GREEN'S FUNCTION FOR LAPLACIAN

The Green's function is a tool to solve non-homogeneous linear equations. We will illustrate this idea for the Laplacian Δ.

Suppose we want to find the solution u of the Poisson equation in a domain $D \subset \mathbb{R}^n$:

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in D$$

subject to some homogeneous boundary condition. Imagine f is the heat source and u is the temperature. The idea of Green's function is that if we know the temperature responding to an impulsive heat source at any point $\mathbf{x}_0 \in D$, then we can just sum up the result with the weight function $f(\mathbf{x}_0)$ (it specifies the strength of the heat at point $\mathbf{x}_0$) to obtain the temperature responding to the heat source $f(\mathbf{x})$ in D. Mathematically, one may express this idea by defining the Green's function as the following:

Let $u = u(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \ldots x_n)$ be the solution of the following problem:

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in D \\ u \text{ satisfies some homogeneous boundary condition along the boundary } \partial D \end{cases} \tag{0.1}$$

Define the Green's function $G = G(\mathbf{x}, \mathbf{x}_0)$ to be the solution of

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \qquad \mathbf{x} \in D \\ G(\mathbf{x}, \mathbf{x}_0) \text{ satisfies the same homogeneous boundary condition as in (0.1)} \end{cases} \tag{0.2}$$

here $\mathbf{x}_0 \in D$ is a fixed point. Then

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, d\mathbf{x}_0 \tag{0.3}$$

should be the solution of the boundary value problem (0.1). We can check this statement for the Dirichlet boundary value problem:

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in D \\ u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \partial D \end{cases} \tag{0.4}$$

We define the Green's function G with Dirichlet BC by

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \qquad \mathbf{x} \in D \\ G(\mathbf{x}, \mathbf{x}_0) = 0 \qquad \mathbf{x} \in \partial D \end{cases} \tag{0.5}$$

Applying $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ to both sides of (0.3), we get

$$\Delta u(\mathbf{x}) = \int_D \Delta G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, d\mathbf{x}_0 = \int_D \delta(\mathbf{x} - \mathbf{x}_0) f(\mathbf{x}_0)\, d\mathbf{x}_0 = f(\mathbf{x})$$

and for $\mathbf{x}$ on the boundary of D, we have $u(\mathbf{x}) = 0$ because $G(\mathbf{x}, \mathbf{x}_0) = 0$ by the definition of G in (0.5). Verification of (0.3) for u and G satisfying Neumann or Robin conditions can be done similarly.

Now let's see how to find the Green's function for some particular domains.

To simplify the discussion, we will be focusing on $D \subset \mathbb{R}^2$, the same idea extends to domains $D \subset \mathbb{R}^n$ for any $n \geq 1$, and to other linear equations. In what follows we let $\mathbf{x} = (x, y) \in \mathbb{R}^2$.

0.1. **The fundamental solution.** We first look for the function $\Gamma(\mathbf{x})$ in the whole space $\mathbb{R}^2$ so that

$$\Delta\Gamma(\mathbf{x}) = \delta(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbb{R}^2$$

Since $\Gamma(\mathbf{x})$ is the responding temperature to the point heat source at the origin, it must be radially symmetric, that is

$$\Gamma(\mathbf{x}) = \Gamma(r), \qquad r = |\mathbf{x}| = \sqrt{x^2 + y^2}$$

We know the Laplacian in polar coordinates gives

$$\Delta\Gamma(r) = \Gamma''(r) + \frac{1}{r}\Gamma'(r)$$

we want $\Delta\Gamma(\mathbf{x}) = \delta(\mathbf{x})$, therefore

$$\Gamma''(r) + \frac{1}{r}\Gamma'(r) = 0 \qquad \text{for } r > 0 \tag{0.6}$$

Solving (0.6), we get

$$\Gamma(r) = c_1 \ln r + c_2$$

for some constants c_1 and c_2. We take $c_2 = 0$. We now want to determine c_1 so that $\Delta\Gamma(\mathbf{x}) = \delta(\mathbf{x})$. We use Divergence Theorem.

Recall divergence theorem states that for any vector field $\mathbf{F}(\mathbf{x}) = (F_1(x, y), F_2(x, y))$ defined on a domain $D \subset \mathbb{R}^2$, we have

$$\iint_D \nabla \cdot F(\mathbf{x})\, d\mathbf{x} = \int_{\partial D} \mathbf{n} \cdot F(\mathbf{x})\, dS \tag{0.7}$$

here $\text{div}F(\mathbf{x}) = \nabla \cdot F(\mathbf{x}) = \partial_x F_1(x, y) + \partial_y F_2(x, y)$, $\mathbf{n}$ is the unit outward normal to the boundary of the domain D, $\int_{\partial D} \dots dS$ is the line integral along the boundary curve ∂D.

Now applying the divergence theorem to $\mathbf{F}(\mathbf{x}) = \nabla u(\mathbf{x}) = (\partial_x u(x, y), \partial_y u(x, y))$. Since $\Delta u = \nabla \cdot \nabla u$, we have

$$\iint_D \Delta u(\mathbf{x})\, d\mathbf{x} = \int_{\partial D} \mathbf{n} \cdot \nabla u(\mathbf{x})\, dS$$

We want $\Delta\Gamma(\mathbf{x}) = \delta(\mathbf{x})$. Therefore, we want for any disk $B(0, R)$ centered at the origin with radius R,

$$1 = \iint \delta(\mathbf{x})\, d\mathbf{x} = \iint \Delta\Gamma(\mathbf{x})\, d\mathbf{x} = \iint_{B(0,R)} \Delta\Gamma(\mathbf{x})\, d\mathbf{x} = \int_{\partial B(0,R)} \mathbf{n} \cdot \nabla\Gamma(\mathbf{x})\, dS$$

Notice that the normal derivative of Γ along the circle $\partial B(0, R)$ is the same as the derivative of Γ in the radial direction evaluated at R:

$$\mathbf{n} \cdot \nabla\Gamma(\mathbf{x}) = \Gamma'(R) \qquad \text{for } |\mathbf{x}| = R$$

and $\Gamma'(R)$ is a constant along the circle $\partial B(0, R)$, therefore

$$\int_{\partial B(0,R)} \mathbf{n} \cdot \nabla\Gamma(\mathbf{x})\, dS = \int_{\partial B(0,R)} \Gamma'(R)\, dS = 2\pi R\Gamma'(R)$$

finally we arrive at

$$1 = 2\pi R\Gamma'(R)$$

this gives that $\Gamma'(R) = \frac{1}{2\pi R}$, therefore $\Gamma(R) = \frac{1}{2\pi}\ln R$. In other wards, an application of divergence theorem also gives us the same answer as above, with the constant $c_1 = \frac{1}{2\pi}$.

Now we find that the function

$$\Gamma(\mathbf{x}) = \frac{1}{2\pi}\ln r, \qquad \text{where } r = |\mathbf{x}| = \sqrt{x^2+y^2}$$

satisfies

$$\Delta\Gamma(\mathbf{x}) = \delta(\mathbf{x})$$

We call this function $\Gamma(\mathbf{x})$ the fundamental solution of the Laplacian in $\mathbb{R}^2$. It is clear that for any point $\mathbf{x}_0 \in \mathbb{R}^2$, $\Delta\Gamma(\mathbf{x}-\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)$.

0.2. **Green's function for bounded domains.** We can construct Green's function for a bounded domain $D \subset \mathbb{R}^2$ using the fundamental solution $\Gamma(\mathbf{x}-\mathbf{x}_0)$.

For example if we want to find the Green's function G with Dirichlet BC on $D \subset \mathbb{R}^2$:

$$\begin{cases} \Delta G = \delta(\mathbf{x}-\mathbf{x}_0) & \mathbf{x} \in D \\ G(\mathbf{x},\mathbf{x}_0) = 0 & \text{for } \mathbf{x} \in \partial D \end{cases} \tag{0.8}$$

here $\mathbf{x}_0 \in D$ is a fixed point, we can construct G as the follows

$$G(\mathbf{x},\mathbf{x}_0) = \Gamma(\mathbf{x}-\mathbf{x}_0) + v(\mathbf{x},\mathbf{x}_0) \tag{0.9}$$

In order for G to satisfy (0.8), v must satisfy

$$\begin{cases} \Delta v(\mathbf{x},\mathbf{x}_0) = 0 & \mathbf{x} \in D \\ v(\mathbf{x},\mathbf{x}_0) = -\Gamma(\mathbf{x}-\mathbf{x}_0), & \text{for } \mathbf{x} \in \partial D \end{cases} \tag{0.10}$$

In other wards, v should be a solution of the Laplace equation in D satisfying a non-homogeneous boundary condition that nullifies the effect of Γ on the boundary of D. Similarly we can construct the Green's function with Neumann BC by setting $G(\mathbf{x},\mathbf{x}_0) = \Gamma(\mathbf{x}-\mathbf{x}_0) + v(\mathbf{x},\mathbf{x}_0)$ where v is a solution of the Laplace equation with a Neumann boundary condition that nullifies the heat flow coming from Γ.

In what follows we construct the Green's functions for the upper half plane and for the unit disk.

We point out that for a general domain D, it is usually difficult to solve explicitly for v, therefore difficult to find an explicit expression for G. Nevertheless, (0.9) is an important expression from which we can draw qualitative informations about the Green's function and therefore the solution of the Poisson and Laplace equations.

We call a solution u of the Laplace equation $\Delta u(\mathbf{x}) = 0$ for $\mathbf{x} \in D$ a harmonic function in D.

0.3. **Green's function for the upper half plane $\{y > 0\}$.** We first construct the Green's function in the upper half plane with the Dirichlet boundary condition:

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \mathbf{x} = (x, y),\ y > 0 \\ G(x, 0, \mathbf{x}_0) = 0 & x \in \mathbb{R} \end{cases} \tag{0.11}$$

here $\mathbf{x}_0 = (x_0, y_0)$ is a fixed point in the upper half plane, i.e. $y_0 > 0$.

We know $\Gamma(\mathbf{x} - \mathbf{x}_0) = \Gamma(x - x_0, y - y_0)$ is the temperature response to the single heat source $\delta(\mathbf{x} - \mathbf{x}_0)$ placed at the point $\mathbf{x}_0$. However $\Gamma(x - x_0, y - y_0) \neq 0$ at the boundary $\{y = 0\}$. In order to construct the temperature response G so that G is zero on the boundary $\{y = 0\}$: $G(x, 0, x_0, y_0) = 0$, we place another heat source $-\delta(\mathbf{x} - \mathbf{x}_0^*)$ of negative strength at the image point $\mathbf{x}_0^* = (x_0, -y_0)$ and consider the temperature response to both of the heat sources at $\mathbf{x}_0$ and $\mathbf{x}_0^*$:

$$G(\mathbf{x}, \mathbf{x}_0) = \Gamma(\mathbf{x} - \mathbf{x}_0) - \Gamma(\mathbf{x} - \mathbf{x}_0^*) \tag{0.12}$$

We claim G as defined in (0.12) is the Green's function satisfying (0.11).

First we know

$$\Delta G(\mathbf{x}, \mathbf{x}_0) = \Delta(\Gamma(\mathbf{x} - \mathbf{x}_0) - \Gamma(\mathbf{x} - \mathbf{x}_0^*)) = \delta(\mathbf{x} - \mathbf{x}_0) - \delta(\mathbf{x} - \mathbf{x}_0^*) \qquad \text{for } \mathbf{x} \in \mathbb{R}^2$$

However $\mathbf{x}_0^*$ is a point in the lower half plane, and we know $\delta(\mathbf{x} - \mathbf{x}_0^*) = 0$ for all $\mathbf{x} \neq \mathbf{x}_0^*$, therefore

$$\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$$

for $\mathbf{x}$ in the upper half plane. And it is easy to check that on the boundary $\{y = 0\}$ of the upper half plane,

$$G(x, 0, \mathbf{x}_0) = \Gamma(x - x_0, -y_0) - \Gamma(x - x_0, y_0) = \frac{1}{2\pi}(\ln\sqrt{(x - x_0)^2 + (-y_0)^2} - \ln\sqrt{(x - x_0)^2 + (y_0)^2}) = 0$$

Therefore we have constructed the Green's function for the Dirichlet BC in the upper half plane:

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\ln|\mathbf{x} - \mathbf{x}_0| - \ln|\mathbf{x} - \mathbf{x}_0^*|) = \frac{1}{4\pi}\ln\frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}$$

Similarly we can construct the Green's function for the upper half plane with the homogeneous Neumann BC. Notice that for the upper half plane, the outward normal derivative at the boundary is in the negative y direction:

$$\frac{\partial G}{\partial \mathbf{n}} = -\frac{\partial G}{\partial y}$$

So what we look for next is the G satisfying

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \mathbf{x} = (x, y),\ y > 0 \\ \frac{\partial G}{\partial y}(x, 0, \mathbf{x}_0) = 0 & x \in \mathbb{R} \end{cases} \tag{0.13}$$

here $\mathbf{x}_0 = (x_0, y_0)$ is a fixed point in the upper half plane.

Now we want to make sure the heat sources are placed in such a way that the flux through the boundary $\{y = 0\}$ is zero. This can be achieved by placing another heat source $\delta(\mathbf{x} - \mathbf{x}_0^*)$

with the same strength at the image point $\mathbf{x}_0^* = (x_0, -y_0)$. We claim the Green's function for (0.13) is

$$G(\mathbf{x}, \mathbf{x}_0) = \Gamma(\mathbf{x} - \mathbf{x}_0) + \Gamma(\mathbf{x} - \mathbf{x}_0^*) = \frac{1}{4\pi} \ln\{((x-x_0)^2 + (y-y_0)^2)((x-x_0)^2 + (y+y_0)^2)\}$$

We leave the checking of this claim to the readers.

0.4. **Green's representation formula.** Now that we have constructed the Green's function for the upper half plane. Before we move on to construct the Green's function for the unit disk, we want to see besides the homogeneous boundary value problem (0.1), what other problems can be solved by the Green's function approach.

First we derive the Green's identity from the divergence theorem.

Let u, v be smooth functions defined on a domain $D \subset \mathbb{R}^2$. Let $\mathbf{F} = u\nabla v - v\nabla u$, and we apply the divergence theorem to $\mathbf{F}$. Notice that

$$\nabla \cdot \mathbf{F} = \nabla u \nabla v + u\Delta v - \nabla v \nabla u - v\Delta u = u\Delta v - v\Delta u$$

we get

$$\iint_D (u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x}))\, d\mathbf{x} = \int_{\partial D} (u\mathbf{n} \cdot \nabla v - v\mathbf{n} \cdot \nabla u)\, dS \qquad (0.14)$$

where $\mathbf{n}$ is the unit outward normal to ∂D. (0.14) is called the Green's identity.

Let G be the Green's function defined in (0.8) satisfying the Dirichlet BC. A quick application of the Green's identity to $u = G(\mathbf{x}, \mathbf{x}_1)$ and $v = G(\mathbf{x}, \mathbf{x}_2)$ gives the symmetry of G:

$$G(\mathbf{x}_1, \mathbf{x}_2) = G(\mathbf{x}_2, \mathbf{x}_1)$$

We now apply the Green's identity to u and $v = G(\mathbf{x}, \mathbf{x}_0)$, where G is the Green's function defined in (0.8). We have

$$\iint_D (u(\mathbf{x})\Delta G(\mathbf{x}, \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}_0)\Delta u(\mathbf{x}))\, d\mathbf{x} = \int_{\partial D} (u(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}} - G(\mathbf{x}, \mathbf{x}_0)\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}})\, dS \qquad (0.15)$$

Therefore

$$u(\mathbf{x}_0) = \iint_D G(\mathbf{x}, \mathbf{x}_0)\Delta u(\mathbf{x})\, d\mathbf{x} + \int_{\partial D} u(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}}\, dS$$

Interchange the notations $\mathbf{x}$ with $\mathbf{x}_0$, we have

$$u(\mathbf{x}) = \iint_D G(\mathbf{x}, \mathbf{x}_0)\Delta u(\mathbf{x}_0)\, d\mathbf{x}_0 + \int_{\partial D} u(\mathbf{x}_0)\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}_0}\, dS_0 \qquad (0.16)$$

In other words,

$$u(\mathbf{x}) = \iint_D G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, d\mathbf{x}_0 + \int_{\partial D} h(\mathbf{x}_0)\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}_0}\, dS_0$$

is the solution formula for the solution of the following non-homogeneous boundary value problem:

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}) & \mathbf{x} \in D \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \partial D \end{cases} \qquad (0.17)$$

(0.16) is called **the Green's representation formula**.

As we know $G(\mathbf{x}, \mathbf{x}_0)$ is the influence function for the source term $f(\mathbf{x})$, while $\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial \mathbf{n}_0}$ is the influence function for the nonhomogeneous boundary condition. We know

$$\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}_0} = \lim_{\epsilon \to 0} \frac{G(\mathbf{x}, \mathbf{x}_0 + \epsilon \mathbf{n}_0) - G(\mathbf{x}, \mathbf{x}_0)}{\epsilon}$$

$\frac{1}{\epsilon}G(\mathbf{x}, \mathbf{x}_0 + \epsilon\mathbf{n}_0)$ is the response to the source located at $\mathbf{x}_0 + \epsilon\mathbf{n}_0$ with strength $\frac{1}{\epsilon}$, while $-\frac{1}{\epsilon}G(\mathbf{x}, \mathbf{x}_0)$ is the response to the source located at $\mathbf{x}_0$ with strength $-\frac{1}{\epsilon}$. So the influence function for the non-homogeneous Dirichlet boundary condition is a response to a **dipole source** distributed along the boundary ∂D.

0.5. **Green's representation formula for upper half plane.** We now apply Green's representation formula (0.16) for the solution of the following Dirichlet problem on the upper half plane:

$$\begin{cases} \partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0 & -\infty < x < \infty, \ y > 0 \\ u(x, 0) = h(x) & -\infty < x < \infty \end{cases} \tag{0.18}$$

We know the Green's function in the upper half plane with homogeneous Dirichlet BC is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\ln|\mathbf{x} - \mathbf{x}_0| - \ln|\mathbf{x} - \mathbf{x}_0^*|) = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}$$

The outward unit normal to the boundary of the upper half plane is in the $-y$ direction. So we calculate

$$\frac{\partial G(x, y, x_0, y_0)}{\partial y_0} = \frac{1}{4\pi}\left(\frac{2(y_0 - y)}{(x - x_0)^2 + (y - y_0)^2} - \frac{2(y_0 + y)}{(x - x_0)^2 + (y + y_0)^2}\right)$$

and

$$\frac{\partial G(x, y, x_0, 0)}{\partial \mathbf{n}_0} = -\frac{\partial G(x, y, x_0, 0)}{\partial y_0} = \frac{1}{\pi} \frac{y}{(x - x_0)^2 + y^2}$$

So the solution of (0.18) is given by

$$u(x, y) = \frac{1}{\pi} \int \frac{y}{(x - x_0)^2 + y^2} h(x_0)\, dx_0 \tag{0.19}$$

The solution formula (0.19) is called the Poisson integral formula and can also be found using the Fourier transform. $\frac{1}{\pi}\frac{y}{(x-x_0)^2+y^2}$ is called Poisson kernel.

0.6. **Green's function for the unit disk.** Let $D = \{|\mathbf{x}| < 1\}$ be the unit disk. For any fixed $\mathbf{x}_0 \in D$, we want to construct the Green's function satisfying the Dirichlet BC:

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \mathbf{x} \in D \\ G(\mathbf{x}, \mathbf{x}_0) = 0 & \text{for } |\mathbf{x}| = 1 \end{cases} \tag{0.20}$$

As it turns out, there is an image point $\mathbf{x}_0^* = \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}$ on the prolonged line segment from 0 to $\mathbf{x}_0$, satisfying that for any $\mathbf{x}$ on the unit circle, i.e. $|\mathbf{x}| = 1$,

$$|\mathbf{x} - \mathbf{x}_0|^2 = |\mathbf{x}_0|^2 |\mathbf{x} - \mathbf{x}_0^*|^2 \tag{0.21}$$

We can check this assertion easily. We have, for any $\mathbf{x}$ such that $|\mathbf{x}| = 1$,

$$|\mathbf{x}-\mathbf{x}_0^*|^2 = |\mathbf{x}|^2+|\mathbf{x}_0^*|^2-2\mathbf{x}\cdot\mathbf{x}_0^* = 1+|\mathbf{x}_0|^{-2}-2\mathbf{x}\cdot\mathbf{x}_0|\mathbf{x}_0|^{-2} = |\mathbf{x}_0|^{-2}(|\mathbf{x}_0|^2+1-2\mathbf{x}\cdot\mathbf{x}_0) = |\mathbf{x}_0|^{-2}|\mathbf{x}-\mathbf{x}_0|^2$$

We can now use the fact (0.21) to construct the Green's function for the unit disk with Dirichlet BC:

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}(\ln |\mathbf{x} - \mathbf{x}_0| - \ln |\mathbf{x} - \mathbf{x}_0^*| - \ln |\mathbf{x}_0|)$$

in polar coordinates, it can be written as

$$G(r, \theta, r_0, \theta_0) = \frac{1}{4\pi} \ln \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{r^2 r_0^2 + 1 - 2rr_0 \cos(\theta - \theta_0)}$$

From here we can use the Green's representation formula (0.16) to find the solution formula for the following Dirichlet boundary value problem in the unit disk D:

$$\begin{cases} \Delta u = 0 & 0 \leq r < 1, \ -\pi \leq \theta \leq \pi \\ u(1, \theta) = h(\theta) & -\pi \leq \theta \leq \pi \end{cases} \tag{0.22}$$

We need to find $\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}_0}$ along the unit circle. We know the unit normal to the circle is in the radial direction, so for $\mathbf{x}_0 = (1, \theta_0)$ on the unit circle,

$$\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial \mathbf{n}_0} = \frac{\partial G(r, \theta, 1, \theta_0)}{\partial r_0} = \frac{1}{2\pi} \frac{1 - r^2}{r^2 + 1 - 2r \cos(\theta - \theta_0)}$$

From the Green's representation formula, we know that the solution of problem (0.22) is

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{r^2 + 1 - 2r \cos(\theta - \theta_0)} h(\theta_0) \, d\theta_0$$

This recovers the Poisson integral formula we derived using separation of variables.