MATH 656 HW 2

1. For the equation

$$u_y = (u_x)^3,$$

(a) Find the solution with $u(x,0) = 2x^{3/2}$;
(b) Show that every solution regular for all x, y is linear.

2. (Legendre transformation, Hodograph method) Let $u(x,y)$ be a solution of a quasi-linear equation of the form

$$a(u_x, u_y)u_{xx} + 2b(u_x, u_y)u_{xy} + c(u_x, u_y)u_{yy} = 0.$$

Introduce new independent variables ξ, η and a new unknown function

$$\xi = u_x(x,y), \qquad \eta = u_y(x,y), \qquad \phi = xu_x + yu_y - u.$$

Prove that ϕ as a function of ξ, η satisfies $x = \phi_\xi$, $y = \phi_\eta$ and the linear differential equation

$$a(\xi,\eta)\phi_{\eta\eta} - 2b(\xi,\eta)\phi_{\xi\eta} + c(\xi,\eta)\phi_{\xi\xi} = 0.$$

3. The following equation describes the motion of an isentropic gas:

$$\begin{cases} \rho(v_t + (v\cdot\nabla)v) + \nabla P = 0, & x \in \mathbb{R}^3, t > 0 \\ \rho_t + \operatorname{div}(\rho v) = 0, & \\ P = P(\rho) & \end{cases} \tag{0.1}$$

where $P'(\rho) > 0$. Find the linearization around the constant solution $(0, \rho_0)$. That is: using the ansatz $(v,\rho) = (\epsilon u, \rho_0 + \epsilon\eta)$ for the solution and keeping the $O(\epsilon)$ terms. This will give you the equation for sound waves, with sound speed $c = \sqrt{P'(\rho_0)}$.

4. (a) Show that for n = 3 the general solution of $\partial_t^2 u - c^2\Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \frac{F(r+ct) + G(r-ct)}{r}, \qquad r = |x|.$$

(b) Show that the solution with initial data

$$u = 0, \qquad u_t = g(r)$$

(g is even in r) is given by

$$u = \frac{1}{2cr}\int_{r-ct}^{r+ct} \rho g(\rho)d\rho \tag{0.2}$$

(c) for

$$g(r) = \begin{cases} 1, & 0 < r < a \\ 0, & r > a, \end{cases} \tag{0.3}$$

find u explicitly from (0.2) in the different regions bounded by the cones $r = a \pm ct$ in the rt-space. Show that u is discontinuous at $(0, a/c)$; (due to the focusing of the discontinuity of u_t at $t = 0, r = a$).

5. For $n = 3$ let $u \in C^2$ be a solution of $\partial_t^2 u - c^2 \Delta u = 0$ for $x \in \mathbb{R}^3, t \geq 0$. Assume moreover that

$$U(t) = \sum_{|\alpha| \leq 2} \int |D^\alpha u(x,t)|\, dx < \infty \qquad \text{for } t = 0$$

(a) Show that there is a constant K independent of u such that

$$|u(x,t)| \leq \frac{K}{t} U(0), \qquad \text{for } t > 1.$$

(hint: in the solution formula write the integrand as

$$\sum_i \{(ct)^{-1}(tg(y) + f(y))(y_i - x_i) + ct f_{y_i}(y)\}\xi_i$$

where the $\xi_i = (y_i - x_i)/ct$ are the direction cosines of the exterior surface normal. Convert the integral over the one over the solid sphere $|y - x| < ct$.)

(b) Show that

$$\lim_{t \to \infty} \frac{U(t)}{t} = 0$$

implies that u vanishes identically. (Hint: apply (a) to the function $v(x,t,T) = u(x, T - t)$ for large T.)