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On the Motion of a Self-Gravitating Incompressible Fluid with Free Boundary

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Abstract: We consider the motion of the interface separating a vacuum from an inviscid, incompressible, and irrotational fluid, subject to the self-gravitational force and neglecting surface tension, in two space dimensions. The fluid motion is described by the Euler–Poisson system in moving bounded simply-connected domains. A family of equilibrium solutions of the system are the perfect balls moving at constant velocity. We show that for smooth data that are small perturbations of size ϵ of these static states, measured in appropriate Sobolev spaces, the solution exists and the perturbation remains of size ϵ on a time interval of length at least $c\epsilon^{-2}$, where c is a constant independent of ϵ. This should be compared with the lifespan $O(\epsilon^{-1})$ provided by local well-posedness. The key ingredient of our proof is finding a two-step nonlinear transformation which removes quadratic terms from the nonlinearity. Compared with the gravity water wave problem, besides the different geometry of the bounded moving domain, an important difference is that the gravity in water waves is a constant vector, while the self-gravity in the Euler–Poisson system depends *nonlinearly* on the interface.

1. Introduction

We consider the motion of the interface separating a vacuum from an inviscid, incompressible, and irrotational fluid subject to self-gravitational force in two spatial dimensions. We assume that the fluid domain is bounded and simply connected and the surface tension is zero. Denoting the fluid domain by $\Omega(t) \subset \mathbb{R}^2$, the fluid velocity by $\mathbf{v}$, and the pressure by P, the evolution is described by the Euler–Poisson system

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$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P - \nabla\phi & \text{in} \quad \Omega(t), t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 0 & \text{in} \quad \Omega(t), t \geq 0, \\ P = 0 & \text{on} \quad \partial\Omega(t), \end{cases} \tag{1.1}$$

where the self-gravity Newtonian potential ϕ satisfies

$$\begin{cases} \Delta\phi = 2\pi\chi_{\Omega(t)}, \\ \nabla\phi(\mathbf{x}) = \iint_{\Omega(t)} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}. \end{cases} \tag{1.2}$$

In the equilibrium case where the total force from the pressure and self-gravity are balanced, a ball in $\mathbb{R}^2$, possibly moving with constant velocity, gives a static solution of the system (1.1)–(1.2). An important stability condition for this problem is the Taylor sign-condition [36]

$$\frac{\partial P}{\partial \mathbf{n}} < 0,$$

where $\mathbf{n}$ is the unit outward pointing normal to the boundary of the fluid region. In two dimensions and without the irrotationality assumption, a-priori estimates for this problem were obtained by Lindblad and Nordgren [29] under the assumption that initially the Taylor sign-condition holds. In three dimensions, Nordgren [31] proved local well-posedness. In our case where the fluid is incompressible and irrotational, the Taylor sign-condition holds automatically. Indeed by taking divergence of the first equation in (1.1) and using the fact that $\Delta\phi = 2\pi$ in $\Omega(t)$ we see that in $\Omega(t)$

$$\Delta P = -\Delta\phi - |\nabla v|^2 = -2\pi - |\nabla v|^2 < 0,$$

so by the Hopf's Maximum principle

$$\frac{\partial P}{\partial \mathbf{n}} < 0.$$

In this paper we show that if $\epsilon \ll 1$ is the size of the difference of the smooth initial data from one of the equilibrium states above, measured in Sobolev norms, a unique solution exists and its lifespan has a lower bound of order $O(\epsilon^{-2})$. This should be compared with the $O(\epsilon^{-1})$ estimate from local well-posedness. The key to obtaining our long-time $O(\epsilon^{-2})$ estimate is to find a new unknown function and a coordinate change such that in the new coordinates the new unknown satisfies an equation with only cubic and higher order nonlinearity.

To state our main theorem more precisely we first discuss the reduction of the system (1.1)–(1.2) to a system on the boundary $\partial\Omega(t)$. We occasionally use the notation $\Omega_t := \Omega(t)$. When there is no risk of confusion we simply write Ω; similarly we occasionally write the parametrization of $\partial\Omega := \partial\Omega(t)$ as $z = z(\cdot)$ instead of $z = z(t, \cdot)$. Moreover, we use the usual identification $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto z = x + iy$ of $\mathbb{R}^2$ with $\mathbb{C}$ to identify Ω with a domain in the complex plane.

Let $z(t, \alpha)$, $\alpha \in \mathbb{R}$, be a counterclockwise and 2π-periodic Lagrangian parametrization of $\partial\Omega$. By this we mean

$$z_t(t, \alpha) = \mathbf{v}(t, z(t, \alpha)),$$

so in particular

$$z_{tt}(t,\alpha) = \mathbf{v}_t(t, z(t,\alpha)) + (\mathbf{v}\cdot\nabla\mathbf{v})(t, z(t,\alpha))$$

is the acceleration. The conditions $\operatorname{div}\mathbf{v} = 0$, $\operatorname{curl}\mathbf{v} = 0$ now imply that $\overline{\mathbf{v}}$ is anti-holomorphic in Ω and therefore $\overline{z}_t$ is the boundary value of a holomorphic function in Ω. It then follows, cf. Proposition A.1 in Appendix A, that

$$\overline{z}_t = H\overline{z}_t,$$

where H denotes the Hilbert transform associated to Ω defined by

$$Hf(z_0) := \frac{\text{p.v.}}{\pi i}\int_{\partial\Omega}\frac{f(z)}{z-z_0}dz := \frac{\text{p.v.}}{\pi i}\int_0^{2\pi}\frac{f(z(t,\beta))}{z(t,\beta)-z(t,\alpha)}z_\beta(t,\beta)d\beta$$

for $z_0 = z(t,\alpha) \in \partial\Omega$. Since z is a counterclockwise parametrization of $\partial\Omega$ the unit exterior normal of this boundary is given by $\mathbf{n} := \frac{-iz_\alpha}{|z_\alpha|}$, and since P is constant on $\partial\Omega$ we can write $\nabla P(t,z) = iaz_\alpha$ for a real-valued function

$$a := -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial\mathbf{n}}.$$

It follows from (1.1)–(1.2), and these observations that z satisfies the fully nonlinear system

$$\begin{cases} z_{tt} + iaz_\alpha = -2\partial_{\overline{z}}\phi, \\ H\overline{z}_t = \overline{z}_t, \end{cases} \tag{1.3}$$

or equivalently

$$\begin{cases} \overline{z}_{tt} - ia\overline{z}_\alpha = -2\partial_z\overline{\phi}, \\ H\overline{z}_t = \overline{z}_t. \end{cases} \tag{1.4}$$

As shown in Lemma 3.1 the gravity term in (1.4) can be written as

$$-2\partial_{\overline{z}}\phi = -\frac{\pi}{2}(I - \overline{H})z, \tag{1.5}$$

which depends *nonlinearly* on the unknown interface. The remainder of this paper is devoted to the study of this equation. Note that once a solution z to (1.3) is found, one can recover $\mathbf{v}$ by solving the Dirichlet problem

$$\begin{cases} \Delta\mathbf{v} = 0, & \text{in } \Omega \\ \mathbf{v} = z_t, & \text{on } \partial\Omega \end{cases}.$$

We can now state the main result of this paper. See also Theorems 3.2 and 6.2 for more quantitative formulations.

Theorem 1.1. *Let Ω_0 be a bounded simply-connected domain in $\mathbb{C}$ with smooth boundary $\partial\Omega_0$ satisfying $|\Omega_0| = \pi$, and denote the associated Hilbert transform by H_0. Suppose $z_0(\alpha) = e^{i\alpha} + \epsilon f(\alpha)$ is a parametrization of $\partial\Omega_0$ and $z_1(\alpha) = v_0 + \epsilon g(\alpha)$ where f and g are smooth and g satisfies $H_0\overline{g} = \overline{g}$, and $v_0 \in \mathbb{C}$ is a constant. Then there is $T > 0$ and a unique classical solution $z(t,\alpha)$ of (1.3) on $[0,T)$ satisfying $(z(0,\alpha), z_t(0,\alpha)) = (z_0(\alpha), z_1(\alpha))$. Moreover, if $\epsilon > 0$ is sufficiently small the solution can be extended at least to $T^* = c\epsilon^{-2}$ where c is a constant independent of ϵ.*

Remark 1.2. The normalization $|\Omega_0| = \pi$ is made only for notational convenience. By the incompressibility of the flow the area of $\Omega(t)$ remains constant during the evolution, and our proof goes through without this assumption by renormalizing the transformations in Sect. 3.

Remark 1.3. The constant $v_0 \in \mathbb{C}$ corresponds to the fact that we consider the stability of the equilibrium solution $e^{i\alpha} + v_0 t$. In practice we work in the center of mass coordinates (see Sect. 3) to reduce the analysis to the case $v_0 = 0$.

We continue with a brief historical survey of developments related to Eqs. (1.1)–(1.2) followed by a discussion of the main difficulties in the proof of Theorem 1.1 and the ideas for resolving them. To the best of our knowledge, the only works on the Cauchy problem for the incompressible Euler–Poisson system are due to Lindblad and Nordgren. Although the proof of local well-posedness in [31] should extend to two dimensions, for the sake of completeness we include a sketch of an alternative proof for the local well-posedness of our Eqs. (1.3)–(1.5) using Riemann mapping in Sect. 7. We discuss our motivation for using Riemann-mapping coordinates in the discussion of the proof below.

There are many more works on the related gravity water-wave problem, in which the gravity is a constant vector. Local well-posedness was proved in [1,2,4,5,7,10–12,20,26–28,30,32,34,38,39,42,44]. For global and almost-global well-posedness of the gravity water-wave problem, first Wu obtained almost-global well-posedness in dimension two in [40]. Then global well-posedness in three dimensions was proved by Wu in [41] and by Germain, Masmoudi, and Shatah in [15]. The 2d result was later extended to global well-posedness by Alazard and Delort in [3] and Ionescu and Pusateri in [22]. We refer the reader to [14,16,21] for other related developments. See also [17–19]. For the long-time lifespan estimates of small smooth solutions in the aforementioned works, the main idea is to use the method of the normal forms to eliminate quadratic nonlinearities, although the exact analysis in carrying out the idea varies. Almost-global and global existence rely on the dispersive property of the equation. The use of normal-form transformations in the study of evolution PDEs has a long history, and here we mention for instance the works [13,33,35].

In this paper, we construct a two-step nonlinear transformation to remove the quadratic nonlinearity in the equation. The advantage of this approach is that once the transformation is derived, the construction of the energy is very natural and the analysis is much simpler. Most importantly, the nonlinear transformation and the equation it satisfies reveal deep structural properties of the equation in exact form, which we believe will be useful for further investigations. An analogous transformation was discovered by the last author in [40,41] for the gravity water-wave problem. The crucial differences between the present problem and the gravity water-wave problem are that the gravity of the water-wave problem is a constant vector while in the Euler–Poisson system (1.1)–(1.2) the self-gravity $\nabla\phi$ depends *nonlinearly* on the interface, and that the fluid domains of the two problems assume *different geometric forms*. From a physical point of view an important difference between the self-gravitating one-body problem (1.1)–(1.2) and the gravity water wave problem is that they are subject to two different types of physical forces, one internal and the other external. As in [40,41], the construction of our nonlinear transformation is non-algorithmic. While the near-identity transformation for water waves in [40] is for perturbations near the flat interface, our near-identity transformation is for perturbations near the unit circle. Our construction is accomplished via a thorough analysis of the structure of our equation, in particular, of the nonlinear gravity and the geometry. We mention that our nonlinear transformation can be easily modified to allow non-constant vorticity, see Appendix B. Finally note that since the fluid domain $\Omega(t)$ is bounded, dispersive tools are not available to prove global well-posedness at the time of writing this paper.

We now turn to the discussion of the main ideas of the proof of Theorem 1.1. The two-step nonlinear transformation described above, consists of finding a new unknown and a coordinate change such that the new unknown satisfies an equation with only cubic and higher-order nonlinearities in the new coordinates. To understand what we mean by cubic we have to specify what kinds of terms are considered small. Recall that we are studying the stability of the static solution[1] $z(t,\alpha) \equiv e^{i\alpha}$, $z_t(t,\alpha) \equiv 0$. It therefore makes sense to consider a quantity depending on z to be small if it is zero when z is the static solution. For instance the quantities $|z|^2 - 1$ and z_t are considered small, and to say that the nonlinearity is "cubic and higher order" means that every term in the nonlinearity is the product of at least three small terms.

The new unknown Let $\varepsilon := |z|^2 - 1$ and denote by δ the projection of ε onto the space of functions that are holomorphic outside Ω (see Appendix A), that is,

$$\delta := (I - H)\varepsilon$$

where H is the Hilbert transform. Then δ satisfies

$$(\partial_t^2 + ia\partial_\alpha - \pi)\delta = (I - H)(\partial_t^2 + ia\partial_\alpha - \pi)\varepsilon - [\partial_t^2 + ia\partial_\alpha, H]\varepsilon. \tag{1.6}$$

The first step in our proof of long-time existence, which is carried out in Proposition 3.15, is to show that the right hand side of this equation contains no quadratic terms, or in other words

$$(\partial_t^2 + ia\partial_\alpha - \pi)\delta = \text{cubic}.$$

This comes from a careful analysis of the holomorphicity structure of the equation satisfied by ε. More precisely, the projection operator $I - H$ already annihilates the holomorphic quadratic nonlinearities in the equation for ε, and the key observation is that the remaining quadratic contributions exactly cancel out with the commutator above. An interesting point, which can be seen from an inspection of the proof of Proposition 3.15, is that the same transformation formally[2] yields a cubic equation even if we turn gravity off by setting $\nabla\phi = 0$. In fact, a large part of the proof of Proposition 3.15 consists of showing that the quadratic terms contributed by the gravity cancel out with each other. As already mentioned, the nonlinear gravity is one of the key differences with the water-wave equation.

The coordinate change. To prove energy estimates for this equation we need to have control on the size of the coefficient a, and the dependency of a on z is nonlinear. Now a careful computation shows that the contribution of the term $ia\partial_\alpha\delta$ to the nonlinearity is quadratic. To remedy this problem, we exploit the remaining freedom in the equation, that is, the choice of coordinates. More precisely, note that the right hand side of (1.6)

[1] More precisely we consider the stability of the solutions $z(t,\alpha) = e^{i\alpha} + v_0 t$ where $v_0 \in \mathbb{C}$ is a constant initial velocity. However, by working in the center of mass frame we are able to reduce to the case $v_0 = 0$. See Sect. 1 below as well as Sect. 3.1 for more details.

[2] We say "formally" because in the absence of gravity our equilibrium solutions have no special significance, so we need to interpret smallness with respect to the equilibrium solutions in the presence of gravity.

is invariant under change of coordinates, whereas $a = \frac{-1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}}$ varies under coordinate transformations $k : \mathbb{R} \to \mathbb{R}$. Given such k and with the notation

$$\chi := \delta \circ k^{-1}, \quad A := (ak_\alpha) \circ k^{-1}, \quad b = k_t \circ k^{-1},$$

we see that χ satisfies

$$\left((\partial_t + b\partial_\alpha)^2 + iA\partial_\beta - \pi\right)\chi(t, \beta) = \text{cubic}.$$

The idea now is to choose k in such a way that b and $A-\pi$ are quadratic. Here in the static case the transformation k is simply the identity and A is the constant π. We will show in Sect. 3 that these conditions will be satisfied if we choose k so that $\bar{z}e^{ik}$ is the boundary value of a holomorphic function $F(t, z)$ in Ω such that $\log F$ is also holomorphic in Ω and in addition F satisfies the normalization $F(t, 0) \in \mathbb{R}_+$. The proof of the existence of k satisfying these conditions reduces to solving a Dirichlet problem in Ω. See Sect. 3, in particular Proposition 3.20 and Remark 3.21, for more details. This non-trivial choice of k comes again from gaining a structural understanding of the equation, and the fact that an extra normalization $F(t, 0) \in \mathbb{R}_+$ is needed is related to the geometry of the fluid domain. Indeed, this is reminiscent of the freedom in the choice of Riemann mappings for bounded simply-connected domains, and should be compared with the normalization in our choice of Riemann mapping in the proof of local well-posedness (see below).

Positivity of the energy. With the above choice of k we have obtained our cubic equation, and we next focus on the energy estimates for the equation

$$\left((\partial_t + b\partial_\alpha)^2 + iA\partial_\beta - \pi\right)\Theta = \text{cubic},$$

where $\Theta = (I - H)f$ for some f. Unfortunately the operator $iA\partial_\beta - \pi$ is not positive even when restricted to the class of functions satisfying $\Theta = (I - H)f$. On the other hand, we observe that if we are in the static case $z(t, \alpha) \equiv e^{i\alpha}$, so that $A = \pi$, then a Fourier expansion shows that $i\partial_\beta - 1$ is indeed positive on the class of functions with only negative frequencies, i.e., functions of the form $\Theta = (I - \mathbb{H})f$ where now $\mathbb{H}$ is the Hilbert transform associated with the unit circle. This suggests that the negative part of $iA\partial_\beta - \pi$ should be higher order with respect to our energy, and in Sect. 5 we show that this is indeed the case. As shown in Lemma 5.7, the most natural way to see this structure is to work with the quantity $(z \circ k^{-1})\Theta$ instead of Θ, but a careful examination of the statement of this lemma shows that the main estimate comes with a loss of a half derivative. Therefore, to control the negative part of the energy without loss of derivatives, a very careful choice of higher-order energies is needed, which is based on the observation that the negative error for the energy for the time derivative of χ is controlled in terms of χ itself. For a more detailed discussion of this point we refer the reader to the paragraph following the proof of Lemma 5.7. The detailed execution of these ideas is contained in Sect. 5, and specifically in Lemmas 5.8 and 5.9 and Corollary 5.10.

A formula for the center of mass. We close our discussion of the proof of long-time existence by describing an extra difficulty that arises from working in the center-of-mass coordinates to handle the constant-velocity motion of the equilibrium balls. It is important for the construction of our transformation that the center of mass of the domain moves along a straight line in $\mathbb{C}$. This fact which is consistent with physical intuition, as no external forces act on the body, is proved in Proposition 3.4. However, to be able to use this fact to obtain estimates for the new unknown χ it is important to obtain an expression for the center of mass in terms of our transformed quantities. This expression which is obtained in Proposition 3.5 plays a crucial role in estimating χ and its first derivative in terms of our energies, as shown in the proof of Proposition 4.20.

Riemann mapping and local well-posedness. In our proof of local well-posedness we use a Riemann mapping to derive an explicit formula for the important quantity $-\frac{\partial P}{\partial \mathbf{n}}$, analyze the interface equation, and derive the quasilinear equation. The advantage of Riemann-mapping coordinates is that it simplifies the structure of the interface equation. In particular the Dirichlet–Neumann operator is simply $|\partial_\alpha|$ in these coordinates, and the holomorphicity condition is expressed in terms of the Hilbert transform associated with the unit disc, which is a linear operator. The importance of Riemann-mapping coordinates in studying free-boundary problems is by now well-established and far-reaching consequences are known, see for example [25,38,43].

We end our discussion of local well-posedness by pointing out an important difference between the water-wave and self-gravitating models, stemming from the different domain geometries. Whereas the requirement that infinity is mapped to infinity in the case of the lower half-plane corresponding to the water-wave problem provides a natural normalization of the Riemann mapping in that case, the choice of normalization is not clear in the self-gravitating model. Indeed, as shown in Proposition 7.3, a judicious choice of normalization is needed to guarantee that the contribution of B does not depend on the highest-order derivatives of the unknown. As we already mentioned above, this should be compared with the required normalization in the choice of coordinate-change k in the proof of long-time existence.

1.1. The case of constant vorticity. In Appendix B at the end of this paper we show that our transformations can be easily modified to allow for nonzero constant vorticity, and a similar energy method as in the irrotational case gives an estimate $T \gtrsim \epsilon^{-2}$ for the lifespan T of solutions with data which are size ϵ perturbations of the equilibrium. Assuming that the fluid vorticity is $2\omega_0$, we find that when $\omega_0^2 < \pi$ the Taylor sign condition $\frac{\partial P}{\partial \mathbf{n}} < 0$ is satisfied and the fluid motion is stable. When $\omega_0^2 > \pi$ we have $\frac{\partial P}{\partial \mathbf{n}} > 0$ if the fluid velocity is close to that of the equilibrium state, leading to instability. Our interest in constant vorticity was sparked by the recent paper [19] of Ifrim and Tataru, where they investigated the gravity water wave equation with constant gravity and *constant vorticity.*

1.2. Organization of the paper. The rest of this paper is organized as follows. In Sect. 2, we collect some analytic tools which are used in the rest of the paper. The proof of the long-time existence statement of Theorem 1.1 is the content of Sects. 3–6. The proof relies on the existence of a local-in-time solution, but as local well-posedness is not the primary focus of this paper the proof of local well-posedness is postponed to Sect. 7, where Riemann mapping coordinates are introduced and the quasilinear structure of the equation revealed. In Sect. 3 we introduce the normal form transformation and obtain the desired cubic equation discussed above. In Sect. 4 we investigate the relation between the original and transformed quantities, and how estimates on one set of quantities translate to estimates for the other set. In Sect. 5 we introduce the energies and carry out the energy estimates, and finally in Sect. 6 we combine the results from the previous three sections to conclude the proof of long-time existence. Appendix A contains a review of basic facts that are used about the Hilbert transform in this paper. As mentioned above, Appendix B is devoted to the case of constant vorticity. For the convenience of the reader we have provided a list of notations we use in this paper, before the references.

2. Analysis Tools

In this section we collect a number general estimates which will be used in the rest of the paper. Most notably we will provide classical estimates on certain singular integral operators adapted to our case. Throughout this section we let

$$\mathfrak{z} : [0, 2\pi] \to \partial\Omega \subseteq \mathbb{C}$$

be a parametrization of the (closed) boundary of a domain Ω in $\mathbb{C}$. We require $\mathfrak{z}$ to be at least C^1, but most of the results in this section hold under the weaker assumption that $\mathfrak{z}$ is Lipchitz. By abuse of notation, for a function $A : \partial\Omega \to \mathbb{C}$, we write $A(\alpha)$ instead of $A(\mathfrak{z}(\alpha))$. In this context $A'(\alpha)$ means $\partial_\alpha A(\mathfrak{z}(\alpha))$, so for instance if $A = |\mathfrak{z}|^2 - 1$ then $A' = 2\mathrm{Re}\,\bar{\mathfrak{z}}\mathfrak{z}_\alpha$. In the proof of local well-posedness $\mathfrak{z}$ will usually be chosen as $\mathfrak{z}(\alpha) = e^{i\alpha}$ or $\mathfrak{z}(t, \alpha) = Z(t, \alpha)$. For the long-time existence we will often consider $\mathfrak{z}(t, \alpha) = \zeta(t, \alpha)$ or $\mathfrak{z}(t, \alpha) = z(t, \alpha)$ (the definitions for Z and ζ will be given in later sections).

Even though the functions in this section depend only on α and not on t, we use the notations L^p_α and H^s_α for the Lebesgue and Sobolev spaces in the variable α to be consistent with the rest of the paper. The following standard Sobolev estimate will be used throughout this work, often without reference.

Lemma 2.1 (Sobolev). *There is a constant C such that for all f in the Sobolev space H^1_α*

$$\|f\|_{L^\infty_\alpha} \le C(\|f\|_{L^2_\alpha} + \|\partial_\alpha f\|_{L^2_\alpha}).$$

We now turn to the main estimates of this section. We are interested in bounding operators of the forms

$$C_1(A, f)(\alpha) := \mathrm{p.v.} \int_0^{2\pi} \frac{\prod_{i\le m} (A_i(\alpha) - A_i(\beta))}{(\mathfrak{z}(\alpha) - \mathfrak{z}(\beta))^{m+1-k}(\bar{\mathfrak{z}}(\alpha) - \bar{\mathfrak{z}}(\beta))^k} f(\beta)d\beta, \qquad k \le m+1, \tag{2.1}$$

and

$$C_2(A, f)(\alpha) := \int_0^{2\pi} \frac{\prod_{i\le m} (A_i(\alpha) - A_i(\beta))}{(\mathfrak{z}(\alpha) - \mathfrak{z}(\beta))^{m-k}\,(\bar{\mathfrak{z}}(\alpha) - \bar{\mathfrak{z}}(\beta))^k} \partial_\beta f(\beta)d\beta, \qquad k \le m. \tag{2.2}$$

The two propositions below are due, in their original forms, to Calderon [6], Coifman, McIntosh, Meyer [9], Coifman, David, and Meyer [8], and here we only provide the straightforward modifications necessary for their application in our periodic setting. See also Wu [40] for the proof of the second part of this proposition using these results and the Tb Theorem.

Proposition 2.2. *Suppose $\mathfrak{z}$ satisfies*

$$\sup_{\alpha\neq\beta}\left|\frac{e^{i\alpha}-e^{i\beta}}{\mathfrak{z}(\alpha)-\mathfrak{z}(\beta)}\right|\leq c_0$$

for some constant c_0. Then there is a constant $C = C(c_0)$ such that the following statements hold.

(1) For any $f \in L^2_\alpha$, $A'_i \in L^\infty_\alpha$, $1 \leq i \leq m$,

$$\|C_1(A, f)\|_{L^2_\alpha} \leq C\|A'_1\|_{L^\infty_\alpha}...\|A'_m\|_{L^\infty_\alpha}\|f\|_{L^2_\alpha}.$$

(2) For any $f \in L^\infty_\alpha$, $A'_i \in L^\infty_\alpha$, $2 \leq i \leq m$, $A'_1 \in L^2_\alpha$,

$$\|C_1(A, f)\|_{L^2_\alpha} \leq C\|A'_1\|_{L^2_\alpha}\|A'_2\|_{L^\infty_\alpha}...\|A'_m\|_{L^\infty_\alpha}\|f\|_{L^\infty_\alpha}.$$

Proof. Propositions 2.2 is a consequence of Propositions 3.2 in [40]. Here we describe the modifications necessary to apply this result to our setting. We restrict attention to the case $m = 1$, $k = 0$, and write A instead of A_1. The general case can be handled in a similar way. With χ denoting the characteristic function of the interval $[0, 2\pi]$ we have

$$\begin{aligned} L &:= \int_0^{2\pi}\left(\int_0^{2\pi}\frac{A(\alpha)-A(\beta)}{(\mathfrak{z}(\alpha)-\mathfrak{z}(\beta))^2}f(\beta)d\beta\right)^2 d\alpha \\ &= \int_{\mathbb{R}}\chi(\alpha)\left(\int_{\mathbb{R}}\frac{A(\alpha)-A(\beta)}{(\mathfrak{z}(\alpha)-\mathfrak{z}(\beta))^2}\chi(\beta)f(\beta)d\beta\right)^2 d\alpha. \end{aligned} \tag{2.3}$$

Since A appears only as $A(\alpha) - A(\beta)$ in this expression, we may assume without loss of generality that $A(0) = A(2\pi) = 0$. We introduce some more notation. First let χ_j, $j = 1, 2, 3$ be the characteristic function of the interval $[\frac{2(j-1)\pi}{3}, \frac{2j\pi}{3}]$. Next define $\tilde{A}$ by $\tilde{A}(\alpha) = A(\alpha)$ if $\alpha \in [-4\pi, 4\pi]$ and $\tilde{A}(\alpha) = 0$ if $\alpha \notin [-4\pi, 4\pi]$. Let

$$K := \{w \in \mathbb{C} \mid w = \frac{\mathfrak{z}(\alpha')-\mathfrak{z}(\beta')}{\alpha'-\beta'} \quad \text{for some} \quad |\alpha'-\beta'| \leq \frac{5\pi}{3}\} \subseteq \mathbb{C}.$$

From the assumptions of Proposition 2.2 it follows that K does not contain the origin $w = 0$ in $\mathbb{C}$. Let $K' \supseteq K$ be a compact set containing K such that $0 \notin K'$, and let ϕ_K be a cut-off function supported in K' and equal to one on K. If follows that the function

$$F(w) := \frac{\phi_K(w)}{w^2}$$

is smooth. With these definitions we have

$$L \lesssim \sum_{i,j=1}^{3}\int_{\mathbb{R}}\chi(\alpha)\chi_i(\alpha)\left(\int_{\mathbb{R}}\frac{\tilde{A}(\alpha)-\tilde{A}(\beta)}{(\mathfrak{z}(\alpha)-\mathfrak{z}(\beta))^2}\chi(\beta)\chi_j(\beta)f(\beta)d\beta\right)^2 d\alpha =: \sum_{i,j=1}^{3} L_{i,j}$$

and we will estimate these integrals separately for different values of i, j. First we treat the case $i = j$, and for simplicity of notation we assume $i = j = 1$:

$$L_{1,1} = \int_{\mathbb{R}} \chi(\alpha)\chi_1(\alpha) \left(\int_{\mathbb{R}} F\left(\frac{\mathfrak{z}(\alpha) - \mathfrak{z}(\beta)}{\alpha - \beta} \right) \frac{\tilde{A}(\alpha) - \tilde{A}(\beta)}{(\alpha - \beta)^2} \chi(\beta)\chi_1(\beta) f(\beta) d\beta \right)^2 d\alpha$$
$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F\left(\frac{\mathfrak{z}(\alpha) - \mathfrak{z}(\beta)}{\alpha - \beta} \right) \frac{\tilde{A}(\alpha) - \tilde{A}(\beta)}{(\alpha - \beta)^2} \chi(\beta)\chi_1(\beta) f(\beta) d\beta \right)^2 d\alpha$$
$$\lesssim_{\|\mathfrak{z}\|_{C^1_\alpha}} \|\tilde{A}'\|^2_{L^\infty_\alpha} \|\chi\chi_1 f\|^2_{L^2_\alpha([0,2\pi])} \lesssim \|A'\|^2_{L^\infty_\alpha} \|f\|^2_{L^2_\alpha([0,2\pi])},$$

where we have used Propositions 3.2 in [40] to pass to the last line. The case where $j = i + 1$ is similar, and we now treat the case $i = 1$, $j = 3$. Using again Propositions 3.2 in [40] and the periodicity of A, f, and $\mathfrak{z}$ we have

$$L_{1,3} = \int_{\mathbb{R}} \chi(\alpha)\chi_1(\alpha) \left(\int_{\mathbb{R}} \frac{A(\alpha) - A(\beta - 2\pi)}{(\mathfrak{z}(\alpha) - \mathfrak{z}(\beta - 2\pi))^2} \chi(\beta)\chi_3(\beta) f(\beta - 2\pi) d\beta \right)^2 d\alpha$$
$$= \int_{\mathbb{R}} \chi(\alpha)\chi_1(\alpha) \left(\int_{\mathbb{R}} \frac{A(\alpha) - A(\beta')}{(\mathfrak{z}(\alpha) - \mathfrak{z}(\beta'))^2} \chi(\beta' + 2\pi)\chi_3(\beta' + 2\pi) f(\beta') d\beta' \right)^2 d\alpha$$
$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F\left(\frac{\mathfrak{z}(\alpha) - \mathfrak{z}(\beta')}{\alpha - \beta'} \right) \frac{\tilde{A}(\alpha) - \tilde{A}(\beta')}{(\alpha - \beta')^2} \chi(\beta' + 2\pi)\chi_3(\beta' + 2\pi) f(\beta') d\beta' \right)^2 d\alpha$$
$$\lesssim_{\|\mathfrak{z}\|_{C^1_\alpha}} \|A'\|^2_{L^\infty_\alpha} \|f\|^2_{L^2_\alpha([0,2\pi])}.$$

The remaining cases can be handled using similar arguments. □

A similar argument as in the proof of Proposition 2.2 allows us to deduce the following result from Proposition 3.3 in [40]. We omit the proof.

Proposition 2.3. *Suppose $\mathfrak{z}$ satisfies*

$$\sup_{\alpha \neq \beta} \left| \frac{e^{i\alpha} - e^{i\beta}}{\mathfrak{z}(\alpha) - \mathfrak{z}(\beta)} \right| \leq c_0$$

for some constant c_0. Then there is a constant $C = C(c_0)$ such that the following statements hold.

(1) For any $f \in L^2_\alpha$, $A'_i \in L^\infty_\alpha$, $1 \leq i \leq m$,

$$\|C_2(A, f)\|_{L^2_\alpha} \leq C\|A'_1\|_{L^\infty_\alpha} \ldots \|A'_m\|_{L^\infty_\alpha} \|f\|_{L^2}.$$

(2) For any $f \in L^\infty_\alpha$, $A'_i \in L^\infty_\alpha$, $2 \leq i \leq m$, $A'_1 \in L^2_\alpha$,

$$\|C_2(A, f)\|_{L^2_\alpha} \leq C\|A'_1\|_{L^2_\alpha} \|A'_2\|_{L^\infty_\alpha} \ldots \|A'_m\|_{L^\infty_\alpha} \|f\|_{L^\infty_\alpha}.$$

The next lemma is a simple computation which is used in estimating derivatives of expressions such as $C_1(A, f)$ and $C_2(A, f)$.

Lemma 2.4. *Suppose*

$$\mathbf{K}f(\alpha) = \text{p.v.} \int_0^{2\pi} K(\alpha,\beta) f(\beta) d\beta$$

where $K(\alpha,\beta)$ *or* $e^{i(\alpha} - e^{i\beta})K(\alpha,\beta)$ *are continuous and* K *is* C^1 *away from the diagonal in* $[0, 2\pi] \times [0, 2\pi]$. *Then*

$$\partial_\alpha \mathbf{K} f(\alpha) = \mathbf{K} f_\alpha(\alpha) + \text{p.v.} \int_0^{2\pi} (\partial_\alpha + \partial_\beta) K(\alpha,\beta) f(\beta) d\beta.$$

Proof. This follows from integration by parts. □

The following two lemmas are important corollaries of Propositions 2.2 and 2.3 and Lemma 2.4. Recall that for a C^1 parametrization $\zeta : [0, 2\pi] \to \partial\Omega$ of the boundary of Ω the Hilbert transform is given by

$$\mathcal{H} f(\alpha) = \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(\beta)}{\zeta(\beta) - \zeta(\alpha)} \zeta_\beta(\beta) d\beta.$$

Lemma 2.5. *Suppose* $\zeta : [0, 2\pi] \to \partial\Omega \subseteq \mathbb{C}$ *satisfies* $\sum_{i=1}^{\ell+1} \|\partial_\alpha^j \zeta\|_{L^2_\alpha} \le c$ *for some nonzero constant* c, *where* $\ell \ge 4$ *is a fixed integer, and*

$$\sup_{\alpha \ne \beta} \left| \frac{e^{i\alpha} - e^{i\beta}}{\mathfrak{z}(\alpha) - \mathfrak{z}(\beta)} \right| \le c_0$$

Then there is a constant $C = C(j, c, c_0)$ *such that for* $4 \le j \le \ell$

$$\sum_{i \le j} \left\| \partial_\alpha^i \int_0^{2\pi} \frac{g(\beta) - g(\alpha)}{\zeta(\beta) - \zeta(\alpha)} f(\beta) d\beta \right\|_{L^2_\alpha} \le C \sum_{i \le j} \|\partial_\alpha^i g\|_{L^2_\alpha} \sum_{i \le j-1} \|\partial_\alpha^i f\|_{L^2_\alpha}.$$

In particular

$$\sum_{i \le j} \left\| \partial_\alpha^i [g, \mathcal{H}] \frac{f}{\zeta_\alpha} \right\|_{L^2_\alpha} \le C \sum_{i \le j} \|\partial_\alpha^i g\|_{L^2_\alpha} \sum_{i \le j-1} \|\partial_\alpha^i f\|_{L^2_\alpha}.$$

Proof. The second estimate follows from the first by writing

$$[g, \mathcal{H}] \frac{f}{\zeta_\alpha} = \frac{1}{\pi i} \int_0^{2\pi} \frac{g(\alpha) - g(\beta)}{\zeta(\beta) - \zeta(\alpha)} f(\beta) d\beta.$$

To prove the first estimate we use Lemma 2.4 to distribute the derivative on f and g. In the case where all derivatives fall on f Proposition 2.3 gives

$$\left\| \int_0^{2\pi} \frac{g(\beta) - g(\alpha)}{\zeta(\beta) - \zeta(\alpha)} \partial_\beta^j f(\beta) d\beta \right\|_{L^2_\alpha} \lesssim \|\partial_\alpha g\|_{L^\infty_\alpha} \|\partial_\alpha^{j-1} f\|_{L^2_\alpha}.$$

When all derivatives fall on g we use the boundedness of the Hilbert transform and Proposition 2.2 to estimate

$$\left\| \int_0^{2\pi} \frac{\partial_\beta^j g(\beta) - \partial_\alpha^j g(\alpha)}{\zeta(\beta) - \zeta(\alpha)} f(\beta) d\beta \right\|_{L^2_\alpha} \lesssim \|\partial_\alpha^j g\|_{L^2_\alpha} \|Hf\|_{L^\infty_\alpha}$$
$$+ \|H(f \partial_\alpha^j g)\|_{L^2_\alpha} \lesssim \|\partial_\alpha^j g\|_{L^2_\alpha} (\|f\|_{L^2_\alpha} + \|\partial_\alpha f\|_{L^2_\alpha}).$$

The case when $j-1$ derivatives fall on g and one derivative on f can be estimated directly by Proposition 2.3. When $j-1$ derivatives fall on g and none on f we have

$$\begin{aligned}&\left\|\int_0^{2\pi}\frac{(\partial_\beta^{j-1}g(\beta)-\partial_\alpha^{j-1}g(\alpha))(\zeta_\beta(\beta)-\zeta_\alpha(\alpha))}{(\zeta(\beta)-\zeta(\alpha))^2}f(\beta)d\beta\right\|_{L^2_\alpha}\\&\lesssim\left\|\int_0^{2\pi}\frac{\partial_\beta^{j-1}g(\beta)-\partial_\beta^{j-1}g(\alpha)}{(\zeta(\beta)-\zeta(\alpha))^2}f(\beta)\zeta_\beta(\beta)d\beta\right\|_{L^2_\alpha}\\&\quad+\left\|\zeta_\alpha(\alpha)\int_0^{2\pi}\frac{\partial_\beta^{j-1}g(\beta)-\partial_\beta^{j-1}g(\alpha)}{(\zeta(\beta)-\zeta(\alpha))^2}f(\beta)d\beta\right\|_{L^2_\alpha}\end{aligned}$$

which can be estimated using Proposition 2.2. All other cases can simply be estimated by bounding the contributions of both f and g in L^∞_α and using the embedding $L^\infty_\alpha \hookrightarrow L^2_\alpha$. □

Lemma 2.6. *Under the assumptions of Lemma 2.5 for any $\ell \geq 4$*

$$\sum_{j\leq\ell}\|\partial_\alpha^j(I-\mathcal{H})f\|_{L^2_\alpha}\leq C\sum_{j\leq\ell}\|\partial_\alpha^j f\|_{L^2_\alpha},$$

where C depends on the H^ℓ_α norm of ζ.

Proof. This follows from Lemma 2.5 and Propositions 2.2 and 2.3 by writing

$$\partial_\alpha^j(I-\mathcal{H})f=(I-\mathcal{H})\partial_\alpha^j f-\sum_{i=1}^j\partial_\alpha^{j-i}[\zeta_\alpha,\mathcal{H}]\frac{\partial_\alpha^i f}{\zeta_\alpha}=(I-\mathcal{H})\partial_\alpha^j f-\sum_{i=1}^j\partial_\alpha^{j-i}[\eta,\mathcal{H}]\frac{\partial_\alpha^i f}{\zeta_\alpha},$$

where $\eta := \zeta_\alpha - i\zeta$, Here to compute the commutator $[\partial_\alpha, \mathcal{H}] = [\zeta_\alpha, \mathcal{H}]\frac{\partial_\alpha}{\zeta_\alpha}$ we have used Lemma 3.7. □

As another corollary of Proposition 2.2 and Lemma 2.4 we get the following L^∞_α estimate for $C_1(A, f)$, which is similar to Proposition 3.4 in [40].

Proposition 2.7. *Suppose $\mathfrak{z}$ satisfies*

$$\sup_{\alpha\neq\beta}\left|\frac{e^{i\alpha}-e^{i\beta}}{\mathfrak{z}(\alpha)-\mathfrak{z}(\beta)}\right|\leq c_0$$

for some constant c_0, and A_i', A_i'', f, and f' are in L^∞_α. Assume further that $\|\mathfrak{z}\|_{H^2_\alpha} \leq M$. Then there exists a constant $C = C(c_0)$ such that

$$\|C_1(A,f)\|_{L^\infty_\alpha}\leq C(1+M)\prod_{i\leq m}\left(\|A_i''\|_{L^\infty_\alpha}+\|A_i'\|_{L^\infty_\alpha}\right)\left(\|f\|_{L^\infty_\alpha}+\|f'\|_{L^\infty_\alpha}\right).$$

Proof. This follows from applying the Sobolev inequality to $C(A, f)$ and using Lemma 2.4 and Proposition 2.2. Note that in view of the embedding $L^\infty_\alpha \hookrightarrow L^2_\alpha$ we may replace the L^2_α norms appearing on the right hand side of the statement of Proposition 2.2 by L^∞_α norms. □

We close this section by stating the following estimates from [42] (see also [38] Lemma 5.2) which are proved using Fourier analysis. The adaptations from the case of the real line to the circle are straightforward and omitted. Here $\mathbb{H}$ is the Hilbert transform on the circle.

Lemma 2.8. *Let* $r \geq 0$, *and* $q > 1/2$. *Then for any smooth functions a and u*

$$\|[a, \mathbb{H}]u\|_{H^r_\alpha} \lesssim \|a\|_{H^{r+p}_\alpha} \|u\|_{H^{q-p}_\alpha}, \quad p \geq 0.$$

3. The Normal Form Transformation

In this section we begin the study of the Cauchy problem of the system (1.3) with small initial data. We start with the following important representation formula for the boundary contribution of the gravity term.

Lemma 3.1. $-2\partial_{\bar{z}}\phi = -\frac{\pi}{2}(I - \overline{H})z = -\pi z + \frac{\pi}{2}(I + \overline{H})z.$

Proof. With $\mathbf{x} = z(\alpha)$ and by the dominated convergence theorem

$$\nabla\phi(\mathbf{x}) = -\int\int_\Omega \nabla_\mathbf{y} \log(|\mathbf{x} - \mathbf{y}|)dy = -\lim_{\epsilon\to 0}\int\int_{\Omega\backslash B_\epsilon} \nabla_\mathbf{y} \log(|\mathbf{x} - \mathbf{y}|)d\mathbf{y}$$

where B_ϵ is a ball of radius ϵ centered at $\mathbf{x}$. We now identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way and abuse notation to write for instance $\nabla\phi = \partial_x\phi + i\partial_y\phi$. Defining the vector fields

$$X = (\log(|\mathbf{x} - \mathbf{y}|), 0), \qquad Y = (0, \log(|\mathbf{x} - \mathbf{y}|)),$$

we have

$$\nabla\phi(\mathbf{x}) = -\lim_{\epsilon\to 0}\int\int_{\Omega\backslash B_\epsilon} (\operatorname{div} X + i \operatorname{div} Y)d\mathbf{y} = -\lim_{\epsilon\to 0}\int_{\partial(\Omega\backslash B_\epsilon)} (X + iY)\cdot N d\sigma(\mathbf{y})$$

where $d\sigma$ is the line element of the boundary and N the outward pointing normal vector. The boundary has two parts: C_ϵ corresponding to ∂B_ϵ and Γ_ϵ corresponding to $\partial\Omega_\epsilon$. We can find $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ which are $O(\epsilon)$ and such that Γ_ϵ is parametrized by $z(\cdot)$: $[0, 2\pi]\backslash[\alpha - \delta_1, \alpha + \delta_2] \to \Gamma_\epsilon$. The outward pointing normal vector is therefore given by $-iz_\alpha/|z_\alpha|$ in complex notation or $\frac{1}{|z_\alpha|}(\operatorname{Im} z_\alpha, -\operatorname{Re} z_\alpha)$ in real notation. Similarly there are numbers $\eta_1(\epsilon) < \eta_2(\epsilon)$ in $(0, 2\pi)$ such that in complex notation C_ϵ is parametrized by $\theta \in (\eta_1, \eta_2) \mapsto \mathbf{x}+\epsilon e^{i\theta}$. It follows form the computation above and the 2π-periodicity of $z(\cdot)$ that

$$\begin{aligned}
\nabla\phi(\mathbf{x}) &= i\lim_{\epsilon\to 0}\int_{\alpha+\delta_1}^{2\pi+\alpha-\delta_2} \log(|z(\alpha) - z(\beta)|)z_\beta(\beta)d\beta \\
&\quad - \lim_{\epsilon\to 0} \epsilon \log|\epsilon| \int_{\eta_1}^{\eta_2} (1, i)\cdot N_{C_\epsilon} d\theta \\
&= i\int_0^{2\pi} \log(|z(\alpha) - z(\beta)|)\partial_\beta(z(\beta) - z(\alpha))d\beta \\
&= -i\int_0^{2\pi} \frac{(z(\alpha) - z(\beta))\operatorname{Re}((z(\alpha) - z(\beta))\bar{z}_\beta(\beta))}{|z(\alpha) - z(\beta)|^2} d\beta
\end{aligned}$$

$$= -\frac{i}{2}\int_0^{2\pi} \frac{z(\alpha)-z(\beta)}{\overline{z}(\alpha)-\overline{z}(\beta)}\overline{z}_\beta(\beta)d\beta - \frac{i}{2}\int_0^{2\pi} z_\beta(\beta)d\beta$$
$$= -\frac{i}{2}\int_0^{2\pi} \frac{z(\alpha)-z(\beta)}{\overline{z}(\alpha)-\overline{z}(\beta)}\overline{z}_\beta(\beta)d\beta = \frac{\pi}{2}(I-\overline{H})z.$$

□

In view of Lemma 3.1 we can replace (1.3) by

$$\begin{cases} z_{tt} + iaz_\alpha = -\frac{\pi}{2}(I-\overline{H})z, \\ H\overline{z}_t = \overline{z}_t. \end{cases} \tag{3.1}$$

The norms in which the data are assumed to be small will be made precise below. Our main objective here to transform Eq. (3.1) to an equation for which the nonlinearity is small of cubic order. Again the exact meaning of the term "cubic" will be clarified below, but roughly speaking we consider a quantity to be 'small' if the corresponding quantity in the case of the static solution $z(t,\alpha) \equiv e^{i\alpha'}$, $z_t(t,\alpha) \equiv 0$ is zero. This implies, for instance, that the quantity $(|z_\alpha|-1)(|z|^2-1)z_t$ is thought of as cubic. However, before we can investigate the structure of (3.1) we need to know the existence of a solution, at least locally in time. Theorem 3.2 on local well-posedness for (3.1) is therefore the first stepping stone in our analysis. Since local well-posedness is not the focus of this work, we postpone the proof of Theorem 3.2 to Sect. 7 and until then we treat it as a black box.

Theorem 3.2. *Let $s \geq 5$. Assume that $z_0 \in H_\alpha^{s+\frac{1}{2}}$ $z_1 \in H_\alpha^{s+\frac{1}{2}}$ and $|z_0(\alpha)-z_0(\beta)| \geq c_0'|e^{i\alpha}-e^{i\beta}|$ for some constant $c_0' > 0$. Then there is $T > 0$, depending on the norm of the initial data, so that (3.1) with initial data $(z, z_t)|_{t=0} = (z_0, z_1)$ has a unique solution $z = z(t,\alpha)$ for $t \in [0,T)$ satisfying for all $j \leq s$,*

$$\partial_\alpha^j z, \partial_\alpha^j z_t \in C\left([0,T], H_\alpha^{\frac{1}{2}}\right),$$
$$\partial_\alpha^j z_{tt} \in C\left([0,T], L_\alpha^2\right),$$

and $|z(t,\alpha)-z(t,\beta)| \geq \frac{c_0'}{2}|e^{i\alpha}-e^{i\beta}|$ for all $\alpha \neq \beta$. Moreover, if T^* is the supremum over all such time T, then either $T^* = \infty$, or

$$\sup_{t<T^*}\left(\|z_{tt}\|_{H_\alpha^4} + \|z_t\|_{H_\alpha^{\frac{9}{2}}}\right) + \sup_{\substack{t<T^* \\ \alpha\neq\beta}}\left|\frac{e^{i\alpha}-e^{i\beta}}{z(t,\alpha)-z(t,\beta)}\right| = \infty.$$

Remark 3.3. Note that to prove local well-posedness we differentiate Eq. (1.3) with respect to time (cf. Eq. (3.4)) to reveal the quasilinear structure, and treat the resulting equation as a second order equation for z_t. The original unknown z is then obtained from z_t by integration, which explains the choice of regularity for the initial data. See Sect. 7 for more details.

In what follows we will use the notation

$$g^a := \frac{\pi}{2}(I+\overline{H})z,$$

for the anti-holomorphic part of the contribution of the gravity and

$$g^h := \overline{g^a} = \frac{\pi}{2}(I + H)\overline{z},$$

for the holomorphic part of the conjugate of the gravity term. We will show later that g^a and g^h are small in an appropriate sense. With this notation we rewrite the equations for z and $\overline{z}$ as

$$z_{tt} + iaz_\alpha = -\pi z + g^a \tag{3.2}$$

and

$$\overline{z}_{tt} - ia\overline{z}_\alpha = -\pi \overline{z} + g^h. \tag{3.3}$$

For future reference we also record the time-differentiated versions of Eqs. (3.2) and (3.3). Differentiation of (3.2) and use of anti-holomorphicity of z_t give

$$z_{ttt} + iaz_{t\alpha} = -ia_t z_\alpha + \frac{\pi}{2}[\overline{z}_t, \overline{H}]\frac{z_\alpha}{\overline{z}_\alpha}. \tag{3.4}$$

Similarly, differentiating (3.3) we get

$$\overline{z}_{ttt} - ia\overline{z}_{t\alpha} = ia_t\overline{z}_\alpha + \frac{\pi}{2}[z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}. \tag{3.5}$$

Since $|z|$ is not expected to be small, we want to linearize these equations about the static solution $z_0(\alpha) := e^{i\alpha}$ in some sense, to exploit the smallness of the initial data. This will be achieved in Sect. 3.3, but before that we will need to establish some basic identities involving H and $\overline{H}$. This will be the content of Sect. 3.2. A final point to keep in mind when thinking about the smallness of the solution is that if we start with the static solution $z_0(\alpha) := e^{i\alpha}$ but with arbitrary constant initial velocity, then the domain will move in the direction of the initial velocity without changing its geometry. Therefore to properly interpret small quantities as those which are small when the static solution is the unit disk centered at zero, we need to appropriately renormalize the solution to account for this motion with constant velocity. It turns out that this issue can be resolved simply by choosing coordinates in which the center of mass is static. We begin the analysis in this section by clarifying this point in Sect. 3.1.

3.1. Center of mass. In this subsection we first show that the center of mass $C = C_\Omega(t)$ moves along a straight line with constant speed, that is, $C_{tt} = 0$, which is consistent with the fact that no external force acts on the system. Then we derive a formula for the center of mass only involving quantities defined on the boundary $\partial\Omega(t)$, which will be useful later. We begin by recalling the definition of the center of mass

$$C_\Omega(t) := \frac{1}{\pi}\iint_\Omega \mathbf{x}\, dxdy. \tag{3.6}$$

Proposition 3.4. *The center of mass $C := C_\Omega$ satisfies*

$$\frac{d^2C}{dt^2} = 0.$$

Proof. We prove that

$$\pi \frac{d^2 C}{dt^2} = -\int_{\partial\Omega} \phi \, \vec{n} \, dS, \tag{3.7}$$

where ϕ is the gravity potential, $\mathbf{n} = -\frac{i z_\alpha}{|z_\alpha|}$ is the exterior unit normal of $\partial\Omega$, and $dS = |z_\alpha| d\alpha$ is the line element of the boundary. We assume (3.7) for the moment and prove that the integral on the right hand side vanishes. Recall that ϕ satisfies $\partial_z \partial_{\bar{z}} \phi = \frac{\pi}{2}$ inside Ω. Integration in z gives $\partial_{\bar{z}} \phi = \frac{\pi}{2} z + A(\bar{z})$ where A is an anti-holomorphic function inside Ω, and another integration in $\bar{z}$ gives $\phi = \frac{\pi}{2} z\bar{z} + A(\bar{z}) + B(z)$ for some holomorphic function B. Moreover, from Lemma 3.1 we know that for points on the boundary $\partial_{\bar{z}} \phi = \partial_{\bar{z}} (\phi - B) = \frac{\pi}{4} (I - \overline{H}) z$. With this notation we rewrite (3.7) as

$$\begin{aligned}
\pi \frac{d^2 C}{dt^2} &= i \int_0^{2\pi} \left(\frac{\pi}{2} z\bar{z} + A(\bar{z}) \right) z_\alpha d\alpha + i \int_{\partial\Omega} B(z) dz \\
&= -\frac{\pi i}{2} \int_0^{2\pi} \bar{z} \, z \, z_\alpha d\alpha - \frac{\pi i}{4} \int_0^{2\pi} ((I - \overline{H}) z) \, z \, \bar{z}_\alpha d\alpha \\
&= \frac{\pi i}{4} \int_0^{2\pi} z (\overline{H} z) \, \bar{z}_\alpha d\alpha = \frac{\pi i}{4} \int_{\partial\Omega} z \overline{H} z d\bar{z}.
\end{aligned} \tag{3.8}$$

Now recall that the (conjugate) Hilbert transform is defined as

$$\overline{H} f(\bar{z}) := -\frac{\text{p.v.}}{\pi i} \int_{\partial\Omega} \frac{f(w)}{\overline{w} - \bar{z}} d\overline{w} := -\frac{1}{\pi i} \lim_{\epsilon \to 0} \int_{\partial\Omega \setminus B_\epsilon(\bar{z})} \frac{f(w)}{\overline{w} - \bar{z}} d\overline{w},$$

where the last limit converges in the L^2 sense. In particular if $f, g \in L^2$ then

$$\begin{aligned}
-\pi i \int_{\partial\Omega} g(z) \overline{H} f(z) d\bar{z} &= \int_{\partial\Omega} g(z) \lim_{\epsilon \to 0} \int_{\partial\Omega \setminus B_\epsilon(z)} \frac{f(w)}{\overline{w} - \bar{z}} d\overline{w} d\bar{z} \\
&= \lim_{\epsilon \to 0} \int_{\partial\Omega} \int_{\partial\Omega - B_\epsilon(z)} \frac{g(z) f(w)}{\overline{w} - \bar{z}} d\overline{w} d\bar{z} \\
&= \lim_{\epsilon \to 0} \int_{\partial\Omega} \int_{\partial\Omega - B_\epsilon(w)} \frac{g(z) f(w)}{\overline{w} - \bar{z}} d\bar{z} d\overline{w} \\
&= \lim_{\epsilon \to 0} \int_{\partial\Omega} \int_{\partial\Omega - B_\epsilon(z)} \frac{g(w) f(z)}{\bar{z} - \overline{w}} d\overline{w} d\bar{z} \\
&= \int_{\partial\Omega} f(z) \lim_{\epsilon \to 0} \int_{\partial\Omega - B_\epsilon(z)} \frac{g(w)}{\bar{z} - \overline{w}} d\overline{w} d\bar{z} = \pi i \int_{\partial\Omega} f(\bar{z}) \overline{H} g(\bar{z}) d\bar{z}.
\end{aligned}$$

Applying this observation to $f(z) = g(z) = z$ we see that $\int_{\partial\Omega} z \overline{H} z d\bar{z} = -\int_{\partial\Omega} z \overline{H} z d\bar{z}$ and therefore in view of (3.8) we get $\frac{d^2 C}{dt^2} = 0$. Finally we establish (3.7) by direct differentiation. For this we denote the flow map by X, that is,

$$X(t, \cdot) : \Omega(0) \to \Omega(t)$$

satisfies $\frac{dX(t, \mathbf{x})}{dt} = V(t, X(t, \mathbf{x}))$, $\frac{d^2 X(t, \mathbf{x})}{dt^2} = -\nabla P(t, X(t, \mathbf{x})) - \nabla \phi(t, X(t, \mathbf{x}))$. Then since the flow is incompressible we have

$$C = \frac{1}{\pi} \iint\limits_{\Omega(0)} X(t, \mathbf{x}') d\mathbf{x}',$$

and hence

$$\begin{aligned} \pi \frac{d^2 C}{dt^2} &= -\iint\limits_{\Omega(0)} \nabla \left(P(t, X(t, \mathbf{x}')) + \nabla \phi(t, X(t, \mathbf{x}')) \right) d\mathbf{x}' \\ &= -\iint\limits_{\Omega(t)} (\nabla P(t, \mathbf{x}) + \nabla \phi(t, \mathbf{x})) d\mathbf{x} = -\int_{\partial \Omega} \phi \, \vec{n} \, dS, \end{aligned}$$

as desired. □

The formula (3.6) is in terms of the domain $\Omega(t)$, but since we work with the boundary Eq. (3.1), it is more convenient to derive a formula for center of mass only involving quantities defined on the boundary. This is achieved in the following proposition.

Proposition 3.5. *Let us denote*

$$\varepsilon := |z|^2 - 1, \quad \delta := (I - H)\varepsilon. \tag{3.9}$$

Then the center of mass (3.6) as a complex number can be written as

$$C_{\Omega(t)} = -\frac{i}{2\pi} \int_0^{2\pi} \varepsilon(t, \alpha) z_\alpha(t, \alpha) d\alpha = -\frac{i}{4\pi} \int_0^{2\pi} \delta(t, \alpha) z_\alpha(t, \alpha) d\alpha \tag{3.10}$$

where $z(t, \alpha)$ is the parametrization of $\partial\Omega(t)$.

Proof. We can write the center of mass as $\frac{1}{\pi} \iint_{\Omega(t)} (x + iy) dx dy$. Using the divergence theorem, we have

$$\begin{aligned} \iint_{\Omega(t)} x dx dy = \iint_{\Omega(t)} \operatorname{div} \left(\frac{x^2}{2}, 0 \right) dx dy &= \int_{\partial\Omega(t)} \left(\frac{x^2}{2}, 0 \right) \cdot \left(\frac{y_\alpha}{|z_\alpha|}, -\frac{x_\alpha}{|z_\alpha|} \right) ds \\ &= \frac{1}{2} \int_0^{2\pi} x^2 y_\alpha d\alpha = -\frac{i}{2} \int_0^{2\pi} x^2 (x_\alpha + i y_\alpha) d\alpha \end{aligned}$$

and

$$\begin{aligned} i \iint_{\Omega(t)} y dx dy = i \iint_{\Omega(t)} \operatorname{div} \left(0, \frac{y^2}{2} \right) dx dy &= i \int_{\partial\Omega(t)} \left(0, \frac{y^2}{2} \right) \cdot \left(\frac{y_\alpha}{|z_\alpha|}, -\frac{x_\alpha}{|z_\alpha|} \right) ds \\ &= -\frac{i}{2} \int_0^{2\pi} y^2 x_\alpha d\alpha = -\frac{i}{2} \int_0^{2\pi} y^2 (x_\alpha + i y_\alpha) d\alpha. \end{aligned}$$

Therefore we have

$$\begin{aligned} \iint_{\Omega(t)} (x + iy) dx dy &= -\frac{i}{2} \int_0^{2\pi} |z|^2 z_\alpha d\alpha = -\frac{i}{2} \int_0^{2\pi} \varepsilon z_\alpha d\alpha \\ &= -\frac{i}{2} \int_0^{2\pi} \left(\frac{I - H}{2} \varepsilon \right) z_\alpha d\alpha = -\frac{i}{4} \int_0^{2\pi} \delta z_\alpha d\alpha. \end{aligned}$$

This completes the proof. □

We have the following corollary.

Corollary 3.6. *Let v_c^0 and c^0 be the initial velocity and position of the center of mass respectively. If $z = z(t, \alpha)$ is a solution to (3.1) then $z(t, \alpha) - c^0 - v_c^0 t$ is also a solution to (3.1). Moreover, $z - c^0 - v_c^0 t$ parametrizes the boundary of a domain whose center of mass is always at the origin.*

Proof. This follows from Proposition 3.4 and the fact that z_{tt}, z_α and H are invariant under the transformation

$$z(\alpha, t) \mapsto z(\alpha, t) - c^0 - v_c^0 t. \tag{3.11}$$

□

In what follows, we will only consider this normalized solution to (3.1), that is we assume that the center of mass is always at the origin, and this assumption is justified by Corollary 3.6. Therefore in view of Proposition 3.5 and Corollary 3.6 we always have

$$\int_0^{2\pi} \varepsilon z_\alpha d\alpha = \int_0^{2\pi} \delta z_\alpha d\alpha = 0.$$

3.2. Basic identities. In this subsection we record some basic identities which will be used in the remainder of this work. A few more standard properties of the Hilbert transform are recalled in Appendix A. In the remainder of this section we assume that the parametrization z of $\partial\Omega(t)$ has regularity $C^2_{t,\alpha}$.

3.2.1. Commutation relations. We compute the commutators of various operators with the Hilbert transform.

Lemma 3.7. *For any 2π-periodic function f in $C^2_{t,\alpha}$*

(i) $[\partial_t, H]f = [z_t, H]\frac{f_\alpha}{z_\alpha}$,

(ii) $[\partial_t^2, H]f = 2[z_t, H]\frac{f_{t\alpha}}{z_\alpha} + [z_{tt}, H]\frac{f_\alpha}{z_\alpha} + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2 f_\beta(\beta)d\beta$,

(iii) $\partial_\alpha Hf = z_\alpha H\frac{f_\alpha}{z_\alpha}$,

(iv) $[a\partial_\alpha, H]f = [az_\alpha, H]\frac{f_\alpha}{z_\alpha}$,

(v) $[\partial_t^2 + ia\partial_\alpha, H]f = -\frac{\pi}{2}[(I-\overline{H})z, H]\frac{f_\alpha}{z_\alpha} + 2[z_t, H]\frac{f_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2 f_\beta(\beta)d\beta.$

Proof of Lemma 3.7. (i)

$$\begin{aligned}[\partial_t, H]f &= \frac{\text{p.v.}}{\pi i}\int_0^{2\pi}\left(\frac{f(\beta)z_{t\beta}(\beta)}{z(\beta)-z(\alpha)} - \frac{(z_t(\beta)-z_t(\alpha))f(\beta)z_\beta(\beta)}{(z(\beta)-z(\alpha))^2}\right)d\beta \\ &= -\frac{\text{p.v.}}{\pi i}\int_0^{2\pi}\frac{(z_t(\beta)-z_t(\alpha))f_\beta(\beta)}{(z(\beta)-z(\alpha))z_\beta(\beta)}z_\beta(\beta)d\beta = [z_t, H]\frac{f_\alpha}{z_\alpha}.\end{aligned}$$

(ii)

$$\begin{aligned}[\partial_t^2, H]f &= \partial_t([z_t, H]\frac{f_\alpha}{z_\alpha}) + \partial_t H\partial_t f - H\partial_t^2 f \\ &= \partial_t([z_t, H]\frac{f_\alpha}{z_\alpha}) + [z_t, H]\frac{f_{t\alpha}}{z_\alpha} \\ &= [z_{tt}, H]\frac{f_\alpha}{z_\alpha} + 2[z_t, H]\frac{f_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2 f_\beta(\beta)d\beta.\end{aligned}$$

(iii)

$$\partial_\alpha Hf = \frac{\text{p.v.}}{\pi i}\int_0^{2\pi} \frac{f(\beta)}{(z(\beta)-z(\alpha))^2} z_\alpha(\alpha) z_\beta(\beta)d\beta$$
$$= z_\alpha(\alpha)\frac{\text{p.v.}}{\pi i}\int_0^{2\pi} \frac{f_\beta(\beta)}{z\beta)-z(\alpha)}d\beta = z_\alpha H\frac{f_\alpha}{z_\alpha}.$$

(iv)

$$[a\partial_\alpha, H]f = az_\alpha H\frac{f_\alpha}{z_\alpha} - H(af_\alpha) = [az_\alpha, H]\frac{f_\alpha}{z_\alpha}.$$

(v) This part is a corollary of the previous parts combined with Eq. (1.3) and Lemma 3.1. □

Lemma 3.8. *For any 2π-periodic function f and g in $C^2_{t,\alpha}$*

$$\partial_t[f, H]g = [f_t, H]g + [f, H]g_t + f[z_t, H]\frac{f_\alpha}{z_\alpha} - [z_t, H]\frac{\partial_\alpha(fg)}{z_\alpha}.$$

Proof. Using part (i) of Lemma 3.7 we get

$$\partial_t[f, H]g = \partial_t(fHg) - \partial_t H(fg) = f_t Hg + fHg_t$$
$$+ f[z_t, H]\frac{g_\alpha}{z_\alpha} - H(f_t g) - H(fg_t) - [z_t, H]\frac{\partial_\alpha(fg)}{z_\alpha}$$
$$= [f_t, H]g + [f, H]g_t + f[z_t, H]\frac{f_\alpha}{z_\alpha} - [z_t, H]\frac{\partial_\alpha(fg)}{z_\alpha}.$$

□

Next we recored the following important computation relating $[\bar{z}, \overline{H}]z$ and the area of Ω.

Lemma 3.9. *If $z : [0, 2\pi] \to \partial\Omega$ is a counterclockwise parametrization then*

$$[\bar{z}, \overline{H}]z = -\frac{2|\Omega|}{\pi} = -2.$$

Proof. Since the parametrization is counterclockwise the exterior normal $\mathbf{n}$ is given by

$$\mathbf{n} = -\frac{iz_\alpha}{|z_\alpha|} = \frac{y_\alpha - ix_\alpha}{|z_\alpha|}$$

in complex notation. It follows that with $z = x + iy$

$$[\bar{z}, \overline{H}]z = \frac{1}{\pi i}\int_0^{2\pi} z(\beta)\bar{z}_\beta(\beta)d\beta = \frac{1}{2\pi i}\int_0^{2\pi}(z(\beta)\bar{z}_\beta(\beta) - \bar{z}(\beta)z_\beta(\beta))d\beta$$
$$= \frac{1}{\pi}\int_0^{2\pi} \text{Im}\,(z(\beta)\bar{z}_\beta(\beta))d\beta = \frac{1}{\pi}\int_0^{2\pi}(x_\beta(\beta)y(\beta) - y_\beta(\beta)x(\beta))d\beta$$
$$= -\frac{1}{\pi}\int_{\partial\Omega}\begin{pmatrix}x\\y\end{pmatrix}\cdot\mathbf{n}\,|z_\beta(\beta)|d\beta = -\frac{1}{\pi}\iint_\Omega \text{div}\begin{pmatrix}x\\y\end{pmatrix}dxdy = -\frac{2|\Omega|}{\pi}.$$

□

Lemma 3.10. *For any 2π-periodic function f in $C^2_{t,\alpha}$*

$$[z, H]\frac{f_\alpha}{z_\alpha} = 0$$

Proof. This is an immediate consequence of the definition of the Hilbert transform and the periodicity of f. □

Lemma 3.11. *For any 2π-periodic function f, g, and h in $C^2_{t,\alpha}$*

$$[fg, H]h = f[g, H]h + [f, H](gh).$$

Proof.

$$[fg, H]h = fgHh - fH(gh) + fH(gh) - H(fgh) = f[g, H]h + [f, H](gh).$$

□

Lemma 3.12. *Suppose f and g are 2π-periodic functions in $C^2_{t,\alpha}$ which are anti-holomorphic inside Ω. Then with the notation $\varepsilon = |z|^2 - 1$*

$$[f, H\frac{1}{z_\alpha} + \overline{H}\frac{1}{\overline{z}_\alpha}]g_\alpha = -\frac{1}{\pi i}\int_0^{2\pi} \frac{(f(\alpha) - f(\beta))g_\beta(\beta)\overline{z}(\alpha)\overline{z}(\beta)\left(\frac{\varepsilon(\alpha)}{\overline{z}(\alpha)} - \frac{\varepsilon(\beta)}{\overline{z}(\beta)}\right)}{|z(\beta) - z(\alpha)|^2} d\beta.$$

Proof.

$$\begin{aligned}
[f, H\frac{1}{z_\alpha}+\overline{H}\frac{1}{\overline{z}_\alpha}]g_\alpha &= \frac{\text{p.v.}}{\pi i}\int_0^{2\pi}\left(\frac{1}{z(\beta) - z(\alpha)} - \frac{1}{\overline{z}(\beta)-\overline{z}(\alpha)}\right)(f(\alpha) - f(\beta))g_\beta(\beta)d\beta \\
&= -[f, \overline{H}]\frac{g_\alpha}{\overline{z}_\alpha}+\overline{z}[f, \overline{H}]\frac{\overline{z}g_\alpha}{\overline{z}_\alpha} - \frac{1}{\pi i}\int_0^{2\pi} \frac{(f(\alpha) - f(\beta))g_\beta(\beta)\overline{z}(\alpha)\overline{z}(\beta)\left(\frac{\varepsilon(\alpha)}{\overline{z}(\alpha)} - \frac{\varepsilon(\beta)}{\overline{z}(\beta)}\right)}{|z(\beta) - z(\alpha)|^2} d\beta.
\end{aligned}$$

Since $\frac{g_\alpha}{\overline{z}_\alpha}$ is anti-holomorphic in side Ω, the first two terms on the last line above are zero, and this proves the lemma. □

3.2.2. The relation between H and $\overline{H}$. In the static case where the boundary of the domain Ω is exactly the unit circle, the corresponding Hilbert transform $\mathbb{H}$ satisfies $\overline{\mathbb{H}} = -\mathbb{H} + 2\text{Av}$ where $\text{Av}(f) := \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha$. Here we prove an important lemma which quantifies the failure of this identity when Ω is a small perturbation of the unit disc.

Lemma 3.13. *For any 2π-periodic function f in $C^2_{t,\alpha}$*

$$\begin{aligned}
\overline{H}f &= -zH\frac{f}{z} + z[\varepsilon, H]\frac{f_\alpha}{z_\alpha} + E(f) \\
&= -Hf - [z, H]\frac{f}{z} + z[\varepsilon, H]\frac{f_\alpha}{z_\alpha} + E(f)
\end{aligned} \tag{3.12}$$

where $\varepsilon := |z|^2 - 1$ *and* $E(f) = E_1(f) + E_2(f) + E_3(f)$ *with*

$$E_1(f) := -\frac{1}{\pi i}\int_0^{2\pi} \frac{f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)} - \frac{\epsilon(\beta)}{z(\beta)}\right)(z(\alpha)z(\beta))^2}{(z(\alpha)-z(\beta))^2}\partial_\beta\left(\frac{\epsilon(\beta)}{z(\beta)}\right)d\beta$$

$$E_2(f) := -\frac{1}{\pi i}\int_0^{2\pi} \frac{f(\beta)\left(\frac{\epsilon(\beta)}{z(\beta)} - \frac{\epsilon(\alpha)}{z(\alpha)}\right)^2(z(\alpha)z(\beta))^2}{(z(\beta)-z(\alpha))|z(\beta)-z(\alpha)|^2}\partial_\beta\left(\frac{\epsilon(\beta)}{z(\beta)}\right)d\beta$$

$$E_3(f) := -\frac{1}{\pi i}\int_0^{2\pi} \frac{f(\beta)\left(\frac{\epsilon(\beta)}{z(\beta)} - \frac{\epsilon(\alpha)}{z(\alpha)}\right)^2(z(\alpha)z(\beta))^2}{(z(\beta)-z(\alpha))|z(\beta)-z(\alpha)|^2}\partial_\beta\left(\frac{1}{z(\beta)}\right)d\beta.$$

Proof. Recall the following relations:

$$\overline{z} = \frac{1+\epsilon}{z}, \quad \overline{z}_\beta(\beta) = \frac{\epsilon_\beta(\beta)z(\beta) - z_\beta(\beta)(1+\epsilon(\beta))}{(z(\beta))^2}.$$

We have

$$\begin{aligned}
(\overline{H}f)(\alpha) &= -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\overline{z}_\beta(\beta)}{\overline{z}(\beta)-\overline{z}(\alpha)}d\beta \\
&= -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)}{\frac{1+\epsilon(\beta)}{z(\beta)} - \frac{1+\epsilon(\alpha)}{z(\alpha)}}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta \\
&= -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)}{\frac{1}{z(\beta)} - \frac{1}{z(\alpha)}}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta \\
&\quad -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)} - \frac{\epsilon(\beta)}{z(\beta)}\right)}{\left(\frac{1}{z(\beta)} - \frac{1}{z(\alpha)}\right)\left(\frac{1}{z(\beta)} - \frac{1}{z(\alpha)} + \frac{\epsilon(\beta)}{z(\beta)} - \frac{\epsilon(\alpha)}{z(\alpha)}\right)}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta \\
&= -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)}{\frac{1}{z(\beta)} - \frac{1}{z(\alpha)}}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta \\
&\quad -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)} - \frac{\epsilon(\beta)}{z(\beta)}\right)}{\left(\frac{1}{z(\beta)} - \frac{1}{z(\alpha)}\right)^2}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta \\
&\quad -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)} - \frac{\epsilon(\beta)}{z(\beta)}\right)^2}{\left(\frac{1}{z(\beta)} - \frac{1}{z(\alpha)}\right)^2\left(\frac{1}{z(\beta)} - \frac{1}{z(\alpha)} + \frac{\epsilon(\beta)}{z(\beta)} - \frac{\epsilon(\alpha)}{z(\alpha)}\right)}\left(\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta - \frac{z_\beta(\beta)}{(z(\beta))^2}\right)d\beta.
\end{aligned} \tag{3.13}$$

The 'constant term' above (the second term in the first line) is

$$\begin{aligned}
&\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)z(\alpha)z(\beta)}{z(\alpha)-z(\beta)}\frac{z_\beta(\beta)}{(z(\beta))^2}d\beta \\
&\quad = \frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)z(\alpha)z_\beta(\beta)}{(z(\alpha)-z(\beta))\,z(\beta)}d\beta = -zH\left(\frac{f}{z}\right).
\end{aligned} \tag{3.14}$$

The 'linear terms' above (the first term in the first line and the second term in the second line) are given by

$$-\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)z(\alpha)z(\beta)}{z(\alpha)-z(\beta)}\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta d\beta. \tag{3.15}$$

and

$$\begin{aligned}&\frac{1}{\pi i}\int_0^{2\pi} f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)}-\frac{\epsilon(\beta)}{z(\beta)}\right)\left(\frac{1}{\frac{1}{z(\beta)}-\frac{1}{z(\alpha)}}\right)_\beta d\beta\\ &\quad= -\frac{1}{\pi i}\int_0^{2\pi} f_\beta(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)}-\frac{\epsilon(\beta)}{z(\beta)}\right)\left(\frac{1}{\frac{1}{z(\beta)}-\frac{1}{z(\alpha)}}\right)d\beta\\ &\qquad+\frac{1}{\pi i}\int_0^{2\pi} f(\beta)\left(\frac{\epsilon(\beta)}{z(\beta)}\right)_\beta\left(\frac{1}{\frac{1}{z(\beta)}-\frac{1}{z(\alpha)}}\right)d\beta.\end{aligned} \tag{3.16}$$

The last term in (3.16) cancels with (3.15). Therefore the 'linear term' in $\overline{H}f$ is given by

$$-\frac{1}{\pi i}\int_0^{2\pi} f_\beta(\beta)\frac{\epsilon(\alpha)z(\beta)}{z(\alpha)-z(\beta)}d\beta+\frac{1}{\pi i}\int_0^{2\pi} f_\beta(\beta)\frac{\epsilon(\beta)z(\alpha)}{z(\alpha)-z(\beta)}d\beta = z[\epsilon, H]\left(\frac{f_\alpha}{z_\alpha}\right). \tag{3.17}$$

where in the last step we used the fact that $f(0)=f(2\pi)$. The remaining terms in $\overline{H}f$ are the first term in the second line and the two terms in the third line of (3.13). The first term in the second line can be written as

$$E_1(f) := -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\alpha)}{z(\alpha)}-\frac{\epsilon(\beta)}{z(\beta)}\right)(z(\alpha)z(\beta))^2}{(z(\alpha)-z(\beta))^2}\partial_\beta\left(\frac{\epsilon(\beta)}{z(\beta)}\right)d\beta. \tag{3.18}$$

The first term in the third line of (3.13) can be written as

$$E_2(f) := -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\beta)}{z(\beta)}-\frac{\epsilon(\alpha)}{z(\alpha)}\right)^2(z(\alpha)z(\beta))^2}{(z(\beta)-z(\alpha))|z(\beta)-z(\alpha)|^2}\partial_\beta\left(\frac{\epsilon(\beta)}{z(\beta)}\right)d\beta. \tag{3.19}$$

The second term in the third line of (3.13) can be written as

$$E_3(f) := -\frac{1}{\pi i}\int_0^{2\pi}\frac{f(\beta)\left(\frac{\epsilon(\beta)}{z(\beta)}-\frac{\epsilon(\alpha)}{z(\alpha)}\right)^2(z(\alpha)z(\beta))^2}{(z(\beta)-z(\alpha))|z(\beta)-z(\alpha)|^2}\partial_\beta\left(\frac{1}{z(\beta)}\right)d\beta. \tag{3.20}$$

□

Remark 3.14. Note that if we measure smallness of quantities by comparison with the static case $z \equiv e^{i\alpha}$, then by Lemma 3.13, $E(f)$ is order of ε^2 smaller than f. This observation will be made precise when we carry out the estimates in Sects. 4 and 5.

3.3. The δ equation. In this section we derive an equation for the small quantity

$$\delta := (I - H)\varepsilon, \tag{3.21}$$

where

$$\varepsilon := |z|^2 - 1. \tag{3.22}$$

Note that in view of our small data assumptions we expect the quantities ε and δ to be (linearly) small. Our main goal here is to show that δ satisfies a constant-coefficient PDE with cubic nonlinearity. This will be accomplished in two steps. In the first step we show that the nonlinear part of $(\partial_t^2 + ia\partial_\alpha)\delta$ is cubic. If we then replace the operator $\partial_t^2 + ia\partial_\alpha$ by $\partial_t^2 + i\pi\partial_\alpha$, corresponding to the value of a in the static case, we will notice that the resulting error is only quadratic. For this reason, in the second step we perform a change of variables $\beta(t, \alpha) = k^{-1}(t, \alpha)$ such that the nonlinearity in the equation for $(\partial_t^2 + i\pi\partial_\beta)\delta$ has no quadratic part. The first step is achieved in the following proposition.

Proposition 3.15. *The quantities $\delta = (I - H)\varepsilon$ and $\delta_t = \partial_t\delta$ satisfy*

$$(\partial_t^2 + ia\partial_\alpha - \pi)\delta = \mathcal{N}_1 := \frac{\pi}{2}[E(z), H]\frac{\varepsilon_\alpha}{z_\alpha} + \frac{\pi}{2}(I - H)E(\varepsilon)$$
$$- 2[z_t, H\frac{1}{z_\alpha} + \overline{H}\frac{1}{\overline{z}_\alpha}]\partial_\alpha(z_t\overline{z}) - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta) - z_t(\alpha)}{z(\beta) - z(\alpha)}\right)^2 \varepsilon_\beta(\beta)d\beta \tag{3.23}$$

and

$$(\partial_t^2 + i\partial_\alpha - \pi)\delta_t = \mathcal{N}_2 := -ia_t\partial_\alpha\delta + \frac{\pi}{2}\left((I - H)\partial_t E(\varepsilon) - [z_t, H]\frac{\partial_\alpha E(\varepsilon)}{z_\alpha}\right)$$
$$+ \frac{\pi}{2}\left([\partial_t E(z), H]\frac{\varepsilon_\alpha}{z_\alpha} + [E(z), H]\partial_t\left(\frac{\varepsilon_\alpha}{z_\alpha}\right) + E(z)[z_t, H]\frac{\partial_\alpha\left(\frac{\varepsilon_\alpha}{z_\alpha}\right)}{z_\alpha} - [z_t, H]\frac{\partial_\alpha\left(E(z)\frac{\varepsilon_\alpha}{z_\alpha}\right)}{z_\alpha}\right)$$
$$+ \frac{2}{\pi i}\partial_t\int_0^{2\pi}\frac{(z_t(\alpha) - z_t(\beta))\partial_\beta(z_t(\beta)\overline{z}(\beta))\overline{z}(\alpha)\overline{z}(\beta)\left(\frac{\varepsilon(\alpha)}{\overline{z}(\alpha)} - \frac{\varepsilon(\beta)}{\overline{z}(\beta)}\right)}{|z(\beta) - z(\alpha)|^2}d\beta$$
$$- \frac{1}{\pi i}\partial_t\int_0^{2\pi}\left(\frac{z_t(\beta) - z_t(\alpha)}{z(\beta) - z(\alpha)}\right)^2\varepsilon_\beta(\beta)d\beta, \tag{3.24}$$

where $E(f)$ is as in Lemma 3.13. Moreover, we can write

$$[z_t, H\frac{1}{z_\alpha} + \overline{H}\frac{1}{\overline{z}_\alpha}]\partial_\alpha(z_t\overline{z}) = -\frac{1}{\pi i}$$
$$\int_0^{2\pi}\frac{(z_t(\alpha) - z_t(\beta))\partial_\beta(z_t(\beta)\overline{z}(\beta))\overline{z}(\alpha)\overline{z}(\beta)\left(\frac{\varepsilon(\alpha)}{\overline{z}(\alpha)} - \frac{\varepsilon(\beta)}{\overline{z}(\beta)}\right)}{|z(\beta) - z(\alpha)|^2}d\beta. \tag{3.25}$$

Proof. We want to apply the last part of Lemma 3.7. To this end we first compute $(\partial_t^2 + ia\partial_\alpha)\varepsilon$.

$$\begin{aligned}(\partial_t^2 + ia\partial_\alpha)\varepsilon &= (z_{tt} + iaz_\alpha)\overline{z} + (\overline{z}_{tt} + ia\overline{z}_\alpha)z + 2z_t\overline{z}_t \\ &= -\frac{\pi}{2}(\overline{z}(I - \overline{H})z - z(I - H)\overline{z}) + 2\partial_t(z\overline{z}_t),\end{aligned} \tag{3.26}$$

and since $z\bar{z}_t$ is holomorphic

$$(I-H)(\partial_t^2 + ia\partial_\alpha)\varepsilon = \frac{\pi}{2}(I-H)\big(z(I-H)\bar{z} - \bar{z}(I-\overline{H})z\big) + 2[z_t, H]\frac{\partial_\alpha(z\bar{z}_t)}{z_\alpha}.$$

Applying Lemma 3.7 we get

$$\begin{aligned}(\partial_t^2 + ia\partial_\alpha)\delta &= \frac{\pi}{2}(I-H)\big(z(I-H)\bar{z} - \bar{z}(I-\overline{H})z\big) + \frac{\pi}{2}[(I-\overline{H})z, H]\frac{\varepsilon_\alpha}{z_\alpha} \\ &\quad - 2[z_t, H]\frac{\partial_\alpha(z_t\bar{z})}{z_\alpha} - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2 \varepsilon_\beta(\beta)d\beta \\ &= \frac{\pi}{2}(I-H)\big(z(I-H)\bar{z} - \bar{z}(I-\overline{H})z\big) + \frac{\pi}{2}[(I-\overline{H})z, H]\frac{\varepsilon_\alpha}{z_\alpha} \\ &\quad - 2[z_t, H\frac{1}{z_\alpha} + \overline{H}\frac{1}{\bar{z}_\alpha}]\partial_\alpha(z_t\bar{z}) - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2 \varepsilon_\beta(\beta)d\beta. \end{aligned} \tag{3.27}$$

The last two terms already have the right form so we concentrate on the first two. Using Lemma 3.9 we write

$$\bar{z}(I-\overline{H})z = (I-\overline{H})(z\bar{z}) - [\bar{z}, \overline{H}]z = \bar{\delta} + \frac{2|\Omega|}{\pi} = \bar{\delta} + 2,$$

and hence

$$\frac{\pi}{2}(I-H)\big(z(I-H)\bar{z} - \bar{z}(I-\overline{H})z\big) = \frac{\pi}{2}(I-H)(\delta - \bar{\delta}) = \pi\delta - \frac{\pi}{2}(I-H)\bar{\delta}, \tag{3.28}$$

where to pass to the last equality we have used the fact that $\left(\frac{1}{2}(I-H)\right)^2 = \frac{1}{2}(I-H)$. To understand the contributions of $\bar{\delta}$ and $(I-\overline{H})z$ we use Lemma 3.13 to replace $\overline{H}$ by H. For $\bar{\delta}$, noting that $H\frac{1}{z} = -\frac{1}{z}$ we get

$$\begin{aligned}\bar{\delta} &= \varepsilon + zH(\bar{z} - \frac{1}{z}) - z[\varepsilon, H]\frac{\varepsilon_\alpha}{z_\alpha} - E(\varepsilon) \\ &= z(I+H)\bar{z} - z[\varepsilon, H]\frac{\varepsilon_\alpha}{z_\alpha} - E(\varepsilon) \\ &= z(I+H)\bar{z} - z\varepsilon(I+H)\frac{\varepsilon_\alpha}{z_\alpha} + z(I+H)\frac{\varepsilon\varepsilon_\alpha}{z_\alpha} - E(\varepsilon),\end{aligned}$$

which implies

$$\begin{aligned}-\frac{\pi}{2}(I-H)\bar{\delta} &= \frac{\pi}{2}(I-H)(z\varepsilon(I+H)(\frac{\varepsilon_\alpha}{z_\alpha})) + \frac{\pi}{2}(I-H)E(\varepsilon) \\ &= \frac{\pi}{4}(I-H)(z\delta(I+H)(\frac{\varepsilon_\alpha}{z_\alpha})) + \frac{\pi}{2}(I-H)E(\varepsilon).\end{aligned} \tag{3.29}$$

Similarly for $(I-\overline{H})z$ we have

$$(I-\overline{H})z = 2z - z[\varepsilon, H]1 + E(z) = 2z - z\delta + E(z).$$

It follows from this and Lemma 3.10 that

$$\frac{\pi}{2}[(I-\overline{H})z, H]\frac{\varepsilon_\alpha}{z_\alpha} = -\frac{\pi}{2}[z\delta, H]\frac{\varepsilon_\alpha}{z_\alpha} + \frac{\pi}{2}[E(z), H]\frac{\varepsilon_\alpha}{z_\alpha}$$
$$= -\frac{\pi}{4}[z\delta, H](I+H)\frac{\epsilon_\alpha}{z_\alpha} - \frac{\pi}{4}[z\delta, H](I-H)\frac{\epsilon_\alpha}{z_\alpha} + \frac{\pi}{2}[E(z), H]\frac{\varepsilon_\alpha}{z_\alpha}. \tag{3.30}$$

By Lemma 3.11, the second term in (3.30) can be written as

$$-\frac{\pi}{4}z[\delta, H](I-H)\frac{\epsilon_\alpha}{z_\alpha} - \frac{\pi}{4}[z, H]\delta(I-H)\left(\frac{\epsilon_\alpha}{z_\alpha}\right). \tag{3.31}$$

The first term in (3.31) can be written as

$$-\frac{\pi}{4}z[\delta, I+H](I-H)\frac{\epsilon_\alpha}{z_\alpha} = \frac{\pi}{4}z(I+H)\left((I-H)\epsilon(I-H)\left(\frac{\epsilon_\alpha}{z_\alpha}\right)\right) = 0.$$

By (iii) in Lemma 3.7, the second term in (3.31) can be written as

$$-\frac{\pi}{4}[z, H]\delta\frac{\delta_\alpha}{z_\alpha} = 0.$$

Combining these observations with the fact that

$$[z\delta, H](I+H)\frac{\varepsilon_\alpha}{z_\alpha} = (I-H)\left(z\delta(I+H)\frac{\varepsilon_\alpha}{z_\alpha}\right)$$

we get

$$-\frac{\pi}{2}(I-H)\overline{\delta} + \frac{\pi}{2}[(I-\overline{H})z, H]\frac{\varepsilon_\alpha}{z_\alpha} = \frac{\pi}{2}(I-H)E(\varepsilon) + \frac{\pi}{2}[E(z), H]\frac{\varepsilon_\alpha}{z_\alpha}.$$

Equation (3.23) now follows from combining this identity with (3.27) and (3.28). Finally, Eqs. (3.24) and (3.25) are direct consequences of Lemmas 3.7, 3.8, and 3.12 and Eq. (3.23). □

By comparing the terms on the right hand sides of the Eqs. (3.23) and (3.24) with their corresponding values in the static case, one can see that the nonlinearity is cubic. This is least clear for the first term involving a_t in the equation for δ_t so in the following lemma we present a formula for a_t which sheds some light the structure of this term.

Lemma 3.16. *Let K^* denote the formal adjoint of $K := \mathrm{Re}\, H = \frac{1}{2}(H+\overline{H})$, i.e.,*

$$K^*g(\alpha) = -\mathrm{Re}\,\frac{\mathrm{p.v.}}{\pi i}\int_0^{2\pi}\frac{z_\alpha(\alpha)}{|z_\alpha(\alpha)|}\frac{|z_\beta(\beta)|}{z(\beta)-z(\alpha)}g(\beta)d\beta = -\mathrm{Re}\left\{\frac{z_\alpha}{|z_\alpha|}H\frac{|z_\beta|g}{z_\beta}\right\}.$$

Then

$$(I+K^*)(a_t|z_\alpha|) = \mathrm{Re}\Bigg[\frac{-iz_\alpha}{|z_\alpha|}\Bigg\{2[z_t, H]\frac{\overline{z}_{tt\alpha}}{z_\alpha} + 2[z_{tt}, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} - [g^a, H]\frac{\overline{z}_{t\alpha}}{z_\alpha}$$
$$+\frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\overline{z}_{t\beta}(\beta)d\beta + \frac{\pi}{2}([z_t, H]\frac{\partial_\alpha g^h}{z_\alpha})\Bigg\}\Bigg].$$

Proof. Using Eqs. (3.2), (3.5) and Lemma 3.7 we have

$$
\begin{aligned}
(I-H)(ia_t\overline{z}_\alpha) &= (I-H)(\overline{z}_{ttt} - ia\overline{z}_{t\alpha} - \frac{\pi}{2}[z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}) \\
&= [\partial_t^2 - ia\partial_\alpha, H]\overline{z}_t - \frac{\pi}{2}(I-H)([z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}) \\
&= 2[z_t, H]\frac{\overline{z}_{tt\alpha}}{z_\alpha} + [z_{tt}, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} - [iaz_\alpha, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} \\
&\quad + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\overline{z}_{t\beta}(\beta)d\beta - \frac{\pi}{2}(I-H)([z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}) \\
&= 2[z_t, H]\frac{\overline{z}_{tt\alpha}}{z_\alpha} + 2[z_{tt}, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} - [g^a, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} \\
&\quad + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\overline{z}_{t\beta}(\beta)d\beta - \frac{\pi}{2}(I-H)([z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}).
\end{aligned}
$$

The lemma now follows by multiplying the two sides of this equation by $\frac{-iz_\alpha}{|z_\alpha|}$ and taking real parts and also observing that

$$
\frac{\pi}{2}(I-H)\left([z_t, H]\frac{\overline{z}_\alpha}{z_\alpha}\right) = -\frac{\pi}{2}(I-H)\left(\partial_t(I-H)\overline{z}\right) = \frac{\pi}{2}[z_t, H]\frac{\partial_\alpha g^h}{z_\alpha}.
$$

□

For future reference we also record the following representation for K^* which is more amenable to estimates.

Lemma 3.17. *For any real valued 2π-periodic function f*

$$
\begin{aligned}
K^*f &= \frac{1}{\pi|z_\alpha|}\int_0^{2\pi} f(\beta)|z_\beta(\beta)|d\beta - \frac{1}{2|z_\alpha|}(H+\overline{H})(|z_\beta|f) \\
&\quad - \mathrm{Re}\left\{\frac{1}{|z_\alpha|}[z_\alpha - iz, H]\frac{|z_\beta|f}{z_\beta}\right\}.
\end{aligned}
$$

Proof. Using the definition $K^*f = -\mathrm{Re}\{\frac{z_\alpha}{|z_\alpha|}H\frac{|z_\beta|}{z_\beta}f\}$ of K^* we have

$$
\begin{aligned}
-2K^*g &= \frac{z_\alpha}{|z_\alpha|}H\frac{|z_\beta|f}{z_\beta} + \frac{\overline{z}_\alpha}{|z_\alpha|}\overline{H}\frac{|z_\alpha|f}{\overline{z}_\alpha} \\
&= 2\frac{\mathrm{Re}}{|z_\alpha|}\left\{[z_\alpha, H]\frac{|z_\beta|f}{z_\beta}\right\} + \frac{1}{|z_\alpha|}(H+\overline{H})(|z_\beta|f) \\
&= 2\mathrm{Re}\left\{\frac{1}{|z_\alpha|}[z_\alpha - iz, H]\frac{|z_\beta|f}{z_\beta}\right\} \\
&\quad + \frac{1}{|z_\alpha|}(H+\overline{H})(|z_\alpha|f) + 2\mathrm{Re}\left\{\frac{i}{|z_\alpha|}[z, H]\frac{|z_\beta|f}{z_\beta}\right\}.
\end{aligned}
$$

The lemma now follows by noting that

$$
\frac{i}{|z_\alpha|}[z, H]\frac{|z_\beta|f}{z_\beta} = -\frac{1}{\pi|z_\alpha|}\int_0^{2\pi}|z_\beta(\beta)|f(\beta)d\beta.
$$

□

We now turn to the left hand side of (3.23). As mentioned above, the nonlinear contribution of a to (3.23) can be seen to be only quadratic, and therefore a change of variables is necessary to retain the cubic structure. More precisely suppose

$$k(t,\cdot):\mathbb{R}\to\mathbb{R}$$

is an increasing function such that $k-\alpha$ is 2π periodic and k is differentiable on $(0,2\pi)$. Let us define

$$\zeta(t,\alpha') := z\circ k^{-1}(t,\alpha'), \qquad \chi(t,\alpha') := \delta\circ k^{-1}(t,\alpha').$$

Then introducing

$$b := k_t\circ k^{-1}, \qquad A = (ak_\alpha)\circ k^{-1}$$

we have

$$z_t\circ k^{-1} = (\partial_t + b\partial_{\alpha'})\zeta, \qquad (az_\alpha)\circ k^{-1} = A\zeta_{\alpha'}.$$

In particular,

$$(\delta_{tt} + ia\delta_\alpha - \pi\delta)\circ k^{-1} = ((\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi)\chi.$$

We wish to choose the change of variables k in such a way that b consists of quadratic and higher order terms, and A has no linear terms. This is achieved in the following three propositions. First in Propositions 3.18 and 3.20 we derive the desired representations for b and A under various assumptions on k. Then in Remark 3.21 we explain how to construct k satisfying these assumptions.

Proposition 3.18. *Suppose that $z(t,\cdot)$ is a simple closed curve containing origin in its interior for each t, and that k is increasing and such that $k-\alpha$ is 2π periodic and $(I-H)(\bar z e^{ik}) = (I-H)(\log\bar z + ik) = 0$. Then*

$$(I-H)k_t = -i(I-H)\frac{\bar z_t\varepsilon}{\bar z} - i[z_t, H]\frac{\left(\log(\bar z e^{ik})\right)_\alpha}{z_\alpha}.$$

$$(I-H)(ak_\alpha) = [z_t, H]\frac{(\bar z_t z)_\alpha}{z_\alpha} - [z_t, H]\bar z_t$$
$$- (I-H)\frac{\bar z_{tt}\varepsilon}{\bar z} + (I-H)\frac{g^h\varepsilon}{\bar z} + [z_{tt} - g^a, H]\frac{\left(\log(\bar z e^{ik})\right)_\alpha}{z_\alpha}.$$

Remark 3.19. The conditions on k in the proposition can be understood in the following way. First note that if we fix a value of $\arg(z(t,0))$ (uniquely determined up to an integer multiple of 2π) then $\log\bar z(t,\cdot)$ is an unambiguously defined *continuous* function of the real variable α for each fixed t. Moreover, if $z(t,\cdot)$ is a simple closed curve surrounding the origin, then by the periodicity assumption on k, the curve $\bar z e^{ik}$ does not contain the origin in its interior. Therefore $\log(\bar z e^{ik})$ is defined unambiguously as a complex logarithm, its value agrees with $\log\bar z + ik$, and for any other choice of $\arg(z(t,0))$ it differs from this by an additive constant, so in particular the condition $(I-H)(\log\bar z + ik) = 0$ is independent of this choice. The conditions on k can now be understood as requiring that $\bar z e^{ik}$ be the boundary value of a holomorphic function F, such that $0\notin\{F(z)|z\in\Omega\}$ and therefore $\log F$ is also well-defined and holomorphic.

Proof of Proposition 3.18. Differentiating $(I-H)\left(\log(\bar{z}e^{ik})\right)=0$ on both sides with respect to t, we have:

$$0=(I-H)\left(\frac{\bar{z}_t}{\bar{z}}+ik_t\right)-[z_t,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha},$$

which implies

$$(I-H)k_t=i(I-H)\left(\frac{\bar{z}_t}{\bar{z}}\right)-i[z_t,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}.$$

In view of the fact $(I-H)(\bar{z}_t z)=0$, the first term on the right hand side above can be written as

$$-i(I-H)\frac{\bar{z}_t\varepsilon}{\bar{z}},$$

which gives the first formula in the proposition. For the second formula, we apply the operator $ia\partial_\alpha$ to the equation $(I-H)\left(\log(\bar{z}e^{ik})\right)=0$, to arrive at

$$0=(I-H)\left(\frac{ia\bar{z}_\alpha}{\bar{z}}-ak_\alpha\right)-i[az_\alpha,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}.$$

Using (3.3) we get

$$\begin{aligned}(I-H)(ak_\alpha)&=(I-H)\left(\frac{\bar{z}_{tt}}{\bar{z}}+\pi-\frac{g^h}{\bar{z}}\right)+[z_{tt}+\pi z-g^a,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}\\&=(I-H)\left(\frac{\bar{z}_{tt}z}{|z|^2}-\frac{zg^h}{|z|^2}\right)+[z_{tt}-g^a,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}\\&=(I-H)(\bar{z}_{tt}z)-(I-H)\frac{\bar{z}_{tt}\varepsilon}{\bar{z}}+(I-H)\frac{g^h\varepsilon}{\bar{z}}+[z_{tt}-g^a,H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}.\end{aligned}$$

Here we used the fact that $(I-H)(zg^h)=0$. The first term above can be written as

$$(I-H)\left((\bar{z}_t z)_t-z_t\bar{z}_t\right)=[z_t,H]\frac{(\bar{z}_t z)_\alpha}{z_\alpha}-[z_t,H]\bar{z}_t,$$

and this completes the proof. □

Now suppose we define k in a way that $(I-H)\bar{z}e^{ik}=0$. Then in view of Proposition 3.18, to prove that b is quadratic and ak_α contains no linear terms we need to understand the invertibility properties of Re $(I-H)$ (note that ak_α and k_t are real). In fact, a proper understanding of this is necessary also for controlling various other quantities, such as ε from our control of δ. A rigorous quantitative treatment of this in the context of small data problem will be given when we carry out the estimates in Sects. 4 and 5, but for now we note that if f is real valued, then regarding the last two terms on the second line of (3.12) in Lemma 3.13 as $O(\varepsilon)$,[3]

$$\mathrm{Re}\,(I-H)f=(I-\frac{1}{2}(H+\overline{H}))f=(I+O(\varepsilon))\,f+\frac{1}{2}\mathrm{Re}\left([z,H]\frac{f}{z}\right).$$

[3] Note that in the static case $z(\alpha)=e^{i\alpha}$ the 'average' $[z,H]\frac{f}{z}$ is real if f is real-valued. We may therefore treat the imaginary part of this average as perturbative.

Therefore, roughly speaking, if ε is small, then we expect Re $(I - H)$ to be invertible on the space of functions in L^2_α which satisfy Re $\mathcal{AV}(f) = 0$, where [4]

$$\mathcal{AV}(f) := \frac{1}{2}[z, H]\frac{f}{z} = -\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\beta)z_\beta(\beta)}{z(\beta)}d\beta. \tag{3.32}$$

With this observation in mind we compute $\mathcal{AV}$ for ε, b, and ak_α in the following proposition.

Proposition 3.20. *Suppose that $z(t,\cdot)$, $t \in I$, for some interval $I \subseteq \mathbb{R}$, is a simple closed curve containing the origin in its interior for each t, $|\Omega| = \pi$, and that $\bar{z}e^{ik}$ is the boundary value of a holomorphic function $F(t,z)$ such that* $\log F(t,z)$ *is also holomorphic and $F(t,0) \in \mathbb{R}_+$, $\forall t \in I$. Then*

$$\begin{aligned}
&\mathcal{AV}(\varepsilon) = 0,\\
&\mathcal{AV}(ak_\alpha) = -\pi + \frac{1}{2\pi i}\int_0^{2\pi} \bar{z}_t z_{t\beta} d\beta + \frac{1}{2\pi i}\int_0^{2\pi} \frac{(\bar{z}_{tt} - g^h)\varepsilon z_\beta}{|z|^2} d\beta\\
&\quad - \frac{1}{2\pi i}\int_0^{2\pi} \left(\frac{z_{tt} - g^a}{z}\right)\partial_\beta \log F d\beta,\\
&\operatorname{Re}\mathcal{AV}(k_t) = \frac{\operatorname{Re}}{2\pi}\int_0^{2\pi} \frac{\bar{z}_t\varepsilon}{|z|^2} z_\beta d\beta - \frac{\operatorname{Re}}{2\pi}\int_0^{2\pi} \log F\left(\frac{z z_{t\beta} - z_t z_\beta}{z^2}\right) d\beta.
\end{aligned}$$

Proof. For ε we have

$$\mathcal{AV}(\varepsilon) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{z_\beta(\beta)}{z(\beta)}d\beta + \frac{1}{2}[z, H]\bar{z} = 0$$

by Lemma 3.9. To compute $\mathcal{AV}(ak_\alpha)$ we write $\bar{z}e^{ik} = F(z)$ where F is as in the statement of the lemma. Differentiating with respect to α and multiplying by ia we get

$$ia\bar{z}_\alpha e^{ik} - ak_\alpha F = iaz_\alpha F_z$$

or

$$ak_\alpha = \frac{ia\bar{z}_\alpha}{\bar{z}} - iaz_\alpha \partial_z \log F.$$

Using Eqs. (3.2) and (3.3) and the relation $\frac{1}{|z|^2} = 1 - \frac{\varepsilon}{|z|^2}$ we get

$$\begin{aligned}
\frac{ak_\alpha}{z} &= \frac{\pi}{z} + \frac{\bar{z}_{tt} - g^h}{|z|^2} + \pi\,\partial_z \log F + \left(\frac{z_{tt} - g^a}{z}\right)\partial_z \log F\\
&= \frac{\pi}{z} + \bar{z}_{tt} - g^h - \frac{(\bar{z}_{tt} - g^h)\varepsilon}{|z|^2} + \pi\,\partial_z \log F + \left(\frac{z_{tt} - g^a}{z}\right)\partial_z \log F.
\end{aligned}$$

[4] Note that the sign convention is such that $\mathcal{AV}(1) = -1$.

It follows that

$$\begin{aligned}\mathcal{AV}(ak_\alpha) &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{ak_\beta z_\beta}{z}d\beta \\ &= -\pi + \frac{1}{2\pi i}\int_0^{2\pi}\overline{z}_t z_{t\beta}d\beta + \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\overline{z}_{tt}-g^h)\varepsilon z_\beta}{|z|^2}d\beta \\ &\quad - \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{z_{tt}-g^a}{z}\right)\partial_\beta \log F d\beta.\end{aligned}$$

The computation for $\mathcal{AV}(k_t)$ is similar. We differentiate the equation $\overline{z}e^{ik} = F(t,z)$ with respect to time to get

$$\overline{z}_t e^{ik} + ik_t F = \partial_t(F)$$

or

$$k_t = \frac{i\overline{z}_t}{\overline{z}} - i\partial_t(\log F).$$

It follows that

$$\frac{k_t}{z}z_\alpha = i\overline{z}_t z_\alpha - \frac{i\overline{z}_t\varepsilon z_\alpha}{|z|^2} - i\partial_t\left(\frac{z_\alpha \log F}{z}\right) + i\log F\left(\frac{zz_{t\alpha}-z_t z_\alpha}{z^2}\right).$$

Therefore, since $\overline{z}_t$ and $\log F$ are holomorphic,

$$\mathcal{AV}(k_t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{z}_t\varepsilon}{|z|^2}z_\beta d\beta - \frac{1}{2\pi}\int_0^{2\pi}\log F\left(\frac{zz_{t\beta}-z_t z_\beta}{z^2}\right)d\beta + \partial_t(i\log F(0,t)).$$

The last equality in the lemma now follows by taking real parts of this expression and noting that $\log F(0,t)\in\mathbb{R},\ \forall t.$ □

It follows from the previous two propositions that if k satisfies the conditions in these propositions then A and b have the desired smallness properties. In the following remark we explain how to construct k satisfying the hypotheses of these propositions. Note that the fact that k is increasing will follow from the definition of k below and the smallness assumptions in our problem. See Proposition 6.1 and the proof of Theorem 6.2 for more details.

Remark 3.21. Suppose $z(t,\cdot)$ is a simple closed curve containing the origin in its simply connected interior for each $t\in I$, where I is some time interval. We explain how to construct a function $k: I\times\mathbb{R}\to\mathbb{R}$ such that $k-\alpha$ is periodic and $\overline{z}e^{ik}$ is the boundary value of a function F such that F and $\log F$ are holomorphic inside Ω, so in particular $(I-H)(\log\overline{z}+ik)=0$. Moreover, we normalize k such that $\log F(t,0)\in\mathbb{R}$ for all $t\in I$.

We fix a choice of the logarithm so that $\log z - i\alpha$ is continuous and 2π periodic. We let u be the solution of the Dirichlet problem in Ω with boundary value $\log|z|$ and let v be the harmonic conjugate of u which exists because the domain is simply connected. It is then easy to see that if $k := v|_{\partial\Omega} + \arg z$ then $\overline{z}e^{ik}$ is the boundary value of a holomorphic function F such that $\log F$ is also holomorphic and $k-\alpha$ is 2π periodic.

It remains to show that k may be chosen such that $\log F(t,0) \in \mathbb{R}$. For this note that $0 \notin \{F(t,z) \mid z \in \Omega\}$ and the function

$$G(t,z) := F(t,z)e^{-i \arg F(t,0)}$$

is also holomorphic and $0 \notin \{G(t,z) \mid z \in \Omega\}$, so $\log G$ is also holomorphic. Moreover, G now satisfies $G(t,0) = |F(t,0)| \in \mathbb{R}$ and the boundary value of G is $\bar{z}e^{ip}$ where $p(t,\alpha) = k(t,\alpha) - \arg F(t,0)$. In other words, we have found p such that $(I-H)(\bar{z}e^{ip}) = (I-H)(\log \bar{z} + ip) = 0$ and $\bar{z}e^{ip}$ is the boundary value of a holomorphic G such that $G(t,0) \in \mathbb{R}$ and $\log G$ is holomorphic.

Corollary 3.22. *Suppose k is defined as in Remark 3.21. Assume also that k is increasing. Then* $\chi := \delta \circ k^{-1}$ *and* $v := \delta_t \circ k^{-1}$ *satisfy the equations*

$$(\partial_t + b\partial_{\alpha'})^2 \chi + iA\partial_{\alpha'}\chi - \pi\chi = N_1 \tag{3.33}$$

and

$$(\partial_t + b\partial_{\alpha'})^2 v + iA\partial_{\alpha'} v - \pi v = N_2 \tag{3.34}$$

where $N_j := \mathcal{N}_j \circ k^{-1}$ *and* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are as defined in Eqs.* (3.23) *and* (3.24) *respectively.*

4. Relations Between Original and Transformed Quantities

In the previous section we derived an equation for the transformed quantities $\delta = (I - H)\varepsilon$ and $\chi = \delta \circ k^{-1}$, defined as in (3.21)–(3.22), where k was chosen according to Remark 3.21. In order to prove energy estimates for this equation it will be important to be able to transfer estimates on δ to estimates on ε and conversely. More precisely we define the following quantities

$$\begin{aligned} &\zeta = z \circ k^{-1}, \quad u = z_t \circ k^{-1}, \quad w = z_{tt} \circ k^{-1} \\ &\chi = \delta \circ k^{-1}, \quad v = (\partial_t \delta) \circ k^{-1} = (\partial_t + b\partial_{\alpha'})\chi, \\ &\mu = \varepsilon \circ k^{-1}, \quad \eta = \zeta_\alpha - i\zeta, \quad \varepsilon = |z|^2 - 1. \end{aligned} \tag{4.1}$$

We will use $\mathcal{H}$ for the Hilbert transform in the variable ζ and H for the Hilbert transform in z. Our goal in this section will be to obtain algebraic and analytic relations between the ‘transformed’ quantities

$$\chi, \ v, \ (\partial_t + b\partial_\alpha)v \tag{4.2}$$

and the ‘original’ quantities

$$\zeta, \ u, \ w, \ \eta, \ \mu. \tag{4.3}$$

Note that by comparison with the static case (where $z \equiv e^{i\alpha}$ and $k \equiv \alpha$) we expect the ‘small quantities’ to be

$$\text{original}: \quad \eta, \ \mu, \ u, \ w, \tag{4.4}$$

$$\text{transformed}: \quad \chi, \ v, \ (\partial_t + b\partial_\alpha)v. \tag{4.5}$$

The analytic relations in this section will be derived under the following bootstrap assumption, where $\ell \geq 5$ is a fixed integer and $M \leq M_0 < \infty$ are small numbers to be fixed:

$$\begin{cases} \sum_{k\leq \ell} \left(\|\partial_{\alpha'}^k w\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^k u\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^k \eta\|_{L^2_{\alpha'}} \right) \leq M < M_0, \\ |\zeta(t,\alpha')|^2 \geq \frac{1}{4} \end{cases} \tag{4.6}$$

for $t \in I$ where I is some interval containing 0.

We start with the following estimates for ζ which will be used in many other computations.

Proposition 4.1. *(1) There exists $\alpha'_0 = \alpha'_0(t)$ such that*

$$\|\zeta(\cdot) - e^{i(\alpha'_0 + \cdot)}\|_{L^\infty_{\alpha'} \cap L^2_{\alpha'}} \leq C\|\eta\|_{L^2_{\alpha'}} \leq CM,$$
$$\|\mu\|_{L^\infty_{\alpha'} \cap L^2_{\alpha'}} \leq C\|\eta\|_{L^2_{\alpha'}} \leq CM.$$

(2) If M_0 in (4.6) is sufficiently small then there exist non-zero constants c and C such that for all $j \leq \ell$ and $k \leq \ell - 1$

$$c \leq \|\partial_{\alpha'}^j \zeta_{\alpha'}\|_{L^2_{\alpha'}}, \|\partial_{\alpha'}^k \zeta_{\alpha'}\|_{L^\infty_{\alpha'}} \leq C.$$

(3) If M_0 in (4.6) is sufficiently small then for all $k \leq \ell$

$$\sum_{j\leq k} \|\partial_{\alpha'}^j \mu_{\alpha'}\|_{L^2_{\alpha'}} \leq C \sum_{j\leq k} \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}, \qquad \sum_{j\leq k-1} \|\partial_{\alpha'}^j \mu_{\alpha'}\|_{L^\infty_{\alpha'}} \leq C \sum_{j\leq k-1} \|\partial_{\alpha'}^j \eta\|_{L^\infty_{\alpha'}}.$$

Proof. (1) Note that since $0 \in \Omega(0)$ and $|\zeta(t,\alpha')| \geq \frac{1}{2}$ for all α', $0 \in \Omega(t)$ as long as the bootstrap assumptions hold. Direct differentiation implies that $f(\alpha) := e^{-i\alpha}\zeta(\alpha)$ satisfies $\|f_{\alpha'}\|_{L^2_{\alpha'}} \leq \|\eta\|_{L^2_{\alpha'}}$. Moreover since the area of Ω is a preserved by the flow, there exists $\gamma \in [0, 2\pi]$ such that $|f(\gamma)| = 1$, or equivalently $f(\gamma) = e^{i\alpha'_0}$ for some α'_0. Now for any other $\alpha' \in [0, 2\pi]$ we have

$$|f(\alpha') - e^{i\alpha'_0}| \leq \int_{\min\{\gamma,\alpha'\}}^{\max\{\gamma,\alpha'\}} |f_{\beta'}(\beta')|d\beta' \leq \sqrt{2\pi}\|\eta\|_{L^2_{\alpha'}},$$

proving the first inequality. The second inequality is a direct consequence of the first and the definition of μ.

(2) From the definition of η we have

$$\partial_{\alpha'}^j \zeta_{\alpha'} = i\partial_{\alpha'}^j \zeta + \partial_{\alpha'}^j \eta.$$

The desired estimates now follow from the previous part by induction on j and use of the Sobolev inequality $\|\partial_{\alpha'}^k \eta\|_{L^\infty_{\alpha'}} \leq C(\|\partial_{\alpha'}^k \eta\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^{k+1} \eta\|_{L^2_{\alpha'}})$.

(3) This estimate is a direct consequence of the previous part and the relation $\mu_{\alpha'} = \overline{\zeta}\eta + \zeta\overline{\eta}$. □

A corollary of Proposition 4.1 is the following result which allows us to use the tools from Sect. 2 in the remainder of this section and in the next section.

Corollary 4.2. *Under the bootstrap assumption* (4.6)*, and if* M_0 *is sufficiently small,*

$$|\zeta(\alpha') - \zeta(\beta')| \geq \frac{1}{10}|e^{i\alpha'} - e^{i\beta'}|.$$

Proof. Since $\zeta(\alpha \pm 2\pi) = \zeta(\alpha)$ and $e^{i(\alpha \pm 2\pi)} = e^{i\alpha}$ it suffices to prove the corollary for α and β such that $|\alpha - \beta| \leq \frac{3\pi}{2}$. Since for this range of α and β we have $|e^{i\alpha} - e^{i\beta}| \gtrsim |\alpha - \beta|$,

$$\begin{aligned}
\left|\zeta(\alpha') - \zeta(\beta')\right| &= \left|\int_{\beta'}^{\alpha'} \zeta_{\alpha''}(\alpha'')d\alpha''\right| = \left|i\int_{\beta'}^{\alpha'} \zeta(\alpha'')d\alpha'' + \int_{\beta}^{\alpha} O(M)d\alpha'\right| \\
&= \left|e^{i\alpha'_0}\int_{\beta'}^{\alpha'} ie^{i\alpha''}d\alpha'' + \int_{\beta'}^{\alpha'} O(M)d\alpha''\right| \geq \left|e^{i\alpha'_0}\left(e^{i\alpha'} - e^{i\beta'}\right)\right| \\
&\quad - \left|\int_{\beta'}^{\alpha'} O(M)d\alpha''\right| \\
&\geq \frac{1}{10}|e^{i\alpha} - e^{i\beta}|.
\end{aligned}$$

if M is sufficiently small. □

As immediate important consequences of Proposition 4.1 and Corollary 4.2 we record the following two corollaries.

Corollary 4.3. *If* M_0 *in* (4.6) *is sufficiently small, then for any* $2 \leq k \leq \ell$*, any* 2π*-periodic function* f*, and with* E *defined as in Lemma* 3.13

$$\sum_{j\leq k} \|\partial_{\alpha'}^j E(f)\|_{L^2_{\alpha'}} \leq C\left(\sum_{j\leq k} \|\partial_{\alpha'}^j \mu\|_{L^2_{\alpha'}}^2\right)\left(\sum_{j\leq k} \|\partial_{\alpha'}^j f\|_{L^2_{\alpha'}}\right) \leq CM^2 \sum_{j\leq k} \|\partial_{\alpha'}^j f\|_{L^2_{\alpha'}}.$$

Proof. From the definition of E this is a direct corollary of Proposition 2.2 and Lemma 2.4. Notice that by Proposition 4.1 $\|\partial_{\alpha'}^{\ell+1}\mu\|_{L^2_{\alpha'}} \leq CM$ and $1 \lesssim \|\partial_{\alpha'}^{\ell+1}\zeta\|_{L^2_{\alpha'}} \lesssim 1$ under the bootstrap assumptions (4.6). □

Corollary 4.4. *If* M_0 *in* (4.6) *is sufficiently small, then for any* $2 \leq k \leq \ell$*,*

$$\sum_{j\leq k+1} \|\partial_{\alpha'}^j \chi\|_{L^2_{\alpha'}} \leq C\sum_{j\leq k} \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}.$$

Proof. Since $\chi = (I - \mathcal{H})\mu$, this follows from Lemma 2.6 and Proposition 4.1. □

Next we record the following algebraic relations.

Proposition 4.5. *With the same notation as* (4.1)

$$\partial_{\alpha'}\chi = \left(I - \zeta_{\alpha'}\mathcal{H}\frac{1}{\zeta_{\alpha'}}\right)\mu_{\alpha'} \tag{4.7}$$

$$v = (\partial_t + b\partial_{\alpha'})\chi = 2u\overline{\zeta} - \left(\mathcal{H} + \overline{\mathcal{H}}\right)u\overline{\zeta} - [u, \mathcal{H}]\frac{\mu_{\alpha'}}{\zeta_{\alpha'}}, \tag{4.8}$$

$$(\partial_t + b\partial_{\alpha'})v = 2w\overline{\zeta} + 2u\overline{u} - (\mathcal{H} + \overline{\mathcal{H}})(w\overline{\zeta} + u\overline{u}) - [u, \mathcal{H}]\frac{(u\overline{\zeta})_{\alpha'}}{\zeta_{\alpha'}} - [\overline{u}, \overline{\mathcal{H}}]\frac{(u\overline{\zeta})_{\alpha'}}{\overline{\zeta}_{\alpha'}}$$
$$- [w, \mathcal{H}]\frac{\mu_{\alpha'}}{\zeta_{\alpha'}} - [u, \mathcal{H}]\frac{(u\overline{\zeta} + \overline{u}\zeta)_{\alpha'}}{\zeta_{\alpha'}}$$
$$- \frac{1}{\pi i}\int_0^{2\pi} \left(\frac{u(t,\alpha') - u(t,\beta')}{\zeta(t,\beta') - \zeta(t,\alpha')}\right)^2 \mu_{\beta'}(t,\beta')d\beta'. \tag{4.9}$$

Proof. First

$$\begin{aligned}
\partial_{\alpha'}\chi &= \left(\zeta\overline{\zeta} - 1\right)_{\alpha'} - \partial_{\alpha'}\left(\mathcal{H}(\zeta\overline{\zeta} - 1)\right) \\
&= (I - \mathcal{H})\left(\zeta\overline{\zeta} - 1\right)_{\alpha'} - [\zeta_{\alpha'}, \mathcal{H}]\,\frac{\left(\zeta\overline{\zeta} - 1\right)_{\alpha'}}{\zeta_{\alpha'}} \\
&= \left(I - \zeta_{\alpha'}\mathcal{H}\frac{1}{\zeta_{\alpha'}}\right)\left((\zeta\overline{\zeta} - 1)_{\alpha'}\right).
\end{aligned}$$

Composing with k^{-1} we get the first identity. Similarly

$$\begin{aligned}
\partial_t\delta &= \partial_t(I - H)(z\overline{z} - 1) = (z\overline{z} - 1)_t - \partial_t\left(H(z\overline{z} - 1)\right) \\
&= (z\overline{z} - 1)_t - H\left((z\overline{z} - 1)_t\right) - [z_t, H]\frac{(z\overline{z} - 1)_\alpha}{z_\alpha} \\
&= z_t\overline{z} - H(z_t\overline{z}) - [z_t, H]\frac{(z\overline{z} - 1)_\alpha}{z_\alpha} \\
&= 2z_t\overline{z} - \left(H + \overline{H}\right)(z_t\overline{z}) - [z_t, H]\frac{(z\overline{z} - 1)_\alpha}{z_\alpha}.
\end{aligned}$$

To derive the third formula, we need to compute a time derivative as follows:

$$\begin{aligned}
\partial_t\left([z_t, H]\frac{f_\alpha}{z_\alpha}\right) &= [z_{tt}, H]\left(\frac{f_\alpha}{z_\alpha}\right) + [z_t, H]\frac{f_{t\alpha}}{z_\alpha} \\
&\quad + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta) - z_t(\alpha)}{z(\beta) - z(\alpha)}\right)^2 f_\beta(\beta)d\beta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_t^2\delta &= 2z_{tt}\overline{z} + 2z_t\overline{z}_t - (H + \overline{H})(z_{tt}\overline{z} + z_t\overline{z}_t) \\
&\quad - [z_t, H]\frac{(z_t\overline{z})_\alpha}{z_\alpha} - [\overline{z}_t, \overline{H}]\frac{(z_t\overline{z})_\alpha}{\overline{z}_\alpha} \\
&\quad - [z_t, H]\frac{(z_t\overline{z} + z\overline{z}_t)_\alpha}{z_\alpha} - [z_{tt}, H]\frac{(z\overline{z})_\alpha}{z_\alpha} \\
&\quad - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta) - z_t(\alpha)}{z(\beta) - z(\alpha)}\right)^2 (z\overline{z})_\beta d\beta.
\end{aligned}$$

The third formula follows by precomposing with k^{-1}. □

The estimates for u and w are given in the following proposition.

Proposition 4.6. *If M_0 in (4.6) is sufficiently small, then there are non-zero constants c and C such that for any $2 \le k \le \ell$*

$$c\sum_{j\le k}\|\partial_{\alpha'}^j v\|_{L^2_{\alpha'}}^2 \le \sum_{j\le k}\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}^2 \le C\sum_{j\le k}\|\partial_{\alpha'}^j v\|_{L^2_{\alpha'}}^2, \tag{4.10}$$

$$\left|\sum_{j\le k}\|\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}} - \sum_{j\le k}\|\partial_\alpha^j w\|_{L^2_{\alpha'}}\right| \le C\sum_{j\le k}\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}^2. \tag{4.11}$$

In particular

$$c\sum_{j\le k}(\|\partial_{\alpha'}^j u\|_{L^2_\alpha} + \|\partial_\alpha^j w\|_{L^2_{\alpha'}}) \le C\sum_{j\le k}(\|\partial_{\alpha'}^j v\|_{L^2_\alpha} + \|\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}})$$
$$\le C\sum_{j\le k}(\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}}).$$

Proof. First we prove (4.10). We begin by rewriting (4.8) as

$$\begin{cases} 2u = \frac{v}{\zeta} + \frac{\zeta}{\zeta}[\mu,\mathcal{H}]\frac{\partial_{\alpha'}(u\bar{\zeta})}{\zeta_{\alpha'}} + \frac{1}{\zeta}[u,\mathcal{H}]\frac{\mu_{\alpha'}}{\zeta_{\alpha'}} + \frac{1}{\zeta}E(u\bar{\zeta}) - \frac{2}{\zeta}\mathcal{AV}(u\bar{\zeta}) \\ v = 2u\bar{\zeta} - \zeta[\mu,\mathcal{H}]\frac{\partial_{\alpha'}(u\bar{\zeta})}{\zeta_{\alpha'}} - [u,\mathcal{H}]\frac{\mu_{\alpha'}}{\zeta_{\alpha'}} - E(u\bar{\zeta}) + 2\mathcal{AV}(u\bar{\zeta}) \end{cases}. \tag{4.12}$$

Now we estimate the terms above in $H^k_{\alpha'}$. First note that by Proposition 4.1 and Sobolev

$$\sum_{j\le k}\|\partial_{\alpha'}^j \frac{v}{\zeta}\|_{L^2_{\alpha'}} \lesssim \sum_{j\le k}\|\partial_{\alpha'}^j v\|_{L^2_{\alpha'}}, \qquad \sum_{j\le k}\|\partial_{\alpha'}^j(u\bar{\zeta})\|_{L^2_{\alpha'}} \lesssim \sum_{j\le k}\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}},$$

so it suffices to bound the contribution of all other terms on the right hand sides by $M\sum_{j\le k}\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}$. The contribution of $E(u\bar{\zeta})$ is already handled in Corollary 4.3. For $\mathcal{AV}(u\bar{\zeta})$ note that since u is anti-holomorphic inside Ω

$$\int_0^{2\pi}\frac{u\bar{\zeta}\zeta_{\alpha'}}{\zeta}d\alpha' = \int_0^{2\pi}\frac{u\mu_{\alpha'}}{\zeta}d\alpha' - \int_0^{2\pi}u\bar{\zeta}_{\alpha'}d\alpha' = \int_0^{2\pi}\frac{u\mu_{\alpha'}}{\zeta}d\alpha',$$

which is bounded by $M\|u\|_{L^\infty}$ (note that $\mathcal{AV}(u\bar{\zeta})$ is a constant as a function of α). The contribution of the other terms is handled by Lemma 2.5.

For (4.11) we use (4.9) and a similar argument as for the proof of (4.10) to bound the contributions of the last integral in (4.9), $|u|^2$, $[u,\mathcal{H}]\frac{(u\bar{\zeta})_\alpha}{\zeta_\alpha}$, and $[\bar{u},\bar{\mathcal{H}}]\frac{(u\bar{\zeta})_\alpha}{\bar{\zeta}_\alpha}$ by $\sum_{j\le k}\|\partial_\alpha^j u\|_{L^2_\alpha}^2$. The contribution of $[w,\mathcal{H}]\frac{\mu_\alpha}{\zeta_\alpha}$ is bounded by $M\sum_{j\le k}\|\partial_\alpha^j w\|_{L^2_\alpha}$, by Lemma 2.5. Finally, applying the identity

$$(\mathcal{H}+\bar{\mathcal{H}})f = -2\mathcal{AV}(f) + E(f) + \zeta[\mu,\mathcal{H}]\frac{f_\alpha}{\zeta_\alpha}$$

to $f = w\bar{\zeta}$ and $f = |u|^2$ and using similar arguments as above we can estimate the contribution of $(\mathcal{H}+\bar{\mathcal{H}})(w\bar{\zeta} + |u|^2)$ by

$$M\sum_{j\le k}\|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}} + \sum_{j\le k}\|\partial_\alpha^j u\|_{L^2_{\alpha'}}^2.$$

Here to estimate $\mathcal{AV}(w\overline{\zeta})$ we have noted that

$$\int_0^{2\pi} \frac{w\overline{\zeta}\zeta_{\alpha'}}{\zeta} d\alpha' = \int_0^{2\pi} \frac{w\mu_{\alpha'}}{\zeta} d\alpha' - \int_0^{2\pi} w\overline{\zeta}_{\alpha'} d\alpha' = \int_0^{2\pi} \frac{w\mu_{\alpha'}}{\zeta} d\alpha' - \int_0^{2\pi} u_{\alpha'}\overline{u}\, d\alpha',$$

where for the last equality we have written $z_t = F(t, \overline{z})$ for some anti-holomorphic function F to get $z_{tt} = F_t + F_{\overline{z}}\overline{z}_t = F_t + \frac{z_{t\alpha}\overline{z}_t}{\overline{z}_\alpha}$. The desired estimates now follow from the bootstrap assumptions (4.6) if M_0 is sufficiently small. Note that the term $\int_0^{2\pi} u_{\alpha'}\overline{u} d\alpha'$ is bounded by $\sum_{j\le k} \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}^2$ because it does not depend on α'. Therefore it vanishes when the spatial derivatives hit it. □

Our next goal is to estimate η and its higher derivatives. To this end we rearrange Eq. (3.2) to get

$$\pi\eta = iw - (A - \pi)\zeta_{\alpha'} - ig^a \circ k^{-1}. \tag{4.13}$$

To use this equation we first need to estimate $A - \pi$. This is accomplished in the next proposition.

Proposition 4.7. *If M_0 in (4.6) is sufficiently small then for any $2 \le k \le \ell$*

$$\sum_{j\le k} \|\partial_{\alpha'}^j (A-\pi)\|_{L^2_{\alpha'}} \le C \left(\sum_{j\le k} \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}^2 + \sum_{j\le k} \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}^2 + \sum_{j\le k} \|\partial_{\alpha'}^j w\|_{L^2_\alpha} \sum_{j\le k} \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}} \right).$$

Proof. Using Propositions 3.18 and 3.20 and the fact that $\partial_{\alpha'} \log(\overline{\zeta} e^{i\alpha'}) = \frac{\overline{\eta}}{\overline{\zeta}}$ we write

$$\begin{cases} (I-\mathcal{H})(A-\pi) = [u,\mathcal{H}]\frac{(\overline{u}\zeta)_{\alpha'}}{\zeta_{\alpha'}} - [u,\mathcal{H}]\overline{u} - (I-\mathcal{H})\frac{\overline{w}\mu}{\overline{\zeta}} + (I-\mathcal{H})\frac{\mu g^h\circ k^{-1}}{\overline{\zeta}} + [w - g^a\circ k^{-1}, \mathcal{H}]\frac{\overline{\eta}}{\zeta_{\alpha'}\overline{\zeta}} \\ \mathcal{AV}(A-\pi) = \frac{1}{2\pi i}\int_0^{2\pi} \overline{u} u_{\beta'} d\beta' + \frac{1}{2\pi i}\int_0^{2\pi} \frac{(\overline{w} - g^h\circ k^{-1})\mu\zeta_{\beta'}}{|\zeta|^2} d\beta' - \frac{1}{2\pi i}\int_0^{2\pi} \frac{\eta(w - g^h\circ k^{-1})}{|\zeta|^2} d\beta' \end{cases}. \tag{4.14}$$

Now since A is real

$$\begin{aligned} A - \pi &= \mathrm{Re}\,(I-\mathcal{H})(A-\pi) + \frac{1}{2}(\mathcal{H}+\overline{\mathcal{H}})(A-\pi) \\ &= \mathrm{Re}\,(I-\mathcal{H})(A-\pi) + \frac{1}{2}\left(\zeta[\mu,\mathcal{H}]\frac{A_{\alpha'}}{\zeta_{\alpha'}} + E(A-\pi) - 2\mathcal{AV}(A-\pi) \right). \end{aligned} \tag{4.15}$$

Moreover, by Lemma 3.13 we can write

$$g^a \circ k^{-1} = \overline{g^h \circ k^{-1}} = \frac{\pi}{2}(\mathcal{H}+\overline{\mathcal{H}})\zeta = \frac{\pi}{2}\zeta\chi + \frac{\pi}{2}E(\zeta).$$

Notice that by Corollary 4.3 and Proposition 4.1

$$\sum_{j\le k} \|\partial_{\alpha'}^j E(\zeta)\|_{L^2_{\alpha'}} \le C \sum_{j\le k} \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}^2,$$

and by Proposition 4.1 if M_0 in (4.6) is sufficiently small

$$\sum_{j\le k}\|\partial_{\alpha'}^j(g^a\circ k^{-1})\|_{L^2_{\alpha'}}\le C\sum_{j\le k}\|\partial_{\alpha'}^j\eta\|_{L^2_{\alpha'}}.$$

It follows from this, Proposition 4.1, Lemma 2.5, and (4.14) that

$$\sum_{j\le k}\|\partial_{\alpha'}^j\mathrm{Re}\,(I-\mathcal{H})(A-\pi)\|_{L^2_{\alpha'}}+\|\mathcal{A}\mathcal{V}(A-\pi)\|_{L^2_{\alpha'}}$$
$$\le C\left(\sum_{j\le k}\|\partial_{\alpha'}^j u\|^2_{L^2_{\alpha'}}+\sum_{j\le k}\|\partial_{\alpha'}^j\eta\|^2_{L^2_{\alpha'}}+\sum_{j\le k}\|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}}\sum_{j\le k}\|\partial_{\alpha'}^j\eta\|_{L^2_{\alpha'}}\right).$$

Similarly the bootstrap assumptions give

$$\sum_{j\le k}\left(\|\partial_{\alpha'}^j(\zeta[\mu,\mathcal{H}]\frac{A_{\alpha'}}{\zeta_{\alpha'}})\|_{L^2_{\alpha'}}+\|\partial_{\alpha'}^j E(A-\pi)\|_{L^2_{\alpha'}}\right)\le CM\sum_{j\le k}\|\partial_{\alpha'}^j(A-\pi)\|_{L^2_{\alpha'}}.$$

Combining these estimates with (4.15) we arrive at the desired conclusion if M is sufficiently small. □

We now go back to the analysis of Eq. (4.13). As observed in the proof of Proposition 4.7 we can write $g^a\circ k^{-1}=\frac{\pi}{2}(\zeta\chi+E(\zeta))$. This shows that Eq. (4.13) by itself is not enough to obtain estimates on η and its higher derivatives in terms of $(\partial_t+b\partial_{\alpha'})\chi$ and $(\partial_t+b\partial_{\alpha'})v$ and their higher derivatives. To get such estimates we will also need to use the original Eq. (3.33), which in turn requires estimates on the right hand side of (3.33). These estimates are also of independent interest in proving energy estimates, so before stating the final estimates for η we state the following estimates on the right hand sides of the Eqs. (3.33) and (3.34).

Proposition 4.8. *Let N_1 and N_2 be as in Corollary 3.22. Then if M_0 in (4.6) is sufficiently small, for any $3\le k\le\ell$*

$$\sum_{j\le k}\left(\|\partial_{\alpha'}^j N_1\|_{L^2_{\alpha'}}+\|\partial_{\alpha'}^j N_2\|_{L^2_{\alpha'}}\right)\lesssim\sum_{j\le k}\left(\|\partial_{\alpha'}^j\eta\|_{L^2_{\alpha'}}+\|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}+\|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}}\right)^3 \tag{4.16}$$

Proof. We begin with N_1. Using Lemmas 2.5 and 2.6 and Corollary 4.3 we can bound the contributions of the first two terms on the right hand side of (3.23) by the right hand side of (4.16). Similarly, in view of Eq. (3.25), the contributions of the last two terms on the right hand side of (3.23) can be bounded by the right hand side of (4.16) by using Lemma 2.4 and Propositions 2.2 and 2.3. This completes the estimates for N_1. The contribution of N_2 can be treated in a similar way. Indeed except for the first term on the right hand side of (3.24) all other terms can be estimated by similar arguments as above using Propositions 2.2 and 2.3 and Lemmas 2.4, 2.5, and 2.6. Here we will also use the observations that

$$\partial_t\left(\frac{A(t,\alpha)-A(t,\beta)}{z(t,\beta)-z(t,\alpha)}\right)=\frac{A_t(t,\alpha)-A_t(t,\beta)}{z(t,\beta)-z(t,\alpha)}-\frac{(A(t,\alpha)-A(t,\beta))(z_t(t,\beta)-z_t(t,\alpha))}{(z(t,\beta)-z(t,\alpha))^2}$$

and $\partial_t \varepsilon = z_t \bar{z} + z\bar{z}_t$. We omit the details. Finally the term a_t is treated independently in the proof of Lemma 5.12 below.[5] □

Using Eq. (4.13), we can now combine Propositions 4.6, 4.7, and 4.8 to prove the following proposition.

Proposition 4.9. *If M_0 in (4.6) is sufficiently small then for any $3 \le k \le \ell$,*

$$\sum_{j\le k}\left(\|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}\right) \le C\sum_{j\le k}\left(\|\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})\chi\|_{L^2_{\alpha'}}\right).$$

Proof. In view of Proposition 4.6 we only need to prove this estimate for η. From Proposition 4.7 we know that $(A-\pi)$ is quadratic. In Eq. (4.13), using Lemma 3.13 we can write the term $g^a \circ k^{-1}$ as

$$g^a \circ k^{-1} = \frac{\pi}{2}(I + \overline{\mathcal{H}})\zeta = \frac{\pi}{2}(\mathcal{H} + \overline{\mathcal{H}})\zeta = \frac{\pi}{2}\zeta\chi + E(\zeta)$$

and Eq. (4.13) can be written as

$$\pi\eta = iw - \frac{\pi i}{2}\zeta\chi - \frac{\pi i}{2}E(\zeta) - (A-\pi)\zeta_{\alpha'} \tag{4.17}$$

The arguments for estimating η itself and its derivatives are different. For η we use Eq. (3.33) and the definition of $v := (\partial_t + b\partial_{\alpha'})\chi$ to get

$$\begin{cases}\pi\chi_{\alpha'} + i\pi\chi = i\,(\partial_t + b\partial_{\alpha'})\,v - (A-\pi)\partial_{\alpha'}\chi - iN_1\\ N_1 := (\partial_t + b\partial_{\alpha'})^2\chi + iA\partial_{\alpha'}\chi - \pi\chi\end{cases}. \tag{4.18}$$

For higher derivatives of η we instead use the following system which is obtained by differentiating (4.17) and the second equation in (4.18)

$$\begin{cases}\pi\partial_{\alpha'}^\ell\eta = i\partial_{\alpha'}^\ell w - \frac{\pi i}{2}\partial_{\alpha'}^{\ell-1}\partial_{\alpha'}(\zeta\chi) - \frac{\pi i}{2}\partial_{\alpha'}^\ell E(\zeta) - \partial_{\alpha'}^\ell((A-\pi)\zeta_{\alpha'}), \quad \ell\ge 1\\ \partial_{\alpha'}(\zeta\chi) = \zeta(\chi_{\alpha'} + i\chi) + \eta\chi = \zeta(\frac{i}{\pi}(\partial_t + b\partial_{\alpha'})v - \frac{1}{\pi}(A-\pi)\chi_{\alpha'} - \frac{i}{\pi}N_1) + \eta\chi\end{cases}. \tag{4.19}$$

We start with the estimates for η itself. In view of Eq. (4.17), we need to obtain an estimate for $\zeta\chi$. On the other hand, by Propositions 4.1, 4.7, 4.8 and Corollary 4.4, the second equation in (4.19) gives us an estimate for $\partial_{\alpha'}(\zeta\chi)$:

[5] We note that the treatment in Lemma 5.12 does not rely on the validity of Proposition 4.8. In fact we only use the estimates for N_1 in the proof of Proposition 4.9 below and the proof of the estimates for a_t in Lemma 5.12 are even independent of this proposition.

$$\|\partial_{\alpha'}(\zeta\chi)\|_{L^2_{\alpha'}} \lesssim \|(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}} + M\|\eta\|_{L^2_{\alpha'}} + M^2\sum_{j\le 3}\left(\|\partial^j_{\alpha'}u\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}w\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}\eta\|_{L^2_{\alpha'}}\right). \tag{4.20}$$

In order to obtain the L^2-estimate for $\zeta\chi$, we still need to know the value of $\zeta\chi$ at least at one point. Note that by Proposition 3.5

$$\int_0^{2\pi} \zeta\chi \cdot \frac{\zeta_{\alpha'}}{\zeta} d\alpha' = 0.$$

Therefore

$$\int_0^{2\pi} \zeta\chi d\alpha' = i\int_0^{2\pi} \chi\eta d\alpha',$$

from which we have

$$\left|\int_0^{2\pi} \mathrm{Re}\,(\zeta\chi)d\alpha'\right|, \quad \left|\int_0^{2\pi} \mathrm{Im}\,(\zeta\chi)d\alpha'\right| \lesssim M\|\eta\|_{L^2_{\alpha'}}.$$

These together with (4.20) imply that

$$\|\zeta\chi\|_{L^2_{\alpha'}} \lesssim \|(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}} + M\|\eta\|_{L^2_{\alpha'}} + M^2\sum_{j\le 3}\left(\|\partial^j_{\alpha}u\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}w\|_{L^2_{\alpha}} + \|\partial^j_{\alpha'}\eta\|_{L^2_{\alpha'}}\right). \tag{4.21}$$

Substituting this into (4.17), using Corollary 4.3 and Proposition 4.7, and taking $M > 0$ sufficiently small we get

$$\|\eta\|_{L^2_{\alpha'}} \lesssim \|(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}} + \|(\partial_t + b\partial_{\alpha'})\chi\|_{L^2_{\alpha'}} + M^2\sum_{j\le 3}\|\partial^j_{\alpha'}\eta\|_{L^2_{\alpha'}} + M^2\sum_{j\le 3}\left(\|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})\chi\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})v\|_{L^2_{\alpha'}}\right). \tag{4.22}$$

Finally applying Propositions 4.7 and 4.8 and Corollary 4.3 to Eq. (4.19), for and $3 \le k \le \ell$, we get the bound

$$\sum_{1\le j\le k}\|\partial^j_{\alpha'}\eta\|^2_{L^2_{\alpha'}} \lesssim \sum_{j\le k}\left(\|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})\chi\|^2_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})\chi\|^2_{L^2_{\alpha'}}\right) + M\sum_{j\le k}\|\partial^j_{\alpha'}\eta\|^2_{L^2_{\alpha'}}.$$

Combining this with (4.22) and choosing M sufficiently small gives

$$\sum_{j\le k}\|\partial^j_{\alpha'}\eta\|^2_{L^2_{\alpha'}} \lesssim \sum_{j\le k}\left(\|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})\chi\|^2_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})\chi\|^2_{L^2_{\alpha'}}\right),$$

which completes the proof of the proposition. □

The next step in our analysis is to obtain estimates for quantities of the form $\|\partial^j_{\alpha'}(\partial_t + b\partial_{\alpha'})f\|_{L^2_{\alpha'}}$ in terms of $\|(\partial_t + b\partial_{\alpha'})\partial^j_{\alpha}f\|_{L^2_{\alpha'}}$, which in turn will be bounded by the higher order energies to be defined in the next section. For this we first obtain estimates on b and its derivatives.

Proposition 4.10. *If M_0 in (4.6) is sufficiently small then for $2 \leq k \leq \ell$*

$$\sum_{j\leq k} \|\partial_{\alpha'}^j b\|_{L^2_{\alpha'}} \leq CM \sum_{j\leq k} \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}}.$$

Proof. The proof is similar to that of Proposition 4.7. Recall from Propositions 3.18 and 3.20 that

$$\begin{cases} b = \mathrm{Re}\{-i(I-\mathcal{H})\frac{\overline{u}\chi}{\overline{\zeta}} - i[u,\mathcal{H}]\frac{\overline{\eta}}{\overline{\zeta}\zeta_{\alpha'}}\} + \frac{1}{2}\zeta[\mu,\mathcal{H}]\frac{b_{\alpha'}}{\zeta_{\alpha'}} + \frac{1}{2}E(b) - \mathcal{AV}(b) \\ \mathcal{AV}(b) = \frac{\mathrm{Re}}{2\pi}\int_0^{2\pi} \frac{\overline{u}\chi}{|\zeta|^2}\zeta_\beta d\beta + \frac{\mathrm{Re}}{2\pi}\int_0^{2\pi} \frac{u\overline{\eta}}{|\zeta|^2} d\beta. \end{cases}.$$

The Proposition now follows from similar arguments as those in the proof of Proposition 4.7. □

An important corollary of Propositions 4.9 and 4.10 is the following result.

Corollary 4.11. *If M_0 in (4.6) is sufficiently small then for $2 \leq k \leq \ell$,*

$$\sum_{j\leq k}\left(\|\partial_\alpha^j w\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j \eta\|_{L^2_{\alpha'}}\right) \leq C\sum_{j\leq k}\left(\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j v\|_{L^2_{\alpha'}} + \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j \chi\|_{L^2_{\alpha'}}\right).$$

Proof. We first note that for any function f

$$\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})f = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j f + \sum_{1\leq i\leq j}\binom{j}{i}\partial_{\alpha'}^i b\,\partial_{\alpha'}^{j+1-i} f,$$

and therefore by Sobolev

$$\|\partial_{\alpha'}^j(\partial_t + b\partial_{\alpha'})f\|_{L^2_{\alpha'}} \leq \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j f\|_{L^2_{\alpha'}} + c\sum_{i\leq \max\{2,j\}}\|\partial_{\alpha'}^i b\|_{L^2_{\alpha'}}\sum_{i\leq j}\|\partial_{\alpha'}^i f\|_{L^2_{\alpha'}}.$$

Summing this estimate over $j \leq k$ for $f = \chi$ and $f = v$ and using Propositions 4.6, 4.9 and 4.10, Corollary 4.4, and the bootstrap assumption (4.6) we get the desired result. □

5. Energy Estimates

In this section we define the energy and prove energy estimates for Eqs. (3.33) and (3.34). The main energy estimates are stated in Proposition 5.15 below. We consider an equation of the form

$$((\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi)\Theta = G. \tag{5.1}$$

In most applications Θ will be the boundary value of a holomorphic function outside Ω decaying to zero as $|z| \to \infty$, that is, $\Theta = (I-\mathcal{H})f$ for some f. More precisely, the relevant choices of Θ are $\chi = (I-\mathcal{H})\mu$ and $v = (\partial_t + b\partial_{\alpha'})\chi$. Since v cannot be written as $(I-\mathcal{H})f$ we define the new unknown

$$\tilde{v} := (I-\mathcal{H})v. \tag{5.2}$$

Associated to (5.1) we define the following basic energy

$$E_0^{\Theta} := \int_0^{2\pi} \frac{|(\partial_t + b\partial_{\alpha'})\Theta|^2}{A} d\alpha' + \int_0^{2\pi} \left(i\Theta_{\alpha'}\overline{\Theta} - \frac{\pi}{A}|\Theta|^2 \right) d\alpha' =: \mathcal{E}_0^{\Theta} + \mathcal{F}_0^{\Theta}.$$

and for the choices of Θ above we let

$$E_0(\chi) := \mathcal{E}_0(\chi) + \mathcal{F}_0(\chi) := \mathcal{E}_0^{\chi} + \mathcal{F}_0^{\chi} = E_0^{\chi},$$
$$E_0(v) := \mathcal{E}_0(v) + \mathcal{F}_0(v) := \mathcal{E}_0^{\tilde{v}} + \mathcal{F}_0^{\tilde{v}} = E_0^{\tilde{v}}.$$

We will show below that if $\Theta = (I - \mathcal{H})f$ for some f then $i\int_0^{2\pi} \Theta_\alpha \overline{\Theta} d\alpha'$ is non-negative. It is not, however, in general true that $\mathcal{F}_0^{\Theta}$ is non-negative even if $\Theta = (I - \mathcal{H})f$, but this is the case if $\partial\Omega$ is a an exact circle. This can be seen by noting that in this case the Fourier expansion of Θ contains only negative frequencies if $\Theta = (I - \mathcal{H})f$, and then carrying out the integration on the frequency side after an application of Plancherel. Therefore, we expect that for small data where $\partial\Omega$ is nearly a circle, $\mathcal{F}_0^{\Theta}$ can be written as a positive term plus 'higher order terms.' This can be achieved for instance by writing $\mathcal{H}$ as the Hilbert transform on the circle plus an error. While this intuition is helpful, we will not use this argument in our applications, but instead explicitly decompose $\mathcal{F}_0^{\Theta}$ as the sum of a positive term and a 'higher order' difference in terms of known quantities for choices of Θ that interest us. We will postpone this computation to after defining the higher order energies and now only prove the following general estimate.

Lemma 5.1. *The integral $i\int_0^{2\pi} \Theta_\alpha \overline{\Theta} d\alpha'$ is real and if $\Theta = (I - \mathcal{H})f$ for some 2π-periodic function f, then*

$$i\int_0^{2\pi} \Theta_{\alpha'}\overline{\Theta} d\alpha' \geq 0.$$

Remark 5.2. Note that this lemma does not apply to the choice $\Theta = v$, which is why we have replaced v by $\tilde{v}$ in the definition of $E_0(v)$.

Proof. Integration by parts shows that the integral $i\int_0^{2\pi} \Theta_{\alpha'}\overline{\Theta} d\alpha'$ is equal to its conjugate and is therefore real, and hence

$$i\int_0^{2\pi} \Theta_{\alpha'}\overline{\Theta} d\alpha' = \mathrm{Re}\left\{ i\int_0^{2\pi} \frac{\Theta_{\alpha'}}{|\zeta_{\alpha'}|}\overline{\Theta}|\zeta_{\alpha'}| d\alpha' \right\}.$$

Now note that if $\Theta = (I - \mathcal{H})f$ then by Proposition A.1 we can write Θ as the boundary value of a function F which is holomorphic in Ω^c and decays as $|\zeta|^{-1}$ when $|\zeta| \to \infty$. A simple computation using the holomorphicity of F in Ω^c and the Cauchy–Riemann equations gives

$$\mathrm{Re}\left\{ \frac{i\Theta_{\alpha'}\overline{\Theta}}{|\zeta_{\alpha'}|} \right\} = -\langle F, \frac{\partial F}{\partial n} \rangle$$

where $\mathbf{n} := -\frac{iz_\alpha}{|z_\alpha|}$ is the exterior normal of Ω and $\langle F, G\rangle := f_1 g_1 + f_2 g_2$ for complex numbers $F = f_1 + if_2$ and $G = g_1 + ig_2$. From Green's formula and with ds denoting the arc-length measure we get

$$i\int_0^{2\pi} \Theta_{\alpha'}\overline{\Theta} d\alpha' = -\int_{\partial\Omega^c} \langle F, \frac{\partial F}{\partial n} \rangle ds = \iint_{\Omega^c} |\nabla F|^2 dx dy \geq 0,$$

where we have used the decay properties of F stated above to justify the use of Green's formula. □

With this basic positivity estimate in place we turn to the following energy identity for E_0^{Θ}.

Lemma 5.3. *Suppose* Θ *satisfies Eq.* (5.1). *Then*

$$\frac{d}{dt}E_0^{\Theta} = 2\int_0^{2\pi} \frac{1}{A}\mathrm{Re}\,\{G(\partial_t + b\partial_{\alpha'})\overline{\Theta}\}d\alpha' \\ - \int_0^{2\pi} \left(\frac{1}{A}\frac{a_t}{a}\circ k^{-1}\right)\left(|(\partial_t + b\partial_{\alpha'})\Theta|^2 - \pi|\Theta|^2\right)d\alpha'.$$

Remark 5.4. Note that if $\Theta = \chi$ or v, then by the results of Sect. 3 the first integral on the right hand side above is of 'order four'. However, in the definition of $E_0(v)$ we have used $\Theta = \tilde{v}$, so to have this smallness we still need to show that $\tilde{v}$ satisfies a 'cubic' equation. This will be accomplished in Proposition 5.11 below.

Proof. Precomposing with k we can rewrite (5.1) as

$$(\partial_t^2 + ia\partial_\alpha - \pi)\theta = g, \qquad \theta := \Theta\circ k,\ g = G\circ k. \tag{5.3}$$

Then

$$E_0^{\Theta} = \int_0^{2\pi} \frac{|\partial_t\theta|^2}{a}d\alpha + \int_0^{2\pi}\left(i\overline{\theta}\partial_\alpha\theta - \frac{\pi}{a}|\theta|^2\right)d\alpha. \tag{5.4}$$

It follows that

$$\frac{d}{dt}E_0^{\Theta} = \int_0^{2\pi}\frac{2}{a}\mathrm{Re}\,\{\partial_t^2\theta\partial_t\overline{\theta}\}d\alpha - \int_0^{2\pi}\frac{a_t}{a^2}|\partial_t\theta|^2d\alpha \\ + \int_0^{2\pi} i\partial_\alpha\theta\partial_t\overline{\theta}d\alpha + \int_0^{2\pi} i\overline{\theta}\partial_{t\alpha}^2\theta d\alpha \\ - \int_0^{2\pi}\frac{2\pi}{a}\mathrm{Re}\,\{\theta\partial_t\overline{\theta}\}d\alpha + \int_0^{2\pi}\frac{\pi a_t}{a^2}|\theta|^2d\alpha \\ = 2\int_0^{2\pi}\frac{1}{a}\mathrm{Re}\,\{g\partial_t\overline{\theta}\}d\alpha - \int_0^{2\pi}\frac{a_t}{a^2}(|\partial_t\theta|^2 - \pi|\theta|^2)d\alpha.$$

Composing back with k^{-1} we get the desired identity. □

We now turn to the higher energy estimates for (5.1). For simplicity of notation we define

$$\mathcal{P} = (\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi,$$

and note that

$$\mathcal{P}\partial_{\alpha'}^j f - \partial_{\alpha'}^j\mathcal{P}f = \sum_{i=1}^{j}\partial_{\alpha'}^{j-i}[\mathcal{P},\partial_{\alpha'}]\partial_{\alpha'}^{i-1}f.$$

Applying this identity to (5.1) we get

$$\begin{cases} \mathcal{P}\partial_{\alpha'}^j\Theta = G_j \\ G_j := \partial_{\alpha'}^j G + \sum_{i=1}^{j} \partial_{\alpha'}^{j-i}[\mathcal{P}, \partial_{\alpha'}]\partial_{\alpha'}^{i-1}\Theta \\ \quad = \partial_{\alpha'}^j G + \sum_{i=1}^{j} \partial_{\alpha'}^{j-i}\left(-b_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i\Theta - (\partial_t + b\partial_{\alpha'})\left(b_{\alpha'}\partial_{\alpha'}^i\Theta\right) - b_{\alpha'}^2\partial_{\alpha'}^i\Theta - iA_{\alpha'}\partial_{\alpha'}^i\Theta\right) \end{cases} . \tag{5.5}$$

The jth order energy is now defined as

$$E_j^{\Theta} := \int_0^{2\pi} \frac{|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j\Theta|^2}{A} d\alpha' + \int_0^{2\pi} \left(i\partial_{\alpha'}^{j+1}\Theta\partial_{\alpha'}^j\overline{\Theta} - \frac{\pi}{A}|\partial_{\alpha'}^j\Theta|^2\right) =: \mathcal{E}_j^{\Theta} + \mathcal{F}_j^{\Theta}.$$

and in analogy with the undifferentiated case we let

$$\begin{aligned} E_j(\chi) &:= \mathcal{E}_j(\chi) + \mathcal{F}_j(\chi) := \mathcal{E}_j^{\chi} + \mathcal{F}_j^{\chi} = E_j^{\chi}, \\ E_j(v) &:= \mathcal{E}_j(v) + \mathcal{F}_j(v) := \mathcal{E}_j^{\tilde{v}} + \mathcal{F}_j^{\tilde{v}} = E_j^{\tilde{v}}. \end{aligned} \tag{5.6}$$

The following lemma follows from a similar argument to the proof of Lemma 5.3.

Lemma 5.5. $\partial_t E_j^{\theta} = R_j(t)$ *where*

$$\begin{aligned} R_j(t) := 2\int_0^{2\pi} \frac{1}{A}\mathrm{Re}\left(G_j(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j\overline{\Theta}\right) d\alpha' \\ - \int_0^{2\pi} \left(\frac{1}{A}\frac{a_t}{a} \circ k^{-1}\right)\left(|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j\Theta|^2 - \pi|\partial_{\alpha'}^j\Theta|^2\right) d\alpha'. \end{aligned}$$

Supposing for the moment that we know how to deal with the non-positive part $\mathcal{F}_j^{\Theta}$ of the energy, we can use Corollary 4.11 and Proposition 4.7 to estimate the quantities appearing in the bootstrap assumption (4.6) in terms of the positive parts of the energy $\mathcal{E}_j(\chi)$ and $\mathcal{E}_j(v)$. The only difficulty with this is that in the definition of $\mathcal{E}_j(v)$ we have replaced v by $\tilde{v}$, so in the next proposition we show that the conclusions of Corollary 4.11 hold with v replaced by $\tilde{v}$.

Proposition 5.6. *If* M_0 *in* (4.6) *is sufficiently small then for* $2 \le k \le \ell$,

$$\sum_{j\le k}\left(\|\partial_{\alpha'}^j w\|_{L_{\alpha'}^2}^2 + \|\partial_{\alpha'}^j u\|_{L_{\alpha'}^2}^2 + \|\partial_{\alpha'}^j \eta\|_{L_{\alpha'}^2}^2\right) \le C\sum_{j\le k}(\mathcal{E}_j^{\chi} + \mathcal{E}_j^{\tilde{v}}).$$

Proof. In view of Corollary 4.11 and Proposition 4.7 we only need to show that under the assumptions of the proposition

$$\begin{aligned} \int_0^{2\pi} |(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j v|^2 d\alpha' \lesssim \int_0^{2\pi} |(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j \tilde{v}|^2 d\alpha' \\ + M\sum_{i\le j}\int_0^{2\pi}\left(|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i\chi|^2 + |(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i v|^2\right) d\alpha'. \end{aligned} \tag{5.7}$$

To see this we first write

$$\tilde{v} \circ k = (I - H)\partial_t \delta = \partial_t (I - H)\delta + [z_t, H]\frac{\delta_\alpha}{z_\alpha} = 2\partial_t \delta + [z_t, H]\frac{\delta_\alpha}{z_\alpha}$$

so

$$\tilde{v} = 2v + [u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}}. \tag{5.8}$$

Now

$$(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j [u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} = \partial_\alpha^j (\partial_t + b\partial_{\alpha'})[u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} - \sum_{i=1}^{j} \binom{j}{i} \partial_{\alpha'}^i b\, \partial_{\alpha'}^{j-i} [u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}}. \tag{5.9}$$

By Corollary 4.11, Proposition 4.10, and Lemma 2.5 the contribution of the last term above can be bounded as

$$\left\| \partial_{\alpha'}^i b\, \partial_{\alpha'}^{j-i} [u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} \right\|_{L^2_{\alpha'}}^2 \lesssim M \sum_{i \le j} \left(\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i v\|_{L^2_{\alpha'}} + \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i \chi\|_{L^2_{\alpha'}} \right). \tag{5.10}$$

To estimate the first term on the right hand side of (5.9) we first note that

$$\partial_t [z_t, H]\frac{\delta_\alpha}{z_\alpha} = [z_{tt}, H]\frac{\delta_\alpha}{z_\alpha} + [z_t, H]\frac{\partial_\alpha \delta_t}{z_\alpha} + \frac{1}{\pi i}\int_0^{2\pi} \left(\frac{z_t(\alpha) - z_t(\beta)}{z(\beta) - z(\alpha)}\right)^2 \delta_\beta(\beta) d\beta,$$

so

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})[u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} &= [w, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} + [u, \mathcal{H}]\frac{\partial_{\alpha'} v}{\zeta_{\alpha'}} \\ &\quad + \frac{1}{\pi i}\int_0^{2\pi} \left(\frac{u(\alpha') - u(\beta')}{\zeta(\beta') - \zeta(\alpha')}\right)^2 \chi_{\beta'}(\beta') d\beta'.\end{aligned}$$

It follows from this, Corollary 4.11, Proposition 4.6, Proposition 2.3, and Lemma 2.4 that

$$\left\| \partial_{\alpha'}^j (\partial_t + b\partial_{\alpha'})[u, \mathcal{H}]\frac{\chi_{\alpha'}}{\zeta_{\alpha'}} \right\|_{L^2_{\alpha'}}^2 \lesssim M \sum_{i \le j} \left(\|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i v\|_{L^2_{\alpha'}} + \|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i \chi\|_{L^2_{\alpha'}} \right). \tag{5.11}$$

Combining (5.8)–(5.11) we get (5.7). □

We now turn to the issue of non-positivity of $\mathcal{F}_j^\Theta$. Note that even if Θ can be written as $(I - \mathcal{H})f$ this will not in general imply that $\partial_{\alpha'}^j \Theta$ is the boundary value of a function holomorphic outside of Ω, so even the first integral in the definition of $\mathcal{F}_j^\Theta$ above may not be non-negative for $j \geq 1$. Nevertheless, as for $\mathcal{F}_0^\Theta$, we are able to show that the negative part of $\mathcal{F}_j^\Theta$ is of higher order for the choices of Θ we need in the energy estimates. The following simple observation is the main step in this direction.

Lemma 5.7. *Suppose* $\Theta := (I-\mathcal{H})f$ *for some* 2π*-periodic function* f*. Then with* $g = \zeta\Theta$

$$i\int_0^{2\pi} \Theta_{\alpha'}\overline{\Theta}d\alpha' - \int_0^{2\pi} |\Theta|^2 d\alpha' = \int_0^{2\pi} g_{\alpha'}\overline{g}d\alpha' - \int_0^{2\pi} (i\zeta_\alpha\overline{\zeta}+1)|\Theta|^2 d\alpha' - \int_0^{2\pi} i\mu\Theta_{\alpha'}\overline{\Theta}d\alpha'$$
$$\geq -\left(\|i\zeta_\alpha\overline{\zeta}+1\|_{L^\infty_{\alpha'}}\|\Theta\|^2_{L^2_{\alpha'}} + \|\mu\|_{L^\infty_{\alpha'}}\|\Theta_{\alpha'}\|_{L^2_{\alpha'}}\|\Theta\|_{L^2_{\alpha'}}\right).$$

Proof. The first equality follows from

$$i\Theta_{\alpha'}\overline{\Theta} - |\Theta|^2 = i\partial_{\alpha'}(\zeta\Theta)(\overline{\zeta\Theta}) - (i\zeta_{\alpha'}\overline{\zeta}+1)|\Theta|^2 - i\mu\Theta_{\alpha'}\overline{\Theta}.$$

To get the inequality it suffices to show that $i\int_0^{2\pi} g_{\alpha'}\overline{g}d\alpha' \geq 0$. For this note that

$$g = (I-\mathcal{H})(\zeta f) - [\zeta,\mathcal{H}]f$$

and that $[\zeta,\mathcal{H}]f$ is independent of α'. It follows that

$$i\int_0^{2\pi} g_{\alpha'}\overline{g}d\alpha' = i\int_0^{2\pi} \partial_{\alpha'}[(I-\mathcal{H})(\zeta f)]\overline{[(I-\mathcal{H})(\zeta f)]}d\alpha'$$

which is non-negative by Lemma 5.1. □

Lemma 5.7 shows that the difference between the energy and a positive term is of higher order. Note however, that the lower order term involves an extra derivative of Θ. This causes a problem only when we consider $\partial^\ell_{\alpha'}v$, where ℓ is the maximum number of derivatives we commute. But in this case we can write

$$\partial^\ell_{\alpha'}v = (\partial_t + b\partial_{\alpha'})\partial^\ell_{\alpha'}\chi + [\partial^\ell_{\alpha'}, \partial_t + b\partial_{\alpha'}]\chi,$$

and the main term here is already bounded by the energy of χ, that is,

$$\|(\partial_t + b\partial_{\alpha'})\partial^\ell_{\alpha'}\chi\|^2_{L^2_{\alpha'}} \lesssim \mathcal{E}^\chi_\ell.$$

Since $\|(\partial_t + b\partial_{\alpha'})\partial^\ell_{\alpha'}\chi\|^2_{L^2_{\alpha'}}$ is precisely the negative term in the energy of $\partial^\ell_{\alpha'}v$ this idea can be used to resolve the issue in the case where we commute the maximum number of derivatives. We will now make this argument more precise, starting with a few important identities stated only for the choices of Θ which will be used in the energy estimates, namely $\Theta = \chi$ and $\tilde{v}$.

Lemma 5.8. *If* χ *and* v *are as in* (4.1) *then*

$$\partial^j_{\alpha'}\chi = (I-\mathcal{H})\partial^j_{\alpha'}\mu - \sum_{i=1}^j \partial^{j-i}_{\alpha'}[\eta,\mathcal{H}]\frac{\partial^i_{\alpha'}\mu}{\zeta_{\alpha'}},$$
$$\partial^j_{\alpha'}\tilde{v} = (I-\mathcal{H})\partial^j_{\alpha'}v - \sum_{i=1}^j \partial^{j-i}_{\alpha'}[\eta,\mathcal{H}]\frac{\partial^i_{\alpha'}v}{\zeta_{\alpha'}}.$$

Proof. The first identity follows from commuting $\partial_{\alpha'}^j$ with $\mathcal{H}$ in the definition $\chi = (I-\mathcal{H})\mu$ of χ and noting that

$$[\partial_{\alpha'}^j, \mathcal{H}]f = \sum_{i=1}^{j} \partial_{\alpha'}^{j-i}[\partial_\alpha, \mathcal{H}]\partial_{\alpha'}^{i-1} f,$$

and

$$[\partial_{\alpha'}, \mathcal{H}]f = [\zeta_{\alpha'}, \mathcal{H}]\frac{f_{\alpha'}}{\zeta_{\alpha'}} = [\eta, \mathcal{H}]\frac{f_{\alpha'}}{\zeta_{\alpha'}}.$$

The proof of the second identity is similar where we use the definition $\tilde{v} = (I-H)v$. □

We can now prove the following positivity estimate.

Lemma 5.9. *(1) If M_0 in (4.6) is sufficiently small then for $\ell \geq 2$*

$$\sum_{i=0}^{\ell} \mathcal{F}_i^{\chi} \geq -C\sum_{i=0}^{\ell}\left(\mathcal{E}_i^{\chi} + \mathcal{E}_i^{\tilde{v}}\right)^{\frac{3}{2}},$$

$$\sum_{i=0}^{\ell-1} \mathcal{F}_i^{\tilde{v}} \geq -C\sum_{i=0}^{\ell}\left(\mathcal{E}_i^{\chi} + \mathcal{E}_i^{\tilde{v}}\right)^{\frac{3}{2}}.$$

(2) If M_0 in (4.6) is sufficiently small then for $\ell \geq 2$

$$\mathcal{F}_\ell^{\tilde{v}} \geq -C\sum_{i=0}^{\ell}\left(\mathcal{E}_i^{\chi} + \mathcal{E}_i^{\tilde{v}}\right)^2 - C\sum_{i=0}^{\ell}(\mathcal{E}_i^{\chi} + \mathcal{E}_i^{\tilde{v}})^{\frac{3}{2}} - C\mathcal{E}_\ell^{\chi}.$$

Proof. (1) We assume M_0 is small enough that Corollary 4.11 holds. We start with the estimate for χ. By Lemma 5.8

$$\partial_{\alpha'}^i \chi = (I-\mathcal{H})\partial_\alpha^i \mu - \sum_{m=1}^{i} \partial_{\alpha'}^{i-m}[\eta, \mathcal{H}]\frac{\partial_{\alpha'}^m \mu}{\zeta_\alpha} =: f_i + g_i.$$

It follows that

$$\begin{aligned}\mathcal{F}_i^{\chi} &= i\int_0^{2\pi} \partial_{\alpha'} f_i \overline{f_i} d\alpha' - \int_0^{2\pi} |f_i|^2 d\alpha' \\ &\quad - 2\mathrm{Re}\, i\int_0^{2\pi} f_i \partial_{\alpha'}\overline{g_i} d\alpha' - 2\mathrm{Re}\int_0^{2\pi} f_i\overline{g_i} d\alpha' + i\int_0^{2\pi} \partial_{\alpha'} g_i \overline{g_i} d\alpha' - \int_0^{2\pi} |g_i|^2 d\alpha'. \end{aligned} \tag{5.12}$$

To estimate the first line above we apply Lemma 5.7 with $\Theta = (I-\mathcal{H})\partial_{\alpha'}^j \mu = f_i$ to get (for $i \leq \ell$)

$$\begin{aligned} i\int_0^{2\pi} \partial_{\alpha'} f_i \overline{f_i} d\alpha' - \int_0^{2\pi} |f_i|^2 d\alpha' &\geq -(\|\zeta\|_{L^\infty_{\alpha'}}\|\eta\|_{L^\infty_{\alpha'}} \\ &\quad + \|\mu\|_{L^\infty_{\alpha'}})\|f_i\|^2_{L^2_{\alpha'}} - \|\mu\|_{L^\infty_{\alpha'}}\|\partial_{\alpha'} f_i\|_{L^2_{\alpha'}}\|f_i\|_{L^2_{\alpha'}} \\ &\geq -C\sum_{j=0}^{i}\left(\mathcal{E}_j^{\chi} + \mathcal{E}_j^{\tilde{v}}\right)^{\frac{3}{2}}, \end{aligned} \tag{5.13}$$

by Corollary 4.11 and Proposition 4.7. Here to estimate $\|\partial_{\alpha'} f_\ell\|_{L^2_{\alpha'}}$ we have noted that

$$\partial_{\alpha'} f_\ell = (I-\mathcal{H})\partial_{\alpha'}^{\ell+1}\mu - [\eta, \mathcal{H}]\frac{\partial_{\alpha'}^{\ell+1}\mu}{\zeta_{\alpha'}} = (I-\mathcal{H})\partial_{\alpha'}^{\ell}(\zeta\overline{\eta} + \overline{\zeta}\eta) - [\eta, \mathcal{H}]\frac{\partial_{\alpha'}^{\ell}(\zeta\overline{\eta} + \overline{\zeta}\eta)}{\zeta_{\alpha'}}. \tag{5.14}$$

To estimate the second line in (5.12) it suffices to show that for $i \leq \ell$

$$\left(\|f_i\|_{L^2_{\alpha'}} + \|g_i\|_{L^2_{\alpha'}}\right)\left(\|g_i\|_{L^2_{\alpha'}} + \|\partial_\alpha g_i\|_{L^2_{\alpha'}}\right) \leq C\sum_{j=0}^{i}\left(\mathcal{E}_j^\chi + \mathcal{E}_j^{\tilde{v}}\right)^{\frac{3}{2}}. \tag{5.15}$$

But (5.15) is a direct consequence of Corollary 4.11 and Lemma 2.5. Combining (5.12), (5.13), and (5.15) we get the estimate for χ.
The estimate for $\tilde{v}$ is similar. Using Lemma 5.8 we write

$$\partial_{\alpha'}^i \tilde{v} = \phi_i + \psi_i$$

where

$$\phi_i := (I-\mathcal{H})\partial_{\alpha'}^i v, \qquad \psi_i = -\sum_{m=1}^{i}\partial_\alpha^{i-m}[\eta, \mathcal{H}]\frac{\partial_{\alpha'}^m v}{\zeta_\alpha}.$$

The argument is now the same as for χ where we replace g_i by ψ_i and f_i by ϕ_i everywhere. The only difference is that (5.14) is now replaced by

$$\partial_{\alpha'}\phi_{\ell-1} = (I-\mathcal{H})\partial_{\alpha'}^{\ell} v - [\eta, \mathcal{H}]\frac{\partial_{\alpha'}^{\ell} v}{\zeta_{\alpha'}},$$

which is responsible for the loss of one derivative.

(2) Note that with $c_i := \binom{\ell}{i}$

$$\partial_{\alpha'}^{\ell} v = (\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^{\ell}\chi + \sum_{i=1}^{\ell} c_i\partial_{\alpha'}^i b\,\partial_{\alpha'}^{\ell+1-i}\chi.$$

It follows from this, Corollary 4.11, Proposition 4.7, and Proposition 4.10 that

$$\|\partial_{\alpha'}^{\ell} v\|_{L^2_{\alpha'}}^2 \leq C\sum_{i=0}^{\ell}(\mathcal{E}_i^\chi + \mathcal{E}_i^{\tilde{v}})^2 + C\mathcal{E}_\ell^\chi,$$

and from the second identity in Lemma 5.8 that

$$\|\partial_{\alpha'}^{\ell}\tilde{v}\|_{L^2_{\alpha'}}^2 \leq C\sum_{i=0}^{\ell}(\mathcal{E}_i^\chi + \mathcal{E}_i^{\tilde{v}})^2 + C\mathcal{E}_\ell^\chi.$$

From the definition of $\mathcal{F}_\ell^{\tilde{v}}$ and in view of Proposition 4.7 if M_0 is sufficiently small it follows that

$$\mathcal{F}_\ell^{\tilde{v}} \geq i\int_0^{2\pi}\partial_{\alpha'}^{j+1}\tilde{v}\partial_{\alpha'}^{j}\overline{\tilde{v}}d\alpha' - C\sum_{i=0}^{\ell}(\mathcal{E}_i^\chi + \mathcal{E}_i^{\tilde{v}})^2 - C\mathcal{E}_\ell^\chi,$$

so it suffice to show

$$i\int_0^{2\pi}\partial_{\alpha'}^{\ell+1}\tilde{v}\partial_\alpha^\ell\overline{\tilde{v}}d\alpha' \geq -C\sum_{i=0}^{\ell}\left(\mathcal{E}_i^\chi+\mathcal{E}_i^{\tilde{v}}\right)^{\frac{3}{2}}. \tag{5.16}$$

For this we use Lemma 5.8 to write

$$\partial_{\alpha'}^\ell\tilde{v} = f+g$$

where

$$f=(I-\mathcal{H})\partial_{\alpha'}^\ell v, \quad g=-\sum_{i=1}^{\ell}\partial_{\alpha'}^{\ell-i}[\eta,\mathcal{H}]\frac{\partial_{\alpha'}^i v}{\zeta_{\alpha'}}.$$

Since

$$i\int_0^{2\pi} f_{\alpha'}\overline{f}d\alpha' \geq 0,$$

arguing as in (5.12) and (5.15) we just need to show that

$$\|g\|_{L^2_{\alpha'}}\|g_{\alpha'}\|_{L^2_{\alpha'}}+\|f\|_{L^2_{\alpha'}}\|g_{\alpha'}\|_{L^2_{\alpha'}}+\|f\|_{L^2_{\alpha'}}\|g\|_{L^2_{\alpha'}} \leq C\sum_{i=0}^{\ell}\left(\mathcal{E}_i^\chi+\mathcal{E}_i^{\tilde{v}}\right)^{\frac{3}{2}}.$$

But this is again a consequence of Corollary 4.11, Proposition 4.6, and Lemma 2.5. This now proves (5.16) which concludes the proof of the Lemma. □

Combining Lemmas 5.5 and 5.9 we see that if M_0 in (4.6) is sufficiently small we can find constants c_1, C_1 and C_2 such that with R_k as in Lemma 5.5

$$\begin{aligned}\sum_{k\leq\ell}\mathcal{E}_k^\chi(t)+\sum_{k\leq\ell-1}\mathcal{E}_k^{\tilde{v}}(t) &\leq \sum_{k\leq\ell}\left(E_k^\chi(0)+E_k^{\tilde{v}}(0)\right)\\ &+C_1\sum_{k\leq\ell}\left(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t)\right)^{\frac{3}{2}}+\sum_{k\leq\ell}\int_0^t|R_k(t)|dt\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_\ell^{\tilde{v}}(t)-c_1\mathcal{E}_\ell^\chi(t) &\leq \sum_{k\leq\ell}\left(E_k^\chi(0)+E_k^{\tilde{v}}(0)\right)\\ &+C_2\sum_{k\leq\ell}\left(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t)\right)^{\frac{3}{2}}+C_2\sum_{k\leq\ell}\left(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t)\right)^2\\ &+\sum_{k\leq\ell}\int_0^t|R_k(t)|dt.\end{aligned}$$

Adding an appropriate multiple of the second estimate to the first we get the following energy estimate.

Corollary 5.10. *If M_0 in (4.6) is sufficiently small then with N_k as in Lemma 5.5*

$$\sum_{k\le\ell}(\mathcal{E}_k^{\chi}(t)+\mathcal{E}_k^{\tilde v}(t)) \le C\sum_{k\le\ell}(E_k^{\chi}(0)+E_k^{\tilde v}(0)) + C\sum_{k\le\ell}(\mathcal{E}_k^{\chi}(t)+\mathcal{E}_k^{\tilde v}(t))^{\frac{3}{2}}$$
$$+ C\sum_{k\le\ell}(\mathcal{E}_k^{\chi}(t)+\mathcal{E}_k^{\tilde v}(t))^2 + C\sum_{k\le\ell}\int_0^t |R_k(t)|dt.$$

We now turn to the estimates for R_k. For notational convenience we define

$$\mathcal{E} := \sum_{i=0}^{\ell}\left(\mathcal{E}_i^{\chi}+\mathcal{E}_i^{\tilde v}\right). \tag{5.17}$$

Our first step will be to compute the equation for $\tilde v$.

Proposition 5.11. *$\tilde v = (I-H)v$ satisfies*

$$(\partial_t^2 + ia\partial_\alpha - \pi)(\tilde v\circ k) = (I-H)(\partial_t^2+ia\partial_\alpha-\pi)\delta_t + 2[z_t, H]\frac{\partial_\alpha(\partial_t^2+ia\partial_\alpha-\pi)\delta}{z_\alpha}$$
$$+\frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(t,\beta)-z_t(t,\alpha)}{z(t,\beta)-z(t,\alpha)}\right)^2\delta_{t\beta}(t,\beta)d\beta$$
$$+2\pi\left[z[\varepsilon,H]\frac{z_{t\alpha}}{z_\alpha}, H\right]\frac{\delta_\alpha}{z_\alpha} + 2\pi[E(z_t),H]\frac{\delta_\alpha}{z_\alpha}$$
$$+\pi\left[z[\varepsilon,H]\frac{z_{t\alpha}}{z_\alpha}, H\right]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta + \pi[E(z_t),H]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$$
$$+ -2[z_t,H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{g^a\delta_\alpha}{z_\alpha}\right) + 2[z_t,H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{z_{tt}\delta_\alpha}{z_\alpha}\right)$$
$$+\frac{\pi}{2}[z\delta,H]\frac{\partial_\alpha}{z_\alpha}[z_t,H]\frac{\varepsilon_\alpha}{z_\alpha} - \frac{\pi}{2}[E(z),H]\frac{\delta_{t\alpha}}{z_\alpha}.$$

Proof. From Lemma 3.7 we have

$$(\partial_t^2+ia\partial_\alpha-\pi)(\tilde v\circ k) = (\partial_t^2+ia\partial_\alpha-\pi)(I-H)\delta_t = (I-H)(\partial_t^2+ia\partial_\alpha-\pi)\delta_t - [\partial_t^2+ia\partial_\alpha, H]\delta_t$$
$$= (I-H)(\partial_t^2+ia\partial_\alpha-\pi)\delta_t - \frac{\pi}{2}[(I-\overline{H})z, H]\frac{\delta_{t\alpha}}{z_\alpha} + 2[z_t,H]\frac{\partial_\alpha\delta_{tt}}{z_\alpha}$$
$$+\frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\delta_{t\beta}(\beta)d\beta$$
$$= -\frac{\pi}{2}[(I-\overline{H})z,H]\frac{\delta_{t\alpha}}{z_\alpha} - 2[z_t,H]\frac{\partial_\alpha(ia\partial_\alpha\delta-\pi\delta)}{z_\alpha}$$
$$+(I-H)(\partial_t^2+ia\partial_\alpha-\pi)\delta_t + 2[z_t,H]\frac{\partial_\alpha(\partial_t^2+ia\partial_\alpha-\pi)\delta}{z_\alpha}$$
$$+\frac{1}{\pi i}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\delta_{t\beta}(\beta)d\beta.$$

The last three terms above already have the right form, so we only need to consider

$$I+II+III := -\frac{\pi}{2}[(I-\overline{H})z,H]\frac{\delta_{t\alpha}}{z_\alpha} - 2[z_t,H]\frac{\partial_\alpha(ia\partial_\alpha\delta)}{z_\alpha} + 2\pi[z_t,H]\frac{\partial_\alpha\delta}{z_\alpha}. \tag{5.18}$$

Note that if g is the boundary value of a decaying holomorphic function F outside of Ω, i.e., $g = (I - H)f_1$ then $\frac{g_\alpha}{z_\alpha}$ is the boundary value of F_z so $\frac{g_\alpha}{z_\alpha} = (I - H)f_2$ for some f_2. We will use this observation repeatedly in the rest of this proof. Applying this observation to III we see that since $\delta = (I - H)\varepsilon$

$$III = \pi[(I + H)z_t, H]\frac{\delta_\alpha}{z_\alpha} = \pi[(H + \overline{H})z_t, H]\frac{\delta_\alpha}{z_\alpha}$$
$$= \pi\left[z[\varepsilon, H]\frac{z_{t\alpha}}{z_\alpha}, H\right]\frac{\delta_\alpha}{z_\alpha} + \pi[E(z_t), H]\frac{\delta_\alpha}{z_\alpha}.$$

For II we use (3.2) to write

$$II = -2[z_t, H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{g^a\delta_\alpha}{z_\alpha}\right) + 2[z_t, H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{z_{tt}\delta_\alpha}{z_\alpha}\right) + 2\pi[z_t, H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{z\delta_\alpha}{z_\alpha}\right).$$

The first two terms have the right form and we can rewrite the last term as

$$2\pi[z_t, H]\frac{\partial_\alpha}{z_\alpha}\left(\frac{z\delta_\alpha}{z_\alpha}\right) = \pi[(I + H)z_t, H]\frac{\delta_\alpha}{z_\alpha} + 2\pi\,[z_t, H]\,z\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$$
$$= \pi\left[z[\varepsilon, H]\frac{z_{t\alpha}}{z_\alpha}, H\right]\frac{\delta_\alpha}{z_\alpha} + \pi[E(z_t), H]\frac{\delta_\alpha}{z_\alpha}$$
$$+ 2\pi z[z_t, H]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta + 2\pi[[z_t, H], z]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$$

The first term in the last line above can be written as

$$\pi z[(I + H)z_t, H]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta = \pi z[(H + \overline{H})z_t, H]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$$
$$= \pi z\left[z[\varepsilon, H]\frac{z_{t\alpha}}{z_\alpha}, H\right]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta + \pi[E(z_t), H]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta.$$

For the second term $2\pi[[z_t, H], z]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$, we use Jacobi identity to write this as

$$-2\pi[[H, z], z_t]\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta = -2\pi[H, z]z_t\left(\frac{\partial_\alpha}{z_\alpha}\right)^2\delta$$
$$= \frac{2}{i}\int_0^{2\pi} z_t(\beta)\partial_\beta\left(\frac{\partial_\beta}{z_\beta}\left((I - H)\varepsilon\right)(\beta)\right)d\beta.$$

By Lemma 3.12, we have

$$(I - H)\varepsilon = (I + \overline{H})\varepsilon - z[\varepsilon, H]\frac{\varepsilon_\alpha}{z_\alpha} - E(\varepsilon).$$

Therefore the contribution we need to consider is

$$\int_0^{2\pi} z_t(\beta)\partial_\beta\left((I + \overline{H})\frac{\varepsilon_\beta}{z_\beta}\right)(\beta)d\beta = \int_0^{2\pi} z_t(\beta)G_{\overline{z}}(\overline{z}(\beta))\overline{z}_\beta(\beta)d\beta = 0.$$

Here $G_{\overline{z}}(\overline{z}(\beta)) = \frac{\partial_\beta}{z_\beta}\left((I + \overline{H})\frac{\varepsilon_\beta}{z_\beta}\right)$ is the boundary value of an anti-holomorphic function $G_{\overline{z}}(\overline{z})$ in Ω. We also used the fact that $z_t(\beta)$ is the boundary value of an anti-holomorphic function in Ω. Finally for I we compute

$$\begin{aligned} I &= -\frac{\pi}{2}[(I - \overline{H})z, H]\frac{\delta_{t\alpha}}{z_\alpha} \\ &= \frac{\pi}{2}[(I + \overline{H})z, H]\frac{\delta_{t\alpha}}{z_\alpha} \\ &= \frac{\pi}{2}[z\delta, H]\frac{\delta_{t\alpha}}{z_\alpha} + \frac{\pi}{2}[E(z), H]\frac{\delta_{t\alpha}}{z_\alpha} \\ &= \frac{\pi}{2}[z\delta, H]\frac{\partial_\alpha}{z_\alpha}(I - H)(z_t\overline{z}) - \frac{\pi}{2}[z\delta, H]\frac{\partial_\alpha}{z_\alpha}[z_t, H]\frac{\varepsilon_\alpha}{z_\alpha} + \frac{\pi}{2}[E(z), H]\frac{\delta_{t\alpha}}{z_\alpha}. \end{aligned}$$

Again the last two terms have the right form and for the first we use Lemma 3.11 with $f = z$, $g = \delta$ and $h = \frac{\partial_\alpha}{z_\alpha}(I - H)(z_t\overline{z})$ and the fact that for and f_1 and f_2

$$[(I - H)f_1, H](I - H)f_2 = 0$$

to write

$$\begin{aligned} \frac{\pi}{2}[z\delta, H]\frac{\partial_\alpha}{z_\alpha}(I - H)(z_t\overline{z}) &= \frac{\pi}{2}[z, H]\delta\frac{\partial_\alpha}{z_\alpha}(I - H)(z_t\overline{z}) \\ &= -\frac{1}{\pi i}\frac{\pi}{2}\int_0^{2\pi} \delta\partial_\alpha(I - H)(z_t\overline{z})d\alpha = 0. \end{aligned}$$

Here for the last step we have used the fact that since δ and $(I - H)(z_t\overline{z})$ are boundary values of holomorphic functions F_1 and F_2, respectively, in Ω^c going to zero as $|z| \to \infty$,

$$\int_0^{2\pi} \delta\partial_\alpha(I - H)(z_t\overline{z})d\alpha = \int_{\partial\Omega^c} F_1(z)\partial_z F_2(z)dz = 0.$$

□

The following estimate is used for estimating the second integral in the definition of N_k.

Lemma 5.12. *If M_0 in (4.6) is sufficiently small*

$$\left\| \frac{1}{A}\frac{a_t}{a} \circ k^{-1} \right\|_{L^\infty_{\alpha'}} \leq C\mathcal{E}.$$

Proof. We use Lemmas 3.16 and 3.17 and the Sobolev embedding $H^1_{\alpha'} \hookrightarrow L^\infty_{\alpha'}$. Recalling that $A = (ak_\alpha) \circ k^{-1}$, from Lemma 3.16 and precomposition with k^{-1} we get

$$\begin{aligned} (I + \mathcal{K}^*)(\frac{a_t}{a} \circ k^{-1} A|\zeta_{\alpha'}|) = \mathrm{Re}\Bigg\{ &- i\frac{\zeta_{\alpha'}}{|\zeta_{\alpha'}|}\Big\{2[u, \mathcal{H}]\frac{\overline{w}_{\alpha'}}{\zeta_{\alpha'}} + [2w - g^a \circ k^{-1}, \mathcal{H}]\frac{\overline{u}_{\alpha'}}{\zeta_{\alpha'}} \\ &- \frac{\pi}{2}(I - \mathcal{H})\left([u, \mathcal{H}]\frac{\overline{\zeta}_{\alpha'}}{\zeta_{\alpha'}}\right) + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{u(t, \beta') - u(t, \alpha')}{\zeta(t, \beta') - \zeta(t, \alpha')}\right)^2 \overline{u}_{\beta'}(t, \beta')d\beta'\Big\}\Bigg\}. \end{aligned}$$

Recalling that $g^1 \circ k^{-1} = \frac{\pi}{2}\zeta\chi + E(\zeta)$, it follows from this, Lemmas 2.5 and 2.6, and Corollary 4.11 that

$$\sum_{i=0}^{1} \|\partial_{\alpha'}^i \Big((I+\mathcal{K}^*)(\frac{a_t}{a}\circ k^{-1}A|\zeta_{\alpha'}|)\Big)\|_{L^2_{\alpha'}} \le C\mathcal{E}. \tag{5.19}$$

On the other hand,

$$\frac{1}{A}\frac{a_t}{a}\circ k^{-1} = \frac{1}{A^2|\zeta_{\alpha'}|}(I+\mathcal{K}^*)(\frac{a_t}{a}\circ k^{-1}A|\zeta_{\alpha'}|) \\ - \frac{1}{A^2|\zeta_{\alpha'}|}\mathcal{K}^*(\frac{a_t}{a}\circ k^{-1}A|\zeta_{\alpha'}|) =: I - \frac{1}{A^2|\zeta_{\alpha'}|}II.$$

By (5.19) and Propositions 4.1 and 4.7

$$\|I\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}I\|_{L^2_{\alpha'}} \le C\mathcal{E},$$

and therefore in view of Propositions 4.1 and 4.7 to complete the proof of the lemma it suffice to show that

$$\|II\|_{L^2_{\alpha'}} + \|\partial_\alpha II\|_{L^2_{\alpha'}} \le CM\left(\left\|\frac{1}{A}\frac{a_t}{a}\circ k^{-1}\right\|_{L^2_{\alpha'}} + \left\|\partial_{\alpha'}\left(\frac{1}{A}\frac{a_t}{a}\circ k^{-1}\right)\right\|_{L^2_{\alpha'}}\right) + C\mathcal{E}. \tag{5.20}$$

For this we use Lemma 3.17. Note that since $K^*f = -\mathrm{Re}\{\frac{z_\alpha}{|z_\alpha|}H\frac{|z_\alpha|f}{z_\alpha}\}$ we may replace z by ζ and z_α by $\zeta_{\alpha'}$ everywhere in formula derived in Lemma 3.17 to get a representation for $\mathcal{K}^*$. Using this observation and Lemma 3.13, 3.17 we get with $f = \frac{a_t}{a}\circ k^{-1}A|\zeta_{\alpha'}|$

$$II = \frac{1}{\pi|\zeta_{\alpha'}|}\int_0^{2\pi} f(\alpha')|\zeta_{\alpha'}(\alpha')|d\alpha' + \frac{\mathcal{AV}(f|\zeta_{\alpha'}|)}{|\zeta_{\alpha'}|} \\ - \frac{\zeta}{2|\zeta_{\alpha'}|}[\mu,\mathcal{H}]\frac{\partial_\alpha(f|\zeta_{\alpha'}|)}{\zeta_{\alpha'}} - \frac{E(f|\zeta_{\alpha'}|)}{2|\zeta_{\alpha'}|} - \mathrm{Re}\{\frac{1}{|\zeta_{\alpha'}|}[\eta,\mathcal{H}]\frac{f|\zeta_{\alpha'}|}{\zeta_{\alpha'}}\}. \tag{5.21}$$

The contribution of the second line above can be bounded by the right hand side of (5.20) using Lemma 2.5, Corollary 4.3, and Proposition 4.1. To estimate the contribution of the first line of (5.21) we go back to Eq. (3.4) which we rewrite as

$$f|\zeta_{\alpha'}| = \frac{a_t}{a}\circ k^{-1}A|\zeta_{\alpha'}|^2 = i\overline{\zeta}_{\alpha'}(\partial_t + b\partial_{\alpha'})w - \pi u_{\alpha'}\overline{\zeta}_{\alpha'} - (A-\pi)u_{\alpha'}\overline{\zeta}_{\alpha'} \\ - \frac{\pi i}{2}[\overline{u},\mathcal{H}\frac{1}{\zeta_{\alpha'}} + \overline{\mathcal{H}}\frac{1}{\overline{\zeta}_\alpha}]\zeta_{\alpha'}. \tag{5.22}$$

Moreover, we can write

$$\mathcal{AV}(g) = \int_0^{2\pi}\frac{\eta g}{\zeta}\,d\alpha' + i\int_0^{2\pi} g\,d\alpha'$$

so to prove (5.20) for the first line of (5.21), it suffices to bound $\int_0^{2\pi} g d\alpha'$ by the right hand side of (5.20) with g replaced by each of the terms on the right hand side of (5.22). For the last two terms of (5.22) the contributions are of the right form in view of Lemma

3.12 and Proposition 4.7. For the second term of (5.22) it suffices to note that since u is anti-holomorphic inside Ω

$$\int_0^{2\pi} u_{\alpha'}\overline{\zeta}_{\alpha'}d\alpha' = \int_0^{2\pi} u_{\alpha'}\overline{\eta}\,d\alpha' + i\int_0^{2\pi} u\overline{\zeta}_{\alpha'}d\alpha' = \int_0^{2\pi} u_{\alpha'}\overline{\eta}\,d\alpha'$$

which can be bounded by the right hand side of (5.20). Finally for the first term of (5.22) we write $z_t = F(t,\overline{z})$ for an anti-holomorphic function to get

$$z_{tt} = F_t + F_{\overline{z}}\overline{z}_t = F_t + \frac{z_{t\alpha}\overline{z}_t}{\overline{z}_\alpha}, \qquad z_{ttt} = F_{tt} + \frac{z_{tt\alpha}}{\overline{z}_\alpha}\overline{z}_t + \frac{(\overline{z}_{tt\alpha}\overline{z}_t + z_{t\alpha}\overline{z}_{tt})\overline{z}_\alpha - \overline{z}_{t\alpha}z_{t\alpha}\overline{z}_t}{\overline{z}_\alpha^2}.$$

Since F_{tt} is anti-holomorphic, it follows that

$$\int_0^{2\pi}(\partial_t + b\partial_{\alpha'})w\overline{\zeta}_{\alpha'}d\alpha' = \int_0^{2\pi} w_{\alpha'}\overline{u}\,d\alpha' + \int_0^{2\pi}\frac{(\overline{w}_{\alpha'}\overline{u} + u_{\alpha'}\overline{w})\overline{\zeta}_{\alpha'} - \overline{u}_{\alpha'}u_{\alpha'}u}{\overline{\zeta}_{\alpha'}}d\alpha'$$

which can be bounded by the right hand side of (5.20). This completes the proof of (5.20) and hence of the lemma. □

Corollary 5.13. *If M_0 in (4.6) is sufficiently small then for all $j \le \ell$ and with $\Theta = \chi$ or v*

$$\int_0^{2\pi}\left|\frac{1}{A}\frac{a_t}{a}\circ k^{-1}\right|\left(|(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^j\Theta|^2 + \pi|\partial_{\alpha'}^j\Theta|^2\right)d\alpha' \le C\mathcal{E}^2.$$

Proof. This is a direct corollary of the definition of $\mathcal{E}$, Lemma 5.12, Proposition 4.6, and Corollaries 4.4 and 4.11. □

The last step before stating the main result of this section is to obtain an expression for the time derivative of b and then estimates for it.

Proposition 5.14. *Suppose that k is given as in Remark 3.21 and that it is increasing. Then $k_{tt} = (\partial_t + b\partial_\alpha)b \circ k$ satisfies*

$$\begin{aligned}(I-H)k_{tt} = &-i(I-H)\frac{\overline{z}_{tt}\varepsilon + \overline{z}_t\varepsilon_t}{\overline{z}} + i(I-H)\frac{\overline{z}_t^2\varepsilon}{\overline{z}^2}\\ &- i[z_t, H]\frac{(\log(\overline{z}e^{ik}))_{t\alpha} + ik_{t\alpha}}{z_\alpha} + i[z_t, H]\frac{1}{z_\alpha}\partial_\alpha\left(\frac{\overline{z}_t\varepsilon}{\overline{z}}\right)\\ &- i[z_{tt}, H]\frac{(\log(\overline{z}e^{ik}))_\alpha}{z_\alpha} - \frac{1}{\pi}\int_0^{2\pi}\left(\frac{z_t(\beta) - z_t(\alpha)}{z(\beta) - z(\alpha)}\right)^2(\log(\overline{z}e^{ik}))_\beta d\beta\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\mathcal{AV}(\partial_t k_t) = &\frac{\operatorname{Im}}{2\pi}\int_0^{2\pi}\left(\frac{z_{t\beta}z - z_t z_\beta}{z^2}\right)k_t d\beta\\ &+ \frac{\operatorname{Re}}{2\pi}\partial_t\int_0^{2\pi}\frac{\overline{z}_t\varepsilon}{|z|^2}z_\beta d\beta + \frac{\operatorname{Re}}{2\pi}\partial_t\int_0^{2\pi}(\log(\overline{z}e^{ik}))_\beta\frac{z_t}{z}d\beta.\end{aligned}$$

Proof. Differentiating the first formula in Proposition 3.18 with respect to time, we obtain

$$\begin{aligned}(I-H)k_{tt} &= \partial_t(I-H)k_t + [z_t, H]\frac{k_{t\alpha}}{z_\alpha}\\ &= -i\partial_t(I-H)\frac{\bar{z}_t\varepsilon}{\bar{z}} - i\partial_t[z_t, H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha} + [z_t, H]\frac{k_{t\alpha}}{z_\alpha}\\ &=: I + II + III.\end{aligned}$$

Direct computations imply that

$$\begin{aligned}I &= -i(I-H)\frac{\bar{z}_{tt}\varepsilon + \bar{z}_t\varepsilon_t}{\bar{z}} + i(I-H)\frac{\bar{z}_t^2\varepsilon}{\bar{z}^2} + i[z_t, H]\frac{1}{z_\alpha}\partial_\alpha\left(\frac{\bar{z}_t\varepsilon}{\bar{z}}\right)\\ II &= -i[z_t, H]\frac{\left(\log(\bar{z}e^{ik})\right)_{t\alpha}}{z_\alpha} - i[z_{tt}, H]\frac{\left(\log(\bar{z}e^{ik})\right)_\alpha}{z_\alpha}\\ &\quad - \frac{1}{\pi}\int_0^{2\pi}\left(\frac{z_t(\beta)-z_t(\alpha)}{z(\beta)-z(\alpha)}\right)^2\left(\log(\bar{z}e^{ik})\right)_\beta\, d\beta.\end{aligned}$$

Putting all these together, the first formula in the proposition follows. The second formula follows from differentiating the last formula in Proposition 3.20 with respect to time. □

We are finally ready to prove the main result of this section.

Proposition 5.15. *If M_0 in (4.6) is sufficiently small then with R_k as in Lemma 5.5*

$$\sum_{k\leq\ell}(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t)) \lesssim \sum_{k\leq\ell}(\mathcal{E}_k^\chi(0)+\mathcal{E}_k^{\tilde{v}}(0)) + \sum_{k\leq\ell}(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t))^{\frac{3}{2}} + \sum_{k\leq\ell}(\mathcal{E}_k^\chi(t)+\mathcal{E}_k^{\tilde{v}}(t))^2 + \sum_{k\leq\ell}\int_0^t(\mathcal{E}_k^\chi(s)+\mathcal{E}_k^{\tilde{v}}(s))^2\, ds.$$

Proof. By Corollary 5.10 we only need to estimate the nonlinear term R_k. Here R_k is defined in Lemma 5.5 and G_j is given in (5.5) as

$$G_j = \partial_{\alpha'}^j G - \sum_{i=1}^j \partial_{\alpha'}^{j-i}\left(b_{\alpha'}(\partial_t + b\partial_{\alpha'})\partial_{\alpha'}^i\Theta + b_{\alpha'}^2\partial_\alpha^i\Theta + (\partial_t + b\partial_{\alpha'})\left(b_{\alpha'}\partial_{\alpha'}^i\Theta\right) + iA_{\alpha'}\partial_{\alpha'}^i\Theta\right). \tag{5.23}$$

It follows from Corollary 5.13 that we only need to consider the first integral in the expression for R_k in Lemma 5.5. In particular we need to show that

$$\|G_j\|_{L^2_{\alpha'}}^2 \lesssim \mathcal{E}^3. \tag{5.24}$$

We begin with the contribution of $\partial_{\alpha'}^j G$. When $\Theta = \chi$ this is already dealt with in Propositions 4.8 and 5.6 and Corollaries 4.11 and 5.13. When $\Theta = \tilde{v}$ we use the equation derived for $\tilde{v}$ in Proposition 5.11. But then in view of Proposition 4.8, the contribution of $\partial_{\alpha'}^j G$ when $\Theta = \tilde{v}$ is also handled by similar arguments as before using Lemmas 2.6, 2.5, 2.4, Propositions 2.2, 2.3, 4.1, 5.6, and Corollaries 4.3, 4.11. We omit the details. To estimate the contribution of the second term on the right hand side of (5.23) we note that

$$(\partial_t + b\partial_{\alpha'})b_{\alpha'} = \partial_{\alpha'}(\partial_t + b\partial_{\alpha'})b - b_{\alpha'}^2$$

and use Proposition 5.14 to express $(\partial_t + b\partial_{\alpha'})b$ in terms of quantities we can already control. Here we also use the observation that

$$(\log(\overline{z}e^{ik}))_{\alpha'} = \frac{\overline{\eta}}{\overline{\zeta}}$$

and that $\partial_t \eta = u_{\alpha'} - iu$. The proof of the proposition can now be completed by appealing to Propositions 4.7, 4.10, 5.6 and Corollary 4.11. □

6. Long Time Well-Posedness

In this final section we prove long-time existence for solutions of the system

$$\begin{cases} z_{tt} + iaz_\alpha = -\frac{\pi}{2}(I - \overline{H})z, \quad \overline{z}_t = H\overline{z}_t \\ z(0, \alpha) = z_0(\alpha), \quad z_t(0, \alpha) = z_1(\alpha) \end{cases}, \tag{6.1}$$

with small initial data. More precisely we will complete the proof of Theorem 6.2. This section is divided into two parts. To use the energy estimates from the previous section we need to transfer the smallness of the data for Eq. (6.1) to the initial smallness of the quantities appearing in the bootstrap assumption (4.6) and the initial energy defined in the previous section. This will be accomplished in Sect. 6.1. Then in Sect. 6.2 we will establish Theorem 6.2, by showing long-time existence of solutions to (6.1) assuming that initially the bootstrap assumptions (4.6) hold and that the energy defined in the previous section is sufficiently small.

6.1. A discussion for initial data. We consider initial data $z_0(\alpha) = e^{i\alpha} + \epsilon f(\alpha)$ and $z_1(\alpha) = \epsilon g(\alpha)$ for the system (6.1) such that z_0 is a simple closed curve containing the origin in the interior, parametrized counterclockwisely, and such that $(f, g) \in H^s_\alpha \times H^s_\alpha$, $s \geq 15$. Furthermore, we assume

$$\sup_{\alpha \neq \beta} |z_0(\alpha) - z_0(\beta)| \geq \lambda |e^{i\alpha} - e^{i\beta}|$$

for some $\lambda > 0$. We let H_0 be the Hilbert transform associated to the initial domain $\Omega(0)$ bounded by z_0 and $k_0(\alpha) = k(0, \alpha)$ be defined according to Remark 3.21. Using Eq. (6.1) we can now uniquely determine initial values z_2 and a_0 for z_{tt} and a respectively. Here to get the initial value for a one can for instance use the Riemann mapping formulation of the problem as discussed in Sect. 7. Alternatively one could use the double-layered potential as in [40], see also [23,24] and [37]. More precisely, let us write (3.1) as

$$i(a - \pi)\overline{z}_\alpha = \overline{z}_{tt} - i\pi(\overline{z}_\alpha + i\overline{z}) - \frac{\pi}{2}(I + H)\overline{z}. \tag{6.2}$$

Applying $(I - H)$ on both sides we obtain

$$i(I - H)\left((a - \pi)\overline{z}_\alpha\right) = (I - H)\overline{z}_{tt} - i\pi(I - H)(\overline{z}_\alpha + i\overline{z}). \tag{6.3}$$

Using the holomorphicity of $\overline{z}_t$ and multiplying both sides of (6.3) by $\frac{-iz_\alpha}{|z_\alpha|}$ then taking the real part, we get

$$(I+K^*)\left((a-\pi)|z_\alpha|\right) = -\mathrm{Re}\left\{\frac{iz_\alpha}{|z_\alpha|}\left([z_t, H]\frac{\overline{z}_{t\alpha}}{z_\alpha} - \pi i(I-H)(\overline{z}_\alpha + i\overline{z})\right)\right\}. \quad (6.4)$$

Note that $z_\alpha = ie^{i\alpha} + \epsilon f_\alpha(\alpha)$. An argument similar to the proof of Lemma 5.12 using (6.2) and (6.4) implies that

$$\|a-\pi\|_{L^2_\alpha} \lesssim \|z_\alpha - iz\|_{L^2_\alpha} + \epsilon\|z_t\|_{L^2_\alpha} \quad (6.5)$$

if ϵ is small enough. The H^s_α estimate for $(a-\pi)$ can be derived similarly:

$$\|(a-\pi)|z_\alpha|\|_{H^s_\alpha} \lesssim \|z_\alpha - iz\|_{H^s_\alpha} + \epsilon\|z_t\|_{H^s_\alpha}. \quad (6.6)$$

As for a the initial value for z_{tt} can be determined and estimated using the Eq. (3.1)

$$\overline{z}_{tt} = i(a-\pi)\overline{z}_\alpha + \pi i(\overline{z}_\alpha + i\overline{z}) + \frac{\pi}{2}(I+H)\overline{z}. \quad (6.7)$$

Finally we let $k_1(\alpha) = \partial_t k(\alpha, 0)$, where k is extended using Theorem 3.2 and Remark 3.21.

Our goal in this subsection is to prove the following proposition.

Proposition 6.1. *Let $z_0,\ z_1,\ f,\ g,\ z_2,\ k_0,\ k_1,\ a_0,$ and H_0 be defined as above and let $M_0 > 0$ and $\ell \in \mathbb{N},\ \ell \leq s-2$ be fixed constants. Then there exists $\epsilon_0 > 0$, depending only on $\|f\|_{H^s_\alpha}$ and $\|g\|_{H^s_\alpha}$, such that if $\epsilon < \epsilon_0$ then k_0 is a diffeomorphism and*

$$\|k_{0,\alpha} - 1\|_{L^\infty_\alpha} \leq \frac{1}{2}, \quad \|k_\alpha - 1\|_{H^{s-1}_\alpha} \lesssim \|z_\alpha - iz\|_{H^{s-1}_\alpha}. \quad (6.8)$$

Moreover, if $\epsilon < \epsilon_0$ and we define

$$\zeta_0 := z_0 \circ k_0^{-1}, \quad \eta_0 := \partial_\alpha\zeta_0 - i\zeta_0, \quad u_0 := z_1 \circ k_0^{-1}, \quad w_0 := z_2 \circ k_0^{-1}, \quad (6.9)$$

then

$$\sum_{j\leq\ell}\left(\|\partial_\alpha^j\eta_0\|_{L^2_\alpha} + \|\partial_\alpha^j u_0\|_{L^2_\alpha} + \|\partial_\alpha^j w_0\|_{L^2_\alpha}\right) \leq \frac{M_0}{2}, \qquad |\zeta_0|^2 \geq \frac{1}{2}. \quad (6.10)$$

Finally if we extend $z_0,\ z_1$ to a local-in-time solution (z, z_t) of (6.1), with the corresponding Hilbert transform H, and we define $b_0 := k_1 \circ k^{-1},\ A_0 =: (a_0\partial_\alpha k_0)\circ k_0^{-1}$, and

$$\varepsilon := |z|^2 - 1, \quad \delta := (I-H)\varepsilon_0, \quad \chi := \delta \circ k^{-1}, \quad v = \delta_t \circ k^{-1}, \quad \tilde{v} = (I-H)v,$$

then if $\epsilon < \epsilon_0$

$$\mathcal{E}(0) := \sum_{j\leq\ell}\left(\int_0^{2\pi}\frac{((\partial_t + b_0\partial_\alpha)\partial_\alpha^j\chi)|_{t=0}}{A_0}d\alpha + \int_0^{2\pi}\frac{((\partial_t + b_0\partial_\alpha)\partial_\alpha^j\tilde{v})|_{t=0}}{A_0}d\alpha\right) \leq R_0\epsilon^2, \quad (6.11)$$

for a fixed $R_0 > 0$ independent of ϵ.

Proof. Let the $F(\cdot)$ be the holomorphic function with the boundary value $\bar{z}_0 e^{ik_0}$. Differentiating the equation $(I - H_0)(\bar{z}_0 e^{ik_0}) = 0$ with respect to α we get

$$(I - H_0)k_{0,\alpha} = i(I - H_0)\frac{\bar{z}_{0,\alpha} + i\bar{z}_0}{\bar{z}_0} - i[z_{0,\alpha} - iz_0, H_0]\frac{\partial_\alpha(\log F)}{z_{0,\alpha}}. \tag{6.12}$$

On the other hand, for the initial data we have

$$\|z_{0,\alpha} - iz_0\|_{H^s_\alpha}, \quad \|z_1\|_{H^s_\alpha}, \quad \|z_2\|_{H^s_\alpha} \leq C_0\epsilon. \tag{6.13}$$

In fact, the first two estimates are straightforward from the construction of z_0 and z_1 and the last one follows from (6.6) and (6.7). Equation (6.13) together with the relation

$$k_\alpha - 1 = \frac{i\bar{z}_\alpha - \bar{z}}{\bar{z}} - i\partial_\alpha(\log F) \tag{6.14}$$

implies that

$$\mathcal{AV}(k_\alpha - 1) \lesssim \|z_\alpha - iz\|_{H^1_\alpha}. \tag{6.15}$$

Here we used the fact that $\|\log F\|_{L^\infty_\alpha}$ is bounded by an absolute constant, which follows from the definition of F. Therefore writing $k_{0,\alpha}$ in terms of $\mathrm{Re}\,(I - H_0)k_{0,\alpha}$ gives the desired estimate (6.8) for k_α. The other statements of the proposition follow from (6.8), the relation

$$\partial_\alpha\left(f \circ k^{-1}\right) = \frac{f_\alpha \circ k^{-1}}{k_\alpha \circ k^{-1}}, \tag{6.16}$$

and arguments similar to those in Sect. 5 □

6.2. Completion of the proof. In view of Proposition 6.1 the proof of long-time well-posedness will be complete once we prove the following theorem.

Theorem 6.2. *Let z_0, z_1 be as in Proposition 6.1 and denote by $z(t,\alpha)$ the local-in-time solution of (6.1). Then there exist constant M_0, c, and ϵ_1 such that if (6.10) and (6.11) hold with $\epsilon < \epsilon_1$ then (6.1) has a unique classical solution in $[0, \frac{c}{\epsilon^2}]$.*

Proof. Let $T^* > 0$ be the maximal time of existence guaranteed by Theorem 3.2. We want to show that $T^* \geq \frac{c}{\epsilon^2}$ for some c independent of ϵ. Let $T \leq T^*$ be defined as

$$T := \sup\left\{t \in [0, T^*) \mid k_\alpha(t,\alpha) > \frac{1}{100},\ \forall\alpha \in [0, 2\pi]\right\}.$$

In particular k is a diffeomorphism and continuous in time for all $t \leq T$. Moreover, the energy $\mathcal{E}(t)$ defined in (5.17) is continuous in $[0, T]$. Next, define $T_{M_0} \leq T$ as

$$T_{M_0} := \sup\left\{t \leq T \mid \sum_{j\leq\ell}\left(\|\partial^j_{\alpha'}\eta\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}u\|_{L^2_{\alpha'}} + \|\partial^j_{\alpha'}w\|_{L^2_{\alpha'}}\right) \leq M_0\right\},$$

and $T_\epsilon \leq T$ as

$$T_\epsilon := \sup\left\{t \leq T \mid \mathcal{E}^{\frac{1}{2}}(t) \leq 2CR_0\epsilon\right\},$$

where C is the constant in Proposition 5.15.

Step 1. We show that $T_\epsilon \leq T_{M_0}$, provided ϵ_1 is sufficiently small. Indeed, if this is not the case then by Corollary 4.11 for all $t \in [0, T_{M_0}]$

$$\sum_{j\leq\ell}\left(\|\partial_{\alpha'}^j\eta\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j u\|_{L^2_{\alpha'}} + \|\partial_{\alpha'}^j w\|_{L^2_{\alpha'}}\right) \leq C_1(M_0)R_0\epsilon,$$

and choosing $\epsilon_1 \leq \frac{M_0}{2C_1(M_0)R_0}$ we get a contradiction with the maximality of T_{M_0}.

Step 2. We show that there exists a constant $c_1 = c_1(M_0, R_0)$ such that if ϵ_1 is sufficiently small and $T \leq T_0 := \frac{c_1}{\epsilon^2}$ then $T_\epsilon = T$, and hence by the previous step $T_{M_0} = T_\epsilon = T$. To see this, assume the contrary and first let ϵ_1 be so small that the conclusion of the previous step holds. Then we can apply Proposition 5.15 with $t = \frac{c_1}{\epsilon^2} \leq T_\epsilon$, and conclude that if ϵ_1 and c_1 are sufficiently small then $\mathcal{E}^{\frac{1}{2}}(t) \leq 2CR_0\epsilon$ proving the claim by contradiction. Here note that since $t \leq T_\epsilon$ the last integral in the statement of Proposition 5.15 can be bounded by $16c_1R_0^4C^4\epsilon^2 < 4C^2R_0^2\epsilon^2$ if c_1 is sufficiently small.

Step 3. We show that there exists $c_2 = c_2(M_0, R_0)$ such that if ϵ_1 is sufficiently small and $T_1 := \frac{c_2}{\epsilon^2} \leq T_0$ then $k_\alpha \geq \frac{1}{100}$ for all $t \in [0, \min\{T^*, T_1\})$. Suppose ϵ_1 is small enough that the conclusions of the previous two steps hold. From the definition of b

$$\partial_t k_\alpha = (b_{\alpha'} \circ k)\, k_\alpha,$$

and hence

$$k_\alpha(t,\alpha) = k_\alpha(0,\alpha)e^{\int_0^t (b_{\alpha'}\circ k)(s,\alpha)ds} \geq k_\alpha(0,\alpha)e^{-\int_0^t \|b_{\alpha'}\|_{L^\infty_{\alpha'}}ds}. \tag{6.17}$$

But then by Proposition 4.10 and Corollary 4.11 if $t \leq \min\{T_1, T^*\}$ and c_2 is sufficiently small it follows that

$$k_\alpha(t,\alpha) \geq k_\alpha(0,\alpha)e^{-c_2C_2(M_0)} \geq \frac{1}{100}.$$

Step 4. Finally we show that $T^* \geq \frac{c}{\epsilon^2}$ for a sufficiently small constant c. By Theorem 3.2 it suffices to show that if $T^* < \frac{c}{\epsilon^2}$ the H^{10}_α norms of z_t and z_{tt} remain bounded for $t < T^*$ and

$$\sup_{0\leq t<T^*}\sup_{\alpha\neq\beta}\left|\frac{e^{i\alpha}-e^{i\beta}}{z(t,\alpha)-z(t,\beta)}\right| < \infty. \tag{6.18}$$

Let ϵ_1 be small enough that the conclusions of the previous steps hold, and let $c = c_2$ be as in Step 3. Then if $T^* < \frac{c_2}{\epsilon}$, it follows from the previous three steps that $T_\epsilon = T_{M_0} = T = T^*$. By Corollary 4.2, $|\zeta_{\alpha'}(t,\alpha)| \geq \frac{1}{2}$ for all $t \leq T^*$ and all $\alpha \in [0, 2\pi]$, and therefore combining with the fact that $k_\alpha \geq \frac{1}{100}$ we get (6.18). Moreover, from the definition of T_{M_0} the $H^{10}_{\alpha'}$ norms of u and w are bounded up to T^*, so by the chain rule, we only need to prove that the derivatives of k up to order 10 remain bounded for $t \in [0, T^*)$. But this follows from Proposition 4.10 and successive differentiation of the first identity in (6.17). □

7. Riemann Mapping Coordinates and Local Well-Posedness

In this section we outline the proof Theorem 3.2 by investigating the quasilinear structure of the equation

$$\bar{z}_{tt} - ia\bar{z}_\alpha = -\frac{\pi}{2}(I - H)\bar{z} = -\pi\bar{z} + \frac{\pi}{2}(I + H)\bar{z}. \tag{7.1}$$

More precisely, we find a quasilinear equation whose well-posedness implies that of Eq. (7.1). This is achieved by differentiating (7.1) with respect to time and exploiting the holomorphicity of various quantities. Once the equivalent quasilinear system is found, the proof of well-posedness is standard and follows for instance from the vanishing-viscosity method in [38]. To avoid repetition we only prove the equivalence of (7.1) with a quasilinear equation and refer the reader to [38] for the details of the vanishing-viscosity method.

To get a quasilinear equation we differentiate (7.1) with respect to time, noting that $(I + H)\bar{z}$ is the boundary value of a holomorphic function in $\Omega(t)$, to get

$$\bar{z}_{ttt} - ia\bar{z}_{t\alpha} = ia_t\bar{z}_\alpha + \frac{\pi}{2}[z_t, H]\frac{\bar{z}_\alpha}{z_\alpha}, \quad H\bar{z}_t = \bar{z}_t. \tag{7.2}$$

Even though the proof of local existence for (7.2) can be carried out in these coordinates, the structure of the equation will be more clear in Riemann mapping coordinates which we now introduce. Since we are interested in local existence, we fix a point $\mathbf{x}_0 \in \Omega(0)$ and define the Riemann mapping for t such that $\mathbf{x}_0 \in \Omega(t)$. For such t we define the Riemann mapping $\Phi(t.\cdot) : \Omega(t) \to \mathbb{D}$ using the normalization $\Phi(t, \mathbf{x}_0) = 0$ and $\Phi_z(t, \mathbf{x}_0) > 0$ (in particular $\Phi_z(t, \mathbf{x}_0)$ is real). To Φ we associate the coordinate change $h : \mathbb{R} \to \mathbb{R}$ defined by $e^{ih(t,\alpha)} = \Phi(t, z(t, \alpha))$. Alternatively let $\chi_1(\cdot) = z(0, \cdot) : [0, 2\pi] \to \partial\Omega(0)$ be the parametrization of the initial boundary and extend the definition of $\chi_1(\cdot)$ to $\mathbb{R}$ periodically. Similarly let $X(t, \cdot) : \Omega(0) \to \Omega(t)$ denote the flow of the velocity vector field, that is, $\dot{X}(t, \cdot) = \mathbf{v}(t, X(t, \cdot))$. Finally let $\chi_2(\cdot) := -i\log(\cdot) : \partial\mathbb{D} \to \mathbb{R}$ be the inverse parametrization of the boundary of the unit disc. In this notation h is the composition change of variables $h := \chi_2 \circ \Phi \circ X \circ \chi_1$,

$$\mathbb{R} \xrightarrow{\chi_1(\cdot):=z(0,\cdot)} \partial\Omega(0) \subseteq \Omega(0) \xrightarrow{X(t,\cdot)} \Omega(t) \xrightarrow{\Phi(t,\cdot)} \mathbb{D} \supseteq \partial\mathbb{D} \xrightarrow{\chi_2(\cdot)=-i\log(\cdot)} \mathbb{R}, \qquad \mathbb{R} \xrightarrow{h(t,\cdot)} \mathbb{R}$$

and the new unknowns in Riemann mapping coordinates are

$$Z(t, \alpha') := z(t, h^{-1}(t, \alpha')), \quad Z_t(t, \alpha') := z_t(t, h^{-1}(t, \alpha')),$$
$$Z_{tt}(t, \alpha') = z_{tt}(t, h^{-1}(\alpha')), \quad Z_{ttt}(t, \alpha') := z_{ttt}(t, h^{-1}(t, \alpha')).$$

To avoid confusion we separate the subscripts corresponding to partial differentiation by a comma, so for instance $Z_{,\alpha'}(t, \alpha') = \partial_{\alpha'}Z(t, \alpha')$. We denote by $\mathbb{H}$ the Hilbert transform on the circle which can be written as

$$
\begin{aligned}
\mathbb{H} f(\alpha') &:= \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(\beta')}{e^{i\beta'} - e^{i\alpha'}} i e^{i\beta'} d\beta' \\
&= \frac{\text{p.v.}}{2\pi i} \int_0^{2\pi} f(\beta') \cot\left(\frac{\beta' - \alpha'}{2}\right) d\beta' + \frac{1}{2\pi} \int_0^{2\pi} f(\beta') d\beta' \\
&= \widetilde{\mathbb{H}} f(\alpha') + \text{Av}(f), \qquad (7.3)
\end{aligned}
$$

where

$$
\widetilde{\mathbb{H}} f(\alpha') := \frac{\text{p.v.}}{2\pi i} \int_0^{2\pi} f(\beta') \cot\left(\frac{\beta' - \alpha'}{2}\right) d\beta', \quad \text{Av}(f) := \frac{1}{2\pi} \int_0^{2\pi} f(\beta') d\beta'.
$$

For notational convenience we also introduce the following new variables and operators in Riemann mapping coordinates:

$$
\mathcal{A} := (a h_\alpha) \circ h^{-1}, \quad G := \frac{\pi}{2}((I + H)\overline{z}) \circ h^{-1}
$$

and

$$
\mathcal{H} f(\alpha') := \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(\beta')}{Z(t, \beta') - Z(t, \alpha')} Z_{,\beta'}(t, \beta') d\beta'.
$$

With this notation, precomposing with $h^{-1}(t, \cdot)$ we can rewrite Eqs. (7.1) and (7.2) as

$$
\overline{Z}_{tt} - i\mathcal{A}\overline{Z}_{,\alpha'} = -\pi \overline{Z} + G \qquad (7.4)
$$

and

$$
\overline{Z}_{ttt} - i\mathcal{A}\overline{Z}_{t,\alpha'} = i\frac{a_t}{a} \circ h^{-1} \mathcal{A}\overline{Z}_{,\alpha'} + \frac{\pi}{2}[Z_t, \mathcal{H}] \frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}, \quad \mathbb{H}\overline{Z}_t = \overline{Z}_t. \qquad (7.5)
$$

Note that if we let

$$
B := h_t \circ h^{-1}
$$

we can rewrite (7.5) as

$$
\left((\partial_t + B\partial_{\alpha'})^2 - i\mathcal{A}\partial_{\alpha'}\right) \overline{Z}_t = i\frac{a_t}{a} \circ h^{-1} \mathcal{A}\overline{Z}_{,\alpha'} + \frac{\pi}{2}[Z_t, \mathcal{H}] \frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}. \qquad (7.6)
$$

To understand the quasilinear structure of this equation we need to compute $\mathcal{A}$, B, and $\frac{a_t}{a} \circ h^{-1}$ in terms of the unknowns. We begin with $\mathcal{A}$, where in addition we verify that $\mathcal{A}$ is in fact a positive quantity so that the Taylor sign-condition holds.

Proposition 7.1. *$\mathcal{A}_1 := \mathcal{A}|Z_{,\alpha'}|^2$ is positive and is given by*

$$
\begin{aligned}
\mathcal{A}_1 &= \text{Im}\,[Z_t, \widetilde{\mathbb{H}}]\overline{Z}_{t,\alpha'} + \pi \,\text{Im}\,[\overline{Z}, \widetilde{\mathbb{H}}] Z_{,\alpha'} \\
&= \frac{1}{8} \int_0^{2\pi} |Z(t, \beta') - Z(t, \alpha')|^2 \csc^2\left(\frac{\beta' - \alpha'}{2}\right) d\beta' \\
&\quad + \frac{1}{8\pi} \int_0^{2\pi} |Z_t(t, \beta') - Z_t(t, \alpha')|^2 \csc^2\left(\frac{\beta' - \alpha'}{2}\right) d\beta' > 0.
\end{aligned}
$$

Proof. Multiplying (7.4) by $Z_{,\alpha'}$ we get

$$i\mathcal{A}_1 = i\mathcal{A}|Z_{,\alpha'}|^2 = \overline{Z}_{tt}Z_{,\alpha'} + \pi\overline{Z}Z_{,\alpha'} - GZ_{,\alpha'}. \tag{7.7}$$

Note that since $\Phi(t, Z(t,\alpha')) = e^{i\alpha'}$ and Φ_z is non-vanishing,

$$Z_{,\alpha'} = \frac{ie^{i\alpha'}}{\Phi_z(t,Z)}$$

is holomorphic inside $\mathbb{D}$. Moreover writing $\overline{z}_t = F(t,z)$ where F is holomorphic inside $\Omega(t)$ we get

$$\overline{z}_{tt} = F_t(t,z) + F_z(t,z)z_t = F_t(t,z) + \frac{\overline{z}_{t\alpha}z_t}{z_\alpha},$$

and hence

$$\overline{Z}_{tt} = F_t(t,Z) + \frac{\overline{Z}_{t,\alpha'}Z_t}{Z_{,\alpha'}}.$$

Therefore we can apply $(I - \mathbb{H})$ to (7.7) to get

$$i(I-\mathbb{H})\mathcal{A}_1 = (I-\mathbb{H})(\overline{Z}_{t,\alpha'}Z_t) + \pi(I-\mathbb{H})(\overline{Z}Z_{,\alpha'}).$$

Taking imaginary parts of the two sides, keeping in mind that $\mathcal{A}_1$ is real, yields

$$\mathcal{A}_1 - \mathrm{Av}(\mathcal{A}_1) = \mathrm{Im}\,[Z_t,\mathbb{H}]\overline{Z}_{t,\alpha'} + \pi\,\mathrm{Im}\,[\overline{Z},\mathbb{H}]Z_{,\alpha'}. \tag{7.8}$$

Note that from (7.7)

$$\mathrm{Av}(\mathcal{A}_1) = -i\,\mathrm{Av}(\overline{Z}_{t,\alpha'}Z_t) - \pi i\,\mathrm{Av}(\overline{Z}Z_{,\alpha'}). \tag{7.9}$$

Also

$$\begin{aligned}[\overline{Z},\mathbb{H}]Z_{,\alpha'} &= \overline{Z}\widetilde{\mathbb{H}}Z_{,\alpha'} + \overline{Z}\mathrm{Av}(Z_{,\alpha'}) - \widetilde{\mathbb{H}}(\overline{Z}Z_{,\alpha'}) - \mathrm{Av}(\overline{Z}Z_{,\alpha'})\\ &= [\overline{Z},\widetilde{\mathbb{H}}]Z_{,\alpha'} - \mathrm{Av}(\overline{Z}Z_{,\alpha'})\end{aligned}$$

and

$$[Z_t,\mathbb{H}]\overline{Z}_{t,\alpha'} = [Z_t,\widetilde{\mathbb{H}}]\overline{Z}_{t,\alpha'} - \mathrm{Av}(\overline{Z}_{t,\alpha'}Z_t).$$

Using the fact that $\mathrm{Av}(\overline{Z}Z_{,\alpha'})$ and $\mathrm{Av}(\overline{Z}_{t,\alpha'}Z_t)$ are purely imaginary, these computations and (7.8) combine to give

$$\mathcal{A}_1 = \mathrm{Im}\,[Z_t,\widetilde{\mathbb{H}}]\overline{Z}_{t,\alpha'} + \pi\,\mathrm{Im}\,[\overline{Z},\widetilde{\mathbb{H}}]Z_{,\alpha'}.$$

To see that the right hand side above is positive note that

$$\begin{aligned}\mathrm{Im}\,([\overline{Z},\widetilde{\mathbb{H}}]Z_{,\alpha'}) = -\frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi} &\left(\overline{Z}(t,\alpha') - \overline{Z}(t,\beta')\right)\cot\left(\frac{\beta'-\alpha'}{2}\right)\partial_{\beta'}\left(Z(t,\beta')\right.\\ &\left.-Z(t,\alpha')\right)d\beta'\end{aligned}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \cot\left(\frac{\beta' - \alpha'}{2}\right) \partial_{\beta'} |Z(t, \beta') - Z(t, \alpha')|^2 d\beta'$$
$$= \frac{1}{8\pi} \int_0^{2\pi} |Z(t, \beta') - Z(t, \alpha')|^2 \csc^2\left(\frac{\beta' - \alpha'}{2}\right) d\beta' > 0$$

and

$$\begin{aligned}\operatorname{Im}[Z_t, \widetilde{\mathbb{H}}]\overline{Z}_{t,\alpha'} &= -\frac{1}{2\pi}\operatorname{Re} \int_0^{2\pi} \left(Z_t(t,\alpha') - Z_t(t,\beta')\right) \cot\left(\frac{\beta'-\alpha'}{2}\right) \partial_{\beta'} \left(\overline{Z}_t(t,\beta') - \overline{Z}_t(t,\alpha')\right) d\beta' \\ &= \frac{1}{4\pi} \int_0^{2\pi} \partial_{\beta'} |Z_t(t, \beta') - Z_t(t, \alpha')|^2 \cot\left(\frac{\beta' - \alpha'}{2}\right) d\beta' \\ &= \frac{1}{8\pi} \int_0^{2\pi} |Z_t(t, \beta') - Z_t(t, \alpha')|^2 \csc^2\left(\frac{\beta' - \alpha'}{2}\right) d\beta' > 0.\end{aligned}$$

□

The computation for $\frac{a_t}{a} \circ h^{-1}$ is more involved. In order to state the result we introduce the notation

$$D_\alpha := \frac{1}{z_\alpha} \partial_\alpha, \qquad D_{\alpha'} := \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \tag{7.10}$$

We also define

$$\begin{aligned}[Z_t, Z_t; D_{\alpha'}\overline{Z}_t] &:= -[Z_t^2, \mathbb{H}]\partial_{\alpha'} D_{\alpha'} \overline{Z}_t + 2[Z_t, \mathbb{H}]\partial_{\alpha'}((D_{\alpha'}\overline{Z}_t)Z_t) \\ &= -\frac{ie^{i\alpha'}}{\pi} \int_0^{2\pi} \left(\frac{Z_t(t, \beta') - Z_t(t, \alpha')}{e^{i\beta'} - e^{i\alpha'}}\right)^2 \frac{e^{i\beta'}}{Z_{\beta'}(t, \beta')} \overline{Z}_{t,\beta'}(t, \beta') d\beta'.\end{aligned}$$

With this notation we state our next proposition.

Proposition 7.2.

$$\begin{aligned}&\frac{a_t}{a} \circ h^{-1} - \frac{1}{\mathcal{A}_1} \operatorname{Av}\left(\mathcal{A}_1 \frac{a_t}{a} \circ h^{-1}\right) \\ &= \frac{1}{\mathcal{A}_1} \operatorname{Im}\left\{ -\frac{\pi}{2} \left[[Z_t, \mathcal{H}] \frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}, \mathbb{H} \right] Z_{,\alpha'} - \frac{\pi}{2} [(I + \overline{\mathcal{H}})Z, \mathbb{H}]\overline{Z}_{t,\alpha'} \right. \\ &\quad \left. + 2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\overline{Z}_t] \right\},\end{aligned}$$

and

$$\begin{aligned}\operatorname{Av}\left(\mathcal{A}_1 \frac{a_t}{a} \circ h^{-1}\right) &= -2i \operatorname{Av}\left(Z_t \partial_{\alpha'} (\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t) Z_t)\right) - i \operatorname{Av}\left(Z_{tt} \partial_{\alpha'} \overline{Z}_t\right) - i \operatorname{Av}\left(Z_t^2 \partial_{\alpha'} D_{\alpha'} \overline{Z}_t\right) \\ &\quad - i \operatorname{Av}\left(\overline{Z}_{t,\alpha'} Z_{tt}\right) + \frac{\pi i}{2} \operatorname{Av}\left(\overline{Z}_{t,\alpha'} (I + \overline{\mathcal{H}}) Z\right) + \frac{\pi i}{2} \operatorname{Av}\left(Z_{,\alpha'} [Z_t, \mathcal{H}] \frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}\right).\end{aligned}$$

Proof. Multiplying (7.5) by $Z_{,\alpha'}$ gives

$$Z_{,\alpha'}\left(\overline{Z}_{ttt} - i\mathcal{A}\overline{Z}_{t,\alpha'} - \frac{\pi}{2}[Z_t, \mathcal{H}]\frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}\right) = i\mathcal{A}_1\frac{a_t}{a}\circ h^{-1}. \tag{7.11}$$

In order to understand the holomorphicity properties of $\overline{Z}_{ttt}$ we recall that

$$\begin{aligned}
&\bar{z}_t(t,\alpha) = F(t, z(t,\alpha)),\\
&\bar{z}_{tt}(t,\alpha) = F_t(t, z(t,\alpha)) + F_z(t, z(t,\alpha))z_t(t,\alpha),\\
&\bar{z}_{ttt} = F_{tt}(t, z(t,\alpha)) + 2F_{tz}(t, z(t,\alpha))z_t(t,\alpha)\\
&\quad + F_z(t, z(t,\alpha))z_{tt}(t,\alpha) + F_{zz}(t, z(t,\alpha))z_t^2(t,\alpha),\\
&\bar{z}_{t\alpha} = F_z(t, z(t,\alpha))z_\alpha(t,\alpha).
\end{aligned} \tag{7.12}$$

With the notation introduced in (7.10),

$$\begin{aligned}
&F_t\circ z = \bar{z}_{tt} - (D_\alpha\bar{z}_t)z_t, \quad F_z\circ z = D_\alpha\bar{z}_t, \quad F_{zz}\circ z = D_\alpha^2\bar{z}_t,\\
&F_{tz}\circ z = D_\alpha(\bar{z}_{tt} - (D_\alpha\bar{z}_t)z_t)
\end{aligned}$$

where the lasts identity follows form differentiating the first with respect to α. Substituting back into the equation for $\bar{z}_{ttt}$ we get

$$\bar{z}_{ttt} = F_{tt}\circ z + 2z_t D_\alpha(\bar{z}_{tt} - (D_\alpha\bar{z}_t)z_t) + z_{tt}D_\alpha\bar{z}_t + z_t^2 D_\alpha^2\bar{z}_t.$$

We now precompose with h^{-1} to get

$$\overline{Z}_{ttt} = F_{tt}\circ Z + 2Z_t D_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + Z_{tt}D_{\alpha'}\overline{Z}_t + Z_t^2 D_{\alpha'}^2\overline{Z}_t. \tag{7.13}$$

We will substitute this into (7.11) and apply $(I-\mathbb{H})$. To this end we first note that if f is holomorphic then since $Z_{,\alpha'}$ is also holomorphic, $(I-\mathbb{H})(Z_{,\alpha'}f) = 0$, which allows us to compute

$$\begin{aligned}
&(I-\mathbb{H})(Z_{,\alpha'}F_{tt}\circ Z) = 0, \quad (I-\mathbb{H})(Z_{,\alpha'}\overline{Z}_t) = 0,\\
&(I-\mathbb{H})(Z_t\partial_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) = [Z_t, \mathbb{H}]\partial_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t),\\
&(I-\mathbb{H})(Z_{tt}\partial_{\alpha'}\overline{Z}_t) = [Z_{tt}, \mathbb{H}]\partial_{\alpha'}\overline{Z}_t,\\
&(I-\mathbb{H})(Z_t^2\partial_{\alpha'}D_{\alpha'}\overline{Z}_t) = [Z_t^2, \mathbb{H}]\partial_{\alpha'}D_{\alpha'}\overline{Z}_t,
\end{aligned}$$

so

$$(I-\mathbb{H})(Z_{,\alpha'}\overline{Z}_{ttt}) = 2[Z_t, \mathbb{H}]\partial_{\alpha'}(\overline{Z}_{tt} - (D_{\alpha'}\overline{Z}_t)Z_t) + [Z_{tt}, \mathbb{H}]\partial_{\alpha'}\overline{Z}_t + [Z_t^2, \mathbb{H}]\partial_{\alpha'}D_{\alpha'}\overline{Z}_t.$$

In view of (7.4) and holomorphicity of $Z\overline{Z}_{t,\alpha'}$,

$$\begin{aligned}
-i(I-\mathbb{H})(\mathcal{A}\overline{Z}_{t,\alpha'}Z_{,\alpha'}) &= (I-\mathbb{H})\left(\overline{Z}_{t,\alpha'}(Z_{tt} + \pi Z - \overline{G})\right)\\
&= [Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} - [\overline{G}, \mathbb{H}]\overline{Z}_{t,\alpha'}.
\end{aligned}$$

Moreover using Lemma 3.7,

$$-\frac{\pi}{2}(I-\mathbb{H})(Z_{,\alpha'}\partial_t(I-\mathcal{H})\overline{Z}) = \frac{\pi}{2}\left[[Z_t, \mathcal{H}]\frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{,\alpha'}.$$

Putting these together and using the notation introduced before the proposition we get

$$i(I-\mathbb{H})(\mathcal{A}_1\frac{a_t}{a}\circ h^{-1}) = -\frac{\pi}{2}\left[[Z_t,\mathcal{H}]\frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}},\mathbb{H}\right]Z_{,\alpha'} - \frac{\pi}{2}[(I+\overline{\mathcal{H}})Z,\mathbb{H}]\overline{Z}_{t,\alpha'}$$
$$+2[Z_t,\mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt},\mathbb{H}]\overline{Z}_{t,\alpha'} - [Z_t,Z_t;D_{\alpha'}\overline{Z}_t].$$

The first statement of the proposition now follows by taking imaginary parts on both sides of this equation. The second statement follows from taking the averages of the two sides of (7.11) and using (7.13) and (7.4) as well as the facts that by the holomorphicity of F and F_{tt}

$$\mathrm{Av}(Z_{,\alpha'}F_{tt}\circ Z) = \frac{1}{2\pi}\int_{\partial\Omega}F_{tt}dz = 0$$

and

$$\mathrm{Av}(\overline{Z}_{t,\alpha'}Z) = -\frac{1}{2\pi}\int_0^{2\pi}\overline{Z}_t Z_{,\alpha'}d\alpha' = \frac{1}{2\pi}\int_{\partial\Gamma}Fdz = 0.$$

□

Finally we turn to $B := h_t\circ h^{-1}$.

Proposition 7.3. *Suppose the Riemann mapping* Φ *satisfies* $\Phi(t,\mathbf{x}_0)=0$, $\Phi_z(t,\mathbf{x}_0)>0$ *for all* $t\leq T$, *where* T *is such that* $\mathbf{x}_0\in\Omega(t)$ *for* $t\leq T$. *Then* B *satisfies*

$$B-\mathrm{Av}(B) = \mathrm{Re}\left(\left[\frac{Z_t}{e^{i\alpha'}},\mathbb{H}\right]\frac{e^{i\alpha'}}{Z_{,\alpha'}}\right),$$

and

$$\mathrm{Av}(B) = \frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi}\frac{Z_t}{Z_{,\alpha'}}d\alpha'.$$

Proof. Differentiating the equation $\Phi(t,Z(t,\alpha'))=e^{i\alpha'}$ with respect to t gives

$$0=\Phi_t\circ Z+\Phi_z\circ Z(Z_t-BZ_{,\alpha'}) = \Phi_t\circ Z + \frac{ie^{i\alpha'}(Z_t-BZ_{,\alpha'})}{Z_{,\alpha'}}$$

which can be rearranged as

$$B=\frac{\Phi_t\circ Z}{ie^{i\alpha'}}+\frac{Z_t}{Z_{,\alpha'}}.$$

Since $\Phi(t,\mathbf{x}_0)=0$ for all $t\in[0,T]$, applying $(I-\mathbb{H})$ gives

$$(I-\mathbb{H})B = (I-\mathbb{H})\left(\frac{Z_t}{Z_{,\alpha'}}\right) = \left[\frac{Z_t}{e^{i\alpha'}},\mathbb{H}\right]\frac{e^{i\alpha'}}{Z_{,\alpha'}}.$$

Taking the real parts on both sides of above gives

$$B-\mathrm{Av}(B) = \mathrm{Re}\left(\left[\frac{Z_t}{e^{i\alpha'}},\mathbb{H}\right]\frac{e^{i\alpha'}}{Z_{,\alpha'}}\right).$$

Note that

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{\Phi_t(t,\Phi^{-1}(t,e^{i\alpha'}))}{ie^{2i\alpha'}}ie^{i\alpha'}d\alpha'=\frac{1}{i}\Phi_{tz}(t,\mathbf{x}_0)(\Phi^{-1})_w(t,0)$$

is purely imaginary by our choice of normalization for the Riemann mapping. Since B is real, it follows that

$$\mathrm{Av}(B)=\frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi}\frac{Z_t}{Z_{,\alpha'}}d\alpha'.$$

□

Summarizing the computations above, we get the following corollary of (7.4), (7.6), and Propositions 7.1, 7.2, and 7.3.

Corollary 7.4. *If z is a solution to (7.1) and the Riemann mapping Φ is defined according to the normalization above, then $Z := z\circ h^{-1}$ satisfies*

$$\begin{cases}(\partial_t+B\partial_{\alpha'})^2\overline{Z}_t-i\mathcal{A}\partial_{\alpha'}\overline{Z}_t=i\frac{a_t}{a}\circ h^{-1}\frac{\mathcal{A}_1}{Z_{,\alpha'}}+\frac{\pi}{2}[Z_t,\mathcal{H}]\frac{\overline{Z}_{,\alpha'}}{Z_{,\alpha'}}:=g\\ \overline{Z}_t=\mathbb{H}\overline{Z}_t\end{cases},\tag{7.14}$$

where

$$\begin{cases}\mathcal{A}=\frac{|\overline{Z}_{tt}+\frac{\pi}{2}(I-\mathcal{H})\overline{Z}|^2}{\mathcal{A}_1}, \quad \overline{Z}_{tt}=(\partial_t+B\partial_{\alpha'})\overline{Z}_t\\ \frac{1}{Z_{,\alpha'}}=\frac{\overline{Z}_{tt}+\frac{\pi}{2}(I-\mathcal{H})\overline{Z}}{i\mathcal{A}_1}\end{cases},\tag{7.15}$$

and $\mathcal{A}_1$, $\frac{a_t}{a}\circ h^{-1}$, and B are given in Propositions 7.1, 7.2, and 7.3 respectively.

Remark 7.5. The significance of (7.15) is that in proving local well-posedness for (7.14) we will use (7.15) as the *definition* of $\mathcal{A}$ and $\frac{1}{Z_{,\alpha'}}$. As we will discuss below, we will separately show that the resulting solution is a solution of the original system (7.4).

We have now seen how to go from the original system to

$$\begin{cases}(\partial_t+B\partial_{\alpha'})^2V+\mathcal{A}|D|V=\frac{a_t}{a}\circ h^{-1}L+\frac{\pi}{2}[\overline{V},\mathcal{H}]\frac{\overline{W_{\alpha'}}}{W_{\alpha'}}=:g\\ (\partial_t+B\partial_{\alpha'})W=\overline{V}\end{cases},\tag{7.16}$$

where

$$\begin{cases}B-\mathrm{Av}(B)=\mathrm{Re}\left[\frac{\overline{V}}{e^{i\alpha'}},\mathbb{H}\right]\frac{e^{i\alpha'}L}{i\mathcal{A}_1},\\ \mathrm{Av}(B)=\frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi}\frac{\overline{V}L}{i\mathcal{A}_1}d\alpha',\\ \mathcal{A}=\frac{|(\partial_t+B\partial_{\alpha'})V+\frac{\pi}{2}(I-\mathcal{H})\overline{W}|^2}{\mathcal{A}_1},\\ L=(\partial_t+B\partial_{\alpha'})V+\frac{\pi}{2}(I-\mathcal{H})\overline{W},\end{cases}\tag{7.17}$$

and $\mathcal{A}_1$ and $\frac{a_t}{a} \circ h^{-1}$ are defined in Propositions 7.1 and 7.2 with Z, $\overline{Z}_t$, and $\overline{Z}_{tt}$ replaced by W, V, and $(\partial_t + B\partial_{\alpha'})V$ respectively. Here $W = Z$, $V = \overline{Z}_t$, $|D| = \sqrt{-\partial_{\alpha'}^2}$ and we have used the fact that if u is the boundary value of a holomorphic function in the disc, then $|D|u = -i\partial_{\alpha'}\mathbb{H}u = -i\partial_{\alpha'}u$. We next discuss how to go back to the original system from (7.16)–(7.17).

Proposition 7.6. *Suppose* (W, V) *is a solution to* (7.16) *and* (7.17) *on some time interval* J *extending from* $t = 0$ *such that* $\mathbf{x}_0 \in \Omega(t)$ *for all* $t \in J$. *Then the following statements hold.*

(1) W *and* V *are boundary values of holomorphic functions and* $L - \frac{i\mathcal{A}_1}{W_{\alpha'}} = 0$, *if initially* W *and* V *are boundary values of holomorphic functions and* $L - \frac{i\mathcal{A}_1}{W_{\alpha'}} = 0$.

(2) If h *is the solution to*

$$\frac{dh}{dt} = B(h, t), \quad h(0, \alpha) = \alpha,$$

then $z := W \circ h$ *satisfies* (7.1).

Proof. (1) We will derive a linear differential system for the quantities $(I - \mathbb{H})V$, $(I - \mathbb{H})W$ and $\frac{i\mathcal{A}_1}{W_{\alpha'}} - L$ for which uniqueness of solutions holds. Since these quantities are zero initially, they must be zero during the evolution. In this process, we will use $\mathcal{R}$ to denote linear terms in these quantities, whose exact definition may change from line to line. If we want to make the dependence precise, we use expressions such as $\mathcal{R}((I - \mathbb{H})V, (I - \mathbb{H})W, \ldots)$. We start with the equation for W. Applying $(I - \mathbb{H})$ on both sides we get

$$\begin{aligned}
&\partial_t ((I - \mathbb{H})W) + B\partial_{\alpha'} ((I - \mathbb{H})W) = -[B, \mathbb{H}]W_{\alpha'} + (I - \mathbb{H})\overline{V} \\
&= -[B, \mathbb{H}]\frac{(I + \mathbb{H})W_{\alpha'}}{2} + (I - \mathbb{H})\overline{V} + \mathcal{R}_1 \\
&= -\left[\frac{I - \mathbb{H}}{2}B, \mathbb{H}\right]\frac{(I + \mathbb{H})}{2}W_{\alpha'} + (I - \mathbb{H})\overline{V} + \mathcal{R}_1 \\
&= (I - \mathbb{H})\overline{V} - \left[\frac{I - \mathbb{H}}{2}\mathrm{Re}\,[\overline{V}e^{-i\alpha'}, \mathbb{H}]\frac{e^{i\alpha'}}{W_{\alpha'}}, \mathbb{H}\right]\frac{I + \mathbb{H}}{2}W_{\alpha'} + \mathcal{R}_1 + \mathcal{R}_2 \\
&= (I - \mathbb{H})\overline{V} - \left[\frac{I - \mathbb{H}}{2}\mathrm{Re}\,[\overline{V}e^{-i\alpha'}, \mathbb{H}]\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}, \mathbb{H}\right]\frac{I + \mathbb{H}}{2}W_{\alpha'} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \\
&= (I - \mathbb{H})\overline{V} - \left[\frac{I - \mathbb{H}}{2}\left(\frac{\overline{V}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right), \mathbb{H}\right]\frac{I + \mathbb{H}}{2}W_{\alpha'} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \\
&= (I - \mathbb{H})\overline{V} - \left[\frac{\overline{V}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}, \mathbb{H}\right]\frac{I + \mathbb{H}}{2}W_{\alpha'} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \\
&= \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3.
\end{aligned} \tag{7.18}$$

where

$$\mathcal{R}_1 = -[B, \mathbb{H}]\frac{(I-\mathbb{H})}{2}W_{\alpha'},$$
$$\mathcal{R}_2 = -\left[\frac{I-\mathbb{H}}{2}\mathrm{Re}\,[\overline{V}e^{-i\alpha'}, \mathbb{H}]e^{i\alpha'}\left(\frac{L}{i\mathcal{A}_1} - \frac{1}{W_{\alpha'}}\right), \mathbb{H}\right]\frac{I+\mathbb{H}}{2}W_{\alpha'},$$
$$\mathcal{R}_3 = -\left[\frac{I-\mathbb{H}}{2}\mathrm{Re}\,[\overline{V}e^{-i\alpha'}, \mathbb{H}]\left(\frac{e^{i\alpha'}}{W_{\alpha'}} - \frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right), \mathbb{H}\right]\frac{I+\mathbb{H}}{2}W_{\alpha'}. \tag{7.19}$$

Note that in view of Lemma 2.8,

$$\|\mathcal{R}_j\|_{H^s_{\alpha'}} \leq C\left(\left\|L - \frac{i\mathcal{A}_1}{W_{\alpha'}}\right\|_{L^2_{\alpha'}} + \|(I-\mathbb{H})W\|_{L^2_{\alpha'}}\right). \tag{7.20}$$

To derive an equation for $(I-\mathbb{H})V$, we introduce the notation $\mathcal{P} := (\partial_t + B\partial_{\alpha'})^2 + \mathcal{A}|D|$. Then the first equation in (7.16) can be written as

$$\mathcal{P}\left(\frac{I-\mathbb{H}}{2}V\right) = -\mathcal{P}\left(\frac{I+\mathbb{H}}{2}V\right) + e^{-i\alpha'}\frac{ie^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\mathcal{A}_1\frac{a_t}{a}\circ h^{-1} + \frac{\pi}{2}[\overline{V}, \mathcal{H}]\frac{\overline{W}_{\alpha'}}{W_{\alpha'}} + \mathcal{R}_4 \tag{7.21}$$

where

$$\begin{aligned}\mathcal{R}_4 = &\left((\partial_t + B\partial_{\alpha'})\left(\frac{I-\mathbb{H}}{2}V\right) + \pi\frac{I-\overline{\mathbb{H}}}{2}\overline{W} + \frac{\pi}{2}(\widetilde{\mathcal{H}} - \mathcal{H})\overline{W}\right)\frac{a_t}{a}\circ h^{-1}\\ &+ e^{-i\alpha'}\left(\frac{\widetilde{L}e^{i\alpha'}}{i\mathcal{A}_1} - \frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right)i\mathcal{A}_1\left(\frac{a_t}{a}\circ h^{-1}\right).\end{aligned} \tag{7.22}$$

Here

$$\widetilde{L} = (\partial_t + B\partial_{\alpha'})\left(\frac{I+\mathbb{H}}{2}V\right) + \frac{\pi}{2}\left(\frac{I+\overline{\mathbb{H}}}{2}\right)\overline{W} - \frac{\pi}{2}(I+\widetilde{\mathcal{H}})\overline{W},$$
$$(\widetilde{\mathcal{H}}f)(\alpha') := \frac{\mathrm{p.v.}}{\pi i}\int_0^{2\pi}\frac{f(\beta')}{\left(\frac{I+\mathbb{H}}{2}\right)W(\beta') - \left(\frac{I+\mathbb{H}}{2}\right)W(\alpha')}\left(\frac{I+\mathbb{H}}{2}\right)W_{\beta'}(\beta')d\beta'.$$

Note that

$$(\widetilde{\mathcal{H}} - \mathcal{H})f = \mathcal{R}((I-\mathbb{H})W, (I-\mathbb{H})W_{\alpha'}),$$
$$L - \widetilde{L} = \mathcal{R}((I-\mathbb{H})V, (I-\mathbb{H})W, (I-\mathbb{H})W_{\alpha'}, (\partial_t + B\partial_{\alpha'})(I-\mathbb{H})V).$$

Claim 7.7. *Given any $f \in H^s_{\alpha'}$, there is a constant $C = C\left(\|f\|_{H^s_{\alpha'}}\right)$, such that*

$$\begin{aligned}&\left\|(I-\mathbb{H})f\left(\frac{\widetilde{L}e^{i\alpha'}}{i\mathcal{A}_1} - \frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right)\right\|_{H^s_{\alpha'}}\\ &\leq C\left(\left\|L - i\mathcal{A}\overline{W}_{\alpha'}\right\|_{L^2_{\alpha'}} + \|(I-\mathbb{H})V\|_{H^s_{\alpha'}} + \|(\partial_t + B\partial_{\alpha'})(I-\mathbb{H})V\|_{L^2_{\alpha'}} + \|(I-\mathbb{H})W\|_{H^s_{\alpha'}}\right).\end{aligned}$$

Proof of Claim 7.7. First we compute

$$
\begin{aligned}
\mathcal{A}_1(I-\mathbb{H})\left(\frac{\widetilde{L}e^{i\alpha'}}{\mathcal{A}_1}\right) &= (I-\mathbb{H})(e^{i\alpha'}\widetilde{L}) + [\mathbb{H},\mathcal{A}_1]\left(\frac{\widetilde{L}e^{i\alpha'}}{\mathcal{A}_1}\right) \\
&= (I-\mathbb{H})(\partial_t + B\partial_{\alpha'})e^{i\alpha'}\left(\frac{I+\mathbb{H}}{2}V\right) - (I-\mathbb{H})\left(iBe^{i\alpha'}\frac{I+\mathbb{H}}{2}V\right) \\
&\quad + \pi(I-\mathbb{H})e^{i\alpha'}\overline{W} + [\mathbb{H},\mathcal{A}_1]\left(\frac{ie^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) + \mathcal{R} \\
&= [B,\mathbb{H}]\left(e^{i\alpha'}\frac{I+\mathbb{H}}{2}V_{\alpha'}\right) + \pi(I-\mathbb{H})e^{i\alpha'}\overline{W} - \left[\frac{i(I-\mathbb{H})}{2}\mathcal{A}_1,\mathbb{H}\right]\left(\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) + \mathcal{R} \\
&= \left[\frac{I-\mathbb{H}}{2}B,\mathbb{H}\right]\left(\frac{I+\mathbb{H}}{2}V_{\alpha'}\cdot e^{i\alpha'}\right) + \pi(I-\mathbb{H})e^{i\alpha'}\frac{I+\overline{\mathbb{H}}}{2}\overline{W} \\
&\quad - \frac{1}{2}\left[(I-\mathbb{H})\left(\frac{I+\overline{\mathbb{H}}}{2}\right)\overline{V}\left(\frac{I+\mathbb{H}}{2}\right)V_{\alpha'},\mathbb{H}\right]\left(\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) \\
&\quad - \frac{\pi}{2}\left[(I-\mathbb{H})\left(\frac{I+\overline{\mathbb{H}}}{2}\right)\overline{W}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'},\mathbb{H}\right]\left(\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) + \mathcal{R} \\
&= \left[\frac{\left(\frac{I+\overline{\mathbb{H}}}{2}\right)\overline{V}}{\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}},\mathbb{H}\right]\left(e^{i\alpha'}\left(\frac{I+\mathbb{H}}{2}\right)V_{\alpha'}\right) \\
&\quad - \left[\left(\frac{I+\overline{\mathbb{H}}}{2}\right)\overline{V}\left(\frac{I+\mathbb{H}}{2}\right)V_{\alpha'},\mathbb{H}\right]\left(\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) + \mathcal{R} = \mathcal{R}.
\end{aligned}
$$

Therefore since $\mathcal{A}_1$ is bounded away from zero

$$
\begin{aligned}
(I-\mathbb{H})f\left(\frac{\widetilde{L}e^{i\alpha'}}{i\mathcal{A}_1} - \frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) &= [f,\mathbb{H}]\left(\frac{I+\mathbb{H}}{2}\right)\left(\frac{\widetilde{L}e^{i\alpha'}}{i\mathcal{A}_1} - \frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}}\right) + \mathcal{R} \\
&= [f,\mathbb{H}]\left(\frac{I+\mathbb{H}}{2}\right)\left(\frac{e^{i\alpha'}L}{i\mathcal{A}_1} - \frac{e^{i\alpha'}}{W_{\alpha'}}\right) + \mathcal{R}.
\end{aligned}
$$

and the claim follows from Lemma 2.8. □

Applying $(I-\mathbb{H})$ to both sides of (7.21) and with

$$
\mathcal{S} := i\mathcal{A}_1\frac{a_t}{a}\circ h^{-1} - \left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\left(\mathcal{P}\left(\frac{I+\mathbb{H}}{2}\right)V - \frac{\pi}{2}[\overline{V},\mathcal{H}]\frac{\overline{W}_{\alpha'}}{W_{\alpha'}}\right)
$$

we obtain

$$
\begin{aligned}
(I-\mathbb{H})\mathcal{P}\left(\frac{I-\mathbb{H}}{2}V\right) &= \left[e^{-i\alpha'}\mathcal{S},\mathbb{H}\right]\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}} + (I-\mathbb{H})\mathcal{R}_4 \\
&= \frac{1}{2}\left[e^{-i\alpha'}(I-\mathbb{H})\mathcal{S},\mathbb{H}\right]\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}} + \frac{1}{2}\left[[e^{-i\alpha'},\mathbb{H}]\mathcal{S},\mathbb{H}\right]\frac{e^{i\alpha'}}{\frac{I+\mathbb{H}}{2}W_{\alpha'}} + (I-\mathbb{H})\mathcal{R}_4.
\end{aligned}
\tag{7.23}
$$

To see that the first two terms in the last line are linear in $(I - \mathbb{H})V$, $(I - \mathbb{H})W$, and $\frac{i\mathcal{A}_1}{W_{\alpha'}} - L$, we want to mimic the proof of Proposition 7.2, for which we need to introduce the Riemann mapping. First let h be the function on $\mathbb{R}$ defined by

$$\frac{dh}{dt} = B(h, t), \quad h(\alpha, 0) = \alpha. \tag{7.24}$$

Since h is a diffeomorphism at $t = 0$ and $h_{\alpha'}$ satisfies the linear ODE

$$\frac{dh_\alpha}{dt} = B_{\alpha'} h_\alpha,$$

h is a diffeomorphism at least for a short time and $\partial_t(f \circ h) = (\partial_t + B\partial_{\alpha'}) f \circ h$ for all time. Let $\widetilde{\Phi}^{-1}(t, \cdot)$ be the holomorphic function with boundary value $\widetilde{\Phi}^{-1}(t, e^{i\alpha'}) = \left(\frac{I+\mathbb{H}}{2} W\right)(t, \alpha')$. Since $\widetilde{\Phi}_w^{-1}(0, \cdot)$ is never zero on the disc $\mathbb{D}$, the same is true for $\widetilde{\Phi}_w^{-1}(t, \cdot)$ for small t by the Cauchy integral formula for the derivative of a holomorphic function. Therefore $\widetilde{\Phi}^{-1}(t, \cdot)$ has an inverse, which we denote by $\widetilde{\Phi}(t, \cdot) : \widetilde{\Phi}^{-1}(t, \mathbb{D}) \to \mathbb{D}$. Note that with this definition $\widetilde{\Phi}(t, \frac{I+\mathbb{H}}{2} W(t, \alpha')) = e^{i\alpha'}$. It follows that if f is the boundary value of a holomorphic function on $\mathbb{D}$, i.e., $f(\alpha') = F(e^{i\alpha'})$ for a holomorphic function F on $\mathbb{D}$, then $f \circ h$ is the boundary value of the holomorphic function $G = F \circ \widetilde{\Phi}$ on $\widetilde{\Phi}^{-1}(t, \mathbb{D})$. Introducing the variable

$$\widetilde{z} := \frac{I + \mathbb{H}}{2} W \circ h$$

we can write

$$\frac{I + \mathbb{H}}{2} V \circ h = \widetilde{F}(t, \widetilde{z}).$$

Now the same argument as in the proof of Proposition 7.2 implies that $(I - \mathbb{H})\mathcal{S}$ and $[e^{-i\alpha'}, \mathbb{H}]\mathcal{S} = 2e^{-i\alpha'}\mathrm{Av}(\mathcal{S})$ are linear in $(I - \mathbb{H})V$, $(I - \mathbb{H})W$, and $\frac{i\mathcal{A}_1}{W_{\alpha'}} - L$. Next we compute the left hand side of (7.23).

$$\begin{aligned}(I - \mathbb{H})\mathcal{P}\left(\frac{I - \mathbb{H}}{2} V\right) &= (\partial_t + B\partial_{\alpha'})^2 ((I - \mathbb{H})V) \\ &\quad + (\partial_t + B\partial_{\alpha'})[B, \mathbb{H}]\partial_{\alpha'}\left(\frac{I - \mathbb{H}}{2} V\right) \\ &\quad + [B, \mathbb{H}]\partial_{\alpha'}(\partial_t + B\partial_{\alpha'})\left(\frac{I - \mathbb{H}}{2} V\right) \\ &= (\partial_t + B\partial_{\alpha'})^2 ((I - \mathbb{H})V) + \mathcal{R}.\end{aligned}$$

Similarly,

$$\begin{aligned}(I - \mathbb{H})\mathcal{A}|D|\left(\frac{I - \mathbb{H}}{2} V\right) &= \mathcal{A}|D|(I - \mathbb{H})V \\ &\quad + [\mathcal{A}, \mathbb{H}]|D|\left(\frac{I - \mathbb{H}}{2} V\right) = \mathcal{A}|D|(I - \mathbb{H})V + \mathcal{R}.\end{aligned}$$

Combining these observations with (7.23), we get

$$\mathcal{P}\left(\frac{I-\mathbb{H}}{2}V\right)=\mathcal{R}. \tag{7.25}$$

Note that by Claim 7.7 and (7.20), to bound, say, the $H^2_{\alpha'}$ norms of $(I-\mathbb{H})V$ and $(I-\mathbb{H})W$, we only need to use the $L^2_{\alpha'}$ norm of $L-i\frac{\mathcal{A}_1}{W_{\alpha'}}$. Therefore to derive the equation for $L-i\frac{\mathcal{A}_1}{W_{\alpha'}}$, we can write terms involving derivatives of $(I-\mathbb{H})V$ and $(I-\mathbb{H})W$ as $\mathcal{R}$. To derive this equation for $L-\frac{i\mathcal{A}_1}{W_{\alpha'}}$. we first note that

$$(\partial_t+B\partial_{\alpha'})(I-\mathcal{H})\overline{W}=[\overline{V},\mathcal{H}]\frac{\overline{W}_{\alpha'}}{W_{\alpha'}}+(I-\mathcal{H})V=[\overline{V},\mathcal{H}]\frac{\overline{W}_{\alpha'}}{W_{\alpha'}}+\mathcal{R},$$

where for the last equality we have used the fact that $\mathcal{H}f-\widetilde{\mathcal{H}}f=\mathcal{R}$.This computation and the fact that $|D|V=-i\partial_{\alpha'}V+|D|(I-\mathbb{H})V$ allow us to write the first equation in (7.16) as

$$\begin{aligned}(\partial_t+B\partial_{\alpha'})\left(L-i\mathcal{A}\partial_{\alpha'}\overline{W}\right)=\frac{i}{W_{\alpha'}}&\left(\mathcal{A}_1\frac{a_t}{a}\circ h^{-1}\right.\\&\left.-\mathcal{A}_1\frac{\mathbf{a}_t}{\mathbf{a}}\circ h^{-1}\right)-\mathcal{A}|D|(I-\mathbb{H})V+\mathcal{R}.\end{aligned} \tag{7.26}$$

Here we have used the notation

$$\mathbf{a}:=\frac{\mathcal{A}\circ h}{h_\alpha}$$

so in particular since $\mathcal{A}=\frac{L\overline{L}}{\mathcal{A}_1}$

$$\mathcal{A}_1\frac{\mathbf{a}_t}{\mathbf{a}}\circ h^{-1}=\frac{\overline{L}(\partial_\alpha+B\partial_{\alpha'})L}{\mathcal{A}}+\frac{L(\partial_t+B\partial_{\alpha'})\overline{L}}{\mathcal{A}}-(\partial_t+B\partial_{\alpha'})\mathcal{A}_1-\mathcal{A}_1B_{\alpha'}. \tag{7.27}$$

To write (7.26) as a homogeneous linear equation in $(I-\mathbb{H})V$, $(I-\mathbb{H})W$, and $L-\frac{i\mathcal{A}_1}{W_{\alpha'}}$ we need to study the right hand side of (7.27) more carefully. Since by (7.16) and with the notation $L_t:=(\partial_t+B\partial_{\alpha'})L$ the quantity $\overline{L}(L_t+\mathcal{A}|D|V)-\frac{\pi}{2}\overline{L}(\widetilde{\mathcal{H}}-\mathcal{H})V$ is purely real,

$$\begin{aligned}\frac{\overline{L}L_t}{\mathcal{A}}+\frac{L\overline{L}_t}{\mathcal{A}}&=2\frac{\overline{L}(L_t+\mathcal{A}|D|V)}{\mathcal{A}}-\overline{L}|D|V-L|D|\overline{V}+\mathcal{R}\\&=2\frac{\overline{L}}{\mathcal{A}}(L_t+\mathcal{A}|D|V)-L|D|\overline{V}-\overline{L}|D|V+\mathcal{R}\\&=2\mathcal{A}_1\frac{a_t}{a}\circ h^{-1}-L|D|\overline{V}-\overline{L}|D|V+\mathcal{R},\end{aligned}$$

which means

$$\mathcal{A}_1\frac{\mathbf{a}_t}{\mathbf{a}}\circ h^{-1}=2\mathcal{A}_1\frac{a_t}{a}\circ h^{-1}-L|D|\overline{V}-\overline{L}|D|V-(\partial_t+B\partial_{\alpha'})\mathcal{A}_1-\mathcal{A}_1B_{\alpha'}.$$

Together with (7.26) and (7.27) this gives

$$
\begin{aligned}
(\partial_t + B\partial_{\alpha'})\left(L - i\mathcal{A}\partial_{\alpha'}\overline{W}\right) &= \frac{i}{W_{\alpha'}}\Big(-\mathcal{A}_1\frac{a_t}{a}\circ h^{-1} + \overline{L}|D|V + L|D|\overline{V} \\
&\quad +(\partial_t + B\partial_{\alpha'})\mathcal{A}_1 + \mathcal{A}_1 B_{\alpha'}\Big) + \mathcal{R} \\
&=: \frac{i}{W_{\alpha'}}\mathcal{T} + \mathcal{R}. \qquad (7.28)
\end{aligned}
$$

Since $\mathcal{T}$ is purely real,

$$
\mathcal{T} = \operatorname{Im}(i\mathcal{T}) = \operatorname{Im}\left((I - \mathbb{H})i\mathcal{T}\right) + \operatorname{Av}(\mathcal{T}).
$$

First we compute

$$
\begin{aligned}
L|D|\overline{V} + \overline{L}|D|V &= L|D|\overline{V} - i\mathcal{A}W_{\alpha'}|D|\left(\frac{I - \mathbb{H}}{2}\right)V - i\mathcal{A}W_{\alpha'}|D|\left(\frac{I+\mathbb{H}}{2}\right)V + \mathcal{R} \\
&= \left((\partial_t + B\partial_{\alpha'})\frac{I - \mathbb{H}}{2}V\right)|D|\overline{V} + \left((\partial_t + B\partial_{\alpha'})\frac{I + \mathbb{H}}{2}V + \frac{\pi}{2}(I - \mathcal{H})\overline{W}\right)|D|\overline{V} \\
&\quad - i\mathcal{A}W_{\alpha'}|D|\left(\frac{I + \mathbb{H}}{2}\right)V + \mathcal{R} \\
&= \left((\partial_t + B\partial_{\alpha'})\frac{I + \mathbb{H}}{2}V + \frac{\pi}{2}(I - \mathcal{H})\overline{W}\right)|D|\overline{V} - i\mathcal{A}W_{\alpha'}|D|\left(\frac{I + \mathbb{H}}{2}\right)V + \mathcal{R}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{i}{W_{\alpha'}}(I - \mathbb{H})\left(L|D|\overline{V} + \overline{L}|D|V\right) \\
&\quad = \frac{i}{W_{\alpha'}}(I - \mathbb{H})\left(\left((\partial_t + B\partial_{\alpha'})\frac{I + \mathbb{H}}{2}V + \frac{\pi}{2}(I - \mathcal{H})\overline{W}\right)|D|\overline{V}\right. \\
&\qquad \left. - i\mathcal{A}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}|D|\left(\frac{I + \mathbb{H}}{2}\right)V\right) + \mathcal{R}. \qquad (7.29)
\end{aligned}
$$

With the notation $\mathcal{U} := (\partial_t + B\partial_{\alpha'})\frac{I+\mathbb{H}}{2}V + \frac{\pi}{2}(I - \mathcal{H})\overline{W}$

$$
\begin{aligned}
\mathcal{U}|D|\overline{V} &= \mathcal{U}|D|\left(\frac{I + \overline{\mathbb{H}}}{2}\right)\overline{V} + \mathcal{U}|D|\left(\frac{I - \overline{\mathbb{H}}}{2}\right)\overline{V} \\
&= \mathcal{U}i\partial_{\alpha'}\frac{I - \mathbb{H}}{2}\overline{V} + \mathcal{R} = i\mathcal{U}\partial_{\alpha'}\left(\frac{I - \mathbb{H}}{2}(\partial_t + B\partial_{\alpha'})W\right) + \mathcal{R} \\
&= i\mathcal{U}\partial_{\alpha'}(\partial_t + B\partial_{\alpha'})\left(\frac{I - \mathbb{H}}{2}\right)W + \frac{i}{2}\mathcal{U}\partial_{\alpha'}[B, \mathbb{H}]W_{\alpha'} + \mathcal{R} \\
&= \frac{1}{2}i\mathcal{U}\partial_{\alpha'}\left(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3\right) + i\mathcal{U}\partial_{\alpha'}(\partial_t + B\partial_{\alpha'})\left(\frac{I + \mathbb{H}}{2}\right)W \\
&\quad - i\mathcal{U}\partial_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)\overline{V} + \mathcal{R} \\
&= i\mathcal{U}B_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'} + i(\partial_t + B\partial_{\alpha'})\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)
\end{aligned}
$$

$$- i \left(\frac{I + \mathbb{H}}{2}\right) W_{\alpha'}(\partial_t + B\partial_{\alpha'})\mathcal{U} + \mathcal{R}. \tag{7.30}$$

Combining (7.28)–(7.30) we get

$$\begin{aligned}(I - \mathbb{H})(i\mathcal{T}) &= i(I - \mathbb{H})\left(-\mathcal{A}_1 \frac{a_t}{a} \circ h^{-1} + L|D|\overline{V} + \overline{L}|D|V + (\partial_t + B\partial_{\alpha'})\mathcal{A}_1 + \mathcal{A}_1 B_{\alpha'}\right)\\
&= -(I - \mathbb{H})\left(\mathcal{U}B_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) - (I - \mathbb{H})\left((\partial_t + B\partial_{\alpha'})\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\right)\\
&\quad + (I - \mathbb{H})\left(\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\left((\partial_t + B\partial_{\alpha'})\mathcal{U} + \mathcal{A}|D|\left(\frac{I + \mathbb{H}}{2}\right)V\right) - i\mathcal{A}_1 \frac{a_t}{a} \circ h^{-1}\right)\\
&\quad + i(I - \mathbb{H})\left((\partial_t + B\partial_{\alpha'})\mathcal{A}_1 + \mathcal{A}_1 B_{\alpha'}\right) + \mathcal{R}\\
&= -(I - \mathbb{H})\left(\mathcal{U}B_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) - (I - \mathbb{H})\left((\partial_t + B\partial_{\alpha'})\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\right)\\
&\quad + i(I - \mathbb{H})\left((\partial_t + B\partial_{\alpha'})\mathcal{A}_1 + \mathcal{A}_1 B_{\alpha'}\right) + \mathcal{R}.\end{aligned}$$

To compute the second term, we first note that

$$\mathcal{U} = (\partial_t + B\partial_{\alpha'})\left(\frac{I + \mathbb{H}}{2}\right)V + \pi\left(\frac{I + \overline{\mathbb{H}}}{2}\right)\overline{W} - \frac{\pi}{2}(I + \widetilde{\mathcal{H}})\overline{W} + \mathcal{R} := \widetilde{\mathcal{U}} + \mathcal{R}.$$

By a computation similar to Proposition 7.1 it follows that

$$\begin{aligned}&- (I - \mathbb{H})\left((\partial_t + B\partial_{\alpha'})\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\right)\\
&= -(\partial_t + B\partial_{\alpha'})\left((I - \mathbb{H})\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\right)\\
&\quad - [B, \mathbb{H}]\partial_{\alpha'}\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\\
&= -(\partial_t + B\partial_{\alpha'})\left(\left[\overline{V}, \mathbb{H}\right]\left(\frac{I + \mathbb{H}}{2}\right)V_{\alpha'} + \pi\left[\overline{W}, \mathbb{H}\right]\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\\
&\quad - [B, \mathbb{H}]\partial_{\alpha'}\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) + \mathcal{R}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Im}\,((I - \mathbb{H})i\mathcal{T}) &= -(\partial_t + B\partial_{\alpha'})\mathrm{Im}\left(\left[\overline{V}, \mathbb{H}\right]\left(\frac{I + \mathbb{H}}{2}\right)V_{\alpha'} + \pi\left[\overline{W}, \mathbb{H}\right]\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\\
&\quad - \mathrm{Im}\,[B, \mathbb{H}]\partial_{\alpha'}\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) + (\partial_t + B\partial_{\alpha'})\mathcal{A}_1 - \partial_t \mathrm{Av}(\mathcal{A}_1) + \mathcal{A}_1 B_{\alpha'} + \mathcal{R}\\
&\quad - \mathrm{Im}\,(I - \mathbb{H})\left(\mathcal{U}B_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right)\\
&= -\mathrm{Im}\,[B, \mathbb{H}]\partial_{\alpha'}\left(\mathcal{U}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) + \mathcal{A}_1 B_{\alpha'} - \mathrm{Im}\,(I - \mathbb{H})\left(\mathcal{U}B_{\alpha'}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) + \mathcal{R}\\
&= -\mathrm{Im}\,[B, \mathbb{H}]\partial_{\alpha'}\left(\widetilde{\mathcal{U}}\left(\frac{I + \mathbb{H}}{2}\right)W_{\alpha'}\right) + \mathcal{A}_1 B_{\alpha'} - \mathrm{Im}\,(I - \mathbb{H})\left(\widetilde{\mathcal{U}}B_{\alpha'}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right) + \mathcal{R}.\end{aligned} \tag{7.31}$$

We compute

$$
\begin{aligned}
&-[B,\mathbb{H}]\partial_{\alpha'}\left(\widetilde{\mathcal{U}}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right)-(I-\mathbb{H})\left(\widetilde{\mathcal{U}}B_{\alpha'}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right)\\
&\quad=-B_{\alpha'}(I-\mathbb{H})\left(\widetilde{\mathcal{U}}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right)-\partial_{\alpha'}[B,\mathbb{H}]\left(\widetilde{\mathcal{U}}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right).
\end{aligned}
$$

Note that

$$
-\mathrm{Im}\left(B_{\alpha'}(I-\mathbb{H})\left(\widetilde{\mathcal{U}}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right)\right)=-\mathcal{A}_1B_{\alpha'}+\mathrm{Av}(\mathcal{A}_1)B_{\alpha'}+\mathcal{R}
$$

and

$$
\begin{aligned}
-\mathrm{Im}\left(\partial_{\alpha'}[B,\mathbb{H}]\left(\widetilde{\mathcal{U}}\left(\frac{I+\mathbb{H}}{2}\right)W_{\alpha'}\right)\right)&=-\mathrm{Im}\ (\partial_{\alpha'}[B,\mathbb{H}](i\mathcal{A}_1))+\mathcal{R}\\
&=-\mathrm{Im}\ (\partial_{\alpha'}[B,\mathrm{Av}](i\mathcal{A}_1))+\mathcal{R}\\
&=-\mathrm{Av}(\mathcal{A}_1)B_{\alpha'}+\mathcal{R}.
\end{aligned}
$$

Combining these observations with (7.31) we obtain

$$
(\partial_t+B\partial_{\alpha'})(L-i\mathcal{A}\overline{W}_{\alpha'})=\mathcal{R}.
$$

(2) This is a direct consequence of the fact that $L=\frac{i\mathcal{A}_1}{W_{\alpha'}}$ and the definition of h. □

The proof of Theorem 3.2 now follows from local the well-posedness of (7.16)–(7.17):

Proof of Theorem 3.2. By Proposition 7.6 it suffices to show local well-posedness of (7.16)–(7.17). The proof of local well-posedness for the system (7.16)–(7.17) is almost identically the same as the proof of Theorem 5.10 in [38] where the vanishing viscosity method us used. In fact the only difference is that unlike in [38], here we also need to control $W=Z$. But by (7.16) W satisfies a transport equation and therefore control of W follows from control of V by integration. We refer the reader to [38] Section 5, and leave the necessary routine modifications to the reader. □

Appendix A. The Hilbert Transform

In this appendix we recall some facts about the Hilbert transform. If Ω is a bounded domain in $\mathbb{C}$ with $C^2_{t,\alpha}$ boundary and f is a function defined on $\partial\Omega$ then the Hilbert transform Hf of f with respect to Ω is defined as

$$
Hf(z_0):=\lim_{\epsilon\to 0^+}\frac{1}{\pi i}\int_{\gamma_\epsilon}\frac{f(w)}{w-z_0}dw,
$$

where γ_ϵ is the portion of $\partial\Omega$ obtained by removing a segment of $\partial\Omega$ which lies within a circle of radius ϵ centered at $z_0\in\partial\Omega$. Given a $C^2_{t,\alpha}$ parametrization $z:[0,2\pi]\to\partial\Omega$ of $\partial\Omega$ we identify 2π-periodic functions on $\mathbb{R}$ with functions on $\partial\Omega$, and for any such function f we write

$$
Hf(\alpha):=\frac{\text{p.v.}}{\pi i}\int_0^{2\pi}\frac{f(\beta)}{z(\beta)-z(\alpha)}z_\beta(\beta)d\beta.
$$

The relevant results from this appendix are summarized in the following proposition.

Proposition A.1. *Suppose that Ω is a bounded domain in $\mathbb{C}$ with C^2 boundary $\partial\Omega$. Let f be a Lipschitz continuous function on $\partial\Omega$ and Hf be its Hilbert transform. Then $Hf = f$ if and only if f is the boundary value of a holomorphic function in Ω and $Hf = -f$ if and only if f is the boundary value of a holomorphic function F in Ω^c satisfying $F(z) \to 0$ as $|z| \to \infty$.*

Proof. Suppose that Ω is a bounded domain in $\mathbb{C}$ and $\gamma := \partial\Omega$ has $C^2_{t,\alpha}$. Let f be a continuous function defined on $\partial\Omega$. The following Cauchy integral

$$C_f(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw \tag{A.1}$$

defines a holomorphic function when $z \notin \gamma$. In this subsection, we will introduce the Hilbert transforms associated to Ω and Ω^c by considering the limit of $C_f(z)$ as z approaches z_0 from Ω and Ω^c where z_0 is a point on $\partial\Omega$. Here all integrals are understood as counterclockwise, unless otherwise stated. Let us first consider the limit from the inside.

$$\lim_{z\to z_0}\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw = \lim_{z\to z_0}\frac{1}{2\pi i}\int_{\gamma_\epsilon+\xi_\epsilon}\frac{f(w)}{w-z}dw = \lim_{\epsilon\to 0^+}\lim_{z\to z_0}\frac{1}{2\pi i}\int_{\gamma_\epsilon+\xi_\epsilon}\frac{f(w)}{w-z}dw, \tag{A.2}$$

where γ_ϵ is the portion of γ obtained by subtracting the segment ξ_ϵ about z_0 which lies within the circle of radius ϵ centered at z_0. We recognize the limit over γ_ϵ as one half of the Hilbert transform of f associated to Ω:

$$\frac{1}{2}Hf(z_0) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{f(w)}{w-z_0}dw = \lim_{\epsilon\to 0^+}\lim_{z\to z_0}\frac{1}{2\pi i}\frac{f(w)}{w-z}dw. \tag{A.3}$$

On the other hand,

$$\begin{aligned}\lim_{\epsilon\to 0^+}\lim_{z\to z_0}\int_{\xi_\epsilon}\frac{f(w)}{w-z}dw &= \lim_{\epsilon\to 0^+}\lim_{z\to z_0}\left(\int_{\xi_\epsilon}\frac{f(w)-f(z_0)}{w-z}dw + \int_{\xi_\epsilon}\frac{f(z_0)}{w-z}dw\right)\\ &= \lim_{\epsilon\to 0^+}\lim_{z\to z_0}\int_{\xi_0}\frac{f(z_0)}{w-z}dw.\end{aligned}$$

Now with C_ϵ denoting the part of the circle of radius ϵ centered at z_0 which lies within Ω we have

$$\begin{aligned}\lim_{\epsilon\to 0^+}\lim_{z\to z_0}\int_{\xi_\varepsilon}\frac{dw}{w-z} &= \lim_{\epsilon\to 0^+}\lim_{\epsilon\to 0^+}\int_{\xi_\epsilon+C_\epsilon}\frac{dw}{w-z} - \lim_{\epsilon\to 0^+}\int_{C_\epsilon}\frac{dw}{w-z_0}\\ &= 2\pi i - \lim_{\epsilon\to 0^+}\int_{\pi+O(\epsilon)}^{2\pi+O(\epsilon)}\frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} = \pi i.\end{aligned} \tag{A.4}$$

Combining this with (A.2) and (A.3) we get

$$Hf(z_0) = 2\lim_{z\to z_0}C_f(z) - f(z_0).$$

Since C_f is a holomorphic function inside Ω, and $\lim_{z\to z_0}C_f(z) = f(z_0)$ if f can be extended to a holomorphic function inside Ω, we conclude that f is the boundary value of a holomorphic function inside Ω if and only if $Hf(z_0) = f(z_0)$ for all $z_0 \in \partial\Omega$.

The computation is similar for the case where $z \to z_0$ from the outside (i.e. $z \in \Omega^c$). In this case in (A.4) we define C_ϵ to be the part of the circle of radius ϵ centered at z_0, parametrized clockwisely, which lies in Ω^c. It then follows that

$$\int_{\xi_\epsilon + C_\epsilon} \frac{dw}{w-z} = -2\pi i,$$

and hence

$$Hf(z_0) = 2 \lim_{z \to z_0} C_f(z) - 3f(z_0),$$

where now the limit is understood to be from the outside. Now notice that from the definition (A.1) of the Cauchy integral that C_f is holomorphic in Ω^c and decays like $\frac{1}{|z|}$ as $|z| \to \infty$. Therefore if f if $Hf = -f$ then f is the boundary value of a holomorphic function in Ω^c decaying like $\frac{1}{|z|}$ as $|z| \to \infty$. Conversely, if f is the boundary value of such a holomorphic function, then defining $U = \{\frac{1}{z} \text{ s.t. } z \in \Omega^c\} \subseteq \mathbb{C}$, we have

$$\lim_{\substack{z \to z_0 \\ z \in \Omega^c}} C_f(z) = \lim_{\substack{z \to z_0 \\ z \in \Omega^c}} \int_{\partial\Omega} \frac{f(w)}{w-z} dw = \lim_{\substack{z \to 1/z_0 \\ z \in U}} \frac{1}{z} \int_{\partial U} \frac{\frac{f(1/u)}{u}}{u - \frac{1}{z}} du = f(z_0),$$

and therefore $Hf(z_0) = -f(z_0)$. □

Appendix B. The Case of Constant Vorticity

As mentioned in the introduction Theorem 1.1 can be extended to allow for constant vorticity. In this appendix we provide the details of this extension. The equation under consideration is now

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P - \nabla\phi & \text{in } \ \Omega(t), t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 2\omega_0 \in \mathbb{R} & \text{in } \ \Omega(t), t \geq 0, \\ P = 0 & \text{on } \ \partial\Omega(t), \end{cases} \tag{B.1}$$

where the self-gravity Newtonian potential satisfies

$$\begin{cases} \Delta\phi = 2\pi \chi_{\Omega(t)}, \\ \nabla\phi(\mathbf{x}) = \iint_{\Omega(t)} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}. \end{cases} \tag{B.2}$$

Note that applying the curl operator to the first line in Eq. (B.1) implies that the vorticity $\omega := \operatorname{curl} \mathbf{v}$ satisfies $\omega_t + \mathbf{v} \cdot \nabla\omega = 0$, and therefore if $\mathbf{v}(0)$ has constant vorticity $2\omega_0$, the same holds for $\mathbf{v}(t)$ for all times at which the latter is defined, that is $\operatorname{curl} \mathbf{v} = 2\omega_0$. As in the rest of the paper and without loss of generality we fix the normalization $|\Omega| = \pi$. We also assume that $\omega_0^2 < \pi$. As we will show below this is necessary to ensure the validity of the Taylor sign-condition which is necessary for local well-posedness. Note that the vector field $\mathbf{v}_0 := \omega_0(y, -x)$ satisfies $\operatorname{curl} \mathbf{v}_0 = 2\omega_0$ and $\operatorname{div} \mathbf{v}_0 = 0$, and therefore $\mathfrak{v} := \mathbf{v} - \mathbf{v}_0$ is curl and divergence free. Using complex variable notation it follows that $z_t + i\omega_0 z$ is the boundary value of an anti-holomorphic function in Ω, or in other words

$(I - \overline{H})(z_t + i\omega_0 z) = 0$. It follows that with the same notation as in the rest of the paper the system (B.1) reduces to the following system on the boundary $\partial\Omega$:

$$\begin{cases} z_{tt} + iaz_\alpha = -\frac{\pi}{2}(I - \overline{H})z \\ H(\bar{z}_t - i\omega_0\bar{z}) = \bar{z}_t - i\omega_0\bar{z} \end{cases}. \tag{B.3}$$

Note that $z(t, \alpha) = e^{-i\omega_0 t + i\alpha}$ is a solution to (B.3) with $a = \pi - \omega_0^2$. It follows that for the Taylor sign-condition $\frac{\partial P}{\partial \mathbf{n}} < 0$ to hold we need to impose the condition $\omega_0^2 < \pi$. The following theorem is the main result of this appendix.

Theorem B.1. *Let Ω_0 be a bounded simply-connected domain in $\mathbb{C}$ with smooth boundary $\partial\Omega_0$ satisfying $|\Omega_0| = \pi$, and denote the associated Hilbert transform by H_0. Suppose $z_0(\alpha) = e^{i\alpha} + \epsilon f(\alpha)$ is a parametrization of $\partial\Omega_0$ and $z_1(\alpha) = v_0 - i\omega_0 e^{i\alpha} + \epsilon g(\alpha)$ where f and g are smooth, $H_0(\bar{z}_1 - i\omega_0\bar{z}_0) = \bar{z}_1 - i\omega_0\bar{z}_0$, $v_0 \in \mathbb{C}$ is a constant, and $\omega_0^2 < \pi$. Then there is $T > 0$ and a unique classical solution $z(t, \alpha)$ of (B.3) on $[0, T)$ satisfying $(z(0, \alpha), z_t(0, \alpha)) = (z_0(\alpha), z_1(\alpha))$. Moreover if $\epsilon > 0$ is sufficiently small the solution can be extended at least to $T^* = c\epsilon^{-2}$ where c is a constant independent of ϵ.*

We introduce the notation $\mathfrak{z}(t, \alpha) = e^{i\omega_0 t} z(t, \alpha)$ and note that $H\bar{\mathfrak{z}}_t = \bar{\mathfrak{z}}_t$. Note that since the factor $e^{i\omega_0 t}$ is independent of α one can replace z by $\mathfrak{z}$ in the definition of the Hilbert transform, and in particular the conclusions of Lemma 3.7 remain valid with z replaced by $\mathfrak{z}$. The system (B.3) is written in terms of $\mathfrak{z}$ as

$$\begin{cases} \mathfrak{z}_{tt} + ia\mathfrak{z}_\alpha = -\frac{\pi}{2}(I - \overline{H})\mathfrak{z} + 2i\omega_0\mathfrak{z}_t + \omega_0^2\mathfrak{z} \\ H\bar{\mathfrak{z}}_t = \bar{\mathfrak{z}}_t \end{cases}. \tag{B.4}$$

We first show the validity of the Taylor sign-condition provided $\omega_0^2 < \pi$, from which local well-posedness follows as in Sect. 7. Let $\widetilde{\Omega}(t)$ be the domain with $\partial\widetilde{\Omega}(t)$ parametrized by $\mathfrak{z}(t, \cdot)$. We introduce the Riemann mapping $\Phi(t, \cdot) : \widetilde{\Omega}(t) \to \mathbb{D}$, the function $h : \mathbb{R} \to \mathbb{R}$ such that

$$e^{ih(t,\alpha)} = \Phi(t, \mathfrak{z}(t, \alpha)), \tag{B.5}$$

and the new unknowns in the Riemann mapping coordinates:

$$\begin{aligned} &\mathfrak{Z}(t, \alpha') = \mathfrak{z}(t, h^{-1}(t, \alpha)), \quad \mathfrak{Z}_t(t, \alpha') = \mathfrak{z}_t(t, h^{-1}(t, \alpha)), \\ &\mathfrak{Z}_{tt}(t, \alpha') = \mathfrak{z}_{tt}(t, h^{-1}(t, \alpha)), \quad \mathfrak{Z}_{ttt}(t, \alpha') = \mathfrak{z}_{ttt}(t, h^{-1}(t, \alpha)). \end{aligned} \tag{B.6}$$

The new unknowns satisfy the following system on the unit circle

$$\begin{cases} \mathfrak{Z}_{tt} + i\mathcal{A}\mathfrak{Z}_{,\alpha'} = -(\pi - \omega_0^2)\mathfrak{Z} + \overline{G}, \\ \mathbb{H}\bar{\mathfrak{Z}}_t = \bar{\mathfrak{Z}}_t. \end{cases} \tag{B.7}$$

where $\mathcal{A} \circ h = ah_\alpha$, $\mathbb{H}$ is the Hilbert transform associated to the unit circle, and G is given by

$$G = \frac{\pi}{2}((I + H)\bar{\mathfrak{z}}) \circ h^{-1} - 2\omega_0 i\bar{\mathfrak{z}}_t \circ h^{-1}.$$

Note that $\overline{G}$ is the boundary value of an anti-holomorphic function in the unit disc, and therefore a similar argument as the proof of Proposition 7.1 gives the following result.

Proposition B.2. *$\mathcal{A}_1 := \mathcal{A}\left|\mathfrak{Z}_{,\alpha'}\right|^2$ is positive and is given by*

$$\mathcal{A}_1 := \frac{1}{8\pi}\int_0^{2\pi}\left|\mathfrak{Z}_t(t,\beta') - \mathfrak{Z}_t(t,\alpha')\right|^2 \csc^2\left(\frac{\beta'-\alpha'}{2}\right)d\beta' + \frac{\pi-\omega_0^2}{8\pi}\int_0^{2\pi}\left|\mathfrak{Z}(t,\beta') - \mathfrak{Z}(t,\alpha')\right|^2 \csc^2\left(\frac{\beta'-\alpha'}{2}\right)d\beta' > 0.$$

We next turn to the proof of long-time existence for small initial data. Note that $\mathfrak{z}_t$ is now the small quantity corresponding to z_t in the irrotational case. The key point for the proof of Theorem B.1 is that $\delta := (I-H)\varepsilon$, where $\varepsilon := |z|^2 - 1 = |\mathfrak{z}|^2 - 1$, still satisfies an equation without quadratic nonlinear terms. The following proposition is the analogue of Proposition 3.15.

Proposition B.3. *The quantities δ and $\delta_t = \partial_t\delta$ satisfy*

$$(\partial_t^2 + ia\partial_\alpha - \pi - 2\omega_0 i\partial_t)\delta = \widetilde{\mathcal{N}}_1 := \frac{\pi}{2}(I-H)E(\varepsilon) + \frac{\pi}{2}[E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} - 2[\mathfrak{z}_t, H\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\bar{\mathfrak{z}}_\alpha}]\partial_\alpha(\mathfrak{z}_t\bar{\mathfrak{z}}) - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{\mathfrak{z}_t(\beta)-\mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)}\right)^2\varepsilon_\beta(\beta)d\beta, \tag{B.8}$$

and

$$(\partial_t^2 + ia\partial_\alpha - \pi - 2\omega_0 i\partial_t)\delta_t = \widetilde{\mathcal{N}}_2 := -ia_t\partial_\alpha\delta + \frac{\pi}{2}\left((I-H)\partial_t E(\varepsilon) - [\mathfrak{z}_t, H]\frac{\partial_\alpha E(\varepsilon)}{\mathfrak{z}_\alpha}\right) + \frac{\pi}{2}[\partial_t E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} + \frac{\pi}{2}[E(\mathfrak{z}), H]\partial_t\left(\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}\right) + \frac{\pi}{2}E(\mathfrak{z})[\mathfrak{z}_t, H]\frac{\partial_\alpha\left(\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}\right)}{\mathfrak{z}_\alpha} - \frac{\pi}{2}[\mathfrak{z}_t, H]\frac{\partial_\alpha\left(E(\mathfrak{z})\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}\right)}{\mathfrak{z}_\alpha} + \frac{2}{\pi i}\partial_t\int_0^{2\pi}\frac{(\mathfrak{z}_t(\alpha)-\mathfrak{z}_t(\beta))\partial_\beta(\mathfrak{z}_t(\beta)\bar{\mathfrak{z}}(\beta))\bar{\mathfrak{z}}(\alpha)\bar{\mathfrak{z}}(\beta)\left(\frac{\varepsilon(\alpha)}{\bar{\mathfrak{z}}(\alpha)} - \frac{\varepsilon(\beta)}{\bar{\mathfrak{z}}(\beta)}\right)}{|\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)|^2}d\beta - \frac{1}{\pi i}\partial_t\int_0^{2\pi}\left(\frac{\mathfrak{z}_t(\beta)-\mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)}\right)^2\varepsilon_\beta(\beta)d\beta, \tag{B.9}$$

Remark B.4. We note that the vorticity $2\omega_0$ gives the extra linear terms $-2\omega_0 i\partial_t\delta$ in (B.8) and $-2\omega_0 i\partial_t\delta_t$ in (B.9).

Proof. Using (B.4), we write the equation satisfied by ε as

$$(\partial_t^2 + ia\partial_\alpha)\varepsilon = \frac{\pi}{2}\left(\mathfrak{z}(I-H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I-\overline{H})\mathfrak{z}\right) + 2\omega_0 i\partial_t\varepsilon + 2(\bar{\mathfrak{z}}_t\mathfrak{z})_t,$$

which implies

$$(I-H)(\partial_t^2 + ia\partial_\alpha)\varepsilon = \frac{\pi}{2}(I-H)\left(\mathfrak{z}(I-H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I-\overline{H})\mathfrak{z}\right) + 2\omega_0 i\partial_t\delta + 2[\mathfrak{z}_t, H]\frac{(\bar{\mathfrak{z}}_t\mathfrak{z})_\alpha}{\mathfrak{z}_\alpha}.$$

Arguing as in Proposition 3.15 and using Lemma 3.7 we get

$$(\partial_t^2 + ia\partial_\alpha - 2\omega_0 i\partial_t)\delta = \frac{\pi}{2}(I-H)\left(\mathfrak{z}(I-H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I-\overline{H})\mathfrak{z}\right) + \frac{\pi}{2}[(I-\overline{H})\mathfrak{z}, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} - 2[\mathfrak{z}_t, H\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\bar{\mathfrak{z}}_\alpha}]\partial_\alpha(\mathfrak{z}_t\bar{\mathfrak{z}}) - \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{\mathfrak{z}_t(\beta)-\mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)}\right)^2 \varepsilon_\beta(\beta)d\beta.$$

Exactly the same computation as in the proof of Proposition 3.15 now shows that

$$\frac{\pi}{2}(I-H)\left(\mathfrak{z}(I-H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I-\overline{H})\mathfrak{z}\right) + \frac{\pi}{2}[(I-\overline{H})\mathfrak{z}, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} = \pi\delta + \frac{\pi}{2}(I-H)E(\varepsilon) + \frac{\pi}{2}[E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}.$$

Combined with the previous identity this gives Eqs. (B.8), and (B.9) follows from differentiating (B.8) and using Lemma 3.7. □

To show that a_t does not contribute quadratic terms to the nonlinearity in the equation for δ_t we record the following analogue of Lemma 3.16.

Lemma B.5. *Let K^* denote the formal adjoint of $K := \operatorname{Re} H = \frac{1}{2}(H + \overline{H})$. Then*

$$(I + K^*)(a_t|\mathfrak{z}_\alpha|) = \operatorname{Re}\left[\frac{-i\mathfrak{z}_\alpha}{|\mathfrak{z}_\alpha|}\left\{2[\mathfrak{z}_t, H]\frac{\bar{\mathfrak{z}}_{tt\alpha}}{\mathfrak{z}_\alpha} + 2[\mathfrak{z}_{tt}, H]\frac{\bar{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} - [e^{it}g^a, H]\frac{\bar{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} + 2\omega_0 i[\mathfrak{z}_t, H]\frac{\bar{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} + \frac{1}{\pi i}\int_0^{2\pi}\left(\frac{\mathfrak{z}_t(\beta)-\mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)}\right)^2 \bar{\mathfrak{z}}_{t\beta}(\beta)d\beta + \frac{\pi}{2}([\mathfrak{z}_t, H]\frac{\partial_\alpha g^h}{\mathfrak{z}_\alpha})\right\}\right].$$

Proof. The proof is the same as that of Lemma 3.16. The only modification is that differentiating (B.4) in the time variable we get the following equation for $\bar{\mathfrak{z}}_t$:

$$\bar{\mathfrak{z}}_{ttt} - ia\bar{\mathfrak{z}}_{t\alpha} = ia_t\bar{\mathfrak{z}}_\alpha + \frac{\pi}{2}[\mathfrak{z}_t, H]\frac{\bar{\mathfrak{z}}_\alpha}{\mathfrak{z}_\alpha} + \bar{\mathfrak{z}}_t - 2\omega_0 i\bar{\mathfrak{z}}_{tt}.$$

The only new term compared to the equation for $\bar{z}_t$ in Lemma 3.16 is the last term on the right hand side. For this note that if $F(t, \mathfrak{z})$ is the holomorphic function with boundary value $\bar{\mathfrak{z}}_t$ then

$$\bar{\mathfrak{z}}_{tt} = F_t + \frac{\bar{\mathfrak{z}}_{t\alpha}\mathfrak{z}_t}{\mathfrak{z}_\alpha} = F_t + \frac{\bar{\mathfrak{z}}_{t\alpha}\mathfrak{z}_t}{\mathfrak{z}_\alpha}.$$

Since $\bar{\mathfrak{z}}_t$ and $\mathfrak{z}$ have the same holomorphicity properties as $\bar{z}_t$ and z in the irrotational case, the rest of the proof is exactly the same as that of Lemma 3.16. □

We now define the change of coordinates k similarly to Remark 3.21 and show that in the new coordinate $\alpha' = k(t, \alpha)$ the equations for δ and δ_t have no quadratic nonlinearities. The precise identities are given in the following proposition.

Proposition B.6. *Suppose $z(t, \cdot)$ is a simple closed curve containing the origin in its simply connected interior for each $t \in I$, where I is some time interval, and let k be as defined in Remark 3.21, but with z replaced by $\mathfrak{z}$, that is $(I - H)(\log(\bar{\mathfrak{z}}e^{ik})) = 0$. Then*

$$
\begin{aligned}
(I-H)k_t &= -i(I-H)\frac{\bar{\mathfrak{z}}_t\varepsilon}{\bar{\mathfrak{z}}} - i[\mathfrak{z}_t, H]\frac{\left(\log(\bar{\mathfrak{z}}e^{ik})\right)_\alpha}{\mathfrak{z}_\alpha},\\
(I-H)k_{tt} &= -i(I-H)\frac{\bar{\mathfrak{z}}_{tt}\varepsilon + \bar{\mathfrak{z}}_t\varepsilon_t}{\bar{\mathfrak{z}}} + i(I-H)\frac{\bar{\mathfrak{z}}_t^2\varepsilon}{\bar{\mathfrak{z}}^2}\\
&\quad - i[\mathfrak{z}_t, H]\frac{(\log(\bar{\mathfrak{z}}e^{ik}))_{t\alpha} + ik_{t\alpha}}{\mathfrak{z}_\alpha} + i[\mathfrak{z}_t, H]\frac{1}{\mathfrak{z}_\alpha}\partial_\alpha\left(\frac{\bar{\mathfrak{z}}_t\varepsilon}{\bar{\mathfrak{z}}}\right)\\
&\quad - i[\mathfrak{z}_{tt}, H]\frac{(\log(\bar{\mathfrak{z}}e^{ik}))_\alpha}{\mathfrak{z}_\alpha} - \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\mathfrak{z}_t(\beta)-\mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta)-\mathfrak{z}(\alpha)}\right)^2(\log(\bar{\mathfrak{z}}e^{ik}))_\beta d\beta,\\
(I-H)(ak_\alpha) &= [\mathfrak{z}_t, H]\frac{(\bar{\mathfrak{z}}_t\mathfrak{z})_\alpha}{\mathfrak{z}_\alpha} - [\mathfrak{z}_t, H]\bar{\mathfrak{z}}_t\\
&\quad - (I-H)\frac{(\bar{\mathfrak{z}}_{tt} + 2\omega_0 i\bar{\mathfrak{z}}_t)\varepsilon}{\bar{\mathfrak{z}}} + (I-H)\frac{e^{-it}g^h\varepsilon}{\bar{\mathfrak{z}}}\\
&\quad + [\mathfrak{z}_{tt} - 2\omega_0 i\mathfrak{z}_t - e^{it}g^a, H]\frac{\left(\log(\bar{\mathfrak{z}}e^{ik})\right)_\alpha}{\mathfrak{z}_\alpha}.
\end{aligned}
$$

Moreover if F is a holomorphic function with boundary value $\bar{\mathfrak{z}}e^{ik}$ and satisfies $F(t,0) \in \mathbb{R}_+$ for all $t \in I$, then with the notation $\mathcal{AV}(f) := \frac{1}{2}[z, H]\frac{f}{z}$,

$$
\begin{aligned}
&\mathcal{AV}(\varepsilon) = 0,\\
&\mathcal{AV}(ak_\alpha) = -(\pi - \omega_0^2) + \frac{\omega_0}{\pi}\int_0^{2\pi}\frac{\bar{\mathfrak{z}}_t\varepsilon\mathfrak{z}_\beta}{|\mathfrak{z}|^2}d\beta\\
&\qquad + \frac{1}{2\pi i}\int_0^{2\pi}\bar{\mathfrak{z}}_t\mathfrak{z}_{t\beta}d\beta + \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\bar{\mathfrak{z}}_{tt} - e^{-it}g^h)\varepsilon\mathfrak{z}_\beta}{|\mathfrak{z}|^2}d\beta,\\
&\qquad - \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{\mathfrak{z}_{tt} - 2i\mathfrak{z}_t - e^{it}g^a}{\mathfrak{z}}\right)\partial_\beta\log F d\beta,\\
&\operatorname{Re}\mathcal{AV}(k_t) = \frac{\operatorname{Re}}{2\pi}\int_0^{2\pi}\frac{\bar{\mathfrak{z}}_t\varepsilon}{|\mathfrak{z}|^2}\mathfrak{z}_\beta d\beta - \frac{\operatorname{Re}}{2\pi}\int_0^{2\pi}\log F\left(\frac{\mathfrak{z}\mathfrak{z}_{t\beta} - \mathfrak{z}_t\mathfrak{z}_\beta}{\mathfrak{z}^2}\right)d\beta,\\
&\operatorname{Re}\mathcal{AV}(k_{tt}) = \frac{\operatorname{Im}}{2\pi}\int_0^{2\pi}\left(\frac{\mathfrak{z}_{t\beta}\mathfrak{z} - \mathfrak{z}_t\mathfrak{z}_\beta}{\mathfrak{z}^2}\right)k_t d\beta + \frac{\operatorname{Re}}{2\pi}\partial_t\int_0^{2\pi}\frac{\bar{\mathfrak{z}}_t\varepsilon}{|\mathfrak{z}|^2}\mathfrak{z}_\beta d\beta\\
&\qquad + \frac{\operatorname{Re}}{2\pi}\partial_t\int_0^{2\pi}(\log(\bar{\mathfrak{z}}e^{ik}))_\beta\frac{\mathfrak{z}_t}{\mathfrak{z}}d\beta.
\end{aligned}
$$

Proof. The proof is the same as those of Propositions 3.18, 3.20, and 5.14. Indeed for the identities involving k_t and k_{tt} it suffices to note that $\mathfrak{z}_t$ and $\mathfrak{z}$ have the same holomorphicity as z_t and z in the irrotational case and the derivation does not rely on the first equations in (B.3) and (B.4). The second identity follows from the same argument as in Proposition 3.18. The only difference is that instead of $-\pi z + g^a$, the right hand side of the first equation in (B.4) can be written as $-(\pi - \omega_0^2)\mathfrak{z} + e^{it}g^a + 2i\mathfrak{z}_t$. The computation of the averages follows from similar modifications of the proof of Proposition 3.20. □

Note that the computations for $\mathcal{AV}(ak_\alpha)$ and for the static solution in the introduction show that a is close to $\pi - \omega_0^2$, whereas the "negative Klein-Gordon" term in the equation for δ is still $-\pi\delta$. This can be simply rectified by introducing the new unknown

$$\tilde{\delta} := e^{-\omega_0 it}\delta.$$

With this definition $\tilde{\delta}$ satisfies

$$\begin{aligned}(\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2))\tilde{\delta} &= \widetilde{\mathcal{M}}_1 := e^{-\omega_0 it}\widetilde{\mathcal{N}}_1,\\ (\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2))\tilde{\delta}_t &= \widetilde{\mathcal{M}}_2 := e^{-\omega_0 it}(\widetilde{\mathcal{N}}_2 + i\widetilde{\mathcal{N}}_1).\end{aligned} \tag{B.10}$$

With the same notation as the rest of the paper and with $\widetilde{N}_j = \widetilde{\mathcal{M}}_j \circ k^{-1},\ j = 1, 2$, we rewrite the equations for $\chi := \tilde{\delta} \circ k^{-1}$ and $v = (\partial_t \tilde{\delta}) \circ k^{-1}$ as

$$\begin{aligned}(\partial_t + b\partial_{\alpha'})^2\chi + iA\partial_{\alpha'}\chi - (\pi - \omega_0^2)\chi &= \widetilde{N}_1,\\ (\partial_t + b\partial_{\alpha'})^2 v + iA\partial_{\alpha'}v - (\pi - \omega_0^2)v &= \widetilde{N}_2.\end{aligned} \tag{B.11}$$

We can now prove Theorem B.1.

Proof of Theorem B.1. Since Eq. (B.11) has the same form as the equation studied in the proof of Theorem 1.1 the energy estimates are exactly the same. In view of Propositions B.3 and B.6 and Lemma B.5 the right hand sides of the equations in (B.11) contain no quadratic terms. Similarly, the identity for k_{tt} in Proposition B.6 shows that the contributions of $(\partial_t + b\partial_{\alpha'})b$ which arise in the higher energy estimates as in Proposition 5.15 do not contain quadratic terms. Therefore the only remaining step in the proof is to verify that $\tilde{v} := (I - H)v$ also satisfies an equation with no quadratic nonlinearities, analogous to the equation derived in Proposition 5.11. The computation here is similar and we only present the necessary modifications. We use the same proof as in Proposition 5.11 replacing z by $\mathfrak{z}$ throughout. In the first step in the commutator $[\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2), H]\tilde{\delta}_t$ we get the extra term $2\omega_0 i[\mathfrak{z}_t, H]\frac{\tilde{\delta}_{t\alpha}}{\mathfrak{z}_\alpha}$. This can be written as

$$[\mathfrak{z}_t, H]\frac{\tilde{\delta}_{t\alpha}}{\mathfrak{z}_\alpha} = -\omega_0 i[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha} + [\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I - H)\varepsilon_t}{\mathfrak{z}_\alpha} - [\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}[\mathfrak{z}_t, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}}{\mathfrak{z}_\alpha}.$$

The last term is already cubic. To see that $[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}$ is also cubic note that

$$e^{\omega_0 it}\tilde{\delta} = (I - H)\varepsilon = (I + \overline{H})\varepsilon - (H + \overline{H})\varepsilon = (I + \overline{H})\varepsilon - \mathfrak{z}[\varepsilon, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} + E(\varepsilon).$$

The contribution of $-\mathfrak{z}[\varepsilon, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} + E(\varepsilon)$ to $[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}$ is clearly cubic and for $(I + \overline{H})\varepsilon$ note that

$$[\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I + \overline{H})\varepsilon}{\mathfrak{z}_\alpha} = [\mathfrak{z}_t, H\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\overline{\mathfrak{z}}_\alpha}]e^{-\omega_0 it}\partial_\alpha(I + \overline{H})\varepsilon,$$

which is cubic. The contribution of $[\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I-H)\varepsilon_t}{\mathfrak{z}_\alpha}$ is shown to be cubic in a similar way. The only other computation which is different from the proof of Proposition 5.11 is that of the term $II := -2[\mathfrak{z}_t, H]\frac{\partial_\alpha(ia\partial_\alpha\tilde{\delta})}{\mathfrak{z}_\alpha}$ in (5.18), where we use Eq. (B.4) for $\mathfrak{z}$ instead of the equation for z. Here the extra terms we get are

$$-2\omega_0 i[\mathfrak{z}_t, H]\frac{\partial_\alpha}{\mathfrak{z}_\alpha}\left(\frac{\mathfrak{z}_t\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}\right) - \omega_0^2[\mathfrak{z}_t, H]\frac{\partial_\alpha}{\mathfrak{z}_\alpha}\left(\frac{\mathfrak{z}\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}\right).$$

The first term is already cubic and the second term is identical, up to a multiplicative constant, to one of the terms already computed in the calculation of II in (5.18). This shows that the equation for $\tilde{v}$ contains no quadratic nonlinearities which completes the proof of Theorem B.1. □

Notations

For the reader's convenience we give the definitions of some of the symbols used commonly in this work.

$$Hf(t,\alpha) = \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(t,\beta)}{z(t,\beta)-z(t,\alpha)} z_\beta(\beta)d\beta.$$

$$\mathcal{H}f(t,\alpha) = \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(t,\beta)}{\mathfrak{z}(t,\beta)-\mathfrak{z}(t,\alpha)} \mathfrak{z}_\beta(\beta)d\beta, \quad \mathfrak{z}(t,\cdot) = z(t, j(t,\cdot)),$$

$j(t,\cdot) : [0,2\pi] \to [0,2\pi]$ a diffeomorphism.

$$\mathbb{H}f(t,\alpha') = \frac{\text{p.v.}}{\pi i} \int_0^{2\pi} \frac{f(t,\beta')}{e^{i\beta'}-e^{i\alpha'}} ie^{i\beta'}d\beta',$$

$$\widetilde{\mathbb{H}}f(t,\alpha) = \frac{\text{p.v.}}{2\pi i} \int_0^{2\pi} f(t,\beta)\cot\left(\frac{\beta-\alpha}{2}\right)d\beta.$$

$$\mathcal{AV}(f) := \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\alpha)}{z(t,\alpha)} z_\alpha(t,\alpha)d\alpha, \qquad \text{Av}(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha.$$

$$Kf = \text{Re}\, H = \frac{1}{2}(H+\overline{H})f, \quad \mathcal{K}f = \text{Re}\,\mathcal{H}f = \frac{1}{2}(\mathcal{H}+\overline{\mathcal{H}})f, \quad f \text{ real valued.}$$

$$K^*f = -\text{Re}\left\{\frac{z_\alpha}{|z_\alpha|} H \frac{|z_\alpha|}{z_\alpha} f\right\}, \quad \mathcal{K}^*f = -\text{Re}\left\{\frac{\mathfrak{z}_\alpha}{|\mathfrak{z}_\alpha|} \mathcal{H} \frac{|\mathfrak{z}_\alpha|}{\mathfrak{z}_\alpha} f\right\}, \quad f \text{ real valued.}$$

$$a = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial n}, \quad \mathbf{n} \text{ unit exterior normal.}$$

Let h be as defined in Figure 7 and k as defined in Remark 3.21.

$$Z(t,\alpha') = z(t, h^{-1}(t,\alpha')), \quad \zeta(t,\alpha) = z(t,k^{-1}(t,\alpha)).$$

$$Z_t(t,\alpha') = z_t(t,h^{-1}(t,\alpha')), \quad Z_{tt}(t,\alpha') = z_{tt}(t,h^{-1}(t,\alpha')),$$

$$Z_{ttt}(t,\alpha') = z_{ttt}(t,h^{-1}(t,\alpha')).$$

$$B = h_t \circ h^{-1}, \qquad b = k_t \circ k^{-1}.$$

$$\mathcal{A} = (ah_\alpha)\circ h^{-1}, \quad \mathcal{A}_1 = \mathcal{A}|Z_{,\alpha'}|^2, \quad A = (ak_{\alpha'})\circ k^{-1}.$$

$$G = (I+\mathcal{H})\overline{Z}.$$

$$D_\alpha = \frac{1}{|z_\alpha|}\partial_\alpha, \quad D_{\alpha'} = \frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}.$$

$$[Z_t, Z_t; D_{\alpha'}\overline{Z}_t] = \frac{ie^{i\alpha'}}{\pi i}\int_0^{2\pi}\left(\frac{Z_t(t,\beta')-Z_t(t,\alpha')}{e^{i\beta'}-e^{i\alpha'}}\right)^2 \frac{e^{i\beta'}}{Z_{\beta'}(t,\beta')}\overline{Z}_{t,\beta'}(t,\beta')d\beta'.$$

$$\mathcal{P} = (\partial_t + b\partial_\alpha)^2 + ia\partial_\alpha - \pi.$$

$$\varepsilon = |z|^2 - 1, \quad \mu = \varepsilon\circ k^{-1}, \quad \delta = (I-H)\varepsilon, \quad \chi = \delta\circ k^{-1}, \quad \eta = \zeta_\alpha - i\zeta.$$

$$u = z_t \circ k^{-1}, \quad w = z_{tt}\circ k^{-1}, \quad v = \delta_t \circ k^{-1}.$$

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