bar{u}_\beta)\,d\beta$$
$$-\frac{2}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^3(\zeta_\beta-\bar{\zeta}_\beta)\,d\beta+i\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\partial_\alpha\chi. \tag{4.33}$$

We now construct the higher order energy for v_1. We know $\Gamma^j v_1$ satisfies

$$((\partial_t+b\partial_\alpha)^2-iA\partial_\alpha)\Gamma^j v_1=G_j^{v_1}, \tag{4.34}$$

where

$$G_j^{v_1}=\Gamma^j\mathcal{P}v_1+\sum_{k=1}^{j}\Gamma^{j-k}[\mathcal{P},\Gamma]\Gamma^{k-1}v_1. \tag{4.35}$$

Let

$$E_j^v=\int\frac{1}{A}|(\partial_t+b\partial_\alpha)\Gamma^j v_1|^2+i\Gamma^j v_1\partial_\alpha\Gamma^j\bar{v}_1\,d\alpha. \tag{4.36}$$

From Lemma 4.1 we have

$$\frac{dE_j^v(t)}{dt}=\int\frac{2}{A}\mathrm{Re}\{(\partial_t+b\partial_\alpha)\Gamma^j v_1\,\bar{G}_j^{v_1}\}-\frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}|(\partial_t+b\partial_\alpha)\Gamma^j v_1|^2\,d\alpha. \tag{4.37}$$

It is evident from (4.37) that $\frac{dE_j^v(t)}{dt}$ is of at least the 4th order. However the second term in the definition of E_j^v may not be nonnegative. We also need the estimates of $\int|(\partial_t+b\partial_\alpha)\Gamma^j v|^2\,d\alpha$. We show in what follows that E_j^v is bounded below by $\int|(\partial_t+b\partial_\alpha)\Gamma^j v|^2\,d\alpha+\int|(\partial_t+b\partial_\alpha)\Gamma^j v_1|^2\,d\alpha$, minus some terms of at least cubic orders.

Let

$$\eta_j^v=\frac{I-\mathcal{H}}{2}\Gamma^j v_1,\qquad R_j^v=\frac{I+\mathcal{H}}{2}\Gamma^j v_1. \tag{4.38}$$

Then

$$\eta_j^v=(I-\mathcal{H})\Gamma^j v-\frac{I-\mathcal{H}}{2}\sum_{k=1}^{j}\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}v,$$
$$R_j^v=-\frac{I+\mathcal{H}}{2}\sum_{k=1}^{j}\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}v, \tag{4.39}$$
$$\partial_\alpha R_j^v=-\frac{I-\mathcal{H}^*}{2}\sum_{k=1}^{j}\partial_\alpha\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}v$$

and

$$E_j^v=\int\frac{1}{A}|(\partial_t+b\partial_\alpha)\Gamma^j v_1|^2+i\eta_j^v\partial_\alpha\bar{\eta}_j^v+i(\eta_j^v\partial_\alpha\bar{R}_j^v+R_j^v\partial_\alpha\bar{\eta}_j^v+R_j^v\partial_\alpha\bar{R}_j^v)\,d\alpha. \tag{4.40}$$

From (3.50) we know

$$v_1=2v+[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi.$$

So

$$\int \frac{1}{A}|(\partial_t + b\partial_\alpha)\Gamma^j v_1|^2\, d\alpha \geq \int \frac{2}{A}|(\partial_t + b\partial_\alpha)\Gamma^j v|^2\, d\alpha - \int \frac{1}{A}\left|(\partial_t + b\partial_\alpha)\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi\right|^2 d\alpha.$$

Further using the second part of Lemma 4.1, we obtain

$$\begin{aligned} E_j^v &\geq \int \frac{1}{A}|(\partial_t + b\partial_\alpha)\Gamma^j v_1|^2\, d\alpha - \left|\int (\eta_j^v \partial_\alpha \bar{R}_j^v + R_j^v \partial_\alpha \bar{\eta}_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right| \\ &\geq \int \frac{2}{A}|(\partial_t + b\partial_\alpha)\Gamma^j v|^2\, d\alpha - \int \frac{1}{A}\left|(\partial_t + b\partial_\alpha)\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi\right|^2 d\alpha \\ &\quad - \left|\int (\eta_j^v \partial_\alpha \bar{R}_j^v + R_j^v \partial_\alpha \bar{\eta}_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right|. \end{aligned} \tag{4.41}$$

Taking the average of the two inequalities in (4.41), we arrive at

$$\begin{aligned} E_j^v \geq &\int \frac{1}{A}|(\partial_t + b\partial_\alpha)\Gamma^j v|^2\, d\alpha + \int \frac{1}{2A}|(\partial_t + b\partial_\alpha)\Gamma^j v_1|^2\, d\alpha \\ &- \int \frac{1}{2A}\left|(\partial_t + b\partial_\alpha)\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi\right|^2 d\alpha \\ &- \left|\int (\eta_j^v \partial_\alpha \bar{R}_j^v + R_j^v \partial_\alpha \bar{\eta}_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right|, \end{aligned} \tag{4.42}$$

for $1 \leq j \leq l$.

Let's now derive an upper bound for the two non-positive terms

$$\int \frac{1}{2A}\left|(\partial_t + b\partial_\alpha)\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi\right|^2 d\alpha + \left|\int (\eta_j^v \partial_\alpha \bar{R}_j^v + R_j^v \partial_\alpha \bar{\eta}_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right|$$

on the right hand side of (4.42) in terms of the type of L^2 integrals in the energies E_j^χ, E_j^v defined in (4.15), (4.36). From Lemma 3.12 (3.45), we have that for $1 \leq j \leq l$,

$$\begin{aligned} &\left\|(\partial_t + b\partial_\alpha)\Gamma^j [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2 \\ &\quad \leq c \sum_{|k|\leq j} \|\Gamma^k b(t)\|_2 \sum_{|k|\leq \max\{j-1,1\}} \left\|\Gamma^k \partial_\alpha [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2 \\ &\qquad + c \sum_{|k|\leq j} \left\|\Gamma^k (\partial_t + b\partial_\alpha)[u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2, \end{aligned} \tag{4.43}$$

where from (3.14) and Propositions 3.2, 3.3, 3.5 that for $2 \leq j \leq l$,

$$\sum_{|k|\leq j-1} \left\|\Gamma^k \partial_\alpha [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2 \leq c(M) \sum_{|k|\leq j} \|\Gamma^k u(t)\|_2 \sum_{|k|\leq j-1} \|\Gamma^k \partial_\alpha \chi(t)\|_2. \tag{4.44}$$

We know

$$
\begin{aligned}
&(\partial_t + b\partial_\alpha)[u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi \\
&= U_k^{-1}\partial_t[z_t, \mathfrak{H}]\frac{\partial_\alpha}{z_\alpha}\Pi \\
&= U_k^{-1}[z_{tt}, \mathfrak{H}]\frac{\partial_\alpha \Pi}{z_\alpha} + U_k^{-1}[z_t, \mathfrak{H}]\frac{\partial_\alpha \partial_t \Pi}{z_\alpha} - \frac{1}{\pi i}U_k^{-1}\int\left(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\right)^2\partial_\beta \Pi\, d\beta \\
&= [w, \mathcal{H}]\frac{\partial_\alpha \chi}{\zeta_\alpha} + [u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}(\partial_t + b\partial_\alpha)\chi - \frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2\partial_\beta \chi\, d\beta. \qquad (4.45)
\end{aligned}
$$

Therefore for $2 \le j \le l$,

$$
\begin{aligned}
&\sum_{|k|\le j}\left\|\Gamma^k(\partial_t + b\partial_\alpha)[u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2 \\
&\quad \le c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_2 \sum_{|k|\le j}\|\Gamma^k \partial_\alpha \chi(t)\|_2 \\
&\qquad + c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\left(\sum_{|k|\le j}\|\Gamma^k(\partial_t + b\partial_\alpha)\chi(t)\|_2 + \sum_{|k|\le j}\|\Gamma^k \partial_\alpha \chi(t)\|_2\right).
\end{aligned}
$$

With a further application of (3.43), (3.46), (3.47), (3.48), we have that for $2 \le j \le l$, $t \in [0, T]$,

$$
\begin{aligned}
&\sum_{|k|\le j}\left\|(\partial_t + b\partial_\alpha)\Gamma^k[u, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\chi(t)\right\|_2 \\
&\quad \le c(M)M\sum_{|k|\le j}(\|(\partial_t + b\partial_\alpha)\Gamma^k v(t)\|_2 + \|(\partial_t + b\partial_\alpha)\Gamma^k \chi(t)\|_2) \qquad (4.46)
\end{aligned}
$$

for some constant $c(M)$ depending on M. Now from (4.39) and Propositions 3.2, 3.3 and 3.5, we have for $1 \le j \le l$,

$$
\begin{aligned}
&\left|\int(\eta_j^v \partial_\alpha \bar{R}_j^v + R_j^v \partial_\alpha \bar{\eta}_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right| \\
&\quad = \left|\int(\eta_j^v \partial_\alpha \bar{R}_j^v - \bar{\eta}_j^v \partial_\alpha R_j^v + R_j^v \partial_\alpha \bar{R}_j^v)\, d\alpha\right| \\
&\quad \le c(M)\left(\sum_{|k|\le \max\{j,2\}}\|\Gamma^k v(t)\|_2\right)^2 \sum_{|k|\le \max\{j,2\}}\|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 \\
&\quad \le c(M)M\left(\sum_{|k|\le \max\{j,2\}}\|\Gamma^k v(t)\|_2\right)^2.
\end{aligned}
$$

Further using (3.47) we get for $2 \le j \le l$, $t \in [0, T]$,

$$\sum_{|k|\le j}\left|\int (\eta_j^v\partial_\alpha \bar{R}_j^v + R_j^v\partial_\alpha\bar{\eta}_j^v + R_j^v\partial_\alpha \bar{R}_j^v)\,d\alpha\right|$$
$$\le c(M)M\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2^2 + \|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2^2). \tag{4.47}$$

Sum up the above calculations, we arrive at

Lemma 4.2 *There exists* $M_0>0$, *such that for* $0<M\le M_0$, *we have*

$$\sum_{|j|\le l}E_j^v \ge \sum_{|j|\le l}\left(\int \frac{1}{A}|(\partial_t+b\partial_\alpha)\Gamma^j v|^2\,d\alpha + \int\frac{1}{2A}|(\partial_t+b\partial_\alpha)\Gamma^j v_1|^2\,d\alpha\right)$$
$$- c(M)M\sum_{|k|\le l}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2^2 + \|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2^2). \tag{4.48}$$

4.1 Construction of the energy functional and the a priori estimates

We are now ready to construct the energy functional for the water wave equations. Let z be a solution of the water wave system (2.1)–(2.2) satisfying (4.1)–(4.4). Let

$$E^\chi(t)=\sum_{|j|\le l}E_j^\chi(t),\qquad E^\lambda(t)=\sum_{|j|\le l-2}E_j^\lambda(t),\qquad E^v(t)=\sum_{|j|\le l}E_j^v(t), \tag{4.49}$$

where E_j^χ, E_j^λ are defined by (4.15) for $\Theta=\chi,\lambda$ respectively, E_j^v is defined by (4.36). Let

$$\mathfrak{E}(t)=E^\chi(t)+E^\lambda(t)+E^v(t),\quad \text{for } t\in[0,T]. \tag{4.50}$$

We have from Lemmas 4.1, 4.2 and (3.37): $\|(A-1)(t)\|_\infty\le\frac{1}{2}$ for small M,

Proposition 4.3 *There exists a* $M_0>0$, *such that if* $M\le M_0$, *then for* $t\in[0,T]$,

$$\mathfrak{E}(t)\ge\frac{1}{4}\sum_{|j|\le l}(\|(\partial_t+b\partial_\alpha)\Gamma^j v(t)\|_2^2+\|(\partial_t+b\partial_\alpha)\Gamma^j\chi(t)\|_2^2)$$
$$+\frac{1}{4}\sum_{|j|\le l}\|(\partial_t+b\partial_\alpha)\Gamma^j v_1(t)\|_2^2+\frac{1}{2}\sum_{|j|\le l-2}\|(\partial_t+b\partial_\alpha)\Gamma^j\lambda(t)\|_2^2. \tag{4.51}$$

In what follows we prove the following a priori estimate for the energy functional $\mathfrak{E}(t)$ defined by (4.50).

Proposition 4.4 *There exists a* $M_0>0$, *such that if* $M\le M_0$, *then*

$$\frac{d}{dt}\mathfrak{E}(t)\le c_1(M)\mathfrak{E}(t)\left(\sum_{|j|\le l-4}\|\Gamma^j\partial_\alpha\lambda(t)\|_\infty+\sum_{|j|\le l-2}\|\Gamma^j\partial_\alpha\chi(t)\|_\infty\right)^2\ln(t+e)$$
$$+c_1(M)\mathfrak{E}(t)^2\frac{1}{t+1}, \tag{4.52}$$

for $t\in[0,T]$, *where* $c_1(M)$ *is a constant depending on* M.

Proof We know from (4.17) and (4.37) that for $\Theta = \chi, \lambda$,

$$\frac{dE_j^\Theta}{dt} = \int \frac{2}{A}\mathrm{Re}((\partial_t + b\partial_\alpha)\Gamma^j\Theta\,\bar{G}_j^\Theta) - \frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}|(\partial_t + b\partial_\alpha)\Gamma^j\Theta|^2\,d\alpha$$
$$-2\mathrm{Re}\int (i\partial_t\eta_j^\Theta\partial_\alpha\bar{R}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{\eta}_j^\Theta + i\partial_t R_j^\Theta\partial_\alpha\bar{R}_j^\Theta)\,d\alpha$$

and for v,

$$\frac{dE_j^v(t)}{dt} = \int \frac{2}{A}\mathrm{Re}\{(\partial_t + b\partial_\alpha)\Gamma^j v_1\,\bar{G}_j^{v_1}\} - \frac{1}{A}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}|(\partial_t + b\partial_\alpha)\Gamma^j v_1|^2\,d\alpha.$$

Using (3.37) we have for sufficiently small M,

$$\begin{aligned}
\frac{d\mathfrak{E}(t)}{dt} &= \frac{dE^\chi(t)}{dt} + \frac{dE^\lambda(t)}{dt} + \frac{dE^v(t)}{dt}\\
&\le 4\sum_{|j|\le l}(\|(\partial_t + b\partial_\alpha)\Gamma^j\chi(t)\|_2\|G_j^\chi(t)\|_2 + \|(\partial_t + b\partial_\alpha)\Gamma^j v_1(t)\|_2\|G_j^{v_1}(t)\|_2)\\
&\quad + 4\sum_{|j|\le l-2}\|(\partial_t + b\partial_\alpha)\Gamma^j\lambda(t)\|_2\|G_j^\lambda(t)\|_2\\
&\quad + 2\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\sum_{|j|\le l}(\|(\partial_t + b\partial_\alpha)\Gamma^j\chi(t)\|_2^2 + \|(\partial_t + b\partial_\alpha)\Gamma^j v_1(t)\|_2^2)\\
&\quad + 2\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\sum_{|j|\le l-2}\|(\partial_t + b\partial_\alpha)\Gamma^j\lambda(t)\|_2^2\\
&\quad + 2\sum_{|j|\le l}\left|\int(\partial_t\eta_j^\chi\partial_\alpha\bar{R}_j^\chi + \partial_t R_j^\chi\partial_\alpha\bar{\eta}_j^\chi + \partial_t R_j^\chi\partial_\alpha\bar{R}_j^\chi)\,d\alpha\right|\\
&\quad + 2\sum_{|j|\le l-2}\left|\int(\partial_t\eta_j^\lambda\partial_\alpha\bar{R}_j^\lambda + \partial_t R_j^\lambda\partial_\alpha\bar{\eta}_j^\lambda + \partial_t R_j^\lambda\partial_\alpha\bar{R}_j^\lambda)\,d\alpha\right|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d\mathfrak{E}(t)}{dt} &\le c\mathfrak{E}(t)^{1/2}\left\{\sum_{|j|\le l}(\|G_j^{v_1}(t)\|_2^2 + \|G_j^\chi(t)\|_2^2) + \sum_{|j|\le l-2}\|G_j^\lambda(t)\|_2^2\right\}^{1/2}\\
&\quad + c\mathfrak{E}(t)\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\\
&\quad + 2\sum_{|j|\le l}\left|\int(\partial_t\eta_j^\chi\partial_\alpha\bar{R}_j^\chi + \partial_t R_j^\chi\partial_\alpha\bar{\eta}_j^\chi + \partial_t R_j^\chi\partial_\alpha\bar{R}_j^\chi)\,d\alpha\right|\\
&\quad + 2\sum_{|j|\le l-2}\left|\int(\partial_t\eta_j^\lambda\partial_\alpha\bar{R}_j^\lambda + \partial_t R_j^\lambda\partial_\alpha\bar{\eta}_j^\lambda + \partial_t R_j^\lambda\partial_\alpha\bar{R}_j^\lambda)\,d\alpha\right|.
\end{aligned} \tag{4.53}$$

We estimate the right hand side of (4.53) term by term through three steps.

Step 1. We estimate the term $|\int(\partial_t\eta_j^{\Theta}\partial_\alpha\bar{R}_j^{\Theta}+\partial_t R_j^{\Theta}\partial_\alpha\bar{\eta}_j^{\Theta}+\partial_t R_j^{\Theta}\partial_\alpha\bar{R}_j^{\Theta})\,d\alpha|$ for $\Theta=\chi,\ \lambda$ in (4.53). We conclude this step with the estimates in (4.70) and (4.71).

For this estimate, we use (4.23), (4.26). Rewriting $\mathcal{H}+\bar{\mathcal{H}}$ as in (3.31), applying Proposition 3.2, we get

$$\begin{aligned}&\left|\int(\partial_t\eta_j^{\Theta}\partial_\alpha\bar{R}_j^{\Theta}+\partial_t R_j^{\Theta}\partial_\alpha\bar{\eta}_j^{\Theta}+\partial_t R_j^{\Theta}\partial_\alpha\bar{R}_j^{\Theta})\,d\alpha\right|\\&\quad\le c(M)\|\mathfrak{y}_\alpha(t)\|_\infty\|\partial_t\Gamma^j\Theta(t)\|_2\|\partial_\alpha R_j^{\Theta}(t)\|_2\\&\qquad+c(M)\|\zeta_t(t)\|_\infty\|\partial_\alpha\Gamma^j\Theta(t)\|_2\|\partial_\alpha R_j^{\Theta}(t)\|_2\\&\qquad+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty\|\partial_t R_j^{\Theta}(t)\|_2\|\partial_\alpha\Gamma^j\Theta(t)\|_2\\&\qquad+\|\partial_t R_j^{\Theta}(t)\|_2\|\partial_\alpha R_j^{\Theta}(t)\|_2.\end{aligned}\tag{4.54}$$

Now to estimate $\|\partial_t R_j^{\Theta}(t)\|_2$ and $\|\partial_\alpha R_j^{\Theta}(t)\|_2$ in (4.54), we use (4.13), and take into consideration of (3.2). We note here that there is no estimates of $\|\partial_t\Gamma^k\lambda(t)\|_\infty$ in terms of the type of L^2 norms in the energy functional $\mathfrak{E}$ with a decay rate $1/t^{1/2}$,[14] therefore we need to avoid estimating $\frac{d\mathfrak{E}}{dt}$ in terms of $\|\partial_t\Gamma^k\lambda(t)\|_\infty$. This is the motivation of the smaller range $|j|\le l-2$ in the definition of the energy E^{λ}. We are only able to estimate $\|\partial_t R_j^{\lambda}(t)\|_2$ for a smaller range $|j|\le l-2$ without using $\|\partial_t\Gamma^k\lambda(t)\|_\infty$.

We first make the following calculations. Let $\Theta=\chi,\ \lambda$. Notice that $\mathcal{H}\Theta=-\Theta$. Using (2.5), (3.9) and (3.12), we have

$$\begin{aligned}\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Theta&=\frac{\partial_\alpha}{\zeta_\alpha}\mathcal{H}\Theta=-\frac{\partial_\alpha}{\zeta_\alpha}\Theta\quad\text{and}\\\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta&=\frac{\partial_\alpha}{\zeta_\alpha}\mathcal{H}\Gamma^{k-1}\Theta=\frac{\partial_\alpha}{\zeta_\alpha}[\mathcal{H},\Gamma^{k-1}]\Theta+\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\mathcal{H}\Theta\\&=-\frac{\partial_\alpha}{\zeta_\alpha}\sum_{m=1}^{k-1}\Gamma^{k-1-m}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{m-1}\Theta-\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta,\quad\text{for }k\ge2.\end{aligned}\tag{4.55}$$

Therefore by applying Propositions 3.5, 3.3, we have for $k+p\le l$, $k\ge1$, $\Theta=\chi,\lambda$,

$$\begin{aligned}\left\|\Gamma^p\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta(t)\right\|_\infty&\le c(M)\sum_{|m|\le k+p-1}\|\Gamma^m\partial_\alpha\Theta(t)\|_\infty\\&\quad+\sum_{|m|\le k+p-1}\|\Gamma^m(\zeta_\alpha-1)(t)\|_\infty\sum_{|m|\le k+p-1}\|\Gamma^m\partial_\alpha\Theta(t)\|_2.\end{aligned}\tag{4.56}$$

We now consider $\|\partial_\alpha R_j^{\Theta}(t)\|_2$ for $\Theta=\chi,\lambda$. We know from (4.13), (3.2)

$$\partial_\alpha R_j^{\Theta}=-\frac12\sum_{k=1}^{j}\partial_\alpha\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta=\sum_{1\le|k|\le j}c_{k,j}\Gamma^{j-k}\partial_\alpha[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta,\tag{4.57}$$

[14]This assumes we use Proposition 3.1. Using Proposition 3.1, $\|\partial_t\Gamma^k\lambda(t)\|_\infty$ would need to be bounded by $\|\Omega_0\partial_t\Gamma^j\lambda(t)\|_2$ and some other terms with a decay rate $1/t^{1/2}$. However $\|\Omega_0\partial_t\Gamma^j\lambda(t)\|_2$ cannot be bounded by the energy $\mathfrak{E}(t)^{1/2}$. See also the part "L^2–L^∞ estimate and the energy inequality" near the end of Sect. 4.

where $c_{k,j}$ are some constants. Now

$$\begin{aligned}
&\partial_\alpha[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta\\
&\quad=[\Gamma\zeta_\alpha,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta+[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha^2}{\zeta_\alpha}\Gamma^{k-1}\Theta\\
&\qquad-\frac{1}{\pi i}\int\frac{(\Gamma'\zeta(\alpha,t)-\Gamma'\zeta(\beta,t))(\zeta_\alpha-\zeta_\beta)}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}\partial_\beta\Gamma^{k-1}\Theta(\beta,t)\,d\beta. \qquad (4.58)
\end{aligned}$$

Here we used (3.13) and (3.14). Therefore using Propositions 3.2, 3.3, we have for $1\le j\le l-2$,

$$\|\partial_\alpha R_j^\lambda(t)\|_2\le c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\le j-1}\|\Gamma^k\partial_\alpha\lambda(t)\|_2. \qquad (4.59)$$

To estimate $\|\partial_\alpha R_j^\chi(t)\|_2$, for $1\le j\le l$, we expand the first term in (4.58),

$$[\Gamma\zeta_\alpha,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\chi=\Gamma(\zeta_\alpha-1)\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\chi-\mathcal{H}\left(\Gamma(\zeta_\alpha-1)\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\chi\right).$$

We get by using (4.56) and Propositions 3.3, 3.2, 3.5 that

$$\begin{aligned}
\|\partial_\alpha R_j^\chi(t)\|_2&\le c\sum_{k=1}^{j}\left\|\Gamma^{j-k}\left(\Gamma(\zeta_\alpha-1)\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\chi\right)(t)\right\|_2\\
&\quad+c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty\\
&\quad+c(M)\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\le j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2\\
&\le c(M)\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty\\
&\quad+c(M)\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\le j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2. \qquad (4.60)
\end{aligned}$$

We estimate $\|\partial_t R_j^\Theta(t)\|_2$ for $\Theta=\chi,\ \lambda$ mostly in similar ways, except that we need to pay attention to not use $\|\partial_t\Gamma^k\lambda(t)\|_\infty$ in the estimates.

We have from (4.13), (3.2) that for $\Theta=\chi,\ \lambda$,

$$\partial_t R_j^\Theta=-\frac12\sum_{k=1}^{j}\partial_t\Gamma^{j-k}[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta=\sum_{1\le|k|\le j}c_{k,j}\Gamma^{j-k}\partial_t[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta \qquad (4.61)$$

for some constant coefficients $c_{k,j}$, and

$$\begin{aligned}
&\partial_t[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta\\
&\quad=[\partial_t\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\Theta+[\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha\partial_t}{\zeta_\alpha}\Gamma^{k-1}\Theta\\
&\qquad-\frac{1}{\pi i}\int\frac{(\Gamma'\zeta(\alpha,t)-\Gamma'\zeta(\beta,t))(\zeta_t(\alpha,t)-\zeta_t(\beta,t))}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}\partial_\beta\Gamma^{k-1}\Theta(\beta,t)\,d\beta, \qquad (4.62)
\end{aligned}$$

where from (3.2),

$$\partial_t \Gamma' = \Gamma \partial_t - \delta \partial_t, \quad \delta = 0, \text{ or } \frac{1}{2}. \tag{4.63}$$

To estimate $\|\partial_t R_j^\chi(t)\|_2$, for $1 \le j \le l$, we expand the first term in (4.62),

$$[\partial_t \Gamma' \zeta, \mathcal{H}] \frac{\partial_\alpha}{\zeta_\alpha} \Gamma^{k-1} \chi = (\partial_t \Gamma' \zeta) \mathcal{H} \frac{\partial_\alpha}{\zeta_\alpha} \Gamma^{k-1} \chi - \mathcal{H}\left(\partial_t \Gamma' \zeta \frac{\partial_\alpha}{\zeta_\alpha} \Gamma^{k-1} \chi \right). \tag{4.64}$$

We have from Propositions 3.2, 3.3 and (4.63) that for $1 \le j \le l$,

$$\begin{aligned}
\|\partial_t R_j^\chi(t)\|_2 \le c \sum_{k=1}^{j} \left\| \Gamma^{j-k} \left((\partial_t \Gamma' \zeta) \mathcal{H} \frac{\partial_\alpha}{\zeta_\alpha} \Gamma^{k-1} \chi \right)(t) \right\|_2 \\
+ c(M) \sum_{|k| \le j} \|\Gamma^k \zeta_t(t)\|_2 \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \partial_\alpha \chi(t)\|_\infty \\
+ c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \zeta_t(t)\|_\infty \sum_{|k| \le j-1} \|\Gamma^k \partial_\alpha \chi(t)\|_2 \\
+ c(M) \sum_{|k| \le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \partial_t \chi(t)\|_\infty \\
+ c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \sum_{|k| \le j-1} \|\partial_t \Gamma^k \chi(t)\|_2.
\end{aligned} \tag{4.65}$$

Now using (4.56), (4.63),

$$\begin{aligned}
&\sum_{k=1}^{j} \left\| \Gamma^{j-k} \left((\partial_t \Gamma' \zeta) \mathcal{H} \frac{\partial_\alpha}{\zeta_\alpha} \Gamma^{k-1} \chi \right)(t) \right\|_2 \\
&\le c(M) \sum_{|k| \le j} \|\Gamma^k \zeta_t(t)\|_2 \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \sum_{|k| \le j-1} \|\Gamma^k \partial_\alpha \chi(t)\|_2 \\
&\quad + c(M) \sum_{|k| \le j} \|\Gamma^k \zeta_t(t)\|_2 \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \partial_\alpha \chi(t)\|_\infty \\
&\quad + c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \zeta_t(t)\|_\infty \sum_{|k| \le j-1} \|\Gamma^k \partial_\alpha \chi(t)\|_2
\end{aligned}$$

and using the fact that

$$\zeta_t = u - b \zeta_\alpha \tag{4.66}$$

and (3.43), we have

$$\sum_{|k| \le l} \|\Gamma^k \zeta_t(t)\|_2 \le c(M). \tag{4.67}$$

So

$$\sum_{k=1}^{j}\left\|\Gamma^{j-k}\left((\partial_t\Gamma'\zeta)\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\chi\right)(t)\right\|_2$$
$$\leq c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\leq j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2$$
$$+c(M)\sum_{|k|\leq j}\|\Gamma^k\zeta_t(t)\|_2\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty$$
$$+c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\zeta_t(t)\|_\infty\sum_{|k|\leq j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2.$$

Therefore

$$\begin{aligned}\|\partial_t R_j^\chi(t)\|_2 &\leq c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\leq j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2\\ &+c(M)\sum_{|k|\leq j}\|\Gamma^k\zeta_t(t)\|_2\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty\\ &+c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\zeta_t(t)\|_\infty\sum_{|k|\leq j-1}\|\Gamma^k\partial_\alpha\chi(t)\|_2\\ &+c(M)\sum_{|k|\leq j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\partial_t\chi(t)\|_\infty\\ &+c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\leq j-1}\|\partial_t\Gamma^k\chi(t)\|_2.\end{aligned}\tag{4.68}$$

Similarly, to estimate $\|\partial_t R_j^\lambda(t)\|_2$, for $1\leq j\leq l-2$, we expand the first term in (4.62),

$$[\partial_t\Gamma'\zeta,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\lambda=(\partial_t\Gamma'\zeta)\mathcal{H}\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\lambda-\mathcal{H}\left(\partial_t\Gamma'\zeta\frac{\partial_\alpha}{\zeta_\alpha}\Gamma^{k-1}\lambda\right).$$

Using (4.56), (4.63), (4.67) and Propositions 3.2, 3.3, 3.5, we get for $1\leq j\leq l-2$,

$$\begin{aligned}\|\partial_t R_j^\lambda(t)\|_2 &\leq c(M)\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\zeta_t(t)\|_\infty\sum_{|k|\leq j}\|\Gamma^k\partial_\alpha\lambda(t)\|_2\\ &+c(M)\sum_{|k|\leq j}\|\Gamma^k\zeta_t(t)\|_2\sum_{|k|\leq[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty\\ &+c(M)\sum_{|k|\leq j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\left(\sum_{|k|\leq j-1}\|\partial_t\Gamma^k\lambda(t)\|_2+\sum_{|k|\leq j-1}\|\Gamma^k\partial_\alpha\lambda(t)\|_2\right).\end{aligned}\tag{4.69}$$

Sum up (4.54), (4.60) and (4.68), we obtain

$$\begin{aligned}
&\sum_{|j|\le l}\left|\int(\partial_t\eta_j^\chi\partial_\alpha\bar{R}_j^\chi+\partial_t R_j^\chi\partial_\alpha\bar{\eta}_j^\chi+\partial_t R_j^\chi\partial_\alpha\bar{R}_j^\chi)\,d\alpha\right|\\
&\quad\le c(M)\sum_{|j|\le[\frac{l}{2}]}(\|\Gamma^j\zeta_t(t)\|_\infty+\|\Gamma^j(\zeta_\alpha-1)(t)\|_\infty+\|\Gamma^j\partial_t\chi(t)\|_\infty+\|\Gamma^j\partial_\alpha\chi(t)\|_\infty)^2\\
&\qquad\times\sum_{|j|\le l}(\|\Gamma^j\zeta_t(t)\|_2+\|\Gamma^j(\zeta_\alpha-1)(t)\|_2+\|\partial_t\Gamma^j\chi(t)\|_2+\|\Gamma^j\partial_\alpha\chi(t)\|_2)^2. \qquad (4.70)
\end{aligned}$$

Sum up (4.54), (4.59) and (4.69), we have

$$\begin{aligned}
&\sum_{|j|\le l-2}\left|\int(\partial_t\eta_j^\lambda\partial_\alpha\bar{R}_j^\lambda+\partial_t R_j^\lambda\partial_\alpha\bar{\eta}_j^\lambda+\partial_t R_j^\lambda\partial_\alpha\bar{R}_j^\lambda)\,d\alpha\right|\\
&\quad\le c(M)\Bigg(\sum_{|j|\le[\frac{l-2}{2}]}\|\Gamma^j\zeta_t(t)\|_\infty+\sum_{|j|\le[\frac{l-2}{2}]}\|\Gamma^j\partial_\alpha\lambda(t)\|_\infty+\sum_{|j|\le l-2}\|\Gamma^j(\zeta_\alpha-1)(t)\|_\infty\Bigg)^2\\
&\qquad\times\sum_{|j|\le l-2}(\|\partial_t\Gamma^j\lambda(t)\|_2+\|\Gamma^j\partial_\alpha\lambda(t)\|_2+\|\Gamma^j\zeta_t(t)\|_2)^2. \qquad (4.71)
\end{aligned}$$

Step 2. We now consider $\|G_j^\Theta(t)\|_2$ for $\Theta=v_1,\ \chi,\ \lambda$, where

$$G_j^\Theta=\Gamma^j\mathcal{P}\Theta+\sum_{k=1}^{j}\Gamma^{j-k}[\mathcal{P},\Gamma]\Gamma^{k-1}\Theta. \qquad (4.72)$$

We first derive the estimate in (4.74), then work on the terms on the right hand side of (4.74) through three sub-steps. Finally, we conclude this step with the estimate in (4.94).

We know from Proposition 3.5 and (3.43) that

$$\begin{aligned}
\sum_{|q|\le l-1}\|\Gamma^q b(t)\|_\infty&\le c\sum_{|q|\le l}\|\Gamma^q b(t)\|_2\\
&\le c(M)\sum_{|q|\le l}\|\Gamma^q u(t)\|_2\le c(M). \qquad (4.73)
\end{aligned}$$

This implies that for $1\le j\le l$, $k\ge 1$,

$$\sum_{p+k-1\le[\frac{j}{2}]}\|\Gamma^p\partial_\alpha(\partial_t+b\partial_\alpha)\Gamma^{k-1}\Theta(t)\|_\infty\le c(M)\sum_{p+k\le[\frac{j}{2}]+1}\|\Gamma^{p+k}\partial_\alpha\Theta(t)\|_\infty.$$

Therefore from (3.10) and Lemma 3.12, we have for $2\le j\le l$,

$$\begin{aligned}
\sum_{|k|\le j}\|G_k^\Theta(t)\|_2&\le c\sum_{|k|\le j}\|\Gamma^k\mathcal{P}\Theta(t)\|_2\\
&\quad+c(M)\sum_{|k|\le[\frac{j}{2}]}(\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_\infty+\|\Gamma^k b(t)\|_\infty+\|\Gamma^k(A-1)(t)\|_\infty)
\end{aligned}$$

$$\times \sum_{|k|\le j-1} (\|\Gamma^k \partial_\alpha \Theta(t)\|_2 + \|(\partial_t + b\partial_\alpha)\Gamma^k \partial_\alpha \Theta(t)\|_2)$$
$$+ c(M) \sum_{|k|\le j} (\|\Gamma^k (\partial_t + b\partial_\alpha) b(t)\|_2 + \|\Gamma^k b(t)\|_2$$
$$+ \|\Gamma^k (A-1)(t)\|_2) \sum_{|k|\le [\frac{j}{2}]+1} \|\Gamma^k \partial_\alpha \Theta(t)\|_\infty. \tag{4.74}$$

We further estimate the right hand side of (4.74) through the following steps.

Step 2.1. We first consider the estimate of

$$\|\Gamma^k (\partial_t + b\partial_\alpha) b(t)\|_\infty + \|\Gamma^k b(t)\|_\infty + \|\Gamma^k (A-1)(t)\|_\infty.$$

in (4.74). We use (2.30), (2.52) and conclude this sub-step with the estimate in (4.79).

From (2.30), we know

$$(I-\mathcal{H})b = -[u,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}, \qquad (I-\mathcal{H})(A-1) = i[u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha} + i[w,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}.$$

Using Lemma 3.8, then Lemma 3.16 (3.61), (3.62), we have for $0 \le j \le l-2$,

$$\begin{aligned}
&\sum_{|k|\le j} \|\Gamma^k b(t)\|_\infty \\
&\le c(M)\Bigg(\sum_{|k|\le j} \left\|\Gamma^k [u,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}(t)\right\|_\infty + \|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \left\|\Gamma^k [u,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}(t)\right\|_2\Bigg) \\
&\le c(M)\Bigg(\sum_{|k|\le j+1} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \ln(t+e) \\
&\quad + \sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \frac{1}{t+1}\Bigg) \\
&\quad + c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \\
&\le c(M)\Bigg(\sum_{|k|\le j+1} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \ln(t+e) \\
&\quad + \sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \frac{1}{t+1}\Bigg).
\end{aligned} \tag{4.75}$$

Similarly, we have for $0 \le j \le l-2$,

$$\begin{aligned}
&\sum_{|k|\le j} \|\Gamma^k (A-1)(t)\|_\infty \\
&\quad \le c(M) \sum_{|k|\le j} \left\|\Gamma^k \left([u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha} + [w,\mathcal{H}]\frac{\bar\zeta_\alpha - 1}{\zeta_\alpha}\right)(t)\right\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\left\|\Gamma^k\left([u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}+[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}\right)(t)\right\|_2\\
&\le c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u_\alpha(t)\|_\infty \ln(t+e)\\
&\quad+\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k u_\alpha(t)\|_2\frac{1}{t+1}\Bigg)\\
&\quad+c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty \ln(t+e)\\
&\quad+\sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\frac{1}{t+1}\Bigg)\\
&\quad+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u_\alpha(t)\|_2\\
&\quad+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2.
\end{aligned}
$$

This implies that for $0\le j\le l-2$,

$$
\begin{aligned}
&\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_\infty\\
&\quad\le c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u_\alpha(t)\|_\infty \ln(t+e)\\
&\qquad+\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k u_\alpha(t)\|_2\frac{1}{t+1}\Bigg)\\
&\qquad+c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty \ln(t+e)\\
&\qquad+\sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\frac{1}{t+1}\Bigg)\\
&\qquad+c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty.
\end{aligned}
\tag{4.76}
$$

In order to estimate $\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_\infty$, we use (2.52):

$$
\begin{aligned}
&(I-\mathcal{H})(\partial_t+b\partial_\alpha)b\\
&\quad=[u,\mathcal{H}]\frac{\partial_\alpha(2b-\bar u)}{\zeta_\alpha}-[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}+\frac{1}{\pi i}\int\left(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\right)^2(\bar\zeta_\beta-1)\,d\beta.
\end{aligned}
$$

We need to first obtain an estimate of $\|\Gamma^k b(t)\|_\infty$ that does not involve $\ln(t+e)$. For this purpose we use (3.40). We have from Lemma 3.8 and (3.40), (3.63) that for $0 \le j \le l-1$,

$$\begin{aligned}
\sum_{|k|\le j} \|\Gamma^k b(t)\|_\infty &\le c(M)\Bigg(\sum_{|k|\le j} \left\|\Gamma^k [u,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}(t)\right\|_\infty \\
&\quad + \|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \left\|\Gamma^k [u,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}(t)\right\|_2\Bigg) \\
&\le c(M)\Bigg(\sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 \\
&\quad + \sum_{|k|\le j+1} \|\Gamma^k u(t)\|_2 \|(\zeta_\alpha - 1)(t)\|_\infty \\
&\quad + \|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k u(t)\|_2 \sum_{|k|\le j+1} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2\Bigg) \\
&\le c(M)\Bigg(\sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty + \|(\zeta_\alpha - 1)(t)\|_\infty\Bigg). \qquad (4.77)
\end{aligned}$$

Now similar to the cases of $\|\Gamma^k b(t)\|_\infty$ and $\|\Gamma^k (A-1)(t)\|_\infty$, we use Lemma 3.8 and (3.61), (3.62) to estimate $\|\Gamma^k(\partial_t + b\partial_\alpha)b(t)\|_\infty$. We have for $0 \le j \le l-2$,

$$\begin{aligned}
&\sum_{|k|\le j} \|\Gamma^k(\partial_t + b\partial_\alpha)b(t)\|_\infty \\
&\le c(M)\Bigg(\sum_{|k|\le j+1} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k(u_\alpha - 2b_\alpha)(t)\|_\infty \ln(t+e) \\
&\quad + \sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k(u_\alpha - 2b_\alpha)(t)\|_2 \frac{1}{t+1}\Bigg) \\
&\quad + c(M)\Bigg(\sum_{|k|\le j+1} \|\Gamma^k w(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_\infty \ln(t+e) \\
&\quad + \sum_{|k|\le j} \|\Gamma^k w(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k(\zeta_\alpha - 1)(t)\|_2 \frac{1}{t+1}\Bigg) \\
&\quad + c(M)\|\mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j+1} \|\Gamma^k u(t)\|_\infty \\
&\quad + \mathcal{A},
\end{aligned}$$

where

$$\mathcal{A} = c(M) \sum_{|k|\le j+1} \left\|\Gamma^k \int \left(\frac{u(\cdot,t) - u(\beta,t)}{\zeta(\cdot,t) - \zeta(\beta,t)}\right)^2 (\bar{\zeta}_\beta - 1)\, d\beta\right\|_2.$$

Further using Proposition 3.2, we have

$$\sum_{|k|\le j+1}\left\|\Gamma^k\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2(\bar\zeta_\beta-1)\,d\beta\right\|_2$$
$$\le c(M)\sum_{|k|\le j+1}\|\Gamma^k u_\alpha(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2.$$

Therefore for $0\le j\le l-2$,

$$\begin{aligned}&\sum_{|k|\le j}\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_\infty\\
&\quad\le c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j+2}\|\Gamma^k u(t)\|_\infty\ln(t+e)\\
&\qquad+\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2\frac{1}{t+1}\Bigg)\\
&\qquad+c(M)\Bigg(\sum_{|k|\le j+1}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\ln(t+e)\\
&\qquad+\sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\frac{1}{t+1}\Bigg)\\
&\qquad+c(M)\|(\zeta_\alpha-1)(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_\infty\ln(t+e).\end{aligned}\tag{4.78}$$

Sum up (4.75), (4.76), (4.78), and further applying Propositions 3.6, 3.11, 3.13 and Proposition 3.5, we have for $1\le j\le l-4$, and sufficiently small $M>0$,

$$\sum_{|k|\le j}(\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_\infty+\|\Gamma^k b(t)\|_\infty+\|\Gamma^k(A-1)(t)\|_\infty)$$
$$\le c(M)\sum_{|k|\le j+2}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)^2\ln(t+e)+c(M)\mathfrak{E}(t)\frac{1}{t+1}.\tag{4.79}$$

Step 2.2. We now estimate the quantity

$$\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_2+\|\Gamma^k b(t)\|_2+\|\Gamma^k(A-1)(t)\|_2$$

in (4.74) for $1\le k\le l$. This sub-step is concluded with the estimate in (4.83).

Using Lemmas 3.8, 3.17 and (2.30) we get that for $1\le j\le l$,

$$\sum_{|k|\le j}\|\Gamma^k b(t)\|_2\le c(M)\sum_{|k|\le j}\left\|\Gamma^k[u,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}(t)\right\|_2$$
$$\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2$$

$$+\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\ln(t+e)$$
$$+\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\frac{1}{t+1}\Bigg)\Bigg] \tag{4.80}$$

and

$$\sum_{|k|\le j}\|\Gamma^k(A-1)(t)\|_2$$
$$\le c(M)\sum_{|k|\le j}\left\|\Gamma^k\left([u,\mathcal{H}]\frac{\bar u_\alpha}{\zeta_\alpha}+[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}\right)(t)\right\|_2$$
$$\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$+\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k u(t)\|_\infty\ln(t+e)+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_2\frac{1}{t+1}\Bigg)\Bigg]$$
$$+c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2$$
$$+\sum_{|k|\le j}\|\Gamma^k w(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k(\zeta_\alpha-1)(t)\|_\infty\ln(t+e)$$
$$+\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k(\zeta_\alpha-1)(t)\|_2\frac{1}{t+1}\Bigg)\Bigg]. \tag{4.81}$$

From (2.52) and Lemma 3.8 we have for $1\le j\le l$,

$$\sum_{|k|\le j}\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_2$$
$$\le c(M)\sum_{|k|\le j}\left\|\Gamma^k[u,\mathcal{H}]\frac{\partial_\alpha(2b-\bar u)}{\zeta_\alpha}\right\|_2$$
$$+c(M)\sum_{|k|\le j}\left\|\Gamma^k[w,\mathcal{H}]\frac{\bar\zeta_\alpha-1}{\zeta_\alpha}\right\|_2$$
$$+c(M)\sum_{|k|\le j}\left\|\Gamma^k\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2(\bar\zeta_\beta-1)\,d\beta\right\|_2,$$

where using Lemma 3.17 and (3.43), (4.77), we obtain

$$\sum_{|k|\le j}\left\|\Gamma^k[u,\mathcal{H}]\frac{\partial_\alpha(2b-\bar u)}{\zeta_\alpha}\right\|_2$$
$$\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$

$$+ \sum_{|k|\le j} \|\Gamma^k u(t)\|_2 \left\{ \left(\sum_{|k|\le[\frac{j}{2}]+2} \|\Gamma^k u(t)\|_\infty + \|\zeta_\alpha - 1\|_\infty \right) \ln(t+e) + \sum_{|k|\le[\frac{j}{2}]+1} \|\Gamma^k u(t)\|_2 \frac{1}{t+1} \right\} \Bigg]$$

from Lemma 3.17,

$$\sum_{|k|\le j} \left\| \Gamma^k [w, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} \right\|_2$$
$$\le c(M) \Bigg[\sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k w(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 + \sum_{|k|\le j} \|\Gamma^k w(t)\|_2 \Bigg(\sum_{|k|\le[\frac{j}{2}]+1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \ln(t+e) + \sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \frac{1}{t+1} \Bigg) \Bigg]$$

and using Propositions 3.2, 3.5,

$$\sum_{|k|\le j} \left\| \Gamma^k \int \left(\frac{u(\cdot,t) - u(\beta,t)}{\zeta(\cdot,t) - \zeta(\beta,t)} \right)^2 (\bar{\zeta}_\beta - 1)\, d\beta \right\|_2$$
$$\le c(M) \Bigg[\sum_{|k|\le j-1} \|\Gamma^k u_\alpha(t)\|_2 \sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty + \sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1} \|\Gamma^k u(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^j u(t)\|_2 \Bigg(\sum_{|k|\le 2} \|\Gamma^k u(t)\|_2 \|(\zeta_\alpha - 1)(t)\|_\infty + \|u_\alpha(t)\|_\infty \sum_{|k|\le 1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \Bigg) \Bigg].$$

Therefore for $1 \le j \le l$,

$$\sum_{|k|\le j} \|\Gamma^k (\partial_t + b\partial_\alpha) b(t)\|_2$$
$$\le c(M) \Bigg[\sum_{|k|\le[\frac{j}{2}]} \|\Gamma^k w(t)\|_\infty \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 + \sum_{|k|\le j} \|\Gamma^k w(t)\|_2 \sum_{|k|\le[\frac{j}{2}]+1} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_\infty \ln(t+e) + \sum_{|k|\le j} \|\Gamma^k w(t)\|_2 \sum_{|k|\le j} \|\Gamma^k (\zeta_\alpha - 1)(t)\|_2 \frac{1}{t+1}$$

$$+\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\Big(\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k u(t)\|_\infty+\|\zeta_\alpha-1\|_\infty\Big)\ln(t+e)$$

$$+\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_2\frac{1}{t+1}\Big]. \tag{4.82}$$

Notice that for $l\ge 11$, $[\frac{l}{2}]+2\le l-4$. Sum up (4.80), (4.81), (4.82), and further applying Propositions 3.6, 3.11, 3.13, we have for $2\le j\le l$, and sufficiently small $M>0$,

$$\sum_{|k|\le j}(\|\Gamma^k(\partial_t+b\partial_\alpha)b(t)\|_2+\|\Gamma^k b(t)\|_2+\|\Gamma^k(A-1)(t)\|_2)$$
$$\le c(M)\mathfrak{E}(t)^{1/2}\sum_{|k|\le[\frac{j}{2}]+2}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)\ln(t+e)+c(M)\mathfrak{E}(t)\frac{1}{t+1}. \tag{4.83}$$

Step 2.3. We estimate $\sum_{|k|\le j}\|\Gamma^k\mathcal{P}\Theta(t)\|_2$ in (4.74) for $\Theta=\chi$, λ and v_1. We use (2.27), (2.29) and (4.32), (4.33), and further divide our efforts into three sub-steps. We conclude the sub-steps respectively with the estimates in (4.84), (4.85) and (4.93).

Step 2.3.1. We first consider $\|\Gamma^j\mathcal{P}\chi(t)\|_2$.

We know

$$\mathcal{P}\chi=\frac{4}{\pi}\int\frac{(u(\alpha,t)-u(\beta,t))(\mathfrak{y}(\alpha,t)-\mathfrak{y}(\beta,t))}{|\zeta(\alpha,t)-\zeta(\beta,t)|^2}u_\beta\,d\beta$$
$$+\frac{2}{\pi}\int\Big(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2\mathfrak{y}_\beta\,d\beta=I_1+I_2.$$

Using Proposition 3.3, we have for $1\le j\le l$,

$$\|\Gamma^j I_1(t)\|_2\le c(M)\Big[\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty$$
$$+\sum_{|k|\le j-1}\|\Gamma^k u_\alpha(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty\Big]$$
$$+c\Big\|\int\frac{(\Gamma^j u(\cdot,t)-\Gamma^j u(\beta,t))(\mathfrak{y}(\cdot,t)-\mathfrak{y}(\beta,t))}{|\zeta(\cdot,t)-\zeta(\beta,t)|^2}u_\beta\,d\beta\Big\|_2,$$

where from Proposition 3.4 by taking $r=(t+1)^2$,

$$\Big\|\int\frac{(\Gamma^j u(\cdot,t)-\Gamma^j u(\beta,t))(\mathfrak{y}(\cdot,t)-\mathfrak{y}(\beta,t))}{|\zeta(\cdot,t)-\zeta(\beta,t)|^2}u_\beta\,d\beta\Big\|_2$$
$$\le c(M)\|\Gamma^j u(t)\|_2$$
$$\times\Big(\sum_{|k|\le 1}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty\ln(t+e)+\|\mathfrak{y}_\alpha(t)\|_\infty\|u_\alpha(t)\|_2\frac{1}{t+1}\Big).$$

Using Proposition 3.2, we have

$$\|\Gamma^j I_2(t)\|_2 \le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_2$$
$$+\sum_{|k|\le j-1}\|\Gamma^k u_\alpha(t)\|_2 \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty\Bigg]$$
$$+c\left\|\int \frac{(\Gamma^j u(\cdot,t)-\Gamma^j u(\beta,t))(u(\cdot,t)-u(\beta,t))}{(\zeta(\cdot,t)-\zeta(\beta,t))^2}\mathfrak{y}_\beta\, d\beta\right\|_2,$$

where from Proposition 3.4,

$$\left\|\int \frac{(\Gamma^j u(\cdot,t)-\Gamma^j u(\beta,t))(u(\cdot,t)-u(\beta,t))}{(\zeta(\cdot,t)-\zeta(\beta,t))^2}\mathfrak{y}_\beta\, d\beta\right\|_2$$
$$\le c(M)\|\Gamma^j u(t)\|_2$$
$$\times\left(\sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \ln(t+e)+\|u_\alpha(t)\|_\infty\|\mathfrak{y}_\alpha(t)\|_2\frac{1}{t+1}\right).$$

Therefore for $1\le j\le l$,

$$\sum_{|k|\le j}\|\Gamma^k \mathcal{P}\chi(t)\|_2$$
$$\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_2$$
$$+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$+\|\Gamma^j u(t)\|_2\Bigg\{\sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \ln(t+e)$$
$$+(\|\mathfrak{y}_\alpha(t)\|_\infty\|u_\alpha(t)\|_2+\|u_\alpha(t)\|_\infty\|\mathfrak{y}_\alpha(t)\|_2)\frac{1}{t+1}\Bigg\}\Bigg].$$

Further applying Propositions 3.6, 3.11, 3.13, and Proposition 3.5, we have for $2\le j\le l$ and sufficiently small $M>0$,

$$\sum_{|k|\le j}\|\Gamma^k \mathcal{P}\chi(t)\|_2$$
$$\le c(M)\mathfrak{E}(t)^{1/2}\sum_{|k|\le[\frac{j}{2}]+2}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)^2\ln(t+e)$$
$$+c(M)\mathfrak{E}(t))^{3/2}\frac{1}{t+1}. \tag{4.84}$$

Step 2.3.2. We estimate $\|\Gamma^j\mathcal{P}\lambda(t)\|_2$ similarly.

We know from (2.29),

$$\mathcal{P}\lambda = -\left[u, \mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right](\bar{\zeta}_\alpha w) + [u, \bar{\mathcal{H}}]\left(\bar{u}\frac{u_\alpha}{\bar{\zeta}_\alpha}\right) + u[u, \mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha}$$
$$- 2[u, \mathcal{H}]\frac{u \cdot u_\alpha}{\zeta_\alpha} + \frac{1}{\pi i}\int \left(\frac{u(\alpha, t) - u(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)}\right)^2 u \cdot \zeta_\beta \, d\beta = I_3 + I_4 + I_5,$$

where

$$I_3 = -\left[u, \mathcal{H}\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}\right](\bar{\zeta}_\alpha w),$$
$$I_4 = [u, \bar{\mathcal{H}}]\left(\bar{u}\frac{u_\alpha}{\bar{\zeta}_\alpha}\right) + u[u, \mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha} - 2[u, \mathcal{H}]\frac{u \cdot u_\alpha}{\zeta_\alpha},$$
$$I_5 = \frac{1}{\pi i}\int \left(\frac{u(\alpha, t) - u(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)}\right)^2 u \cdot \zeta_\beta \, d\beta$$

we have by using Propositions 3.2, 3.4 that for $1 \le j \le l - 2$,

$$\begin{aligned}
&\|\Gamma^j I_3(t)\|_2 \\
&\quad \le c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k| \le j} \|\Gamma^k w(t)\|_2 \\
&\qquad + c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le j} \|\Gamma^k \mathfrak{y}_\alpha(t)\|_2 \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k w(t)\|_\infty \\
&\qquad + c(M) \sum_{|k| \le j} \|\Gamma^k u(t)\|_2 \Bigg(\sum_{|k| \le [\frac{j}{2}]+1} \|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]+1} \|\Gamma^k w(t)\|_\infty \ln(t + e) \\
&\qquad + \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k w(t)\|_2 \frac{1}{t + 1} \Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\|\Gamma^j I_4(t)\|_2 \\
&\quad \le c(M) \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]+1} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le j+1} \|\Gamma^k u(t)\|_2 \\
&\qquad + c(M) \sum_{|k| \le j} \|\Gamma^k u(t)\|_2 \Bigg(\sum_{|k| \le [\frac{j}{2}]+1} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]+2} \|\Gamma^k u(t)\|_\infty \ln(t + e) \\
&\qquad + \sum_{|k| \le [\frac{j}{2}]} \|\Gamma^k u(t)\|_\infty \sum_{|k| \le [\frac{j}{2}]+1} \|\Gamma^k u(t)\|_2 \frac{1}{t + 1} \Bigg).
\end{aligned}$$

Using Proposition 3.2, we get

$$
\begin{aligned}
&\|\Gamma^j I_5(t)\|_2 \\
&\quad \le c(M)\Bigg(\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k u(t)\|_2 \\
&\qquad + \sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2 \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty\Bigg).
\end{aligned}
$$

Therefore we have for $1 \le j \le l-2$,

$$
\begin{aligned}
&\sum_{|k|\le j}\|\Gamma^k \mathcal{P}\lambda(t)\|_2 \\
&\quad \le c(M)\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k w(t)\|_2 \\
&\qquad + c(M)\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_2 \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k w(t)\|_\infty \\
&\qquad + c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k w(t)\|_\infty \ln(t+e) \\
&\qquad + \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k w(t)\|_2 \frac{1}{t+1}\Bigg) \\
&\qquad + c(M)\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2 \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k u(t)\|_\infty \ln(t+e) \\
&\qquad + c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2 \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_2 \frac{1}{t+1}.
\end{aligned}
$$

Further applying Propositions 3.6, 3.11, 3.13, and Proposition 3.5, we have for $2 \le j \le l-2$ and sufficiently small $M > 0$,

$$
\begin{aligned}
&\sum_{|k|\le j}\|\Gamma^k \mathcal{P}\lambda(t)\|_2 \\
&\quad \le c(M)\mathfrak{E}(t)^{1/2}\sum_{|k|\le[\frac{j}{2}]+2}(\|\Gamma^k \partial_\alpha \chi(t)\|_\infty + \|\Gamma^k \partial_\alpha \lambda(t)\|_\infty)^2 \ln(t+e) \\
&\qquad + c(M)\mathfrak{E}(t)^{3/2}\frac{1}{t+1}.
\end{aligned} \tag{4.85}
$$

Step 2.3.3. We now estimate $\sum_{|k|\le j}\|\Gamma^k \mathcal{P} v_1(t)\|_2$ for $1 \le j \le l$.
We start from (4.32) and (4.33). We have by using Propositions 3.2, 3.3,

$$
\begin{aligned}
\|\Gamma^j \mathcal{P} v_1(t)\|_2 &\le c(M)\sum_{|k|\le j}\|\Gamma^k \mathcal{P} v(t)\|_2 \\
&\quad + c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2 \sum_{|k|\le \max(j,2)}\|\Gamma^k \mathcal{P}\chi(t)\|_2
\end{aligned}
$$

$$+2\left\|\Gamma^j[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\left(w\frac{\partial_\alpha}{\zeta_\alpha}\chi\right)\right\|_2+\left\|\Gamma^j[(\bar{\mathcal{H}}+\mathcal{H})u,\mathcal{H}]\left(\frac{\partial_\alpha}{\zeta_\alpha}\right)^2\chi\right\|_2$$
$$+\frac{1}{\pi}\left\|\Gamma^j\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2\partial_\beta v\,d\beta\right\|_2, \tag{4.86}$$

where by further using Proposition 3.4,

$$\begin{aligned}
&2\left\|\Gamma^j[u,\mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}\left(w\frac{\partial_\alpha}{\zeta_\alpha}\chi\right)\right\|_2\\
&\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u_\alpha(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k(w\partial_\alpha\chi)(t)\|_2\\
&\quad+\sum_{|k|\le j-1}\|\Gamma^k u_\alpha(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k(w\partial_\alpha\chi)(t)\|_\infty\\
&\quad+\|\Gamma^j u(t)\|_2\left(\sum_{|k|\le 2}\|\Gamma^k(w\partial_\alpha\chi)(t)\|_\infty\ln(t+e)+\sum_{|k|\le 1}\|\Gamma^k(w\partial_\alpha\chi)(t)\|_2\frac{1}{t+1}\right)\Bigg],
\end{aligned} \tag{4.87}$$

using the fact that $\mathcal{H}(\frac{\partial_\alpha}{\zeta_\alpha})^2\chi=(\frac{\partial_\alpha}{\zeta_\alpha})^2\mathcal{H}\chi=-(\frac{\partial_\alpha}{\zeta_\alpha})^2\chi$,

$$\begin{aligned}
&\left\|\Gamma^j[(\bar{\mathcal{H}}+\mathcal{H})u,\mathcal{H}]\left(\frac{\partial_\alpha}{\zeta_\alpha}\right)^2\chi\right\|_2\\
&\le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha(\bar{\mathcal{H}}+\mathcal{H})u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k\partial_\alpha\chi(t)\|_2\\
&\quad+\sum_{|k|\le j-1}\|\Gamma^k\partial_\alpha(\bar{\mathcal{H}}+\mathcal{H})u(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty\\
&\quad+\|\Gamma^j(\bar{\mathcal{H}}+\mathcal{H})u(t)\|_2\left\|\left(\frac{\partial_\alpha}{\zeta_\alpha}\right)^2\chi(t)\right\|_\infty\Bigg]\\
&\le c(M)\Bigg(\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k u(t)\|_\infty\ln(t+e)\\
&\quad+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_2\frac{1}{t+1}\Bigg)\sum_{|k|\le j}\|\Gamma^k\partial_\alpha\chi(t)\|_2\\
&\quad+c(M)\Bigg(\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\\
&\quad+\sum_{|k|\le j}\|\Gamma^k\mathfrak{y}_\alpha(t)\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty\Bigg)\sum_{|k|\le\max([\frac{j}{2}],1)}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty,
\end{aligned} \tag{4.88}$$

and

$$\frac{1}{\pi}\left\|\Gamma^j \int \left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2 \partial_\beta v \, d\beta\right\|_2$$
$$\le c(M)\Bigg[\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\Bigg)^2 \sum_{|k|\le j}\|\Gamma^k v(t)\|_2$$
$$+\sum_{|k|\le j-1}\|\Gamma^k\partial_\alpha u(t)\|_2 \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha v(t)\|_\infty$$
$$+c(M)\Bigg[\|\Gamma^j u(t)\|_2\Bigg(\sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k\partial_\alpha v(t)\|_\infty \ln(t+e)$$
$$+\|u_\alpha(t)\|_\infty\|\partial_\alpha v(t)\|_2\frac{1}{t+1}\Bigg)\Bigg]. \tag{4.89}$$

Similar to the calculations above, we have from (4.33) that for $1\le j\le l$,

$$\|\Gamma^j \mathcal{P} v(t)\|_2 \le c(M)\Bigg[\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty \sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k w(t)\|_2$$
$$+\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k w(t)\|_\infty \sum_{|k|\le j}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_2 \sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty$$
$$+\|\Gamma^j w(t)\|_2\Bigg\{\sum_{|k|\le 1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty \ln(t+e)$$
$$+(\|\mathfrak{y}_\alpha(t)\|_\infty\|u_\alpha(t)\|_2+\|u_\alpha(t)\|_\infty\|\mathfrak{y}_\alpha(t)\|_2)\frac{1}{t+1}\Bigg\}$$
$$+\|\Gamma^j u(t)\|_2\Bigg\{\sum_{|k|\le 1}\|\Gamma^k \mathfrak{y}_\alpha(t)\|_\infty \sum_{|k|\le 1}\|\Gamma^k w_\alpha(t)\|_\infty \ln(t+e)$$
$$+(\|\mathfrak{y}_\alpha(t)\|_\infty\|w_\alpha(t)\|_2+\|w_\alpha(t)\|_\infty\|\mathfrak{y}_\alpha(t)\|_2)\frac{1}{t+1}\Bigg\}$$
$$+\Bigg(\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\Bigg)^2 \sum_{|k|\le j}\|\Gamma^k u(t)\|_2$$
$$+\|\Gamma^j u(t)\|_2\Bigg\{\Bigg(\sum_{|k|\le 1}\|\Gamma^k u_\alpha(t)\|_\infty\Bigg)^2 \ln(t+e)+\|u_\alpha(t)\|_\infty\|u_\alpha(t)\|_2\frac{1}{t+1}\Bigg\}$$

$$+\sum_{|k|\le[\frac{j}{2}]}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\sum_{|k|\le j}\|\Gamma^k\partial_\alpha\chi(t)\|_2$$

$$+\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_2\sum_{|k|\le[\frac{j}{2}]}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty\Bigg]. \tag{4.90}$$

We need the estimates of $\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\|_\infty$ and $\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\|_2$. We use (2.32) and Lemma 3.15. We have by applying Lemma 3.16 (3.61), (3.62) that for $0\le j\le l-2$ and sufficiently small $M>0$,

$$\begin{aligned}
&\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\\
&\quad\le c(M)\sum_{|k|\le j+2}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j+2}\|\Gamma^k u(t)\|_\infty\ln(t+e)\\
&\qquad+c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2\frac{1}{t+1}\\
&\qquad+c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j+1}\|\Gamma^k w(t)\|_2\frac{1}{t+1}\\
&\qquad+c(M)(\|w(t)\|_\infty+\|\mathfrak{y}_\alpha(t)\|_\infty)\sum_{|k|\le j+1}(\|\Gamma^k w(t)\|_\infty+\|\Gamma^k u(t)\|_\infty)\\
&\qquad+\mathcal{B},
\end{aligned}$$

where

$$\mathcal{B}=c(M)\sum_{|k|\le j+1}\left\|\Gamma^k\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2\bar{u}_\beta\,d\beta\right\|_2.$$

Using Proposition 3.3, we get

$$\begin{aligned}
&\sum_{|k|\le j+1}\left\|\Gamma^k\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2\bar{u}_\beta\,d\beta\right\|_2\\
&\quad\le c(M)\sum_{|k|\le j+1}\|\Gamma^k u_\alpha(t)\|_\infty^2\sum_{|k|\le j+1}\|\Gamma^k u(t)\|_2.
\end{aligned}$$

Further applying Propositions 3.6, 3.11, 3.13 and Proposition 3.5, we have for $1\le j\le l-4$ and sufficiently small $M>0$,

$$\begin{aligned}
&\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_\infty\\
&\quad\le c(M)\sum_{|k|\le j+2}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)^2\ln(t+e)\\
&\qquad+c(M)\sum_{|k|\le j+1}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2)^2\frac{1}{t+1}.
\end{aligned} \tag{4.91}$$

Also from (2.32) and Lemma 3.15 we have for $1 \le j \le l$,

$$\begin{aligned}&\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_2\\&\quad\le c(M)\sum_{|k|\le j}\left(\left\|\Gamma^k[w,\mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha}\right\|_2+\left\|\Gamma^k[u,\mathcal{H}]\frac{\bar{w}_\alpha}{\zeta_\alpha}\right\|_2\right)\\&\qquad+c(M)\sum_{|k|\le j}\left\|\Gamma^k\int\left(\frac{u(\cdot,t)-u(\beta,t)}{\zeta(\cdot,t)-\zeta(\beta,t)}\right)^2\bar{u}_\beta\,d\beta\right\|_2.\end{aligned}$$

Using Lemma 3.17 (3.65), Propositions 3.3, 3.5, we obtain for $3 \le j \le l$,

$$\begin{aligned}&\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_2\\&\quad\le c(M)\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k w(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\\&\qquad+c(M)\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k w(t)\|_2\\&\qquad+c(M)\sum_{|k|\le j}\|\Gamma^k w(t)\|_2\sum_{|k|\le[\frac{j}{2}]+2}\left(\|\Gamma^k u(t)\|_\infty\ln(t+e)+\|\Gamma^k u(t)\|_2\frac{1}{t+1}\right)\\&\qquad+c(M)\sum_{|k|\le j}\|\Gamma^k u(t)\|_2\sum_{|k|\le[\frac{j}{2}]+2}\left(\|\Gamma^k w(t)\|_\infty\ln(t+e)+\|\Gamma^k w(t)\|_2\frac{1}{t+1}\right)\\&\qquad+c(M)\sum_{|k|\le[\frac{j}{2}]+1}\|\Gamma^k u(t)\|_\infty\sum_{|k|\le j}\|\Gamma^k u(t)\|_2^2.\end{aligned}$$

Further applying Propositions 3.6, 3.11, 3.13, we have for $3 \le j \le l$, and sufficiently small $M > 0$,

$$\begin{aligned}&\sum_{|k|\le j}\left\|\Gamma^k\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}(t)\right\|_2\\&\quad\le c(M)\sum_{|k|\le[\frac{j}{2}]+2}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)\\&\qquad\times\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2)\ln(t+e)\\&\qquad+c(M)\sum_{|k|\le j}(\|(\partial_t+b\partial_\alpha)\Gamma^k v(t)\|_2+\|(\partial_t+b\partial_\alpha)\Gamma^k\chi(t)\|_2)^2\frac{1}{t+1}.\end{aligned}\tag{4.92}$$

Now sum up the inequalities from (4.86) through (4.92), using (4.84), applying Propositions 3.6, 3.11, 3.13, 3.14, and Proposition 3.5, we have for $3 \le j \le l$ and sufficiently small

$M > 0$,

$$\sum_{|k|\le j} \|\Gamma^k \mathcal{P} v_1(t)\|_2 \le c(M)\mathfrak{E}(t)^{1/2} \sum_{|k|\le[\frac{j}{2}]+2} (\|\Gamma^k \partial_\alpha \chi(t)\|_\infty + \|\Gamma^k \partial_\alpha \lambda(t)\|_\infty)^2 \ln(t+e) + c(M)\mathfrak{E}(t)^{3/2} \frac{1}{t+1}. \tag{4.93}$$

Finally from (4.74), using (4.79), (4.83), (4.84), (4.85), (4.93) and Propositions 3.14, 3.5, and notice that for $l \ge 11$, $[\frac{l}{2}] + 2 \le l - 4$. We get

$$\sum_{|j|\le l} (\|G_j^{v_1}(t)\|_2 + \|G_j^{\chi}(t)\|_2) + \sum_{|j|\le l-2} \|G_j^{\lambda}(t)\|_2 \le c(M)\mathfrak{E}(t)^{1/2} \left(\sum_{|j|\le l-4} \|\Gamma^j \partial_\alpha \lambda(t)\|_\infty + \sum_{|j|\le l-2} \|\Gamma^j \partial_\alpha \chi(t)\|_\infty \right)^2 \ln(t+e) + c(M)\mathfrak{E}(t)^{3/2} \frac{1}{t+1}. \tag{4.94}$$

This finishes Step 2.

Step 3. The final step for the energy estimate (4.52) and Proposition 4.4.

Notice that we need to further estimate a few terms on the right hand sides of (4.70), (4.71). In particular, we have from $\zeta_t = u - b\zeta_\alpha$ and using (4.77) for $0 \le j \le l - 1$,

$$\|\Gamma^j \zeta_t(t)\|_\infty \le \|\Gamma^j u(t)\|_\infty + c(M) \sum_{|k|\le j} \|\Gamma^k b(t)\|_\infty \le c(M) \left(\sum_{|k|\le j} \|\Gamma^k u(t)\|_\infty + \|(\zeta_\alpha - 1)(t)\|_\infty \right)$$

and using (3.43) for $1 \le j \le l$,

$$\|\Gamma^j \zeta_t(t)\|_2 \le \|\Gamma^j u(t)\|_2 + c(M) \sum_{|k|\le j} \|\Gamma^k b(t)\|_2 \le c(M) \sum_{|k|\le j} \|\Gamma^k u(t)\|_2.$$

Furthermore for $0 \le j \le l$, $\Theta = \chi,\ \lambda$,

$$\|\partial_t \Gamma^j \Theta(t)\|_2 \le \|(\partial_t + b\partial_\alpha)\Gamma^j \Theta(t)\|_2 + c(M)\|\partial_\alpha \Gamma^j \Theta(t)\|_2.$$

Sum up the inequalities (4.53), (4.70), (4.71), (4.91), (4.94), applying Propositions 3.6, 3.11, 3.13, 3.14, and using the fact that for $l \ge 11$, $[\frac{l}{2}] + 2 \le l - 4$. We conclude

$$\frac{d}{dt}\mathfrak{E}(t) \le c_1(M)\mathfrak{E}(t) \left(\sum_{|j|\le l-4} \|\Gamma^j \partial_\alpha \lambda(t)\|_\infty + \sum_{|j|\le l-2} \|\Gamma^j \partial_\alpha \chi(t)\|_\infty \right)^2 \ln(t+e) + c_1(M)\mathfrak{E}(t)^2 \frac{1}{t+1} \tag{4.95}$$

for some constant $c_1(M)$ depending on M. This proves Proposition 4.4. □

4.2 L^2–L^∞ estimate and the energy inequality

We now derive an estimate of the quantity $\sum_{|j|\le l-4}\|\Gamma^j\partial_\alpha\lambda(t)\|_\infty+\sum_{|j|\le l-2}\|\Gamma^j\partial_\alpha\chi(t)\|_\infty$ in the right hand side of (4.95) (or (4.52)) in turns of the energy $\mathfrak{E}(t)$. We continue assuming (4.1)–(4.4), with a M_0 so small that all the inequalities and estimates derived so far, including Propositions 4.3, 4.4 hold.

Let $\Theta=\chi,\ \lambda$. From Proposition 3.1 we know for $\alpha\in R$, $0\le j\le l-2$,

$$(1+t)^{1/2}|\Gamma^j\partial_\alpha\Theta(\alpha,t)|\le c\left(\sum_{|k|\le 2}\|\Gamma^k\Gamma^j\partial_\alpha\Theta(t)\|_2+\sum_{|k|\le 1}\|\Omega_0\Gamma^k\Gamma^j\partial_\alpha\Theta(t)\|_2\right).$$

Applying (3.3) to the second term on the right, we get

$$(1+t)^{1/2}|\Gamma^j\partial_\alpha\Theta(\alpha,t)|\le c\left(\sum_{|k|\le 2+j}\|\Gamma^k\partial_\alpha\Theta(t)\|_2+\sum_{|k|\le 1+j}\|\Gamma^k\Omega_0\partial_\alpha\Theta(t)\|_2\right).$$

Further using the relation (1.31):

$$\Omega_0\partial_\alpha=L_0\partial_t-\frac{1}{2}t(\partial_t^2-i\partial_\alpha)$$

we obtain

$$\begin{aligned}(1+t)^{1/2}|\Gamma^j\partial_\alpha\Theta(\alpha,t)|\le c\Bigg(&\sum_{|k|\le 2+j}\|\Gamma^k\partial_\alpha\Theta(t)\|_2+\sum_{|k|\le 1+j}\|\Gamma^kL_0\partial_t\Theta(t)\|_2\\&+t\sum_{|k|\le 1+j}\|\Gamma^k(\partial_t^2-i\partial_\alpha)\Theta(t)\|_2\Bigg).\end{aligned}\tag{4.96}$$

Notice that from (3.2) and (3.43), we have

$$\begin{aligned}\sum_{|k|\le 1+j}\|\Gamma^kL_0\partial_t\Theta(t)\|_2&\le c\sum_{|k|\le 2+j}\|\partial_t\Gamma^k\Theta(t)\|_2\\&\le c(M)\sum_{|k|\le 2+j}(\|(\partial_t+b\partial_\alpha)\Gamma^k\Theta(t)\|_2+\|\Gamma^k\partial_\alpha\Theta(t)\|_2).\end{aligned}\tag{4.97}$$

We now focus our attention on the quantity $\sum_{|k|\le 1+j}\|\Gamma^k(\partial_t^2-i\partial_\alpha)\Theta(t)\|_2$ in (4.96). We know

$$\partial_t^2-i\partial_\alpha=\mathcal{P}-(\partial_t+b\partial_\alpha)b\partial_\alpha-b\partial_t\partial_\alpha+i(A-1)\partial_\alpha.$$

We have, by using (3.43) that for $0\le j\le l-2$,

$$\begin{aligned}&\sum_{|k|\le 1+j}\|\Gamma^k(\partial_t^2-i\partial_\alpha)\Theta(t)\|_2\\&\quad\le\sum_{|k|\le 1+j}\|\Gamma^k\mathcal{P}\Theta(t)\|_2\\&\qquad+c(M)\sum_{|k|\le 2+j}\|\Gamma^k(b\partial_\alpha\Theta)(t)\|_2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|k|\le 1+j}(\|\Gamma^k(b\partial_t\partial_\alpha\Theta)(t)\|_2+\|\Gamma^k((A-1)\partial_\alpha\Theta)(t)\|_2)\\
&\le \sum_{|k|\le 1+j}\|\Gamma^k\mathcal{P}\Theta(t)\|_2\\
&\quad + c(M)\sum_{|k|\le[\frac{j}{2}]}(\|\Gamma^k b(t)\|_\infty+\|\Gamma^k(A-1)(t)\|_\infty)\sum_{|k|\le 2+j}\|\Gamma^k\partial_\alpha\Theta(t)\|_2\\
&\quad + c(M)\sum_{|k|\le 2+j}(\|\Gamma^k b(t)\|_2+\|\Gamma^k(A-1)(t)\|_2)\sum_{|k|\le[\frac{j}{2}]+2}\|\Gamma^k\partial_\alpha\Theta(t)\|_\infty.
\end{aligned}
$$

Now using (4.79), (4.83), (4.84), (4.85) and Propositions 3.14, 3.5, we have for $\Theta=\chi$ and $2\le j\le l-2$, and respectively for $\Theta=\lambda$ and $2\le j\le l-4$,

$$
\begin{aligned}
&\sum_{|k|\le 1+j}\|\Gamma^k(\partial_t^2-i\partial_\alpha)\Theta(t)\|_2\\
&\quad\le c(M)\mathfrak{E}(t)^{1/2}\sum_{|k|\le[\frac{j}{2}]+3}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)^2\ln(t+e)+c(M)\mathfrak{E}(t)^{3/2}\frac{1}{t+1}.
\end{aligned}
\tag{4.98}
$$

Using the fact that for $l\ge 11$, $[\frac{l}{2}]+2\le l-4$, combining (4.96)–(4.98) we obtain that for $M\le M_0$,

$$
\begin{aligned}
&(1+t)^{1/2}\Bigg(\sum_{|k|\le l-2}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\sum_{|k|\le l-4}\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty\Bigg)\\
&\quad\le c_2(M)\mathfrak{E}(t)^{1/2}+c_2(M)\mathfrak{E}(t)^{3/2}\\
&\qquad + c_2(M)\mathfrak{E}(t)^{1/2}\ln(t+e)\Bigg[(1+t)^{1/2}\sum_{|k|\le l-4}(\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty)\Bigg]^2,
\end{aligned}
\tag{4.99}
$$

where $c_2(M)$ is a constant depending on M. Let

$$
X(t)=(1+t)^{1/2}\Bigg(\sum_{|k|\le l-2}\|\Gamma^k\partial_\alpha\chi(t)\|_\infty+\sum_{|k|\le l-4}\|\Gamma^k\partial_\alpha\lambda(t)\|_\infty\Bigg).
\tag{4.100}
$$

We know (4.99) implies that $X(t)$ satisfies

$$
X(t)\le c_2(M)\mathfrak{E}(t)^{1/2}+c_2(M)\mathfrak{E}(t)^{3/2}+c_2(M)\mathfrak{E}(t)^{1/2}\ln(t+e)X(t)^2.
$$

We have the following simple calculus lemma.

Lemma 4.5 *Let* $X:[0,\infty)\to R$ *be continuous, satisfying*

$$
X(t)\le X(t)^2+a(t)
\tag{4.101}
$$

for some function $a(t)$. *Assume that* $a(t)<1/4$ *for* $t\in[0,T]$, *and assume* $X(0)\le 1/2$. *Then*

$$
X(t)\le 2a(t)\quad \textit{for all}\ \ t\in[0,T].
$$

Proof We know the inequality $X(t)^2 + a(t) \geq X(t)$ has two disconnected branches of solutions:

$$X(t) \leq \frac{1-\sqrt{1-4a(t)}}{2}, \quad \text{or} \quad X(t) \geq \frac{1+\sqrt{1-4a(t)}}{2}.$$

Since $X : [0, \infty) \to R$ is continuous, and since $X(0) \leq 1/2$. We have $X(t) \leq \frac{1-\sqrt{1-4a(t)}}{2} < 1/2$ for all $t \in [0, T]$. Therefore from (4.101),

$$X(t) \leq \frac{a(t)}{1-X(t)} \leq 2a(t) \quad \text{for all } t \in [0, T]. \qquad \square$$

Combining Lemma 4.5 and (4.99) with Proposition 4.4, we have the following Proposition.

Let $M_0 > 0$ be the constant such that (4.52), (4.99) and all the other inequalities and estimates derived so far hold.

Proposition 4.6 *Let $l \geq 11$. Assume that (4.1)–(4.4) hold on $[0, T]$, with $M = M_0$. Let $X(t)$ be as defined in (4.100). Assume $X(0) \leq 1/2$, and for $t \in [0, T]$,*

$$c_2(M_0)\mathfrak{E}(t)^{1/2}\ln(t+e) \leq 1, \quad \textit{and} \quad \mathfrak{E}(t)^{1/2} < \min\{1/8c_2(M_0), 1\}. \tag{4.102}$$

Then for $t \in [0, T]$,

$$X(t) \leq 4c_2(M_0)\mathfrak{E}(t)^{1/2}; \tag{4.103}$$

and

$$\mathfrak{E}(t) \leq \frac{\mathfrak{E}(0)}{1 - c_3(M_0)\ln^2(t+e)\mathfrak{E}(0)} \tag{4.104}$$

provided $c_3(M_0)\ln^2(t+e)\mathfrak{E}(0) < 1$ on $[0, T]$. Here $c_3(M_0) = \frac{(16c_1(M_0)c_2(M_0)^2 + c_1(M_0))e}{2}$.

Proof Under the assumption of Proposition 4.6, we know from (4.99) and Lemma 4.5 that for $t \in [0, T]$, $X(t) \leq 4c_2(M_0)\mathfrak{E}(t)^{1/2}$. So from Proposition 4.4, we have

$$\frac{d}{dt}\mathfrak{E}(t) \leq (16c_1(M_0)c_2(M_0)^2 + c_1(M_0))\mathfrak{E}(t)^2\frac{\ln(t+e)}{t+1}.$$

Let $c_3(M_0) = \frac{(16c_1(M_0)c_2(M_0)^2 + c_1(M_0))e}{2}$. We get

$$\frac{d}{dt}\mathfrak{E}(t) \leq 2c_3(M_0)\mathfrak{E}(t)^2\frac{\ln(t+e)}{t+e} \tag{4.105}$$

for $t \in [0, T]$. Solving the differential inequality (4.105), we get (4.104). $\square$

5 Almost global well-posedness of the water wave equation

In this section we prove by a continuity argument and Proposition 4.6 that the water wave system (1.7) and equivalently (1.1)–(1.4) has a unique classical solution for the time period $[0, e^{c/\epsilon}]$ for data of form $\epsilon\Psi$, with c depending only on Ψ.

As we know, the initial data describing the water wave motion should satisfy some compatibility conditions [38, 39]. Since the compatibility conditions given in [39] are primarily for the 3D water wave, we give here a brief derivation of the compatibility conditions for the 2D case. This derivation is very much the same as in [39].

We know the water wave motion is described by the system (1.7):

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha, \\ \bar{z}_t = \mathfrak{H}\bar{z}_t, \end{cases} \tag{5.1}$$

where $\mathfrak{a}$ is real valued. So

$$-i(\mathfrak{a}-1)\bar{z}_\alpha = \bar{z}_{tt} + i(\bar{z}_\alpha - 1).$$

From $\bar{z}_t = \mathfrak{H}\bar{z}_t$ and (2.3), we have $(I-\mathfrak{H})\bar{z}_{tt} = [\partial_t, \mathfrak{H}]\bar{z}_t = [z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha}$, therefore

$$-i(I-\mathfrak{H})\{(\mathfrak{a}-1)\bar{z}_\alpha\} = [z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + i(I-\mathfrak{H})(\bar{z}_\alpha - 1). \tag{5.2}$$

Multiplying both sides of (5.2) by $\frac{iz_\alpha}{|z_\alpha|}$ then take the real value, we obtain

$$(\mathfrak{a}-1)|z_\alpha| = (I+\mathfrak{K}^*)^{-1}\mathrm{Re}\left\{\frac{iz_\alpha}{|z_\alpha|}\left([z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + i(I-\mathfrak{H})(\bar{z}_\alpha - 1)\right)\right\}, \tag{5.3}$$

where $\mathfrak{K}^*$ is the adjoint of the double layered potential operator as defined in (1.11). We see from (5.3) that $\mathfrak{a}$ is completely determined by the position $z = z(\cdot, t)$ and velocity $z_t(\cdot, t)$ of the interface. Consequently from

$$z_{tt} = i\mathfrak{a} z_\alpha - i \tag{5.4}$$

the acceleration z_{tt} is also determined by the position and velocity of the interface. Specifying at the initial time $t = 0$, (5.2)–(5.4) provides a compatibility condition for the initial data.

Let $H^{1/2}(R) = \{f \mid (I+|D|)^{1/2} f \in L^2(R)\}$, where $|D| = \sqrt{-\partial_\alpha^2}$ and $\|f\|_{H^{1/2}} = \|(I+|D|)^{1/2} f\|_{L^2}$.

Assume that the initial interface $\Sigma(0)$ separates R^2 into two simply connected, unbounded C^2 domains, $\Sigma(0)$ approaches the x-axis at infinity, and the water occupies the lower region $\Omega(0)$. Take a parametrization of $\Sigma(0) : z = z^0(\alpha)$, $\alpha \in R$ such that $z^0_\alpha(\alpha) - 1 \to 0$ as $|\alpha| \to \infty$, and $z = z^0(\alpha)$ traverses the boundary of $\Omega(0)$ in the clockwise sense, and

$$|z^0(\alpha) - z^0(\beta)| \ge \mu|\alpha - \beta| \quad \text{for all } \alpha, \beta \in R \tag{5.5}$$

for some constant $\mu > 0$. Let $z = z(\alpha, t)$, $\alpha \in R$ be the equation of the free interface $\Sigma(t)$ at time t in Lagrangian coordinate α, and

$$z(\cdot, 0) = z^0(\cdot), \qquad z_t(\cdot, 0) = \mathfrak{u}^0(\cdot), \qquad z_{tt}(\cdot, 0) = \mathfrak{w}^0(\cdot), \tag{5.6}$$

where

$$\mathfrak{w}^0 = i\mathfrak{a}^0 z^0_\alpha - i, \qquad \bar{\mathfrak{u}}^0 = \mathfrak{H}_0\bar{\mathfrak{u}}^0, \tag{5.7}$$

$\mathfrak{a}^0$ is given by

$$-i(I-\mathfrak{H}_0)\{(\mathfrak{a}^0 - 1)\bar{z}^0_\alpha\} = [\mathfrak{u}^0, \mathfrak{H}_0]\frac{\bar{\mathfrak{u}}^0_\alpha}{z^0_\alpha} + i(I-\mathfrak{H}_0)(\bar{z}^0_\alpha - 1), \tag{5.8}$$

and $\mathfrak{H}_0 f(\alpha) = \frac{1}{\pi i}\int \frac{z^0_\beta(\beta)}{z^0(\alpha)-z^0(\beta)} f(\beta)\, d\beta$ is the Hilbert transform associated to the initial interface $\Sigma(0) : z = z^0(\cdot)$. Let $s \ge 5$. Assume that

$$\sum_{|j|\le s-1} \|\Gamma^j (z^0_\alpha - 1)\|_{H^{1/2}} < \infty, \qquad \sum_{|j|\le s-1} (\|\Gamma^j \mathfrak{u}^0\|_{H^{1/2}} + \|\Gamma^j \partial_\alpha \mathfrak{u}^0\|_{H^{1/2}}) < \infty$$
$$\sum_{|j|\le s-1} (\|\Gamma^j \mathfrak{w}^0\|_{L^2} + \|\Gamma^j \partial_\alpha \mathfrak{w}^0\|_{L^2}) < \infty \quad \text{for } \Gamma = \partial_\alpha,\ L_0. \tag{5.9}$$

We have the following local in time well-posedness result of the initial value problem (5.1)–(5.6).

Theorem 5.1 (local existence) *There is a $T > 0$, depending on the norm of the initial data, so that the initial value problem (5.1)–(5.6) has a unique solution $z = z(\alpha, t)$ for $t \in [0, T]$, satisfying for all $|j| \le s - 1$, $\Gamma = \partial_\alpha,\ L_0$,*

$$\Gamma^j (z_\alpha - 1),\ \Gamma^j z_t,\ \Gamma^j \partial_\alpha z_t \in C([0, T], H^{1/2}(R)),$$
$$\Gamma^j z_{tt},\ \Gamma^j \partial_\alpha z_{tt} \in C([0, T], L^2(R)), \tag{5.10}$$

and $|z(\alpha, t) - z(\beta, t)| \ge \nu|\alpha - \beta|$ for all $\alpha,\ \beta \in R$ and $t \in [0, T]$, for some constant $\nu > 0$.

Moreover, if T^* is the supremum over all such times T, then either $T^* = \infty$, or

$$\sum_{\substack{|j|\le[\frac{s}{2}]+2\\ \Gamma=\partial_\alpha, L_0}} (\|\Gamma^j z_{tt}(t)\|_{L^2(R)} + \|\Gamma^j z_t(t)\|_{H^{1/2}(R)})$$
$$+ \sup_{\alpha\ne\beta} \left| \frac{\alpha - \beta}{z(\alpha, t) - z(\beta, t)} \right| \notin L^\infty[0, T^*). \tag{5.11}$$

Theorem 5.1 is proved very much in the same way as in [39] using the quasilinear system (1.12)–(1.10)–(1.11). The main modification is to use both vector fields $\Gamma = \partial_\alpha$ and L_0 instead of using only ∂_α as in [39]. We omit the proof.

By taking successive derivatives to t to the system (5.1) and an inductive argument, we have that the solution obtained in Theorem 5.1 in fact satisfies that for all $|j| \le s - 1$ and $\Gamma = \partial_t,\ \partial_\alpha,\ L_0$,

$$\Gamma^j (z_\alpha - 1),\ \Gamma^j z_t,\ \Gamma^j \partial_\alpha z_t \in C([0, T], H^{1/2}(R)),$$
$$\Gamma^j z_{tt},\ \Gamma^j \partial_\alpha z_{tt},\ \Gamma^j \partial_t z_{tt} \in C([0, T], L^2(R)). \tag{5.12}$$

In what follows we resume the convention that $\Gamma = \partial_t,\ \partial_\alpha,\ L_0$.

To prove that the maximal existence time T^* is at least $e^{c/\epsilon}$ for solutions obtained in Theorem 5.1 when the norm of the data is at most ϵ (to be specified later in this section), we use Proposition 4.6 and a continuity argument. This requires us to recast the solution in Theorem 5.1 in the new coordinate system k as defined in (2.18). We need to first understand whether the function k defined in (2.18) is a diffeomorphism, i.e. satisfies $k_\alpha(\alpha, t) > 0$ for all α. We first present the following preparatory Lemma. This Lemma is a more general version of the first parts of Lemmas 3.8 and 3.15.

Lemma 5.2 *Let $z = z(\cdot, \tau)$, $\tau \in [0, T]$ be a family of smooth and nonself-intersecting curves. Let $t \in [0, T]$ be fixed. Assume that*

$$|z(\alpha, t) - z(\beta, t)| \geq \nu|\alpha - \beta| \quad \text{for all } \alpha, \beta \in R, \tag{5.13}$$

for some constant $\nu > 0$. Let m_1, m_2, l be given, so that $m_1 \leq l$, $m_2 \leq l$ and $l \geq 2$. Assume that

$$\sum_{\substack{k_1 \leq m_1,\ k_2 \leq m_2 \\ k_1+k_2+k_3 \leq l}} \|\partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3} (z_\alpha - 1)(t)\|_{L^2(R)} \leq N(t) < \infty. \tag{5.14}$$

1. *If $(I - \mathfrak{H}) f = g$, and f is real valued, then there exists a constant $c(N(t))$ depending only on $N(t)$ and ν, such that*

$$\sum_{\substack{k_1 \leq m_1,\ k_2 \leq m_2 \\ k_1+k_2+k_3 \leq l}} \|\partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3} f(t)\|_{L^2(R)} \leq c(N(t)) \sum_{\substack{k_1 \leq m_1,\ k_2 \leq m_2 \\ k_1+k_2+k_3 \leq l}} \|\partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3} g(t)\|_{L^2(R)}. \tag{5.15}$$

2. *If $(I - \mathfrak{H})(f \bar{z}_\alpha) = g$, and f is real valued, then there exists a constant $c(N(t))$ depending only on $N(t)$ and ν, such that*

$$\sum_{\substack{k_1 \leq m_1,\ k_2 \leq m_2 \\ k_1+k_2+k_3 \leq l}} \|\partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3} f(t)\|_{L^2(R)} \leq c(N(t)) \sum_{\substack{k_1 \leq m_1,\ k_2 \leq m_2 \\ k_1+k_2+k_3 \leq l}} \|\partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3} g(t)\|_{L^2(R)}. \tag{5.16}$$

Proof Lemma 5.2 is proved in the same way as that of the first parts of Lemmas 3.8 and 3.15, i.e. for part 1 of Lemma 5.2, we use the identity (3.34)

$$(I - \mathfrak{H})\Gamma^j f = \sum_{k=1}^{j} \Gamma^{j-k} [\Gamma' z, \mathfrak{H}] \frac{\partial_\alpha}{z_\alpha} \Gamma^{k-1} f + \Gamma^j g$$

with $\Gamma^j = \partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3}$, $k_1 + k_2 + k_3 = j$, $j \leq l$, $k_i \leq m_i$, $i = 1, 2$, Propositions 3.3, 3.5 and an inductive argument. For part 2 of Lemma 5.2 we use the identity (3.60)

$$(I - \mathfrak{H})(\bar{z}_\alpha \Gamma^j f) = \sum_{k=1}^{j} \Gamma^{j-k} [\Gamma' z, \mathfrak{H}] \frac{\partial_\alpha}{z_\alpha} \Gamma^{k-1} (f \bar{z}_\alpha) + \Gamma^j g - (I - \mathfrak{H})(\Gamma^j (f \bar{z}_\alpha) - \bar{z}_\alpha \Gamma^j f)$$

with $\Gamma^j = \partial_t^{k_1} L_0^{k_2} \partial_\alpha^{k_3}$, $k_1 + k_2 + k_3 = j$, $j \leq l$, $k_i \leq m_i$, $i = 1, 2$, Proposition 3.3, 3.5 and an inductive argument. □

We now consider the question of when the function k defined by (2.18) is a diffeomorphism, and present some estimates of k in terms of the interface z.

For $t \in [0, T]$, let the free interface $\Sigma(t) : z = z(\alpha, t) = x(\alpha, t) + i\, y(\alpha, t)$, $-\infty < \alpha < \infty$ traverse the boundary of the simply connected C^2 domain $\Omega(t)$ in the clockwise sense. Let $k(\alpha, t) = 2x(\alpha, t) - h(\alpha, t)$ be defined as in (2.18). We have

Lemma 5.3 *Let $t \in [0, T]$ be fixed.*

1. *Assume that*

$$2(x \circ h^{-1})_\alpha(\alpha, t) - 1 > 0, \quad \textit{for all } \alpha \in R. \tag{5.17}$$

Then the $k(\cdot, t) = 2x(\cdot, t) - h(\cdot, t) : R \to R$ defined by (2.18) is a diffeomorphism, $k_\alpha(\cdot, t) > 0$, and

$$(2(x \circ k^{-1})_\alpha - 1) = \frac{1}{(k \circ h^{-1})_\alpha \circ h \circ k^{-1}}. \tag{5.18}$$

2. *Assume that*

$$\sum_{j \le 2} \|\partial_\alpha^j (z_\alpha - 1)(t)\|_{L^2} \le N(t) < \infty. \tag{5.19}$$

There exists a $N_0 > 0$, such that for $0 < N(t) \le N_0$, we have

$$\|(k_\alpha - 1)(t)\|_{L^\infty} \le \frac{3}{4}. \tag{5.20}$$

3. *Let $l \ge 2$. Assume that $\sum_{|j| \le l} \|\Gamma^j (z_\alpha - 1)(t)\|_{L^2} \le L(t) < \infty$, and*

$$|z(\alpha, t) - z(\beta, t)| \ge \nu |\alpha - \beta|, \quad \textit{for all } \alpha, \ \beta \in R$$

for some constant $\nu > 0$. Then

$$\sum_{|j| \le l} \|\Gamma^j (k_\alpha - 1)(t)\|_{L^2} \le c(L(t), \nu) \sum_{|j| \le l} \|\Gamma^j (z_\alpha - 1)(t)\|_{L^2}, \tag{5.21}$$

where $c(L(t), \nu)$ is a constant depending on $L(t)$ and ν. Moreover if $\Gamma^j (z_\alpha - 1) \in C([0, T], L^2(R))$, for all $|j| \le l$, then $\Gamma^j (k_\alpha - 1) \in C([0, T], L^2(R))$, for all $|j| \le l$.

Proof Let $\Phi(\cdot, t) : \Omega(t) \to P_-$ be the Riemann mapping from $\Omega(t)$ to the lower half plane P_-, satisfying $\lim_{z \to \infty} \Phi_z(z, t) = 1$ and $\Phi(z(0, t), t) = 2x(0, t)$. Let $h(\alpha, t) = \Phi(z(\alpha, t), t)$. We know $h : R \to R$ is a diffeomorphism and $h_\alpha > 0$. From (2.18):

$$k(\alpha, t) = 2x(\alpha, t) - h(\alpha, t),$$

so $k \circ h^{-1}(\alpha, t) = 2x \circ h^{-1}(\alpha, t) - 1$, and

$$(k \circ h^{-1})_\alpha(\alpha, t) = 2(x \circ h^{-1})_\alpha(\alpha, t) - 1. \tag{5.22}$$

Therefore the assumption (5.17) implies that $(k \circ h^{-1})_\alpha(\alpha, t) > 0$ for $\alpha \in R$, so $k_\alpha > 0$ and $k(\cdot, t) : R \to R$ is a diffeomorphism.

Now from (2.18) and that $k : R \to R$ being a diffeomorphism, $1 = 2x \circ k^{-1} - h \circ k^{-1}$, so

$$(h \circ k^{-1})_\alpha(\alpha, t) = 2(x \circ k^{-1})_\alpha(\alpha, t) - 1. \tag{5.23}$$

We know $k \circ h^{-1} \circ h \circ k^{-1} = I$. This together with (5.22), (5.23) gives us (5.18).

We now prove parts 2 and 3 of Lemma 5.3. We know $\frac{h_\alpha}{z_\alpha}(\alpha, t) = \Phi_z(z(\alpha, t), t)$ is the boundary value of the holomorphic function Φ_z, and $\frac{h_\alpha}{z_\alpha}(\alpha, t) - 1 \to 0$ as $|\alpha| \to \infty$. So

$(I-\mathfrak{H})(\frac{h_\alpha}{z_\alpha}-1)=0$, or

$$(I-\mathfrak{H})\left(\frac{h_\alpha-1}{|z_\alpha|^2}\bar{z}_\alpha\right)=(I-\mathfrak{H})\left(\frac{z_\alpha-1}{z_\alpha}\right).$$

From Proposition 3.5, we know there exists $N_1>0$, so that for $N(t)\le N_1$,

$$\|(z_\alpha-1)(t)\|_{L^\infty}\le c\sum_{j\le 1}\|\partial_\alpha^j(z_\alpha-1)(t)\|_{L^2}\le\frac{1}{4}. \tag{5.24}$$

Applying Lemma 5.2 for $m_1=m_2=0$, and $l=2$, and Propositions 3.2, 3.3, 3.5. We have

$$\|(h_\alpha-1)(t)\|_{L^\infty}\le c\sum_{j\le 2}\|\partial_\alpha^j(h_\alpha-1)(t)\|_{L^2}\le c(N(t))\sum_{j\le 2}\|\partial_\alpha^j(z_\alpha-1)(t)\|_{L^2}. \tag{5.25}$$

So there exists $0<N_0\le N_1$, so that for $N(t)\le N_0$,

$$\|(h_\alpha-1)(t)\|_{L^\infty}\le\frac{1}{4}.$$

From $k_\alpha=2x_\alpha-h_\alpha$, we obtain

$$\|(k_\alpha-1)(t)\|_{L^\infty}\le 2\|(z_\alpha-1)(t)\|_{L^\infty}+\|(h_\alpha-1)(t)\|_{L^\infty}\le\frac{3}{4}.$$

This proves (5.20).

For part 3, we apply Lemma 5.2 with $m_1=m_2=l$, and Propositions 3.2, 3.3, 3.5. We obtain

$$\sum_{|j|\le l}\|\Gamma^j(h_\alpha-1)(t)\|_{L^2}\le c(L(t))\sum_{|j|\le l}\|\Gamma^j(z_\alpha-1)(t)\|_{L^2}. \tag{5.26}$$

(5.21) therefore follows from $k_\alpha=2x_\alpha-h_\alpha$. The continuity of $\Gamma^j(k_\alpha-1)$ is proved similarly. We omit the details. □

Remark 1 It follows from part 2 of Lemma 5.3 that if $\sum_{j\le 2}\|\partial_\alpha^j(z_\alpha-1)(t)\|_{L^2}\le N_0$, then $k(\cdot,t)=2x(\cdot,t)-h(\cdot,t)$ defines a diffeomorphism, and $1/4\le k_\alpha(\alpha,t)\le 7/4$.

Remark 2 Let $2\le l\le s-1$, $z=z(\alpha,t)$, $t\in[0,T]$ with $T<T^*$ be the solution obtained in Theorem 5.1, $L(t)$ be as in part 3 of Lemma 5.3. Then for $t\in[0,T]$,

$$\sum_{|j|\le l}\|\Gamma^j k_t(t)\|_{L^2}\le c(L(t),\nu)\sum_{|j|\le l}\|\Gamma^j(z_\alpha-1)(t)\|_{L^2}\sum_{|j|\le l}\|\Gamma^j z_t(t)\|_{L^2}, \tag{5.27}$$

and $\Gamma^j k_t\in C([0,T],L^2(R))$ for all $|j|\le l$.

Proof Recall the solution z satisfies (2.23):

$$-(I-\mathfrak{H})k_t=[z_t,\mathfrak{H}]\frac{\bar{z}_\alpha-k_\alpha}{z_\alpha}.$$

Notice that k_t is real valued. (5.27) and the continuity of $\Gamma^j k_t$ for $|j|\le l$ therefore follow from Lemma 5.2 ($m_1=m_2=l$), Propositions 3.2, 3.5 and part 3 of Lemma 5.3. □

We now present the following

Lemma 5.4 *Let $q \geq 0$ be an integer and $0 \leq T < \infty$. Assume that for each $t \in [0, T]$, $k(\cdot, t) : R \to R$ is a diffeomorphism, $k(0, t) = 0$, and there are constants $\mu_1,\ \mu_2 > 0$, such that $\mu_1 \leq k_\alpha(\alpha, t) \leq \mu_2$ for all $\alpha \in R$ and $t \in [0, T]$; and $k_t \in L^\infty([0, T] \times R)$. When $q \geq 2$, we assume in addition that*

$$\sum_{|j| \leq q-1} \|\Gamma^j (k_\alpha - 1)(t)\|_{L^2} \leq L, \qquad \sum_{|j| \leq q-1} \|\Gamma^j k_t(t)\|_{L^2} \leq L \tag{5.28}$$

for all $t \in [0, T]$. Then

$$\sup_{[0,T]} \sum_{|j| \leq q} \|\Gamma^j (f \circ k^{-1})(t)\|_{L^2} \leq c_1 \sup_{[0,T]} \sum_{|j| \leq q} \|\Gamma^j f(t)\|_{L^2} \tag{5.29}$$

and

$$\sup_{[0,T]} \sum_{|j| \leq q} \|\Gamma^j (f \circ k)(t)\|_{L^2} \leq c_2 \sup_{[0,T]} \sum_{|j| \leq q} \|\Gamma^j f(t)\|_{L^2}, \tag{5.30}$$

where c_1, c_2 are constants depending on T, μ_1, μ_2 and $\|k_t\|_{L^\infty}$ for $q = 0, 1$; c_1, c_2 depending additionally on L for $q \geq 2$.

Remark Lemma 5.4 still holds if the closed interval $[0, T]$ is replaced by the half open interval $[0, T)$.

Proof The inequalities (5.29), (5.30) are straightforward when $q = 0$. Let $q \geq 1$. From the chain rule, we have

$$\partial_\alpha (f \circ k^{-1}) = \frac{f_\alpha \circ k^{-1}}{k_\alpha \circ k^{-1}}, \qquad \partial_t (f \circ k^{-1}) = f_t \circ k^{-1} - k_t \circ k^{-1} \frac{f_\alpha \circ k^{-1}}{k_\alpha \circ k^{-1}}, \tag{5.31}$$

and

$$L_0 (f \circ k^{-1}) = \frac{1}{2} t \partial_t (f \circ k^{-1}) + \alpha \partial_\alpha (f \circ k^{-1}) = \frac{1}{2} t \partial_t (f \circ k^{-1}) + \frac{k f_\alpha}{k_\alpha} \circ k^{-1}. \tag{5.32}$$

Now for the quantity $k f_\alpha$ in (5.32), and for $0 \leq |j| < q$,

$$\begin{aligned} \Gamma^j (k f_\alpha)(\alpha, t) &= \sum_{|m|+|k|=j} c_{m,k} \Gamma^m k(\alpha, t) \Gamma^k f_\alpha(\alpha, t) \\ &= \sum_{\substack{|m|+|k|=j \\ |m| \geq 1}} c_{m,k} \Gamma^m k(\alpha, t) \Gamma^k f_\alpha(\alpha, t) + \frac{k(\alpha, t)}{\alpha} \alpha \Gamma^j f_\alpha(\alpha, t), \end{aligned}$$

where $c_{m,k}$ are some constants. Notice that for $|m| \geq 1$ and by (3.2), we have

$$\partial_t \Gamma^{m-1} k(\alpha, t) = \sum_{|\iota| \leq |m-1|} c_\iota^1 \Gamma^\iota k_t(\alpha, t), \qquad \partial_\alpha \Gamma^{m-1} k(\alpha, t) = \sum_{|\iota| \leq |m-1|} c_\iota^2 \Gamma^\iota k_\alpha(\alpha, t),$$

with $c_\iota^i = 0,\ \pm \frac{1}{2^p}$, p some non-negative integers and

$$L_0 \Gamma^{m-1} k(\alpha, t) = \frac{1}{2} t \partial_t \Gamma^{m-1} k(\alpha, t) + \alpha \partial_\alpha \Gamma^{m-1} k(\alpha, t).$$

On the other hand, we have from (3.2),

$$\alpha\Gamma^k f_\alpha(\alpha,t)=\sum_{|\iota|\le|k|} c_\iota\alpha\partial_\alpha\Gamma^\iota f(\alpha,t)=\sum_{|\iota|\le|k|} c_\iota L_0\Gamma^\iota f(\alpha,t)-\frac{1}{2}t\sum_{|\iota|\le|k|} c_\iota\partial_t\Gamma^\iota f(\alpha,t),$$

where $c_\iota=0$ or ±1. And we know

$$\left|\frac{k(\alpha,t)}{\alpha}\right|=\left|\frac{k(\alpha,t)-k(0,t)}{\alpha}\right|\le\|k_\alpha(t)\|_{L^\infty}.$$

So for $0\le|j|<q$,

$$\|\Gamma^j(kf_\alpha)(t)\|_{L^2}\le c\sum_{|m|\le|j|+1}\|\Gamma^m f(t)\|_{L^2} \tag{5.33}$$

where c depends on μ_2, T when $q=1$, c depends on L and T when $q\ge 2$. (5.29) follows from (5.31), (5.32), (5.33) and an inductive argument.

Notice that for the inverse map k^{-1},

$$(k^{-1})_\alpha=\frac{1}{k_\alpha\circ k^{-1}},\qquad (k^{-1})_t=-\frac{k_t\circ k^{-1}}{k_\alpha\circ k^{-1}}.$$

Using (5.29) we have when $q\ge2$,

$$\sum_{|j|\le q-1}\|\Gamma^j((k^{-1})_\alpha-1)(t)\|_{L^2}\le c(L,T),\qquad \sum_{|j|\le q-1}\|\Gamma^j(k^{-1})_t(t)\|_{L^2}\le c(L,T)$$

for all $t\in[0,T]$ and some constant $c(L,T)$ depending on μ_i, $i=1,2$, $\|k_t\|_{L^\infty}$, L and T. (5.30) is therefore a consequence of (5.29). □

We now present and prove the following almost global well-posedness result.

Let $s\ge12$, $\max\{[\frac{s}{2}]+3,11\}\le l\le s-1$, N_0 be the constant in Lemma 5.3, and M_0 be the constant in Proposition 4.6, in particular, M_0 is the constant such that all the inequalities and estimates proved upto and including Proposition 4.6 hold. Let the initial interface $\Sigma(0)$ be a graph, given by

$$z^0(\alpha)=\alpha+i\,y^0(\alpha),\qquad \alpha\in R. \tag{5.34}$$

Assume that the initial data satisfies the compatibility and regularity assumptions as given by (5.6) to (5.9). Assume further that

$$\sum_{|j|\le l-2}\|\Gamma^j y^0\|_{L^2}<\infty,\qquad \sum_{j\le2}\|\partial_\alpha^j y_\alpha^0\|_{L^2}\le N_0, \tag{5.35}$$

so $k(\cdot,0)$ is a diffeomorphism, and $1/4\le k_\alpha(\alpha,0)\le 7/4$ for all $\alpha\in R$. Assume

$$\begin{aligned}&\sum_{|j|\le l}(\|\Gamma^j(\zeta_\alpha-1)(0)\|_2+\|\Gamma^j u(0)\|_2+\|\Gamma^j w(0)\|_2)\le\frac{M_0}{2},\\ &\sum_{|j|\le l-4}\|\Gamma^j\partial_\alpha\lambda(0)\|_{L^\infty}+\sum_{|j|\le l-2}\|\Gamma^j\partial_\alpha\chi(0)\|_{L^\infty}\le 1/2.\end{aligned} \tag{5.36}$$

Let $\mathfrak{E}(t)$ be defined by (4.50).

Theorem 5.5 *Assume that* $\mathfrak{E}(0)^{1/2} \le H\epsilon < \infty$. *There exist* ϵ_0 *and* $c > 0$, *depending on* H *and* M_0, *such that for* $0 < \epsilon \le \epsilon_0$, *the initial value problem* (5.1)–(5.6) *has a unique classical solution for the time period* $[0, e^{c/\epsilon}]$. *During this time period, the solution is regular in the sense of* (5.12), *the interface* $z = z(\cdot, t)$ *is a graph, and*

$$\mathfrak{E}(t)^{1/2} \le 4H\epsilon, \qquad \sum_{|j|\le l} (\|\Gamma^j(\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le M_0. \tag{5.37}$$

We give a discussion on the initial data satisfying the assumptions of Theorem 5.5 at the end of this section.

Proof We know from the local existence Theorem 5.1 that (5.1)–(5.6) has a unique solution satisfying (5.12) for some positive time period. Let T^* be the maximum existence time given in Theorem 5.1. Let $T < T^*$ be such that on $[0, T]$,

$$k_\alpha(\alpha, t) \ge \frac{1}{100} \quad \text{for all } \alpha \in R. \tag{5.38}$$

Therefore for $t \in [0, T]$, $k(\cdot, t)$ is a diffeomorphism, is continuous in time, and by Lemma 5.3 part 3 and the Remark 2 of Lemma 5.3, and Lemma 5.4, for all $|j| \le s - 1$,

$$\Gamma^j(\zeta_\alpha - 1),\ \Gamma^j u,\ \Gamma^j w,\ \Gamma^j(\partial_t + b\partial_\alpha)w,\ \Gamma^j\partial_\alpha u,\ \Gamma^j\partial_\alpha w \in C([0, T], L^2(R)).$$

Moreover by (5.18), and the fact that $|\zeta(\alpha, t) - \zeta(\beta, t)| \ge |x \circ k^{-1}(\alpha, t) - x \circ k^{-1}(\beta, t)|$, we have

$$|\zeta(\alpha, t) - \zeta(\beta, t)| \ge \frac{1}{2}|\alpha - \beta| \quad \text{for } \alpha,\ \beta \in R,\ t \in [0, T]. \tag{5.39}$$

Now from the assumption (5.35) and the fact that $z(\alpha, t) = \alpha + i\, y^0(\alpha) + \int_0^t z_t(\alpha, \tau)\, d\tau$, we know $\Gamma^j y \in C([0, T], L^2(R))$ for $|j| \le l - 2$. From (2.51), applying Propositions 3.2, 3.3, 3.5, we have $\Gamma^j(\partial_t + b\partial_\alpha)\lambda \in C([0, T], L^2(R))$ for $|j| \le l - 2$. Furthermore from the fact that $\lambda(\alpha, t) = \lambda(\alpha, 0) + \int_0^t \partial_t\lambda(\alpha, \tau)\, d\tau$, (3.54), and the assumption $\mathfrak{E}(0) < \infty$, we have $\Gamma^j\lambda \in C([0, T], \dot{H}^{1/2}(R))$ for $|j| \le l - 2$ and $E^\lambda \in C([0, T])$. Using Proposition 3.7 (3.30), Lemma 3.12, (3.43), (3.46), (3.49), and Propositions 2.5, 2.6, 3.2, 3.3, 3.5, and the fact that $\chi(\alpha, t) = \chi(\alpha, 0) + \int_0^t \partial_t\chi(\alpha, \tau)\, d\tau$, $\mathfrak{E}(0) < \infty$, we know $E^\chi, E^v \in C([0, T])$, therefore the energy $\mathfrak{E}(t)$ is defined and continuous on $[0, T]$. We note that the aforementioned statements hold without the smallness assumptions on M.

Let $T_m \le T$ be the largest such that on $[0, T_m]$,

$$\sum_{|j|\le l} (\|\Gamma^j(\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le M_0. \tag{5.40}$$

Let $T_\epsilon \le T$ be the largest such that on $[0, T_\epsilon]$,

$$\mathfrak{E}(t)^{1/2} \le 4H\epsilon. \tag{5.41}$$

Step 1. We want to show that $T_m \ge T_\epsilon$, provided ϵ is small.

If not, $T_m < T_\epsilon$. We know on $[0, T_m]$, Propositions 4.3 and 3.13 hold. So there is a constant $c_4(M_0)$ depending on M_0, such that for $t \in [0, T_m]$,

$$\sum_{|j|\le l} (\|\Gamma^j(\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le c_4(M_0)\mathfrak{E}(t)^{1/2} \le 4Hc_4(M_0)\epsilon. \tag{5.42}$$

Take

$$\epsilon_1 = \frac{M_0}{8Hc_4(M_0)}. \tag{5.43}$$

Therefore for $0 < \epsilon \le \epsilon_1$, we have that if on the interval $[0, T_m] \subset [0, T_\epsilon)$ (5.40) holds, then in fact on $[0, T_m]$,

$$\sum_{|j|\le l}(\|\Gamma^j(\zeta_\alpha - 1)(t)\|_2 + \|\Gamma^j u(t)\|_2 + \|\Gamma^j w(t)\|_2) \le \frac{M_0}{2}.$$

This contradicts with the assumption that T_m is the largest number $\le T$ such that (5.40) holds. Therefore $T_m \ge T_\epsilon$, provided $0 < \epsilon \le \epsilon_1$.

Step 2. We want to show there are constants ϵ_2, c_1, depending on M_0 and H, such that if $0 < \epsilon \le \epsilon_2$, and $T \le e^{c_1/\epsilon}$, then $T_\epsilon = T$.

Let

$$\epsilon_2 = \min\left\{\epsilon_1, \frac{1}{4H}, \frac{1}{32Hc_2(M_0)}, \frac{1}{4H\sqrt{c_3(M_0)}}\right\} \tag{5.44}$$

and $\epsilon \le \epsilon_2$, where $c_2(M_0)$, $c_3(M_0)$ are as given in Proposition 4.6. From Step 1, we know $T_m \ge T_\epsilon$. We want to use Proposition 4.6. Let

$$T_1 = \min\left\{e^{\frac{1}{5c_2(M_0)H\epsilon}}, e^{\frac{1}{4\sqrt{c_3(M_0)}H\epsilon}}\right\}. \tag{5.45}$$

Assume $T \le T_1$. Therefore the assumptions of Proposition 4.6 hold on $[0, T_\epsilon]$, and we have for $t \in [0, T_\epsilon]$

$$\mathfrak{E}(t) \le \frac{\mathfrak{E}(0)}{1 - c_3(M_0)\ln^2(t+e)\mathfrak{E}(0)} \le \frac{H^2\epsilon^2}{1 - c_3(M_0)\ln^2(t+e)H^2\epsilon^2} \le 2H^2\epsilon^2,$$

so $\mathfrak{E}(t)^{1/2} \le 2H\epsilon$ on $[0, T_\epsilon]$. Because $T_\epsilon \le T$ is the largest such that (5.41) holds, we must have $T_\epsilon = T$.

Step 3. We want to show there exist $0 < \epsilon_0 \le \epsilon_2$, and $T_0 = e^{c/\epsilon} \le T_1$, where ϵ_0 and c depend on H and M_0 only, such that for $\epsilon \le \epsilon_0$, (5.38) holds for all $t \in [0, \min\{T^*, T_0\})$.

Assume $\epsilon \le \epsilon_2$. Let $T \in [0, \min\{T^*, T_1\})$ be such that on $[0, T]$ (5.38) holds. From steps 1 and 2 we know $T_m = T_\epsilon = T$, and the assumptions of Proposition 4.6 and the inequality (4.79) hold on $[0, T]$. From (4.79) and Proposition 4.6 (4.103), there exists a constant $c_5(M_0) > 0$, such that for $t \in [0, T]$,

$$\|b_\alpha(t)\|_{L^\infty} \le c_5(M_0)\frac{\ln(t+e)}{t+e}\mathfrak{E}(t) \le (4H\epsilon)^2 c_5(M_0)\frac{\ln(t+e)}{t+e}. \tag{5.46}$$

On the other hand, from $b = k_t \circ k^{-1}$ we have $k_t = b \circ k$. So

$$\partial_t k_\alpha = b_\alpha \circ k\, k_\alpha. \tag{5.47}$$

Therefore

$$k_\alpha(\alpha, t) = k_\alpha(\alpha, 0)e^{\int_0^t b_\alpha \circ k(\alpha,\tau)\,d\tau}. \tag{5.48}$$

This implies that

$$k_\alpha(\alpha, t) \ge k_\alpha(\alpha, 0)e^{-\int_0^T \|b_\alpha(\tau)\|_{L^\infty} d\tau} \quad \text{on } [0, T].$$

From (5.46) we know

$$\int_0^T \|b_\alpha(\tau)\|_{L^\infty}\, d\tau \le 8(H\epsilon \ln(T+e))^2 c_5(M_0).$$

Let

$$\epsilon_0 = \min\left\{\epsilon_2, \frac{1}{4H\sqrt{c_5(M_0)}}\right\}, \qquad T_0 = \min\{T_1, e^{\frac{1}{4H\sqrt{c_5(M_0)}\epsilon}}\}. \tag{5.49}$$

Assume $\epsilon \le \epsilon_0$. If $T < \min\{T^*, T_0\}$ is such that on $[0, T]$ (5.38) holds, then

$$\int_0^T \|b_\alpha(\tau)\|_{L^\infty}\, d\tau \le 2 \le \ln\frac{25}{2}$$

consequently

$$k_\alpha(\alpha, t) \ge \frac{2}{25} k_\alpha(\alpha, 0) \ge \frac{1}{50} \quad \text{on } [0, T].$$

This implies that the set of $T \in [0, \min\{T_0, T^*\})$ such that (5.38) holds is open in $[0, \min\{T^*, T_0\})$. It is clear that the set of such T is also closed and nonempty in $[0, \min\{T^*, T_0\})$. Therefore (5.38) holds for all $t \in [0, \min\{T^*, T_0\})$.

Step 4. Assume $\epsilon \le \epsilon_0$. We want to show $T^* > T_0 = e^{c/\epsilon}$, where

$$c = \min\left\{\frac{1}{5c_2(M_0)H}, \frac{1}{4\sqrt{c_3(M_0)}H}, \frac{1}{4H\sqrt{c_5(M_0)}}\right\},$$

and for $t \in [0, T_0]$, $z = z(\cdot, t)$ is a graph.

Assume $\epsilon \le \epsilon_0$. Let $T_* = \min\{T^*, T_0\}$. We know from steps 1–3 that on $[0, T_*)$, $k(\cdot, t) : R \to R$ is a diffeomorphism and (5.38), (5.40) and (5.41) hold. Furthermore from (3.43) and Proposition 3.13, we know

$$\sum_{|j|\le l} \|\Gamma^j b(t)\|_2 \le c(M_0)\mathfrak{E}(t)^{1/2} \le 4H\epsilon c(M_0) \quad \text{for } t \in [0, T_*) \tag{5.50}$$

for some constant $c(M_0)$ depending on M_0. Using (5.48) and (5.50), we now show that the assumptions of Lemma 5.4 (with $q = l$) hold on $[0, T_*)$, with L depending only on $N_0,\ M_0,\ H,\ T_*$ and ϵ. First, from (5.50) and Proposition 3.5, we know for $t \in [0, T_*)$,

$$\|k_t(t)\|_{L^\infty(R)} = \|b(t)\|_{L^\infty(R)} \le c \sum_{|j|\le l} \|\Gamma^j b(t)\|_2 \le 4cH\epsilon c(M_0) \tag{5.51}$$

and

$$\|b_\alpha(t)\|_{L^\infty(R)} \le c \sum_{|j|\le l} \|\Gamma^j b(t)\|_2 \le 4cH\epsilon c(M_0).$$

Since $1/4 \le k_\alpha(\alpha, 0) \le 7/4$, therefore for $t \in [0, T_*)$,

$$\frac{1}{4} e^{-4cH\epsilon c(M_0)T_*} \le k_\alpha(\alpha, t) = k_\alpha(\alpha, 0) e^{\int_0^t b_\alpha \circ k(\alpha,\tau)\, d\tau} \le \frac{7}{4} e^{4cH\epsilon c(M_0)T_*}. \tag{5.52}$$

Using Lemma 5.4 ($q=1$) we have for $t\in[0,T_*)$,

$$\sum_{|j|\le 1}\|\Gamma^j k_t(t)\|_{L^2}=\sum_{|j|\le 1}\|\Gamma^j(b\circ k)(t)\|_{L^2}\le c_2\sum_{|j|\le 1}\|\Gamma^j b(t)\|_2\le 4c_2 H\epsilon c(M_0),$$

with c_2 depending on $H,\ M_0,\ T_*$ and ϵ. On the other hand, for any function f,

$$\Gamma\int_0^t f(\alpha,\tau)\,d\tau=\begin{cases}\int_0^t \partial_\alpha f(\alpha,\tau)\,d\tau, & \text{if } \Gamma=\partial_\alpha,\\ f(\alpha,t), & \text{if } \Gamma=\partial_t,\\ \int_0^t (L_0 f(\alpha,\tau)+\frac{1}{2}f(\alpha,\tau))\,d\tau, & \text{if } \Gamma=L_0.\end{cases}\tag{5.53}$$

From Lemma 5.3, we know $\sum_{|j|\le l}\|\Gamma^j(k_\alpha-1)(0)\|_2<\infty$. Therefore from (5.48), (5.50) and Lemma 5.4 ($q=1$) we have for $t\in[0,T_*)$,

$$\begin{aligned}\sum_{|j|\le 1}\|\Gamma^j(k_\alpha-1)(t)\|_2&\le c\Big(\sum_{|j|\le 1}\|\Gamma^j(k_\alpha-1)(0)\|_2+1\Big)\Big(\sum_{|j|\le 1}\|\Gamma^j(e^{\int_0^t b_\alpha\circ k(\tau)\,d\tau}-1)\|_2+1\Big)\\&\le c(N_0,M_0,H,\epsilon,T_*)\end{aligned}$$

with c an absolute constant, $c(N_0,M_0,H,\epsilon,T_*)$ depending on $N_0,\ M_0,\ H,\ \epsilon,\ T_*$. Using (5.48), (5.50), (5.53), Lemma 5.4 and an inductive argument, we see that the assumption (5.28) (with $q=l$) of Lemma 5.4 holds on $[0,T_*)$, with $L=L(N_0,M_0,H,\epsilon,T_*)$ depending only on $N_0,\ M_0,\ H,\ \epsilon,\ T_*$. Therefore by using Lemma 5.4 again we have that

$$\begin{aligned}&\sup_{[0,T_*)}\sum_{|j|\le[\frac{s}{2}]+2}(\|\Gamma^j z_{tt}(t)\|_{L^2(R)}+\|\Gamma^j z_t(t)\|_{H^{1/2}(R)})\\&\le\sup_{[0,T_*)}\sum_{|j|\le l}(\|\Gamma^j(w\circ k)(t)\|_{L^2(R)}+\|\Gamma^j(u\circ k)(t)\|_{L^2(R)})\\&\le c(N_0,M_0,H,\epsilon,T_*)\sup_{[0,T_*)}\sum_{|j|\le l}(\|\Gamma^j w(t)\|_{L^2(R)}+\|\Gamma^j u(t)\|_{L^2(R)})\\&\le c(N_0,M_0,H,\epsilon,T_*)M_0<\infty,\end{aligned}$$

here $c(N_0,M_0,H,\epsilon,T_*)$ is a constant depending on $N_0,\ M_0,\ H,\ \epsilon,\ T_*$. Now because on $[0,T_*)$ (5.38) holds, and by the definition of the diffeomorphism k, $2x=k+h$, therefore $2x_\alpha(\alpha,t)=k_\alpha(\alpha,t)+h_\alpha(\alpha,t)\ge 1/100$, so for $t\in[0,T_*)$, $z=z(\cdot,t)$ is a graph, with

$$|z(\alpha,t)-z(\beta,t)|\ge|x(\alpha,t)-x(\beta,t)|\ge\frac{1}{200}|\alpha-\beta|\quad\text{for }\alpha,\ \beta\in R.$$

This proves that $T_*<T^*$, for otherwise it contradicts with (5.11). Therefore $T_0=e^{c/\epsilon}<T^*$. This proves Theorem 5.5. □

5.1 A discussion of the initial data

We now give some sufficient conditions on the initial data so that the assumptions (5.35), (5.36) and $\mathfrak{E}(0)\le H^2\epsilon^2$ of Theorem 5.5 hold. Let

$$z^0(\alpha) = \alpha + i\, y^0(\alpha) = \alpha + i\,\epsilon f(\alpha),$$
$$z_t(\alpha, 0) = \mathfrak{u}^0(\alpha) = \epsilon g(\alpha), \qquad \mathbf{v}(z, 0) = \epsilon \mathbf{g}(z), \tag{5.54}$$

where $\mathbf{v}(\cdot, 0)$ is the initial velocity field. So $\bar{\mathbf{v}}(\cdot, 0)$ or $\bar{\mathbf{g}}$ is a holomorphic function in the initial fluid domain $\Omega(0)$ and $\mathbf{g}(z^0(\alpha)) = g(\alpha)$. Assume that the initial data satisfies (5.6)–(5.9). Assume

$$\sum_{|j|\le l} (\|\Gamma^j f\|_{L^2} + \|\Gamma^j f_\alpha\|_{L^2}) = N_1 < \infty, \qquad \sum_{|j|\le l} (\|\Gamma^j g\|_{L^2} + \|\Gamma^j g_\alpha\|_{L^2}) = N_2 < \infty,$$
$$\sum_{j\le l-2} \|(z\partial_z)^j \bar{\mathbf{g}}\|_{L^2(\Omega(0))} = N_3 < \infty, \quad \text{for } \Gamma = \partial_\alpha, L_0. \tag{5.55}$$

Notice that at $t = 0$, $L_0 = \alpha\partial_\alpha$. We now show that there exists $\epsilon_0 = \epsilon(N_0, N_1, N_2)$, a constant depending on N_0, N_1, N_2, such that for $\epsilon \le \epsilon_0$, (5.35), (5.36) hold. Moreover, there exists a constant $H = H(N_1, N_2, N_3)$ depending only on N_1, N_2, N_3, such that $\mathfrak{E}(0) \le H^2\epsilon^2$.

We start by assuming $0 < \epsilon \le \min\{1, N_0/N_1\}$. This implies $\sum_{j\le 2} \|\partial_\alpha^2 y_\alpha^0\|_{L^2} \le N_0$, so $1/4 \le k_\alpha(\alpha, 0) \le 7/4$. Now from the assumption (5.54), we know

$$|z^0(\alpha) - z^0(\beta)| \ge |\alpha - \beta| \quad \text{for all } \alpha, \beta \in R. \tag{5.56}$$

Using the compatibility condition (5.7)–(5.8) and Lemma 5.2 ($m_1 = 0$), (and (3.9), (3.12), (3.14) and Propositions 3.2, 3.3, 3.5) we have

$$\begin{aligned}
&\sum_{|j|\le l} \|\Gamma^j \mathfrak{w}^0\|_2 \\
&\quad \le \sum_{|j|\le l} (\|\Gamma^j((\mathfrak{a}^0 - 1)z_\alpha^0)\|_2 + \|\Gamma^j(z_\alpha^0 - 1)\|_2) \\
&\quad \le c(N_1, N_2)\left(\sum_{|j|\le l} (\|\Gamma^j \mathfrak{u}^0\|_2^2 + \|\Gamma^j(z_\alpha^0 - 1)\|_2\right) \le \epsilon c(N_1, N_2) \quad \text{for } \Gamma = L_0, \partial_\alpha,
\end{aligned}$$

here $c(N_1, N_2)$ are constants depending on N_1, N_2. By taking successive derivatives to t to the system (5.1) and using an inductive argument, Lemma 5.2 and the assumption (5.55), we obtain

$$\sum_{|j|\le l} (\|\Gamma^j(z_\alpha - 1)(0)\|_2 + \|\Gamma^j z_t(0)\|_2 + \|\Gamma^j z_{tt}(0)\|_2) \le \epsilon c(N_1, N_2) \quad \text{for } \Gamma = \partial_t, L_0, \partial_\alpha$$

for some constant $c(N_1, N_2)$. We resume the convention that $\Gamma = L_0, \partial_t, \partial_\alpha$. Using Lemma 5.3 part 3 and it's Remark 2, we have that

$$\sum_{|j|\le l} \|\Gamma^j(k_\alpha - 1)(0)\|_2 \le \epsilon c(N_1, N_2), \qquad \sum_{|j|\le l} \|\Gamma^j k_t(0)\|_2 \le \epsilon c(N_1, N_2) \tag{5.57}$$

Therefore from Lemma 5.4, we get

$$\sum_{|j|\le l} (\|\Gamma^j(\zeta_\alpha - 1)(0)\|_2 + \|\Gamma^j u(0)\|_2 + \|\Gamma^j w(0)\|_2) \le \epsilon c_1(N_1, N_2)$$

for some constant $c_1(N_1, N_2)$ depending only on N_1, N_2. From Proposition 3.5, (3.46), (3.54), notice that (3.46), (3.54) hold without any smallness assumptions on M, we have

$$\sum_{|j|\le l-4} \|\Gamma^j \partial_\alpha \lambda(0)\|_{L^\infty} + \sum_{|j|\le l-2} \|\Gamma^j \partial_\alpha \chi(0)\|_{L^\infty}$$
$$\le c\left(\sum_{|j|\le l-3} \|\Gamma^j \partial_\alpha \lambda(0)\|_{L^2} + \sum_{|j|\le l-1} \|\Gamma^j \partial_\alpha \chi(0)\|_{L^2}\right) \le \epsilon c_2(N_1, N_2)$$

for some constant $c_2(N_1, N_2)$ depending only on N_1, N_2. Take

$$\epsilon_0 = \min\left\{1, \frac{N_0}{N_1}, \frac{M_0}{2c_1(N_1, N_2)}, \frac{1}{2c_2(N_1, N_2)}\right\}.$$

We have that for $0 < \epsilon \le \epsilon_0$, (5.35), (5.36) hold.

Assume $0 < \epsilon \le \epsilon_0$. We now consider $\mathfrak{E}(0) = E^v(0) + E^\chi(0) + E^\lambda(0)$. Notice that from (3.49), (2.5) we have

$$\partial_\alpha v_1 = \left(I - \zeta_\alpha \mathcal{H} \frac{1}{\zeta_\alpha}\right) \partial_\alpha v \tag{5.58}$$

and from (2.35), (2.5), (3.14),

$$\partial_\alpha v = 2\partial_\alpha u - \left(\zeta_\alpha \mathcal{H} \frac{1}{\zeta_\alpha} + \bar{\zeta}_\alpha \bar{\mathcal{H}} \frac{1}{\bar{\zeta}_\alpha}\right) \partial_\alpha u - [\partial_\alpha u, \mathcal{H}] \frac{\zeta_\alpha - \bar{\zeta}_\alpha}{\zeta_\alpha}$$
$$- [u, \mathcal{H}] \frac{\partial_\alpha(\zeta_\alpha - \bar{\zeta}_\alpha)}{\zeta_\alpha} + \frac{1}{\pi i} \int \frac{(u(\alpha,t) - u(\beta,t)(\zeta_\alpha(\alpha,t) - \zeta_\beta(\beta,t))}{(\zeta(\alpha,t) - \zeta(\beta,t))^2} (\zeta_\beta - \bar{\zeta}_\beta)\, d\beta. \tag{5.59}$$

Using Proposition 3.7 (3.30), Lemma 3.12, (3.37), (3.43), (3.46), (3.50), (3.54), (2.51), (5.58), (5.59) and Propositions 3.2, 3.3, 3.5, we have

$$E^v(0) + E^\chi(0) + \sum_{|j|\le l-2} \int \frac{1}{A(\alpha, 0)} |(\partial_t + b\partial_\alpha)\Gamma^j \lambda(\alpha, 0)|^2\, d\alpha \le \epsilon^2 c(N_1, N_2), \tag{5.60}$$

for some constant $c(N_1, N_2)$ depending on N_1, N_2. Furthermore, we have from (2.51), (2.36) and (3.2), Propositions 3.2, 3.3, 3.5 that for $\eta_j^\lambda = \frac{I-\mathcal{H}}{2} \partial_t^{j_1} \partial_\alpha^{j_2} L_0^{j_3} \lambda$, with $j = j_1 + j_2 + j_3 \le l - 2$, and $j_1 + j_2 > 0$,

$$\left| \int i\eta_j^\lambda(\alpha, 0) \partial_\alpha \bar{\eta}_j^\lambda(\alpha, 0)\, d\alpha \right| \le \|\eta_j^\lambda(0)\|_2 \|\partial_\alpha \eta_j^\lambda(0)\|_2 \le \epsilon^2 c(N_1, N_2) \tag{5.61}$$

for some constant $c(N_1, N_2)$ depending on N_1, N_2.

The only term left to be estimated is $|\int i\eta_j^\lambda(\alpha, 0)\partial_\alpha \bar{\eta}_j^\lambda(\alpha, 0)\, d\alpha|$ in $E^\lambda(0)$ where

$$\eta_j^\lambda = \frac{I - \mathcal{H}}{2} L_0^j \lambda, \quad j \le l - 2. \tag{5.62}$$

For this estimate we need N_3. Notice that for our domain $\Omega(t)$, the Riemann Mapping $\Phi(\cdot, t) : \Omega(t) \to P_-$ defined in (2.18) has the property that $\Phi(z, t) - z$ is holomorphic in $\Omega(t)$ and there exists a real number $d(t)$, such that

$$\Phi(z(\alpha, t), t) - z(\alpha, t) = h(\alpha, t) - x(\alpha, t) - iy(\alpha, t) \to d(t) \quad \text{as } \alpha \to \pm\infty$$

therefore $x(\alpha,t)-k(\alpha,t)-d(t)-iy(\alpha,t)=\Phi(z(\alpha,t),t)-z(\alpha,t)-d(t)\to 0$ as $|\alpha|\to\infty$ and

$$(I-\mathfrak{H})(x-k-d-iy)=0.$$

Using Lemma 5.2 ($m_1=0$, at $t=0$), and (5.55) we have

$$\sum_{|j|\le l}\|\Gamma^j(k-x-d)(0)\|_2\le c(N_1)\sum_{|j|\le l}\|\Gamma^j y^0\|_2\le\epsilon c(N_1)\quad\text{for }\Gamma=L_0,\partial_\alpha. \tag{5.63}$$

Notice that at $t=0$, $x(\alpha,0)=\alpha$, and

$$L_0(k-x-d)(\alpha,0)=\alpha k_\alpha(\alpha,0)-\alpha=\alpha(k_\alpha(\alpha,0)-1). \tag{5.64}$$

We have further that

$$\begin{aligned}|d(0)|&=\left|\int_0^\infty(k_\alpha(\alpha,0)-1)\,d\alpha\right|\le\left|\int_0^1(k_\alpha(\alpha,0)-1)\,d\alpha\right|+\left|\int_1^\infty(k_\alpha(\alpha,0)-1)\,d\alpha\right|\\&\le\|k_\alpha-1\|_2+\|L_0(k-x-d)(0)\|_2\le\epsilon c(N_1,N_2).\end{aligned} \tag{5.65}$$

We need some further facts.

Claim 1 *Let* $\mathfrak{K}_0=\frac{\mathfrak{H}_0+\bar{\mathfrak{H}}_0}{2}$ *be the double layered potential operator. Let* $z^\tau(\alpha)=(1-\tau)z^0(\alpha)+\tau\bar{z}^0(\alpha)$. *Then*

$$2\mathfrak{K}_0 f=(\mathfrak{H}_0+\bar{\mathfrak{H}}_0)f=\int_0^1[2iy^0,\mathfrak{H}_{z^\tau}]\frac{\partial_\alpha f}{\partial_\alpha z^\tau}\,d\tau. \tag{5.66}$$

Here $\mathfrak{H}_{z^\tau}$ *is the Hilbert transform associated to the curve* z^τ.

Notice that $\mathfrak{H}_0=\mathfrak{H}_{z^0}$ and $-\bar{\mathfrak{H}}_0=\mathfrak{H}_{z^1}$. (5.66) is a simple consequence of the fundamental theorem of calculus and (2.3):

$$(\mathfrak{H}_0+\bar{\mathfrak{H}}_0)f=-(\mathfrak{H}_{z^1}-\mathfrak{H}_{z^0})f=-\int_0^1[\partial_\tau,\mathfrak{H}_{z^\tau}]f\,d\tau=\int_0^1[2iy^0,\mathfrak{H}_{z^\tau}]\frac{\partial_\alpha f}{\partial_\alpha z^\tau}\,d\tau.$$

Let $\phi(\cdot,0)$ be the velocity potential at $t=0$ and $\tilde{\phi}$ it's complex conjugate, so $\Xi(z)=\phi(z,0)+i\tilde{\phi}(z)$ is holomorphic in $\Omega(0)$, and $\bar{\mathbf{v}}(z,0)=\partial_z\Xi(z)$. Let $\psi^0(\alpha)=\psi(\alpha,0)=\phi(z^0(\alpha),0)$. Notice that $(I+\mathfrak{H}_0)(I+\mathfrak{K}_0)^{-1}\psi^0$ is the boundary value of a holomorphic function in $\Omega(0)$ and $\mathrm{Re}\{(I+\mathfrak{H}_0)(I+\mathfrak{K}_0)^{-1}\psi^0\}=\psi^0$. Without loss of generality we choose $\tilde{\phi}$ so that

$$\Xi\circ z^0=(I+\mathfrak{H}_0)(I+\mathfrak{K}_0)^{-1}\psi^0. \tag{5.67}$$

Recall $\Lambda(\cdot,0)=\lambda\circ k(\cdot,0)=(I-\mathfrak{H}_0)\psi^0$. We have

Claim 2

$$\Lambda(\cdot,0)=\overline{\Xi}\circ z^0-(I+\mathfrak{H}_0)(I+\mathfrak{K}_0)^{-1}\mathfrak{K}_0\psi^0. \tag{5.68}$$

(5.68) follows from the following calculation:

$$
\begin{aligned}
&(I-\mathfrak{H})-(I+\bar{\mathfrak{H}})(I+\mathfrak{K})^{-1}\\
&\quad=\left[\frac{1}{2}(I-\mathfrak{H})(2I+\mathfrak{H}+\bar{\mathfrak{H}})-(I+\bar{\mathfrak{H}})\right](I+\mathfrak{K})^{-1}\\
&\quad=\left[\frac{1}{2}(I-\mathfrak{H})(I+\bar{\mathfrak{H}})-(I+\bar{\mathfrak{H}})\right](I+\mathfrak{K})^{-1}=-\frac{1}{2}(I+\mathfrak{H})(I+\bar{\mathfrak{H}})(I+\mathfrak{K})^{-1}\\
&\quad=-\frac{1}{2}(I+\mathfrak{H})(\mathfrak{H}+\bar{\mathfrak{H}})(I+\mathfrak{K})^{-1}=-(I+\mathfrak{H})(I+\mathfrak{K})^{-1}\mathfrak{K}.
\end{aligned}
$$

In this calculation, we used repeatedly the fact that $\mathfrak{H}^2=I$. This gives us (5.68).

Claim 3 *We have*

$$
P^j-Q^j=\sum_{k=1}^{j}P^{j-k}(P-Q)Q^{k-1}. \tag{5.69}
$$

(5.69) is straightforward.

We now consider $|\int i\eta_j^\lambda(\alpha,0)\partial_\alpha\bar{\eta}_j^\lambda(\alpha,0)\,d\alpha|$, where η_j^λ is as given in (5.62). With the change of variable k, we have

$$
\int i\eta_j^\lambda(\alpha,0)\partial_\alpha\bar{\eta}_j^\lambda(\alpha,0)\,d\alpha=\int i(U_k\eta_j^\lambda)(\alpha,0)\partial_\alpha(U_k\bar{\eta}_j^\lambda)(\alpha,0)\,d\alpha
$$

and

$$
U_k\eta_j^\lambda=\frac{I-\mathfrak{H}}{2}(U_kL_0U_k^{-1})^j\Lambda.
$$

At $t=0$, we know

$$
U_kL_0U_k^{-1}f(\alpha,0)=\frac{k(\alpha,0)}{k_\alpha(\alpha,0)}\partial_\alpha f(\alpha,0)=\left(\frac{k(\alpha,0)}{k_\alpha(\alpha,0)}-\alpha\right)\partial_\alpha f(\alpha,0)+L_0f(\alpha,0). \tag{5.70}
$$

Let

$$
B_1(\alpha)=\frac{k(\alpha,0)}{k_\alpha(\alpha,0)}-\alpha.
$$

So

$$
B_1(\alpha)-\frac{d(0)}{k_\alpha(\alpha,0)}=\frac{k(\alpha,0)-\alpha-d(0)}{k_\alpha(\alpha,0)}-\frac{\alpha(k_\alpha(\alpha,0)-1)}{k_\alpha(\alpha,0)}.
$$

From (5.57), (5.63), (5.64), we have

$$
\sum_{|j|\le l-1}\left\|\Gamma^j\left(B_1-\frac{d(0)}{k_\alpha}\right)\right\|_{L^2(R)}\le\epsilon c(N_1,N_2)\quad\text{for }\Gamma=\alpha\partial_\alpha,\partial_\alpha. \tag{5.71}
$$

Now using (5.69), we get

$$
U_k\eta_j^\lambda=\frac{I-\mathfrak{H}}{2}L_0^j\Lambda+\sum_{k=1}^{j}\frac{I-\mathfrak{H}}{2}(U_kL_0U_k^{-1})^{j-k}(B_1\partial_\alpha)L_0^{k-1}\Lambda\quad\text{at }t=0.
$$

Let

$$\eta_j^1 = \frac{I - \mathfrak{H}}{2} L_0^j \Lambda.$$

Using (3.2), (2.36), (5.70), (5.71), (5.57), and Propositions 3.2, 3.3, 3.5, we have that there exists a constant $c(N_1, N_2)$, such that for $j \le l - 2$,

$$\begin{aligned}
&\left| \int [i\eta_j^\lambda(\alpha, 0)\partial_\alpha \bar{\eta}_j^\lambda(\alpha, 0) - i\eta_j^1(\alpha, 0)\partial_\alpha \bar{\eta}_j^1(\alpha, 0)]\, d\alpha \right| \\
&\quad = \left| \int [(iU_k\eta_j^\lambda)(\alpha, 0)\partial_\alpha (U_k\bar{\eta}_j^\lambda)(\alpha, 0) - i\eta_j^1(\alpha, 0)\partial_\alpha \bar{\eta}_j^1(\alpha, 0)]\, d\alpha \right| \le \epsilon^2 c(N_1, N_2). \quad (5.72)
\end{aligned}$$

To estimate of the term $|\int i\eta_j^1(\alpha, 0)\partial_\alpha \bar{\eta}_j^1(\alpha, 0)\, d\alpha|$, we decompose further η_j^1. Let

$$\mathbf{R} = -(I + \mathfrak{H}_0)(I + \mathfrak{K}_0)^{-1}\mathfrak{K}_0 \psi^0.$$

Therefore by (5.68), $\Lambda(\cdot, 0) = \overline{\Xi} \circ z^0 + \mathbf{R}$. Now

$$\begin{aligned}
\eta_j^1(\cdot, 0) &= \frac{I - \mathfrak{H}_0}{2} L_0^j \Lambda(\cdot, 0) = \frac{I + \bar{\mathfrak{H}}_0}{2} L_0^j \Lambda(\cdot, 0) - \mathfrak{K}_0 L_0^j \Lambda(\cdot, 0) \\
&= \frac{I + \bar{\mathfrak{H}}_0}{2} L_0^j \overline{\Xi} \circ z^0 + \frac{I + \bar{\mathfrak{H}}_0}{2} L_0^j \mathbf{R} - \mathfrak{K}_0 L_0^j \Lambda(\cdot, 0).
\end{aligned}$$

Let

$$\eta_j^2 = \frac{I + \bar{\mathfrak{H}}_0}{2} L_0^j \overline{\Xi} \circ z^0.$$

Using the fact that $\partial_\alpha \psi^0 = \mathrm{Re}\{\bar{\mathfrak{u}}^0 z_\alpha^0\}$ and (5.66), (3.2), (2.36), and Propositions 3.2, 3.3, 3.5, we have that there exists a constant $c(N_1, N_2)$, such that for $j \le l - 2$

$$\left| \int [i\eta_j^1(\alpha, 0)\partial_\alpha \bar{\eta}_j^1(\alpha, 0) - i\eta_j^2(\alpha)\partial_\alpha \bar{\eta}_j^2(\alpha)]\, d\alpha \right| \le \epsilon^2 c(N_1, N_2). \quad (5.73)$$

Since at $t = 0$, $L_0 = \alpha\partial_\alpha$, we have for any holomorphic function ξ in $\Omega(0)$,

$$\begin{aligned}
L_0(\xi \circ z^0)(\alpha) &= \alpha z_\alpha^0(\alpha)(\partial_z \xi) \circ z^0(\alpha) \\
&= (\alpha z_\alpha^0(\alpha) - z^0(\alpha))(\partial_z \xi) \circ z^0(\alpha) + (z\partial_z \xi) \circ z^0(\alpha) \\
&= \frac{\alpha z_\alpha^0(\alpha) - z^0(\alpha)}{z_\alpha^0(\alpha)} \partial_\alpha (\xi \circ z^0)(\alpha) + (z\partial_z \xi) \circ z^0(\alpha). \quad (5.74)
\end{aligned}$$

Notice that in (5.74), the function $z\partial_z \xi(z)$ is also holomorphic in $\Omega(0)$. We know $\alpha z_\alpha^0(\alpha) - z^0(\alpha) = i(\alpha\partial_\alpha y^0(\alpha) - y^0(\alpha))$. Let

$$B_2(\alpha) = \frac{\alpha z_\alpha^0(\alpha) - z^0(\alpha)}{z_\alpha^0(\alpha)}.$$

We have

$$\sum_{|j| \le l-1} \|\Gamma^j B_2\|_{L^2(R)} \le \epsilon c(N_1) \quad \text{for } \Gamma = L_0, \partial_\alpha \quad (5.75)$$

for some constant $c(N_1)$ depending only on N_1. We decompose η_j^2 as follows:

$$\begin{aligned}\bar{\eta}_j^2 &= \frac{I+\mathfrak{H}_0}{2}(L_0^j - (z\partial_z)^j)\,\Xi\circ z^0 + (z\partial_z)^j\,\Xi\circ z^0\\ &= \sum_{k=1}^{j}\frac{I+\mathfrak{H}_0}{2}L_0^{j-k}(B_2\partial_\alpha)(z\partial_z)^{k-1}\,\Xi\circ z^0 + (z\partial_z)^j\,\Xi\circ z^0\\ &= \sum_{k=1}^{j}\frac{I+\mathfrak{H}_0}{2}L_0^{j-k}(B_2\partial_\alpha)(L_0 - B_2\partial_\alpha)^{k-1}\,\Xi\circ z^0 + (z\partial_z)^j\,\Xi\circ z^0.\end{aligned}$$

Here we used the fact that $(z\partial_z)^j\,\Xi$ is holomorphic in $\Omega(0)$ and (5.69). Using (3.2), (5.75), the fact that $\partial_\alpha(\Xi\circ z^0)(\alpha) = z_\alpha^0(\alpha)\bar{\mathfrak{u}}^0(\alpha)$, and Propositions 3.2, 3.3, 3.5, we have for $j \le l-2$,

$$\left|\int [i\eta_j^2(\alpha)\partial_\alpha\bar{\eta}_j^2(\alpha) - i\overline{(z\partial_z)^j\,\Xi}\circ z^0(\alpha)\partial_\alpha\{(z\partial_z)^j\,\Xi\circ z^0(\alpha)\}]\,d\alpha\right| \le \epsilon^2 c(N_1, N_2) \tag{5.76}$$

for some constant $c(N_1, N_2)$ depending only on N_1, N_2. Now using Green's identity, we have

$$\begin{aligned}&\int i\overline{(z\partial_z)^j\,\Xi}\circ z^0(\alpha)\partial_\alpha\{(z\partial_z)^j\,\Xi\circ z^0(\alpha)\}\,d\alpha\\ &\quad= 2\int_{\Omega(0)}|\partial_z(z\partial_z)^j\,\Xi(z)|^2\,dxdy = 2\int_{\Omega(0)}|(\partial_z z)^j\,\bar{\mathbf{v}}(z,0)|^2\,dxdy\\ &\quad\le 2^{2j+1}\epsilon^2 N_3^2.\end{aligned} \tag{5.77}$$

Sum up (5.60), (5.61), (5.72), (5.73), (5.76), (5.77), we conclude that there exists $H(N_1, N_2, N_3)$, a constant depending only on N_1, N_2, N_3, such that

$$\mathfrak{E}(0) \le H^2\epsilon^2.$$

Therefore the initial data given by (5.54), (5.55) satisfy the assumptions of Theorem 5.5.

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