Relation (6.22) implies for any a,b,y the "conservation law"

$$0=\frac{d}{dy}\int_a^b R(u(x,y))\,dx+S(u(b,y))-S(u(a,y)). \tag{6.24}$$

Conversely (6.22) follows from (6.24) for any $u\in C^1$. Now (6.24) makes sense for more general u and can serve to define "weak" solutions of (6.22). In particular we consider the case where u is a C^1-solution of (6.22) in each of two regions in the xy-plane separated by a curve $x=\xi(y)$, across which the value of u shall undergo a jump ("shock"). Denoting the limits of u from the left and right respectively by u^- and u^+, we find from (6.24) for $a<\xi(y)<b$

$$\begin{aligned}0&=S(u(b,y))-S(u(a,y))+\frac{d}{dy}\left(\int_a^{\xi}R(u)\,dx+\int_{\xi}^{b}R(u)\,dx\right)\\&=S(u(b,y))-S(u(a,y))+\xi'R(u^-)-\xi'R(u^+)\\&\quad-\int_a^{\xi}\frac{\partial S(u)}{\partial x}dx-\int_{\xi}^{b}\frac{\partial S(u)}{\partial x}dx\\&=-(R(u^+)-R(u^-))\xi'-S(u^-)+S(u^+).\end{aligned}$$

Hence we find the relation ("shock condition")

$$\frac{d\xi}{dy}=\frac{S(u^+)-S(u^-)}{R(u^+)-R(u^-)} \tag{6.25}$$

connecting the speed of propagation $d\xi/dy$ of the discontinuity with the amounts by which R and S jump. We observe that (6.25) depends not only on the original partial differential equation (6.12) but also on our choice of the functions $R(u)$, $S(u)$ satisfying (6.23). [Compare with Burgers' equation, p. 214.]

Problems

1. Solve the following initial-value problems and verify your solution:
   (a) $u_x+u_y=u^2$, $u(x,0)=h(x)$
   (b) $u_y=xuu_x$, $u(x,0)=x$
   (Answer: $x=ue^{-yu}$ implicitly.)
   (c) $xu_x+yu_y+u_z=u$, $u(x,y,0)=h(x,y)$
   (d) $xu_y-yu_x=u$, $u(x,0)=h(x)$
   (Answer: $u=h(\sqrt{x^2+y^2}\,)e^{\arctan(y/x)}$.)
2. (Picone). Let u be a solution of
$$a(x,y)u_x+b(x,y)u_y=-u$$
of class C^1 in the closed unit disk Ω in the xy-plane. Let $a(x,y)x+b(x,y)y>0$ on the boundary of Ω. Prove that u vanishes identically. (Hint: Show that $\max_\Omega u\leqslant 0$, $\min_\Omega u\geqslant 0$, using conditions for a maximum at a boundary point.)

3. Let u be a C^1-solution of (6.12) in each of two regions separated by a curve $x=\xi(y)$. Let u be continuous, but u_x have a jump discontinuity on the curve. Prove that
$$\frac{d\xi}{dy}=u$$
and hence that the curve is a characteristic. (Hint: By (6.12)
$$(u_y^+-u_y^-)+u(u_x^+-u_x^-)=0.$$
Moreover $u(\xi(y),y)$ and $(d/dy)u(\xi(y),y)$ are continuous on the curve.)
4. Show that the function $u(x,y)$ defined for $y\geqslant 0$ by
$$u=-\tfrac{2}{3}\left(y+\sqrt{3x+y^2}\,\right)\quad\text{for } 4x+y^2>0$$
$$u=0\quad\text{for } 4x+y^2<0$$
is a weak solution of (6.22) for the choice $R(u)=u, S(u)=\frac{1}{2}u^2$.
5. Define a *weak* solution $u(x,y)$ of (6.22) as a function for which the relation
$$\iint (R(u)\phi_y+S(u)\phi_x)\,dx\,dy=0 \tag{6.26}$$
holds for any function $\phi(x,y)$ of class C_0^∞ (Relation (6.26) follows formally from (6.22) by integration by parts.) Show that this definition of weak solution also leads to the jump condition (6.25).
6. Show that the solution u of the quasi-linear partial differential equation
$$u_y+a(u)u_x=0 \tag{6.27}$$
with initial condition $u(x,0)=h(x)$ is given implicitly by
$$u=h(x-a(u)y) \tag{6.28}$$
Show that the solution becomes singular for some positive y, unless $a(h(s))$ is a nondecreasing function of s.

7. The General First-Order Equation for a Function of Two Variables

The general first-order partial differential equation for a function $z=u(x,y)$ has the form

$$F(x,y,z,p,q)=0, \tag{7.1}$$

where $p=u_x, q=u_y$. We assume that F where considered has continuous second derivatives with respect to its arguments x,y,z,p,q. Surprisingly enough the problem of solving the general equation (7.1) reduces to that of solving a system of ordinary differential equations. This reduction is suggested by the geometric interpretation of (7.1) as a condition on the integral surface $z=u(x,y)$ in xyz-space determined by a solution $u(x,y)$.