MATH 656 HW 4

1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $u \in C^2(\Omega)) \cap C(\bar{\Omega})$ be a solution of the equation

$$\Delta u + c(x)u = 0$$

where $c(x) \leq 0$ in Ω. Show that u = 0 on $\partial\Omega$ implies u = 0 in Ω.

2. (a) Let $\mu > 0$ be a constant, $u = u(x,t)$ be a positive solution of class C^2 of

$$u_t = \mu u_{xx}, \qquad \text{for } x \in \mathbb{R}, t > 0.$$

Show that $\theta = -2\mu u_x/u$ satisfies the viscous Burgers' equation

$$\theta_t + \theta\theta_x = \mu\theta_{xx}, \qquad \text{for } x \in \mathbb{R}, t > 0. \tag{0.1}$$

(b)For $\phi \in C_0^2(\mathbb{R})$ find a solution of (0.1) with initial data $\theta(x,0) = \phi(x)$, for which

$$\lim_{t\to\infty} \theta(x,t) = 0.$$

(The viscous term μu_{xx} prevents singularities that would occur, compares with Burgers' equation $\theta_t + \theta\theta_x = 0$.)

3. Write down an explicit formula for a solution of the initial value problem

$$\begin{cases} u_t - \Delta u + cu = f, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \end{cases} \tag{0.2}$$

where $c \in \mathbb{R}$ is a constant.

4. Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula for a solution of the initial boundary value problem

$$\begin{cases} u_t - \Delta u = 0, & x > 0, t > 0 \\ u(0,t) = g(t), & t \geq 0 \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{0.3}$$

(Hint: let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection. Solution $u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds$.)

5. (Duhamel principle) Let $P(D) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$ be a linear differential operator of order m, with coefficients independent of t. Assume that we know how to solve the homogeneous problem

$$\begin{cases} (\partial_t - P(D))u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = f(x). \end{cases} \tag{0.4}$$

Find a solution formula for the inhomogeneous equation

$$\begin{cases} (\partial_t - P(D))u = F(x,t), & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = 0, \end{cases} \tag{0.5}$$

and check (prove) your assertion. (Hint: use the Duhamel principle for the heat equation to guess what the solution formula is.)