MATH 656 HW 3

1. Prove that $\Delta u(x) = 0$ also implies that

$$\Delta(|x|^{2-n} u(x/|x|^2)) = 0$$

for $x/|x|^2$ in the domain of definition of u.

2. By Newton's law the gravitational attraction exerted on a unit mass located at $\xi \in \mathbb{R}^3$ by a solid Ω with density $\mu = \mu(x)$ is given by

$$F(\xi) = \gamma \iiint_\Omega \frac{\mu(x)(x-\xi)}{|x-\xi|^3}\, dx$$

(γ is the universal gravitational constant.)

(a) Prove that $F = \nabla u$, where the "potential" u is given by

$$u(\xi) = \gamma \iiint \frac{\mu(x)}{|x-\xi|}\, dx.$$

(b) Prove that the attraction $F(\xi)$ exerted by Ω on a far away unit mass is approximately the same as if the total mass of Ω were concentrated at its center of gravity

$$x^0 = \frac{\iiint_\Omega \mu(x) x\, dx}{\iiint_\Omega \mu(x)\, dx}.$$

(Hint: approximate $|x-\xi|^{-3}$ by $|x^0-\xi|^{-3}$ for large $|\xi|$.)

(c) Calculate the potential u and attraction F of a solid sphere Ω of radius a with center at the origin and of constant density μ. Use here that u must have spherical symmetry, must be harmonic outside Ω, satisfy Poisson's equation in Ω, be of class C^1 everywhere, and vanish at ∞.

3. Let $n = 2$ and Ω be the upper half plane. Prove that $\sup_\Omega u = \sup_{\partial\Omega} u$ for $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ which is harmonic in Ω under the additional assumption that u is bounded from above in $\bar{\Omega}$. (The additional assumption is needed to exclude examples like $u = x_2$) (Hint: take for $\epsilon > 0$, the function $u(x) - \epsilon \ln |x + (0,1)|^2$ and apply Maximum Principle to a region $\Omega \cap B_R(0)$. Let $\epsilon \to 0$, $R \to \infty$.)

4. (Harnack's inequality) Let $u \in C^2(B_a(0)) \cap C(\overline{B_a(0)})$, and $u(x) \geq 0$, $\Delta u(x) = 0$ for $x \in B_a(0)$. Show that for $|x| < a$,

$$\frac{a^{n-2}(a-|x|)}{(a+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{a^{n-2}(a+|x|)}{(a-|x|)^{n-1}} u(0).$$

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Show that the Green's function $G(x,y)$ (for the Dirichlet problem of the Laplace equation for domain Ω) satisfies

(a) $G(y,x) \leq 0, \qquad \forall x, y \in \Omega, x \neq y.$

(b) $\frac{\partial}{\partial n_y} G(y, x) \geq 0, \qquad \forall x \in \Omega, y \in \partial\Omega.$

(c) $\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x, y)\, dS_y = 1.$

Here n_y is the outward unit normal to Ω.