

WELLPOSEDNESS AND SINGULARITIES OF THE WATER WAVE EQUATIONS

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ABSTRACT. A class of water wave problems concerns the dynamics of the free interface separating an inviscid, incompressible and irrotational fluid, under the influence of gravity, from a zero-density region. In this note, we present some recent methods and ideas developed concerning the local and global wellposedness of these problems, the focus is on the structural aspect of the equations.

1. INTRODUCTION

A class of water wave problems concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n -dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where \mathbf{k} is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\left\{ \begin{array}{ll} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{k} - \nabla P & \text{on } \Omega(t), t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 0, & \text{on } \Omega(t), t \geq 0, \\ P = 0, & \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } & (t, \Sigma(t)), \end{array} \right. \quad (1.1)$$

where \mathbf{v} is the fluid velocity, P is the fluid pressure. There is an important condition for these problems:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0 \quad (1.2)$$

pointwise on the interface, where \mathbf{n} is the outward unit normal to the fluid interface $\Sigma(t)$ [36]; it is well known that when surface tension is neglected and the Taylor sign condition (1.2) fails, the water wave motion can be subject to the Taylor instability [36, 7, 5].

The study on water waves dates back centuries. Early mathematical works include Stokes [35], Levi-Civita [28], and G.I. Taylor [36]. Nalimov [30], Yosihara [45] and Craig [14] proved local in time existence and uniqueness of solutions for the 2d water wave equation (1.1) for small initial data. In [39, 40], we showed that for dimensions $n \geq 2$, the strong Taylor sign condition

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \quad (1.3)$$

always holds for the infinite depth water wave problem (1.1), as long as the interface is non-self-intersecting and smooth; and the initial value problem of equation (1.1) is locally well-posed in Sobolev spaces H^s , $s \geq 4$ for arbitrary given data. Since then, local wellposedness for water waves with additional effects such as the surface tension, bottom and non-zero vorticity, under the assumption (1.3)¹ were obtained (c.f. [4, 10, 11, 21, 27, 29, 31, 34, 46]). Alazard, Burq & Zuily [1, 2] proved local wellposedness of (1.1) in low regularity Sobolev

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¹When there is surface tension, or bottom, or vorticity, (1.3) does not hold, it needs to be assumed.

spaces where the interfaces are only in $C^{3/2}$. Recently, we proved the almost global and global wellposedness of (1.1) for small, smooth and localized initial data for dimensions $n \geq 2$ [41, 42]; Germain, Masmudi & Shatah obtained global existence for 3d water waves for a different class of small, smooth and localized data [16]. Our 2d almost global existence result has now been extended to global by Ionescu & Pusateri [22] and independently Alazard & Delort [3], see [18, 20] for an alternative proof. Finally, we mention our most recent work on two dimensional water waves with angled crests [43, 25, 44], in which we showed that for water waves with angled crests, only the degenerate Taylor sign condition (1.2) holds, with degeneracy at the singularities on the interface. We proved an a priori estimate [25] and local existence [44] for the 2d water wave equation (1.1) in this framework.

The advances of water waves theory rely crucially on the understanding of the structure of the water wave equations. Indeed, (1.1) is a nonlinear equation defined on moving domains, it is difficult to study it directly. A classical approach is to reduce from (1.1) to an equation on the interface, and study solutions of the interface equation. Then use the incompressibility and irrotationality of the velocity field to recover the velocity in the fluid domain by solving a boundary value problem for the Laplace equation. However the fluid interface equation is itself a fully nonlinear and nonlocal equation, its structure is not easy to understand. It is by achieving better understandings of the structure of this equation that has enabled us to apply analytical tools to deduce informations on the nature of the fluid motion.

In this note we describe the approach in [39, 40, 41, 42]. Our focus is on the structural aspect of the work. It is clear that the 2d case is structurally simpler than 3d. Our strategy has been to first understand the two dimensional case, taking advantage of complex analysis tools, in particular the Riemann mapping theorem, then use Clifford analysis to extend the 2d results to 3d. We note that although Riemann mapping is not available in 3d, it is by using it that has enabled us to understand the 2d case well enough to develop an approach that extends to all dimensions.

We consider solutions of the water wave equation (1.1) in the setting where

$$\mathbf{v}(\xi, t) \rightarrow 0, \quad \text{as } |\xi| \rightarrow \infty$$

and the interface $\Sigma(t)$ tends to the horizontal plane at infinity.²

In section 2, we discuss the local wellposedness of (1.1), the focus is on deriving the quasilinear structure of equation (1.1), c.f. [39, 40, 25]. In section 3, we consider the global in time behavior of solutions of (1.1) in the regime of small waves, the focus is on understanding the nature of the nonlinearity of equation (1.1), c.f. [41, 42]. We give some preparatory materials in the Appendices. In Appendix A, we give some basic analysis tools such as estimates for commutators and operators involved in the equations. These inequalities tell us how a certain term behaves in terms of estimates, whether it is of higher order, or lower order etc., and guide us in our derivations of the structure of the equation. In Appendix B we give some commutator identities that is used in our derivations.

We use the following notations and conventions: $[A, B] := AB - BA$ is the commutator of operators A and B ; compositions are always in terms of the spatial variables and we write for $f = f(\cdot, t)$, $g = g(\cdot, t)$, $f(g(\cdot, t), t) := f \circ g(\cdot, t) := U_g f(\cdot, t)$.

2. LOCAL WELLPOSEDNESS OF THE WATER WAVE EQUATIONS

From basic PDE theory we expect that in general, the Cauchy problem for a hyperbolic type PDE is locally solvable in Sobolev spaces, and we can solve it, for a short time, by

²The problem with velocity $\mathbf{v}(\xi, t) \rightarrow (c, 0)$ as $|\xi| \rightarrow \infty$ can be reduced to one with \mathbf{v} tends to 0 at infinity by the following observation: if (\mathbf{v}, P) with $\Sigma(t) : \xi = \xi(\cdot, t)$ is a solution of (1.1), then

$$\mathbf{V} = \mathbf{v}(\zeta + (c, 0)t, t) - c, \quad \mathbf{P} = P(\zeta + (c, 0)t, t), \quad \text{with } \Sigma(t) - (c, 0)t : \zeta = \xi(\cdot, t) - (c, 0)t$$

is also a solution of (1.1).

energy estimates and an iterative argument; while the Cauchy problem for an elliptic type PDE is ill-posed. Hence in order to understand whether the water wave equation (1.1) is uniquely solvable for a positive time period for arbitrary given Cauchy data, it is crucial to understand its quasi-linear structure. In this section, we derive the quasi-linear structure of the water wave equation (1.1), our focus is on the 2d case, since it is through a thorough understanding of this case that has enabled us to extend our work to 3d. We only give a brief description of how to extend the 2d derivations to 3d. We show that the strong Taylor inequality (1.3) always holds for $C^{1,\gamma}$ interfaces, while for singular interfaces only the weak Taylor inequality (1.2) holds; and this implies that the quasilinear structure of the water wave equation (1.1) is of hyperbolic type in the regime of $C^{1,\gamma}$ interfaces and respectively of degenerate hyperbolic type in the regime that includes singular free surfaces. The derivation given here is based on that in [39, 40]; due to the scope of this lecture note, we will only discuss the structural aspect of the work [39, 40], and leave out the proof for the local wellposedness of (1.1). The interested reader may consult [39, 40] for the proof.

2.1. The equation of the fluid interface in two space dimensions. In two space dimensions, we identify (x, y) with the complex number $x + iy$; $\operatorname{Re} z$, $\operatorname{Im} z$ are the real and imaginary parts of z ; $\bar{z} = \operatorname{Re} z - i \operatorname{Im} z$ is the complex conjugate.

Let the interface $\Sigma(t) : z = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be given by Lagrangian parameter α , so $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t); t)$ is the velocity of the fluid particles on the interface, $z_{tt}(\alpha, t) = \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v}(z(\alpha, t); t)$ is the acceleration; notice that $P = 0$ on $\Sigma(t)$ implies that ∇P is normal to $\Sigma(t)$, therefore $\nabla P = -i \mathbf{a} z_\alpha$, where $\mathbf{a} = -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial \mathbf{n}}$; we have from the first and third equation of (1.1) that

$$z_{tt} + i = i \mathbf{a} z_\alpha \quad (2.1)$$

The second equation of (1.1): $\operatorname{div} \mathbf{v} = \operatorname{curl} \mathbf{v} = 0$ implies that $\bar{\mathbf{v}}$ is holomorphic in the fluid domain $\Omega(t)$; hence \bar{z}_t is the boundary value of a holomorphic function in $\Omega(t)$.

Let $\Omega \subset \mathbb{C}$ be a domain with boundary $\Sigma : z = z(\alpha)$, $\alpha \in I$, oriented clockwise. Let \mathfrak{H} be the Hilbert transform associated to Ω :

$$\mathfrak{H}f(\alpha) = \frac{1}{\pi i} \operatorname{pv.} \int \frac{z_\beta(\beta)}{z(\alpha) - z(\beta)} f(\beta) d\beta \quad (2.2)$$

We have the following characterization of the trace of a holomorphic function on Ω .

Proposition 2.1. [23] *a. Let $g \in L^p$ for some $1 < p < \infty$. Then g is the boundary value of a holomorphic function G on Ω with $G(z) \rightarrow 0$ at infinity if and only if*

$$(I - \mathfrak{H})g = 0. \quad (2.3)$$

b. Let $f \in L^p$ for some $1 < p < \infty$. Then $\mathbb{P}_H f := \frac{1}{2}(I + \mathfrak{H})f$ is the boundary value of a holomorphic function \mathfrak{G} on Ω , with $\mathfrak{G}(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

c. $\mathfrak{H}1 = 0$.

Observe Proposition 2.1 gives that $\mathfrak{H}^2 = I$ on L^p .

From Proposition 2.1 the second equation of (1.1) is equivalent to $\bar{z}_t = \mathfrak{H}\bar{z}_t$. Therefore the motion of the fluid interface $\Sigma(t) : z = z(\alpha, t)$ is given by ³

$$\begin{cases} z_{tt} + i = i \mathbf{a} z_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \quad (2.4)$$

(2.4) is a fully nonlinear equation. To understand whether the equation is well posed, a usual strategy is to quasi-linearize the equation by differentiating. Notice that it can be hard to analyze the Hilbert transform \mathfrak{H} in the second equation since it depends nonlinear nonlocally on the interface $z = z(\alpha, t)$, this motivates us to use the Riemann mapping (c.f.[39]).⁴

³Equation (1.1) and equation (2.4) are equivalent, see [39, 40].

⁴In [30, 45], a quasilinear equation for the interface was derived for small and smooth waves in terms of the Lagrangian coordinates. In [39], by using the Riemann mapping, a more concise quasilinear equation

2.2. Riemann mappings and the quasi-linear structure of 2d water waves. Let $\Phi(\cdot, t) : \Omega(t) \rightarrow P_-$ be the Riemann mapping taking $\Omega(t)$ to the lower half plane P_- , satisfying $\lim_{z \rightarrow \infty} \Phi_z(z, t) = 1$. Let

$$h(\alpha; t) := \Phi(z(\alpha, t), t),$$

so $h : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism. Let h^{-1} be defined by

$$h(h^{-1}(\alpha', t), t) = \alpha', \quad \alpha' \in \mathbb{R};$$

and

$$Z(\alpha', t) := z \circ h^{-1}(\alpha', t), \quad Z_t(\alpha', t) := z_t \circ h^{-1}(\alpha', t), \quad Z_{tt}(\alpha', t) := z_{tt} \circ h^{-1}(\alpha', t)$$

be the reparametrization of the position, velocity and acceleration of the interface in the Riemann mapping variable α' . Let

$$Z_{,\alpha'}(\alpha', t) := \partial_{\alpha'} Z(\alpha', t), \quad Z_{t,\alpha'}(\alpha', t) := \partial_{\alpha'} Z_t(\alpha', t), \quad \text{etc.}$$

We note that $\Phi^{-1}(\alpha', t) = Z(\alpha', t)$. Notice that $\bar{\mathbf{v}} \circ \Phi^{-1} : P_- \rightarrow \mathbb{C}$ is holomorphic in the lower half plane P_- with $\bar{\mathbf{v}} \circ \Phi^{-1}(\alpha', t) = \bar{Z}_t(\alpha', t)$. Precomposing (2.1) with h^{-1} and applying Proposition 2.1 to $\bar{\mathbf{v}} \circ \Phi^{-1}$ in P_- , we have the free surface equation in the Riemann mapping variable:

$$\begin{cases} Z_{tt} + i = i\mathcal{A}Z_{,\alpha'} \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \quad (2.5)$$

where $\mathcal{A} \circ h = \mathbf{a}h_\alpha$ and \mathbb{H} is the Hilbert transform associated with the lower half plane P_- :

$$\mathbb{H}f(\alpha') = \frac{1}{\pi i} \text{pv.} \int \frac{1}{\alpha' - \beta'} f(\beta') d\beta'.$$

⁵ As has been shown in [39], the quasi-linearization of (2.4) or (2.5) can be accomplished by just taking one derivative to t to equation (2.1).

Taking one derivative with respect to t to (2.1), we get

$$\bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha} = -i\mathbf{a}_t\bar{z}_\alpha = \frac{\mathbf{a}_t}{\mathbf{a}}(\bar{z}_{tt} - i), \quad (2.6)$$

the free surface equation is now (2.6) with the constraint $\bar{z}_t = \mathfrak{H}\bar{z}_t$. Precomposing with h^{-1} on both sides of (2.6) we have in the Riemann mapping variable the free surface equation⁶

$$\begin{cases} \bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'} = \frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \quad (2.7)$$

We note that from the chain rule that for any function f ,

$$U_{h^{-1}}\partial_t U_h f(\alpha', t) = (\partial_t + \mathcal{B}\partial_{\alpha'})f(\alpha', t)$$

where $\mathcal{B} = h_t \circ h^{-1}$. Hence $\bar{Z}_{tt} = (\partial_t + \mathcal{B}\partial_{\alpha'})\bar{Z}_t$ and $\bar{Z}_{ttt} = (\partial_t + \mathcal{B}\partial_{\alpha'})^2\bar{Z}_t$. We will find \mathcal{B} , \mathcal{A} , and $\frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}$ in terms of \bar{Z}_t and \bar{Z}_{tt} and show that (2.7) is a quasilinear equation for

was derived for all non-self-intersecting waves. The approach described here is inspired by that in [39], it was used when we extended the work in [39] to 3d [40]. It is similar to that in [39], with the difference that in [39] the derivation is in terms of the real component and here it is in both components.

⁵The advantage of using the Riemann mapping is that in P_- , the boundary value of a holomorphic function is characterized by $g = \mathbb{H}g$, see Proposition 2.1. The kernel of the Hilbert transform \mathbb{H} is purely imaginary and is independent of the interface, it is easier to use \mathbb{H} to understand the relations among various quantities and hence the quasi-linear structure of the free surface equation.

⁶(2.6) and (2.7) are equivalent to (2.1) and (2.5) provided the initial data for (2.6) and (2.7) satisfy (2.1) or (2.5) at $t = 0$.

$\mathcal{U} = \bar{Z}_t$ with the right hand side consisting of lower order terms.⁷ ⁸ To this end, we need some basic estimates for commutators, we leave these and some other preparatory materials in the Appendices. The reader may want to consult the Appendices before continuing.

2.2.1. The quantity \mathcal{A} and the Taylor sign condition. The basic idea of deriving the formulas for \mathcal{A} , \mathcal{B} and $\frac{\mathbf{a}}{\alpha} \circ h^{-1}$ is to use the holomorphicity or almost holomorphicity of our quantities, and the fact that $\text{Re}(I - \mathbb{H})f = f$ for real valued functions f . We will often write $(I - \mathbb{H})(fg)$ as a commutator: $(I - \mathbb{H})(fg) = [f, \mathbb{H}]g$ when g satisfies $(I - \mathbb{H})g = 0$, since commutators are favorable in terms of estimates, see Appendix A.

Let $D_\alpha := \frac{1}{z_\alpha} \partial_\alpha$, and $D_{\alpha'} := \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}$. Notice that for any holomorphic function G in $\Omega(t)$ with boundary value $g(\alpha, t) = G(z(\alpha, t), t)$,

$$D_\alpha g(\alpha, t) = G_z(z(\alpha, t), t)$$

and for any function f ,

$$(D_\alpha f) \circ h^{-1} = D_{\alpha'}(f \circ h^{-1}).$$

We note that \mathcal{A} is related to the important quantity $-\frac{\partial P}{\partial \mathbf{n}}$ by $\mathcal{A} \circ h = \mathbf{a}h_\alpha$ and $\mathbf{a} = -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial \mathbf{n}}$, therefore

$$-\frac{\partial P}{\partial \mathbf{n}} \Big|_{z=z(\alpha, t)} = (\mathcal{A}|Z_{,\alpha}|) \circ h. \quad (2.8)$$

In this subsection we derive a formula for the quantity \mathcal{A} . This formula was first derived in [39] to show that the strong Taylor sign condition (1.3) always holds for smooth non-self-intersecting interfaces. It has also played a key role in our recent work on 2d water waves with angled crests [25, 44].

Taking complex conjugate of the first equation in (2.5) then multiply by $Z_{,\alpha'}$ yields

$$Z_{,\alpha'}(\bar{Z}_{tt} - i) = -i\mathcal{A}|Z_{,\alpha'}|^2 := -iA_1. \quad (2.9)$$

The left hand side of (2.9) is almost holomorphic since $Z_{,\alpha'}$ is the boundary value of the holomorphic function $(\Phi^{-1})_{z'}$ and \bar{z}_{tt} is the time derivative of the holomorphic function \bar{z}_t . We explore the almost holomorphicity of \bar{z}_{tt} by expanding. Let $F = \bar{\mathbf{v}}$, we know F is holomorphic in $\Omega(t)$, and $\bar{z}_t = F(z(\alpha, t), t)$, so

$$\bar{z}_{tt} = F_t(z(\alpha, t), t) + F_z(z(\alpha, t), t)z_t(\alpha, t), \quad \bar{z}_{t\alpha} = F_z(z(\alpha, t), t)z_\alpha(\alpha, t) \quad (2.10)$$

therefore

$$\bar{z}_{tt} = F_t \circ z + \frac{\bar{z}_{t\alpha}}{z_\alpha} z_t. \quad (2.11)$$

Precomposing with h^{-1} , subtracting $-i$, then multiplying by $Z_{,\alpha'}$, we have

$$Z_{,\alpha'}(\bar{Z}_{tt} - i) = Z_{,\alpha'}F_t \circ Z + Z_t \bar{Z}_{t,\alpha'} - iZ_{,\alpha'} = -iA_1$$

Apply $(I - \mathbb{H})$ to both sides of the equation. Notice that $F_t \circ Z$ is the boundary value of the holomorphic function $F_t \circ \Phi^{-1}$, so $(I - \mathbb{H})(Z_{,\alpha'}F_t \circ Z) = 0$, $(I - \mathbb{H})Z_{,\alpha'} = 1$,⁹ therefore

$$-i(I - \mathbb{H})A_1 = (I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'}) - i$$

⁷From Proposition 2.1, $(I - \mathfrak{H})f$ measures how un-holomorphic a function f is. The reason (2.6) is quasi-linear with the right hand side of lower order is that the two terms on the left hand side are "almost holomorphic" in the sense that $(I - \mathfrak{H})(\bar{z}_{ttt} + i\mathbf{a}\partial_\alpha \bar{z}_t)$ is a commutator – since by $(I - \mathfrak{H})\bar{z}_t = 0$, $(I - \mathfrak{H})(\bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha}) = [\partial_t^2 + i\mathbf{a}\partial_\alpha, \mathfrak{H}]\bar{z}_t$, while the conjugate of the right hand side points in the normal direction. We know a commutator is of lower order (c.f. Appendices A and B); and a holomorphic function with real part zero must be a constant; similarly an anti-holomorphic function with zero tangential part on the boundary must be a constant. If the left hand side of (2.6) were holomorphic, then it would have to be zero. Now the left hand side is almost holomorphic with $(I - \mathfrak{H})(\bar{z}_{ttt} + i\mathbf{a}\partial_\alpha \bar{z}_t)$ a lower order term, then the right hand side must be a lower order term. We get this insight after our work in [39], not before.

⁸We sometimes abuse notation and say a function f is holomorphic if f is the boundary value of a holomorphic function in $\Omega(t)$.

⁹We know $(\Phi^{-1})_{z'} \rightarrow 1$ as $z' \rightarrow \infty$; we assume a priori that $(\Phi^{-1})_{z'} F_t \circ \Phi^{-1} \rightarrow 0$ as $z' \rightarrow \infty$. As proved in [39] equation (2.5) is well posed in this regime.

Taking imaginary parts on both sides and using the fact $(I - \mathbb{H})\bar{Z}_{t,\alpha'} = 0$ ¹⁰ to rewrite $(I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'})$ as $[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'}$ yields

$$A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} \quad (2.12)$$

Compute

$$\begin{aligned} -\text{Im}[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} &= -\frac{1}{\pi} \text{Re} \int \frac{(Z_t(\alpha', t) - Z_t(\beta', t))\partial_{\beta'}(\bar{Z}_t(\alpha', t) - \bar{Z}_t(\beta', t))}{\alpha' - \beta'} d\beta \\ &= -\frac{1}{2\pi} \int \frac{\partial_{\beta'} |Z_t(\alpha', t) - Z_t(\beta', t)|^2}{\alpha' - \beta'} d\beta \\ &= \frac{1}{2\pi} \int \frac{|Z_t(\alpha', t) - Z_t(\beta', t)|^2}{(\alpha' - \beta')^2} d\beta' \end{aligned} \quad (2.13)$$

where in the last step we used integration by parts. We conclude

Proposition 2.2. (c.f. [39], Lemma 3.1) *We have 1.*

$$A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi} \int \frac{|Z_t(\alpha', t) - Z_t(\beta', t)|^2}{(\alpha' - \beta')^2} d\beta' \geq 1. \quad (2.14)$$

2.

$$-\frac{\partial P}{\partial \mathbf{n}} = \frac{A_1}{|Z_{,\alpha'}|}; \quad (2.15)$$

in particular if the interface $\Sigma(t) \in C^{1,\gamma}$ for some $\gamma > 0$, then the strong Taylor sign condition (1.3) holds.

From (2.9) we have

$$\mathcal{A} = \frac{A_1}{|Z_{,\alpha'}|^2} = \frac{|\bar{Z}_{tt} - i|^2}{A_1} \quad (2.16)$$

with A_1 given by (2.14).

2.2.2. *The quantity $\mathcal{B} = h_t \circ h^{-1}$.* The quantity $\mathcal{B} = h_t \circ h^{-1}$ can be calculated similarly. Recall $h(\alpha, t) = \Phi(z(\alpha, t), t)$, so

$$h_t = \Phi_t \circ z + (\Phi_z \circ z)z_t, \quad h_\alpha = (\Phi_z \circ z)z_\alpha$$

hence $h_t = \Phi_t \circ z + \frac{h_\alpha}{z_\alpha} z_t$. Precomposing with h^{-1} yields

$$h_t \circ h^{-1} = \Phi_t \circ Z + \frac{Z_t}{Z_{,\alpha'}}. \quad (2.17)$$

Now $(I - \mathbb{H})\Phi_t \circ Z = (I - \mathbb{H})(\frac{1}{Z_{,\alpha'}} - 1) = 0$ since $\Phi_t \circ Z$ and $\frac{1}{Z_{,\alpha'}} - 1$ are boundary values of the holomorphic functions $\Phi_t \circ \Phi^{-1}$ and $\frac{1}{(\Phi^{-1})_{z'}} - 1$ respectively.¹¹ Apply $(I - \mathbb{H})$ to both sides of (2.17) then take the real parts; rewriting $(I - \mathbb{H})(Z_t(\frac{1}{Z_{,\alpha'}} - 1))$ as $[Z_t, \mathbb{H}](\frac{1}{Z_{,\alpha'}} - 1)$, we get

$$\mathcal{B} = h_t \circ h^{-1} = \text{Re}([Z_t, \mathbb{H}](\frac{1}{Z_{,\alpha'}} - 1)) + 2 \text{Re} Z_t. \quad (2.18)$$

Here we used the fact that $\text{Re}(I - \mathbb{H})Z_t = 2 \text{Re} Z_t$, since $(I + \mathbb{H})Z_t = \overline{(I - \mathbb{H})\bar{Z}_t} = 0$.

¹⁰Because $(I - \mathbb{H})\bar{Z}_t = 0$.

¹¹Again, we assume a priori that $\Phi_t \circ \Phi^{-1}(z', t) \rightarrow 0$ as $z' \rightarrow \infty$. It has been proved in [39] that such solutions exist.

2.2.3. *The quantity $\frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}$.* We analyze $\frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}$ similarly. Start with (2.6) or the first equation of (2.7), and expand the left hand side using $\bar{z}_t = F(z(\alpha, t), t)$. Taking one more derivative to t to the first equation in (2.10), we get

$$\bar{z}_{ttt} = (F_{zz} \circ z)z_t^2 + 2(F_{tz} \circ z)z_t + (F_z \circ z)z_{tt} + F_{tt} \circ z.$$

From (2.11), the second equation of (2.10), and the holomorphicity of F_z, F_t , we have

$$F_z \circ z = D_\alpha \bar{z}_t, \quad F_{zz} \circ z = D_\alpha^2 \bar{z}_t, \quad \text{and } F_{tz} \circ z = D_\alpha(\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t)$$

therefore

$$\bar{z}_{ttt} = (D_\alpha^2 \bar{z}_t)z_t^2 + 2z_t D_\alpha(\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t) + (D_\alpha \bar{z}_t)z_{tt} + F_{tt} \circ z.$$

Precomposing with h^{-1} yields

$$\bar{Z}_{ttt} = (D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z. \quad (2.19)$$

Now multiply the first equation of (2.7) by $Z_{,\alpha'}$, then substitute in (2.19). We have, by (2.9),

$$\begin{aligned} Z_{,\alpha'} \{ (D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z + i\mathcal{A}\bar{Z}_{t,\alpha'} \} \\ = -iA_1 \frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}. \end{aligned} \quad (2.20)$$

Apply $(I - \mathbb{H})$ to both sides of (2.20). Using the fact that $(I - \mathbb{H})(\partial_{\alpha'} D_{\alpha'} \bar{Z}_t) = 0$, $(I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) = 0$, $(I - \mathbb{H})\partial_{\alpha'} \bar{Z}_t = 0$ and $(I - \mathbb{H})F_{tt} \circ Z = 0$ because of the holomorphicity of $\partial_{z'}(F_z \circ \Phi^{-1})$, $\partial_{z'}(F_t \circ \Phi^{-1})$, $\partial_{z'}(F \circ \Phi^{-1})$ and $F_{tt} \circ \Phi^{-1}$, and the identity $Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}$ (2.5), we rewrite each term on the left as commutator and get

$$\begin{aligned} [Z_t^2, \mathbb{H}]\partial_{\alpha'} D_{\alpha'} \bar{Z}_t + 2[Z_t, \mathbb{H}]\partial_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + 2[Z_{tt}, \mathbb{H}]\partial_{\alpha'} \bar{Z}_t \\ = (I - \mathbb{H})(-iA_1 \frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}). \end{aligned} \quad (2.21)$$

For the sake of estimates we need to further rewrite the first term and the second part of the second term. Using integration by parts on each term, after cancelations we have

$$\begin{aligned} [Z_t^2, \mathbb{H}]\partial_{\alpha'} D_{\alpha'} \bar{Z}_t - 2[Z_t, \mathbb{H}]\partial_{\alpha'}((D_{\alpha'} \bar{Z}_t)Z_t) &= -\frac{1}{\pi i} \int \frac{(Z_t(\alpha', t) - Z_t(\beta', t))^2}{(\alpha' - \beta')^2} D_{\beta'} \bar{Z}_t(\beta', t) d\beta' \\ &:= -[Z_t, Z_t; D_{\alpha'} \bar{Z}_t]. \end{aligned}$$

Therefore we can write (2.21) as

$$2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\partial_{\alpha'} \bar{Z}_t - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t] = (I - \mathbb{H})(-iA_1 \frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1}). \quad (2.22)$$

Taking imaginary parts on both sides and dividing by $-A_1$ yields

$$\frac{\mathbf{a}_t}{\mathbf{a}} \circ h^{-1} = \frac{-\text{Im}(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\partial_{\alpha'} \bar{Z}_t - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t])}{A_1}. \quad (2.23)$$

2.2.4. *The quasilinear equation.* Sum up (2.7), (2.14), (2.16), (2.18) and (2.23), we have the equation for the free interface:

$$\begin{cases} (\partial_t + \mathcal{B}\partial_{\alpha'})^2 \bar{Z}_t + i\mathcal{A}\partial_{\alpha'} \bar{Z}_t = g \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \quad (2.24)$$

where

$$\begin{cases} \mathcal{A} = \frac{|\bar{Z}_{tt} - i|^2}{A_1}, \quad Z_{tt} = (\partial_t + \mathcal{B}\partial_{\alpha'})Z_t \\ A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi} \int \frac{|Z_t(\alpha', t) - Z_t(\beta', t)|^2}{(\alpha' - \beta')^2} d\beta' \\ \mathcal{B} = h_t \circ h^{-1} = \text{Re}([Z_t, \mathbb{H}](\frac{1}{Z_{,\alpha'}} - 1)) + 2 \text{Re } Z_t \\ \frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1} \\ g = (\bar{Z}_{tt} - i) - \frac{\text{Im}(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\partial_{\alpha'} \bar{Z}_t - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t])}{A_1}. \end{cases} \quad (2.25)$$

From the inequalities in Appendix A, we know in the regime where the interface is $C^{1,\gamma}$, $\gamma > 0$, (2.24)-(2.25) is a quasilinear equation of the conjugate velocity \bar{Z}_t , with the right hand side of (2.24) consisting of lower order terms. (2.24) is of hyperbolic type since

$$\mathcal{A} = \frac{A_1}{|Z_{,\alpha'}|^2} \geq c_0 > 0$$

for $C^{1,\gamma}$ interfaces, and $i\partial_{\alpha'}\bar{Z}_t = |\partial_{\alpha'}|\bar{Z}_t$.¹² Local well-posedness of (2.24)-(2.25) for $(\bar{Z}_t, \bar{Z}_{tt}) \in C([0, T], H^{s+1/2} \times H^s)$, $s \geq 4$ has been proved by the energy method and an iterative argument, we refer the reader to [39] for details.

We make the following remarks concerning some recent works.

Remark 2.3. 1. The difference between the quasilinear equation (2.24)-(2.25) and the quasilinear equation (4.6)-(4.7) of [39] is that (2.24)-(2.25) is in terms of both components of Z_t and (4.6)-(4.7) of [39] is in terms of the real component of Z_t . (4.6)-(4.7) of [39] has been written in a way so that it is easy to prove its equivalence with the interface equation (2.4) or equation (1.7)-(1.8) of [39]; see §6 of [39].

2. (2.24)-(2.25) is an equation for the conjugate velocity \bar{Z}_t and conjugate acceleration \bar{Z}_{tt} , the interface doesn't appear explicitly, so a solution of (2.24)-(2.25) can exist even when $Z = Z(\cdot, t)$ becomes self-intersecting.¹³ Checking through the derivation above we see that we arrived at (2.24)-(2.25) from (1.1) using only the following properties of the domain: 1. there is a conformal mapping taking the fluid region $\Omega(t)$ to P_- ; 2. $P = 0$ on $\Sigma(t)$. We note that $z \rightarrow z^{1/2}$ is a conformal map that takes the region $\mathbb{C} \setminus \{z = x + i0, x > 0\}$ to the upper half plane; a domain with its boundary self-intersecting at the positive real axis can therefore be mapped conformally onto the lower half plane P_- . Taking such a domain as the initial fluid domain, assuming $P = 0$ on $\Sigma(t)$ even when $\Sigma(t)$ self-intersects,¹⁴ one can still solve equation (2.24)-(2.25) for a short time. Indeed this is the main idea in the work of [8]. Using this idea and the time reversibility of the water wave equation, by choosing an appropriate initial velocity field that pulls the initial domain apart, [8] proved the existence of "splash" and "splat" singularities starting from a smooth non-self-intersecting fluid region.

3. The above derivation applies to fluid domains with arbitrary non-self-intersecting boundaries. We have from (2.15) and (2.14) that the Taylor sign condition (1.2) always holds, as long as the interface is non-self-intersecting. Assume that $\Sigma(t)$ is non-self-intersecting with angled crests, assume the interior angle at a crest is ν . We know the Riemann mapping Φ^{-1} (we move the singular point to the origin) behaves like

$$\Phi^{-1}(z') \approx (z')^r, \quad \text{with } \nu = r\pi$$

near the crest, so $Z_{,\alpha'} \approx (\alpha')^{r-1}$ near the crest. From (2.9) and the fact $A_1 \geq 1$, we can conclude that the interior angle at the crest must be $\leq \pi$ if the acceleration $|Z_{tt}| \neq \infty$; we can also conclude that $-\frac{\partial P}{\partial \mathbf{n}} = 0$ at the singularities where the interior angles are $< \pi$, therefore in the regime that includes singular free surfaces, the quasilinear equation (2.24)-(2.25) is degenerate hyperbolic, c.f. [25, 44]. In [25, 44] we proved an a priori estimate and the local existence for 2d water waves in the regime including interfaces with angled crests, showing that the water wave equation (1.1) admit such solutions.

4. The quasi-linear equation for water waves in the periodic setting can be derived similarly, see [25].

5. The Riemann mapping variable is used in recent work [18, 20].

¹²This is because $i\partial_{\alpha'}\bar{Z}_t = i\partial_{\alpha'}\mathbb{H}\bar{Z}_t = |\partial_{\alpha'}|\bar{Z}_t$.

¹³ $Z = Z(\cdot, t)$ is defined by $z(\cdot, t) = z(\cdot, 0) + \int_0^t z_s(\cdot, s) ds$, where $z_t = Z_t \circ h$, $Z = z \circ h^{-1}$; and $h_t = \mathcal{B}(h, t)$.

¹⁴We note that when $\Sigma(t)$ self-intersects, the condition $P = 0$ on $\Sigma(t)$ is unphysical.

2.3. The quasilinear equation in Lagrangian coordinates. In order to extend our work for 2d water waves to 3d, we need a derivation that does not rely on the Riemann mapping. Upon checking the derivation in §2.2, we see that all we have done is to apply $\text{Re}(I - \mathfrak{H})$ to calculate the parameters in the equations. We certainly can do the same calculations with $(I - \mathfrak{H})$, Riemann mapping is not needed.

We now use this idea to analyze $\mathbf{a}_t|z_\alpha|$ and show that indeed (2.6) is quasilinear with the right hand side of lower order.

First by (2.1) and Proposition 2.2 we have

$$\mathbf{a}|z_\alpha| = |z_{tt} + i|, \quad \text{and} \quad -i \frac{\bar{z}_\alpha}{|z_\alpha|} = \frac{\bar{z}_{tt} - i}{|z_{tt} + i|}. \quad (2.26)$$

We apply $(I - \mathfrak{H})$ to (2.6). Using $\bar{z}_t = \mathfrak{H}\bar{z}_t$ and Proposition B.1 in Appendix B, we have

$$\begin{aligned} (I - \mathfrak{H})(-i\mathbf{a}_t\bar{z}_\alpha) &= (I - \mathfrak{H})(\bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha}) \\ &= [\partial_t^2 + i\mathbf{a}\partial_\alpha, \mathfrak{H}]\bar{z}_t \\ &= 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 \bar{z}_{t\beta} d\beta \end{aligned} \quad (2.27)$$

Multiply both sides of (2.27) by $i\frac{z_\alpha}{|z_\alpha|}$ and take the real parts. Since \mathbf{a} and \mathbf{a}_t are real valued, we have

$$\begin{aligned} (I + \mathfrak{K}^*)(\mathbf{a}_t|z_\alpha|) &= \\ \text{Re}\left(\frac{iz_\alpha}{|z_\alpha|}\left\{2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 \bar{z}_{t\beta} d\beta\right\}\right) & \end{aligned} \quad (2.28)$$

where

$$\mathfrak{K}^* f(\alpha, t) = p.v. \int \text{Re}\left\{\frac{-1}{\pi i} \frac{z_\alpha}{|z_\alpha|} \frac{|z_\beta(\beta, t)|}{z(\alpha, t) - z(\beta, t)}\right\} f(\beta, t) d\beta$$

is the adjoint of the double layer potential operator \mathfrak{K} in $L^2(\Sigma(t), dS)$. We know $I + \mathfrak{K}^*$ is invertible on $L^2(\Sigma(t), dS)$ (cf. [15, 24]). The second equation of (2.26), (2.28) and the estimates in Appendix A show that $\mathbf{a}_t|z_\alpha|$ has the same regularity as that of \bar{z}_{tt} and \bar{z}_t .

We rewrite (2.6) with the constraint $\bar{z}_t = \mathfrak{H}\bar{z}_t$ as

$$\begin{cases} \bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha} = \frac{\bar{z}_{tt} - i}{|z_{tt} + i|} \mathbf{a}_t|z_\alpha| \\ \bar{z}_t = \mathfrak{H}\bar{z}_t, \end{cases} \quad (2.29)$$

where $\mathbf{a}|z_\alpha|$ is given by (2.26) and $\mathbf{a}_t|z_\alpha|$ is given by (2.28). Since \bar{z}_t is holomorphic, $i\frac{1}{|z_\alpha|}\partial_\alpha\bar{z}_t = \nabla_{\mathbf{n}}\bar{z}_t$. By Green's identity, the Dirichlet-Neumann operator $\nabla_{\mathbf{n}}$ is a positive operator. From Proposition 2.2 we know $\mathbf{a}|z_\alpha| = -\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$ in the regime of $C^{1,\gamma}$ interfaces. Therefore (2.29) is a quasilinear system of hyperbolic type, with the right hand side of the first equation in (2.29) consisting of terms of lower order derivatives of \bar{z}_t . The local in time wellposedness of (2.29)-(2.28)-(2.26) in Sobolev spaces (with $(z_t, z_{tt}) \in C([0, T], H^{s+1/2} \times H^s)$, $s \geq 4$) can then be proved by energy estimates and a fixed point iteration argument.¹⁵

This derivation has been extended to 3d using Clifford analysis, cf. [40], we will give a brief discussion on how to do this in §2.4.

Before ending this subsection, we mention that the quasilinear system (2.29)-(2.28)-(2.26) is coordinate invariant.

For fixed t , let $k = k(\alpha, t) : \mathbb{R} \rightarrow \mathbb{R}$ be a diffeomorphism with $k_\alpha > 0$. Let k^{-1} be such that $k \circ k^{-1}(\alpha, t) = \alpha$. Define

$$\zeta := z \circ k^{-1}, \quad b := k_t \circ k^{-1} \quad \text{and} \quad A \circ k := \mathbf{a}k_\alpha. \quad (2.30)$$

¹⁵A proof of the local wellposedness of the 3d counterpart of (2.29)-(2.28)-(2.26) is carried out in [40].

Let

$$D_t := U_k^{-1} \partial_t U_k := \partial_t + b \partial_\alpha \quad (2.31)$$

be the material derivative. By a simple application of the chain rule, we have

$$U_k^{-1} (\partial_t^2 + i a \partial_\alpha) U_k = D_t^2 + i A \partial_\alpha,$$

and equation (2.29) becomes

$$\begin{cases} (D_t^2 + i A \partial_\alpha) \overline{D_t \zeta} = (\mathbf{a}_t | z_\alpha |) \circ k^{-1} \frac{\overline{D_t^2 \zeta} - i}{|D_t^2 \zeta + i|} \\ \overline{D_t \zeta} = \mathcal{H} \overline{D_t \zeta} \end{cases} \quad (2.32)$$

with

$$\begin{aligned} (I + \mathcal{K}^*)((\mathbf{a}_t | z_\alpha |) \circ k^{-1}) = \\ \operatorname{Re} \left(\frac{i \zeta_\alpha}{|\zeta_\alpha|} \left\{ 2 [D_t^2 \zeta, \mathcal{H}] \frac{\partial_\alpha \overline{D_t \zeta}}{\zeta_\alpha} + 2 [D_t \zeta, \mathcal{H}] \frac{\partial_\alpha \overline{D_t^2 \zeta}}{\zeta_\alpha} \right. \right. \\ \left. \left. - \frac{1}{\pi i} \int \left(\frac{D_t \zeta(\alpha, t) - D_t \zeta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 \partial_\beta \overline{D_t \zeta}(\beta, t) d\beta \right\} \right) \end{aligned} \quad (2.33)$$

and

$$\mathcal{H} f(\alpha, t) = U_k^{-1} \mathfrak{H} U_k f(\alpha, t) = \frac{1}{\pi i} p.v. \int \frac{f(\beta, t) \zeta_\beta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} d\beta, \quad (2.34)$$

$$\mathcal{K}^* f(\alpha, t) = p.v. \int \operatorname{Re} \left\{ \frac{-1}{\pi i} \frac{\zeta_\alpha}{|\zeta_\alpha|} \frac{|\zeta_\beta(\beta, t)|}{(\zeta(\alpha, t) - \zeta(\beta, t))} \right\} f(\beta, t) d\beta. \quad (2.35)$$

Notice the remarkable similarities between equations (2.29)-(2.28) and (2.32)-(2.33). In particular, the structures of the terms in (2.29)-(2.28) do not change under the change of variables. This makes it convenient for us to work in another coordinate system and to choose a different coordinate system when there is advantage to do so. In fact, this has been used in our study of the global in time behavior of water waves [41, 42].

2.4. The quasi-linear equation for 3d water waves. We derive the quasi-linear equation for 3d water waves by carrying out the same procedure as for 2d. We first need to write down the 3d counterpart of the interface equation (2.4). While equation (2.1) is readily available in 3d, to write down the second equation, we need a suitable counterpart in 3d of the equation for the trace on the interface of the velocity field \mathbf{v} that satisfies $\operatorname{div} \mathbf{v} = 0$ and $\operatorname{curl} \mathbf{v} = 0$. This leads us to Clifford analysis.

Let's recall the basics of Clifford algebra, or in other words, the algebra of quaternions $\mathcal{C}(V_2)$ (c.f. [17]). Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$, satisfying

$$e_i^2 = -1, \quad \text{for } i = 1, 2, 3, \quad e_i e_j = -e_j e_i, \quad i \neq j, \quad e_3 = e_1 e_2. \quad (2.36)$$

Let $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$. By definition, a Clifford-valued function $F : \Omega \subset \mathbb{R}^3 \rightarrow \mathcal{C}(V_2)$ is Clifford analytic in domain Ω iff $\mathcal{D}F = 0$ in Ω . Therefore $F = \sum_{i=1}^3 f_i e_i$ is Clifford analytic in Ω if and only if $\operatorname{div} F = 0$ and $\operatorname{curl} F = 0$ in Ω . Furthermore a function F is the trace of a Clifford analytic function in Ω if and only if $F = \mathfrak{H}_\Sigma F$, where

$$\mathfrak{H}_\Sigma g(\alpha, \beta) = p.v. \iint K(\eta(\alpha', \beta') - \eta(\alpha, \beta)) (\eta'_{\alpha'} \times \eta'_{\beta'}) g(\alpha', \beta') d\alpha' d\beta' \quad (2.37)$$

is the 3d version of the Hilbert transform on $\Sigma = \partial\Omega : \eta = \eta(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$, with normal $\eta_\alpha \times \eta_\beta$ pointing out of Ω , and

$$\Gamma(\eta) = -\frac{1}{\omega_3 |\eta|}, \quad K(\eta) = -2\mathcal{D}\Gamma(\eta) = -\frac{2}{\omega_3} \frac{\eta}{|\eta|^3},$$

ω_3 is the surface area of the unit sphere in \mathbb{R}^3 .¹⁶ As in the 2d case, if $\xi = \xi(\alpha, \beta, t)$, $(\alpha, \beta) \in \mathbb{R}^2$ is the free interface $\Sigma(t)$ in Lagrangian coordinates (α, β) at time t , with $N = \xi_\alpha \times \xi_\beta$ pointing out of the fluid domain, we can rewrite the 3D water wave system (1.1) ($n = 3$) as

$$\begin{cases} \xi_{tt} + e_3 = \mathbf{a}N \\ \xi_t = \mathfrak{H}_{\Sigma(t)}\xi_t \end{cases} \quad (2.38)$$

where $\mathbf{a} = -\frac{1}{|N|} \frac{\partial P}{\partial \mathbf{n}}$.

Differentiating the first equation with respect to t yields

$$\begin{cases} \xi_{ttt} - \mathbf{a}N_t = \mathbf{a}_tN \\ \xi_t = \mathfrak{H}_{\Sigma(t)}\xi_t. \end{cases} \quad (2.39)$$

This is the 3d counterpart of the 2d quasilinear equation (2.6) with constraint $\bar{z}_t = \mathfrak{H}\bar{z}_t$. It has been proved in [40] that $N_t = -|N|\nabla_{\mathbf{n}}\xi_t$,¹⁷ and similar to the calculation (2.27), (2.28) for 2d, an expression for $\mathbf{a}_t|N|$ in terms of $(I - \mathfrak{H}_{\Sigma(t)})(\xi_{ttt} + \mathbf{a}|N|\nabla_{\mathbf{n}}\xi_t) = [\partial_t^2 + a|N|\nabla_{\mathbf{n}}, \mathfrak{H}_{\Sigma(t)}]\xi_t$ has been derived.

The Taylor sign condition (1.3) also holds for C^2 interfaces of the water wave problem (1.1) in 3d. This was proved in [40] by an application of the Green's identity. Here we give a heuristic argument via the maximum principle:

Applying div to both sides of the Euler equation and using the assumption that $\operatorname{curl} \mathbf{v} = 0$ yields

$$\Delta P = -|\nabla \mathbf{v}|^2 \leq 0 \quad \text{in } \Omega(t).$$

Therefore from $P = 0$ on the interface $\Sigma(t)$ and the maximum principle, we have $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$.

(2.39) is then a quasilinear equation of hyperbolic type with the right hand side consisting of lower order terms. The local in time wellposedness of (2.38) is proved in [40] by applying energy estimates and an iterative argument to (2.39).

Remark 2.4. An analogous derivation of the quasilinear structure can now be performed directly on (1.1). Let $D_t = \partial_t + \mathbf{v} \cdot \nabla$ be the material derivative, we can rewrite the first equation in (1.1) as

$$D_t \mathbf{v} + \mathbf{k} = -\nabla P.$$

Apply D_t to both sides to find

$$D_t^2 \mathbf{v} + [D_t, \nabla]P = -\nabla D_t P. \quad (2.40)$$

From the third equation: $P = 0$ on $\Sigma(t)$, we have $D_t P = 0$ on $\Sigma(t)$, so $\nabla D_t P$ points in the normal direction on the interface. We note that

$$[D_t, \partial_{x_i}]P = -\partial_{x_i} \mathbf{v} \cdot \nabla P = -(\nabla P \cdot \nabla) \mathbf{v}_i = \mathbf{a}|N|\nabla_{\mathbf{n}} \mathbf{v}_i$$

on the fluid interface $\Sigma(t)$,¹⁸ so it corresponds to the term $-\mathbf{a}N_t = \mathbf{a}|N|\nabla_{\mathbf{n}}\xi_t$ in equation (2.39), therefore (2.40) is the counterpart of our quasilinear equation (2.39) in the entire fluid region. In [10, 29], the authors used (2.40) to study a more general case where the vorticity $\operatorname{curl} \mathbf{v}$ need not be zero.¹⁹

We now turn to the question of long time behavior of solutions for the water wave equation (1.1) for small initial data.

¹⁶(2.37) is similar to the 2d Hilbert transform (2.2) in the sense that in 2d, the fundamental solution for Laplace equation is $\Gamma(z) = \frac{1}{2\pi} \ln |z|$, $K(z) = -2(\partial_x - i\partial_y)(\frac{1}{2\pi} \ln |z|) = -\frac{1}{\pi z}$, the outward normal is iz_α , and $K(z(\alpha) - z(\beta))(iz_\beta) = \frac{1}{\pi i} \frac{iz_\beta}{z(\alpha) - z(\beta)}$ is the kernel of the Hilbert transform (2.2).

¹⁷In 2d, we know iz_α is a normal vector to $\Sigma(t)$ pointing out of the fluid domain, and $\partial_t(\overline{iz_\alpha}) = -i\partial_\alpha \bar{z}_t = -|z_\alpha|\nabla_{\mathbf{n}} \bar{z}_t$.

¹⁸In the second equality we used $\operatorname{curl} \mathbf{v} = 0$.

¹⁹In [10, 29], the strong Taylor condition (1.3) is assumed to hold.

3. GLOBAL AND ALMOST GLOBAL WELLPOSEDNESS OF THE WATER WAVE EQUATIONS

To understand the global in time behavior of the water wave motion, we need to understand the dispersion, the nature of the nonlinearity of the water waves and their interaction.

In [41, 42] we studied the water wave equation (1.1) in two and three space dimensions for small data, we found a nonlinear transformation for the unknowns and a nonlinear change of the coordinates, so that the transformed quantities in the new coordinate system satisfy equations containing no quadratic nonlinear terms.²⁰ Using these canonical equations, we showed that for small, smooth and localized data of size ϵ , the solution of the 2d water wave equation (1.1) remain small and smooth for time $0 < t < e^{c/\epsilon}$ and for similar data, the solution of the 3d water wave equation (1.1) remain small and smooth for all time.

Let's give a brief explanation of the dispersion of the water waves, the structural advantage of our canonical equations and how we found the transformations.

Let $\mathbf{u} = \bar{z}_t$ (or $\mathbf{u} = \xi_t$). Linearizing the quasi-linear system (2.29)-(2.28)-(2.26) (or (2.39)) at the zero solution gives

$$\partial_t^2 \mathbf{u} + |D|\mathbf{u} = F(\mathbf{u}_t, |D|\mathbf{u}), \quad (\alpha, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad (3.1)$$

where $|D| = \sqrt{-\Delta}$, Δ is the Laplacian in \mathbb{R}^{n-1} for n-dimensional water waves, F consists of the nonlinear terms. We know the dispersion relation of the linear water wave equation

$$\partial_t^2 \mathbf{u} + |D|\mathbf{u} = 0 \quad (3.2)$$

is

$$\omega^2 = |k| \quad (3.3)$$

for plane wave solution $\mathbf{u} = e^{i(k \cdot \alpha + \omega t)}$, so waves of wave number k travel with phase velocity $\frac{\omega}{|k|} \hat{k}$, where $\hat{k} = \frac{k}{|k|}$, equation (3.1) is dispersive. For a large class \mathfrak{B} of smooth initial data, the solution of the linear equation (3.2) exists for all time and remains smooth, and its L^∞ norm decays with rate $1/t^{\frac{n-1}{2}}$. The question is for small data in \mathfrak{B} , for how long does the solution of the nonlinear equation (3.1) remain smooth. We know nonlinear interactions can cause blow-up of the solutions at finite time. So to answer this question, we need to know for how long does the linear part of the equation (3.1) remain dominant. The weaker the nonlinear interaction, the longer the solution remains smooth. For small data, quadratic interactions are in general stronger than cubic and higher order interactions.

To understand these assertions in qualitative terms, let's consider the following model equation with a $(p+1)$ th-order nonlinearity:

$$\partial_t^2 \mathbf{u} + |D|\mathbf{u} = (\partial_t \mathbf{u})^{p+1}, \quad (\alpha, t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \quad (3.4)$$

Suppose we can prove decay estimates for the solution: for $i \leq s-10$,

$$|\partial^i \partial_t \mathbf{u}(t)|_{L^\infty} \lesssim (1+t)^{-\frac{n-1}{2}} E_s(t)^{1/2},$$

where ∂ is some kind of derivatives,

$$E_s(t) = \sum_{|j| \leq s} \int |\partial^j \partial_t \mathbf{u}(\alpha, t)|^2 + |\partial^j |D|^{1/2} \mathbf{u}(\alpha, t)|^2 d\alpha.$$

Then we can derive energy estimates for large enough s :

$$\frac{d}{dt} E_s(t) \lesssim (1+t)^{-\frac{(n-1)p}{2}} E_s(t)^{p/2+1},$$

therefore

$$E_s(0)^{-p/2} - E_s(T)^{-p/2} \lesssim \int_0^T (1+t)^{-\frac{(n-1)p}{2}} dt. \quad (3.5)$$

²⁰When understood appropriately.

Heuristically, we expect to prove existence of solutions of (3.4) for as long as the energy $E_s(t)$ remains finite; by (3.5), this can be achieved if

$$\int_0^T (1+t)^{-\frac{(n-1)p}{2}} dt \lesssim E_s(0)^{-p/2}.$$

Now if $p = 1$, i.e. if the nonlinear term in (3.4) is quadratic, then for both $n = 2$ and $n = 3$, the integral $\int_0^\infty (1+t)^{-\frac{(n-1)p}{2}} dt = \infty$, and we would not be able to conclude solutions exist for all time for small initial data from this analysis. In fact for $p = 1$ and $n = 2$, the expected existence time is of order $O(\epsilon^{-2})$ for data of size ϵ . If $p \geq 2$, i.e. if there is no quadratic nonlinearity, then we can expect to prove longer time existence for solutions of (3.4) for small initial data. In fact, we can expect for $n = 2, p = 2$ an existence time period of $[0, e^{c/\epsilon^2}]$; and for $n = 3, p = 2$ an existence time period $[0, \infty)$ for data of size ϵ , when ϵ is sufficiently small.

Now for the water wave equation (3.1), the nonlinearity $F(\mathbf{u}_t, |D|\mathbf{u})$ contains quadratic terms, so appears too strong to conclude a global existence result. The question is whether there is another unknown \mathbf{v} that satisfies an equation of the type

$$\partial_t^2 \mathbf{v} + |D|\mathbf{v} = F_1(\mathbf{v}_t, |D|\mathbf{v}, \mathbf{u}_t, |D|\mathbf{u}) \quad (3.6)$$

with F_1 containing no quadratic nonlinearities and $\|\mathbf{v}\| \approx \|\mathbf{u}\|$ in various norms $\|\cdot\|$ involved in the analysis. The idea of finding such an unknown \mathbf{v} is the so called method of normal forms, originally introduced by Poincaré to solve ordinary differential equations. Certainly in most cases, one should not expect such a new unknown \mathbf{v} exist. For quadratic Klein-Gordon equation however, Simon and Shatah [32, 33] succeeded in finding a bilinear normal form transformation of the type

$$\mathbf{v} = \mathbf{u} + B(\mathbf{u}, \mathbf{u}) \quad (3.7)$$

with $B(\mathbf{u}, \mathbf{u})$ bilinear, canceling out the quadratic nonlinear terms in the Klein-Gordon equation, and satisfying the norm equivalence $\|\mathbf{u} + B(\mathbf{u}, \mathbf{u})\| \approx \|\mathbf{u}\|$ for small $\|\mathbf{u}\|$.

For water wave equations (2.29)-(2.28)-(2.26) (or (2.39)) however, a transformation of the type (3.7) doesn't quite work since it has a small divisor; working with the velocity potential and the Bernoulli equation²¹, a bilinear transform of the type (3.7) has a loss of derivatives (for detailed calculations and discussions, see Appendix C of [42]). What we did in [41, 42] was to further introduce a change of the coordinates. Indeed this makes sense since when one applies a method from ODE to PDE, it is reasonable to also take into consideration the spatial variables.

We give in the next two subsections the transforms for the 2d and 3d water waves. The transforms are fully nonlinear. We found the transforms by first considering the 2d quasilinear equation (2.29)-(2.28)-(2.26), starting with the ansatz (3.7), taking into considerations of the coordinate invariance of (2.29), looking for a coordinate invariant transformation.²² This entails much further efforts in understanding the bilinear transformation. What we finally arrived at is a fully nonlinear transform of the unknown function, coupled with a coordinate change. The process of finding the transforms is non-algorithmic. The transforms for the 3d water waves (2.39) is obtained by naturally extending the 2d version via Clifford analysis, c.f. [42].

3.1. The transformations for the 2D water waves. We give here the transformations we constructed in [41] for the 2D water waves. Let $\Sigma(t) : z = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be the interface in Lagrangian coordinate α .

²¹Or equivalently the Zakharov-Craig-Sulem equation for the interface.

²²The bilinear transform of the type (3.7) is not coordinate invariant.

Proposition 3.1 (Proposition 2.3 of [41]). *Let $z = z(\alpha, t)$ be a solution of the 2d water wave system (2.4). Let $\Pi := (I - \mathfrak{H})(z - \bar{z})$; let $k = k(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary diffeomorphism. Let $\zeta := z \circ k^{-1}$, $D_t \zeta := z_t \circ k^{-1}$ etc. be as in (2.30)-(2.31). Then*

$$\begin{aligned} & (D_t^2 - iA\partial_\alpha)(\Pi \circ k^{-1}) \\ &= \frac{4}{\pi} \int \frac{(D_t \zeta(\alpha, t) - D_t \zeta(\beta, t))(\operatorname{Im} \zeta(\alpha, t) - \operatorname{Im} \zeta(\beta, t))}{|\zeta(\alpha, t) - \zeta(\beta, t)|^2} \partial_\beta D_t \zeta(\beta, t) d\beta \\ &+ \frac{2}{\pi} \int \left(\frac{D_t \zeta(\alpha, t) - D_t \zeta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 \partial_\beta \operatorname{Im} \zeta(\beta, t) d\beta \end{aligned} \quad (3.8)$$

Notice that the right hand side of equation (3.8) is cubically small if the velocity $D_t \zeta$, the height function $\operatorname{Im} \zeta$ of the interface and their derivatives are small, but the left hand side of (3.8) still contains quadratic nonlinearities. Naturally we ask if there is a coordinate change k , so that $b = k_t \circ k^{-1}$ and $A - 1 = (\mathbf{a}k_\alpha) \circ k^{-1} - 1$ are quadratic. We need not look far, equation (2.18) suggests that we choose

$$k(\alpha, t) = 2 \operatorname{Re} z(\alpha, t) - h(\alpha, t), \quad \alpha \in \mathbb{R}. \quad (3.9)$$

Proposition 3.2 (Proposition 2.4 of [41]). *Let k be as given by (3.9), $b = k_t \circ k^{-1}$ and $A = (\mathbf{a}k_\alpha) \circ k^{-1}$. Let \mathcal{H} be defined by (2.34). We have*

$$\begin{aligned} (I - \mathcal{H})b &= -[D_t \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} \\ (I - \mathcal{H})(A - 1) &= i[D_t \zeta, \mathcal{H}] \frac{\partial_\alpha \bar{D}_t \zeta}{\zeta_\alpha} + i[D_t^2 \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} \end{aligned} \quad (3.10)$$

Propositions 3.1 and 3.2 show that the quantity $\theta := \Pi \circ k^{-1} = (I - \mathcal{H})(\zeta - \bar{\zeta})$ with the coordinate change k given by (3.9) satisfies an equation of the type

$$\begin{cases} (\partial_t^2 - i\partial_\alpha)\theta = \mathcal{G} \\ (I + \mathcal{H})\theta = 0 \end{cases} \quad (3.11)$$

with \mathcal{G} containing only nonlinear terms of cubic and higher order. We make the following remarks:

Remark 3.3. 1. The transformation $I - \mathfrak{H}$ and the coordinate change k as given in (3.9) are fully nonlinear in terms of the unknown function z and its derivatives.²³

2. The bilinear part of the quantity $\Pi := (I - \mathfrak{H})(z - \bar{z})$ has a bounded Fourier symbol. The coordinate change k takes care of the small divisor in the Fourier symbol of the bilinear normal form transformation. For a detailed explanation see Appendix C of [42].

3. For θ satisfying $(I + \mathcal{H})\theta = 0$, $(\partial_t^2 - i\partial_\alpha)\theta = \partial_t^2 \theta + i\partial_\alpha \mathcal{H}\theta = (\partial_t^2 + |D|)\theta + \text{quadratic} + \dots$. The quadratic nonlinearity comes from \mathcal{H} , which depends nonlinearly on the unknown ζ .

4. Let ϕ be the velocity potential, $\psi(\alpha, t) := \phi(z(\alpha, t), t)$ be the trace of ϕ on the free interface. It has been shown in [41] that the quantities $U_{k^{-1}}(I - \mathfrak{H})\psi$, $U_{k^{-1}}\partial_t \Pi$ also satisfy equations of the type (3.11), with their equations given in Proposition 2.3 of [41]. $\partial_\alpha U_{k^{-1}}(I - \mathfrak{H})\psi$, $U_{k^{-1}}\partial_t \Pi$, $\frac{i}{2}\partial_\alpha U_{k^{-1}}\Pi$ and $\operatorname{Im} \partial_\alpha U_{k^{-1}}\Pi$ are near identity transforms of the velocity $D_t \zeta$, $2D_t \zeta$, the acceleration $D_t^2 \zeta$ and $2 \operatorname{Im} \partial_\alpha \zeta$, see Propositions 2.5 and 2.6 of [41].

5. (3.8)-(3.9) is used in [38] to give a rigorous justification of the NLS from the 2d water wave equation (1.1).

6. The idea of changing coordinates is subsequently used in [19] to remove the quadratic nonlinear terms in the Burgers-Hilbert equation.

We now give the proofs of Propositions 3.1, 3.2.²⁴

²³That is, $I - \mathfrak{H}$ and k are not finite sums of multi-linear operators of z and its derivatives.

²⁴The proofs are taken from [41].

Proof. We first prove (3.8). Let $z = z(\cdot, t)$ be a solution of the water wave equation (2.4). Apply $(\partial_t^2 - i\mathfrak{a}\partial_\alpha)$ to $\Pi := (I - \mathfrak{H})(z - \bar{z})$ and commute $\partial_t^2 - i\mathfrak{a}\partial_\alpha$ with $I - \mathfrak{H}$ to find

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})(z - \bar{z})\} = (I - \mathfrak{H})\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(z - \bar{z})\} - [\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}](z - \bar{z}). \quad (3.12)$$

Use (2.4) to find $(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(z - \bar{z}) = -2\bar{z}_{tt}$ then use $\bar{z}_t = \mathfrak{H}\bar{z}_t$ to write $(I - \mathfrak{H})\bar{z}_{tt}$ as the commutator $[\partial_t, \mathfrak{H}]\bar{z}_t$ and applying Lemma B.1 yields

$$(I - \mathfrak{H})\{(\partial_t^2 - i\mathfrak{a}\partial_\alpha)(z - \bar{z})\} = (I - \mathfrak{H})(-2\bar{z}_{tt}) = -2[z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha}. \quad (3.13)$$

Applying Lemma B.1 to the second term gives

$$[\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}](z - \bar{z}) = 2[z_t, \mathfrak{H}]\frac{z_{t\alpha} - \bar{z}_{t\alpha}}{z_\alpha} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 \partial_\beta(z(\beta, t) - \bar{z}(\beta, t)) d\beta. \quad (3.14)$$

Subtract (3.14) from (3.13). After cancelation this leaves, from (3.12),

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})(z - \bar{z})\} = -2[z_t, \mathfrak{H}]\frac{z_{t\alpha}}{z_\alpha} + \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z - \bar{z})_\beta d\beta. \quad (3.15)$$

Because \bar{z}_t and $\frac{\bar{z}_{t\alpha}}{z_\alpha}$ are holomorphic, from part 1 of Lemma B.2

$$[z_t, \bar{\mathfrak{H}}]\frac{1}{\bar{z}_\alpha} z_{t\alpha} = [z_t, \bar{\mathfrak{H}}]\frac{z_{t\alpha}}{\bar{z}_\alpha} = 0.$$

In the first term of the right hand side of (3.15) insert $[z_t, \bar{\mathfrak{H}}]\frac{1}{\bar{z}_\alpha} z_{t\alpha}$ to make it cubic. We have

$$\begin{aligned} & (\partial_t^2 - i\mathfrak{a}\partial_\alpha)\{(I - \mathfrak{H})(z - \bar{z})\} \\ &= -2[z_t, \mathfrak{H}]\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha} z_{t\alpha} + \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z_\beta - \bar{z}_\beta) d\beta. \end{aligned} \quad (3.16)$$

Precomposing with k^{-1} and expanding the two terms on the right gives (3.8). \square

Proof. We prove Proposition 3.2. We have from (3.9)

$$k - \bar{z} = z - h.$$

Recall $h(\alpha, t) = \Phi(z(\alpha, t), t)$ where $\Phi(\cdot, t) : \Omega(t) \rightarrow P_-$ is the Riemann mapping satisfying $\lim_{z \rightarrow \infty} \Phi_z(z, t) = 1$. We have

$$h_t = \Phi_t \circ z + (\Phi_z \circ z)z_t, \quad h_\alpha = (\Phi_z \circ z)z_\alpha,$$

therefore

$$\bar{z}_t - k_t = \Phi_t \circ z + (\Phi_z \circ z - 1)z_t, \quad \bar{z}_\alpha - k_\alpha = (\Phi_z \circ z - 1)z_\alpha. \quad (3.17)$$

Apply $(I - \mathfrak{H})$ to the first equality in (3.17). Because Φ_t, Φ_z are holomorphic in $\Omega(t)$ with $\lim_{z \rightarrow \infty} \Phi_z(z, t) = 1$, using Proposition 2.1 and rewriting $(I - \mathfrak{H})\{(\Phi_z \circ z - 1)z_t\}$ as $[z_t, \mathfrak{H}](\Phi_z \circ z - 1)$ yields

$$\begin{aligned} -(I - \mathfrak{H})k_t &= (I - \mathfrak{H})(\bar{z}_t - k_t) \\ &= (I - \mathfrak{H})\{(\Phi_z \circ z - 1)z_t\} = [z_t, \mathfrak{H}](\Phi_z \circ z - 1) = [z_t, \mathfrak{H}]\frac{\bar{z}_\alpha - k_\alpha}{z_\alpha}. \end{aligned} \quad (3.18)$$

Precomposing with k^{-1} gives the first equality of (3.10).

Now multiply $i\mathfrak{a}$ then apply $(I - \mathfrak{H})$ to the second equality in (3.17). Using (2.4) and the fact that $(I - \mathfrak{H})(\Phi_z \circ z - 1) = 0$, we also have

$$(I - \mathfrak{H})(i\mathfrak{a}\bar{z}_\alpha - i\mathfrak{a}k_\alpha) = (I - \mathfrak{H})(i\mathfrak{a}z_\alpha(\Phi_z \circ z - 1)) = [z_{tt}, \mathfrak{H}](\Phi_z \circ z - 1) = [z_{tt}, \mathfrak{H}]\frac{\bar{z}_\alpha - k_\alpha}{z_\alpha}.$$

Use (2.4) and Lemma B.1 to calculate

$$(I - \mathfrak{H})(i\mathfrak{a}\bar{z}_\alpha) = (I - \mathfrak{H})(-\bar{z}_{tt} + i) = i - [\partial_t, \mathfrak{H}]\bar{z}_t = i - [z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha}$$

so

$$\begin{aligned} -(I - \mathfrak{H})(i\mathfrak{a}k_\alpha) &= -(I - \mathfrak{H})(i\mathfrak{a}\bar{z}_\alpha) + [z_{tt}, \mathfrak{H}]\frac{\bar{z}_\alpha - k_\alpha}{z_\alpha} \\ &= -i + [z_t, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + [z_{tt}, \mathfrak{H}]\frac{\bar{z}_\alpha - k_\alpha}{z_\alpha}. \end{aligned} \quad (3.19)$$

Precomposing with k^{-1} yields the second equality of (3.10). \square

To extend the 2d coordinate change (3.9) to 3d, we need an expression that does not rely on the Riemann mapping. Observe that for the diffeomorphism k given by (3.9), $k - \bar{z} = z - h$ and $z - h$ is holomorphic with $\text{Im}(z - h) = \text{Im} z$, so we can replace $z - h$ by $\frac{1}{2}(I + \mathfrak{H})(I + \mathfrak{K})^{-1}(z - \bar{z})$, where $\mathfrak{K} = \text{Re } \mathfrak{H}$ is the double layer potential operator, and

$$k = \bar{z} + \frac{1}{2}(I + \mathfrak{H})(I + \mathfrak{K})^{-1}(z - \bar{z}) \quad (3.20)$$

modulo a real constant.²⁵ The expression (3.20) is directly extendable to 3d.

3.2. The transformation for the 3D water waves. We extend the 2d transformations to 3d in the framework of the Clifford algebra $\mathcal{C}(V_2)$. Besides those in subsection 2.4, we need some additional notations.

An element $\sigma \in \mathcal{C}(V_2)$ can be represented uniquely by $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$, with $\sigma_i \in \mathbb{R}$ for $0 \leq i \leq 3$. Define $\text{Re } \sigma := \sigma_0$ and call it the real part of σ . We call σ a vector if $\sigma_0 = 0$. If not specified, we always assume in an expression $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$ that $\sigma_i \in \mathbb{R}$ for $0 \leq i \leq 3$. Define $\bar{\sigma} := e_3 \sigma e_3$, the conjugate of σ . We identify a point or a vector $\xi = (x_1, x_2, y) \in \mathbb{R}^3$ with its $\mathcal{C}(V_2)$ counterpart $\xi = x_1 e_1 + x_2 e_2 + y e_3$. For vectors $\xi, \eta \in \mathcal{C}(V_2)$, we know

$$\xi \eta = -\xi \cdot \eta + \xi \times \eta, \quad (3.21)$$

where $\xi \cdot \eta$ is the dot product, $\xi \times \eta$ the cross product. For vectors ξ, ζ, η , $\xi(\zeta \times \eta)$ is obtained by first finding the cross product $\zeta \times \eta$, then regard it as a Clifford vector and calculating its multiplication with ξ by the rule (2.36). We write $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_y)$. We abbreviate notations such as

$$\begin{aligned} \mathfrak{H}_\Sigma f(\alpha, \beta) &= \iint K(\eta(\alpha', \beta') - \eta(\alpha, \beta)) (\eta'_{\alpha'} \times \eta'_{\beta'}) f(\alpha', \beta') d\alpha' d\beta' \\ &:= \iint K(\eta' - \eta) (\eta'_{\alpha'} \times \eta'_{\beta'}) f' d\alpha' d\beta' := \iint K N' f' d\alpha' d\beta'. \end{aligned}$$

As in the 2d case, $\mathfrak{H}_\Sigma^2 = I$ in L^2 , and $\mathfrak{H}_\Sigma 1 = 0$.

We give the transformation for the 3D water wave equation (2.38). Let the free interface $\Sigma(t)$ be given by $\xi = \xi(\alpha, \beta, t) = x_1(\alpha, \beta, t)e_1 + x_2(\alpha, \beta, t)e_2 + y(\alpha, \beta, t)e_3$ in Lagrangian coordinates (α, β) with $N = \xi_\alpha \times \xi_\beta$ pointing out of the fluid domain $\Omega(t)$. For fixed t , let $k = k(\cdot, t) = k_1 e_1 + k_2 e_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism with Jacobian $J(k(t)) > 0$. Let k^{-1} be such that $k \circ k^{-1}(\alpha, \beta, t) = \alpha e_1 + \beta e_2$. Define

$$\zeta := \xi \circ k^{-1}, \quad b := k_t \circ k^{-1}, \quad A \circ k e_3 := \mathfrak{a} J(k) e_3 := \mathfrak{a} k_\alpha \times k_\beta \quad (3.22)$$

Let $D_t := U_k^{-1} \partial_t U_k$ be the material derivative, $\mathcal{N} := \zeta_\alpha \times \zeta_\beta$. By the chain rule, we know

$$D_t = \partial_t + b \cdot (\partial_\alpha, \partial_\beta), \quad U_k^{-1} (\mathfrak{a} N \times \nabla) U_k = A N \times \nabla = A (\zeta_\beta \partial_\alpha - \zeta_\alpha \partial_\beta), \quad (3.23)$$

²⁵ $z - h$ and $\frac{1}{2}(I + \mathfrak{H})(I + \mathfrak{K})^{-1}(z - \bar{z})$ are holomorphic with the same imaginary part, so the difference between them is a constant in \mathbb{R} .

and $U_k^{-1}\mathfrak{H}_{\Sigma(t)}U_k := \mathcal{H}_{\Sigma(t)}$, with

$$\mathcal{H}_{\Sigma(t)}f(\alpha, \beta, t) = \iint K(\zeta(\alpha', \beta', t) - \zeta(\alpha, \beta, t))(\zeta'_{\alpha'} \times \zeta'_{\beta'})f(\alpha', \beta', t) d\alpha' d\beta'. \quad (3.24)$$

We have

Proposition 3.4 (Proposition 1.3 of [42]). *Let $\xi = \xi(\alpha, \beta, t)$ be a solution of the 3d water wave system (2.38). Let $\Pi = (I - \mathfrak{H}_{\Sigma(t)})(\xi - \bar{\xi})$, and for fixed t , $k(\cdot, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism. We have*

$$\begin{aligned} & (D_t^2 - \mathcal{A}\mathcal{N} \times \nabla)(\Pi \circ k^{-1}) \\ &= 2 \iint K(\zeta' - \zeta) (D_t\zeta - D_t'\zeta') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\overline{D_t'\zeta'} d\alpha' d\beta' \\ & - \iint K(\zeta' - \zeta) (D_t\zeta - D_t'\zeta') \times ((D_t'\zeta')_{\beta'}\partial_{\alpha'} - (D_t'\zeta')_{\alpha'}\partial_{\beta'}) (\zeta' - \bar{\zeta}') d\alpha' d\beta' \\ & - \iint D_t K(\zeta' - \zeta) (D_t\zeta - D_t'\zeta') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) (\zeta' - \bar{\zeta}') d\alpha' d\beta' \end{aligned} \quad (3.25)$$

Observe that the second and third terms in the right hand side of (3.25) are cubically small provided the velocity $D_t\zeta$ and the steepness of the height function $\partial_{\alpha}(\zeta - \bar{\zeta})$, $\partial_{\beta}(\zeta - \bar{\zeta})$ are small, while the first term appears to be only quadratically small. Unlike the 2D case, multiplications of Clifford analytic functions are not necessarily analytic, so we cannot reduce the first term at the right hand side of equation (3.25) into a cubic form. However we note that the first term is almost analytic in the fluid domain $\Omega(t)$, while the left hand side of (3.25) is almost analytic in the air region. The orthogonality of the projections $\frac{1}{2}(I - \mathcal{H}_{\Sigma(t)})$ and $\frac{1}{2}(I + \mathcal{H}_{\Sigma(t)})$ allows us to reduce the first term into a cubic in energy estimates, see [42].

Now the left hand side of (3.25) still contains quadratic terms. As in the 2D case, we resolve this difficulty by choosing an appropriate coordinate change k . Let

$$k = k(\alpha, \beta, t) = \xi(\alpha, \beta, t) - (I + \mathfrak{H}_{\Sigma(t)})y(\alpha, \beta, t)e_3 + \mathfrak{K}_{\Sigma(t)}y(\alpha, \beta, t)e_3 \quad (3.26)$$

Here $\mathfrak{K}_{\Sigma(t)} = \text{Re } \mathfrak{H}_{\Sigma(t)}$:

$$\mathfrak{K}_{\Sigma(t)}f(\alpha, \beta, t) = - \iint K(\xi(\alpha', \beta', t) - \xi(\alpha, \beta, t)) \cdot N'f(\alpha', \beta', t) d\alpha' d\beta' \quad (3.27)$$

is the double layer potential operator. It is clear that the e_3 component of k as defined in (3.26) is zero. In addition, the real part of k is also zero. This is because

$$\begin{aligned} & \iint K(\xi' - \xi) \times (\xi'_{\alpha'} \times \xi'_{\beta'}) y' e_3 d\alpha' d\beta' \\ &= \iint (\xi'_{\alpha'} \xi'_{\beta'} \cdot K - \xi'_{\beta'} \xi'_{\alpha'} \cdot K) y' e_3 d\alpha' d\beta' \\ &= -2 \iint (\xi'_{\alpha'} \partial_{\beta'} \Gamma(\xi' - \xi) - \xi'_{\beta'} \partial_{\alpha'} \Gamma(\xi' - \xi)) y' e_3 d\alpha' d\beta' \\ &= 2 \iint \Gamma(\xi' - \xi) (\xi'_{\alpha'} y_{\beta'} - \xi'_{\beta'} y_{\alpha'}) e_3 d\alpha' d\beta' \\ &= 2 \iint \Gamma(\xi' - \xi) (N'_1 e_1 + N'_2 e_2) d\alpha' d\beta' \end{aligned}$$

So

$$\mathfrak{H}_{\Sigma(t)}y e_3 = \mathfrak{K}_{\Sigma(t)}y e_3 + 2 \iint \Gamma(\xi' - \xi) (N'_1 e_1 + N'_2 e_2) d\alpha' d\beta' \quad (3.28)$$

This shows that the mapping k defined in (3.26) has only the e_1 and e_2 components $k = (k_1, k_2) = k_1 e_1 + k_2 e_2$. If $\Sigma(t)$ is a graph with small steepness, i.e. if y_{α} and y_{β} are small, then the Jacobian of $k = k(\cdot, t)$: $J(k) = J(k(t)) = \partial_{\alpha} k_1 \partial_{\beta} k_2 - \partial_{\alpha} k_2 \partial_{\beta} k_1 > 0$ and $k(\cdot, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defines a valid coordinate change (c.f. [42]).

The following proposition shows that if k is as given in (3.26), then b and $A - 1$ are quadratic. Let

$$\mathcal{K}_{\Sigma(t)} := \operatorname{Re} \mathcal{H}_{\Sigma(t)} =: U_k^{-1} \mathfrak{K}_{\Sigma(t)} U_k, \quad P := \alpha e_1 + \beta e_2, \quad \text{and} \quad \zeta := P + \lambda. \quad (3.29)$$

Proposition 3.5 (Proposition 1.4 of [42]). *Let k be as given in (3.26). Let $b = k_t \circ k^{-1}$ and $A \circ k = \mathfrak{a}J(k)$. We have*

$$\begin{aligned} b &= \frac{1}{2}(\mathcal{H}_{\Sigma(t)} - \overline{\mathcal{H}_{\Sigma(t)}}) \overline{D_t \zeta} - \frac{1}{2}[D_t, \mathcal{H}_{\Sigma(t)} - \mathcal{K}_{\Sigma(t)}](\zeta - \bar{\zeta}) + \frac{1}{2}\mathcal{K}_{\Sigma(t)}(D_t \zeta - \overline{D_t \zeta}) \\ (A - 1)e_3 &= \frac{1}{2}(-\mathcal{H}_{\Sigma(t)} + \overline{\mathcal{H}_{\Sigma(t)}}) \overline{D_t^2 \zeta} + \frac{1}{2}([D_t, \mathcal{H}_{\Sigma(t)}] D_t \zeta - \overline{[D_t, \mathcal{H}_{\Sigma(t)}] D_t \zeta}) \\ &\quad + \frac{1}{2}[A\mathcal{N} \times \nabla, \mathcal{H}_{\Sigma(t)}](\zeta - \bar{\zeta}) - \frac{1}{2}A\check{\zeta}_\beta \times (\partial_\alpha \mathcal{K}_{\Sigma(t)}(\zeta - \bar{\zeta})) \\ &\quad + \frac{1}{2}A\check{\zeta}_\alpha \times (\partial_\beta \mathcal{K}_{\Sigma(t)}(\zeta - \bar{\zeta})) + A\partial_\alpha \lambda \times \partial_\beta \lambda \end{aligned} \quad (3.30)$$

Here $\overline{\mathcal{H}_{\Sigma(t)}} = e_3 \mathcal{H}_{\Sigma(t)} e_3$.

Let $\chi = \Pi \circ k^{-1}$ with k be given by (3.26). The left hand side of equation (3.25) is

$$(\partial_t^2 - e_2 \partial_\alpha + e_1 \partial_\beta) \chi - \partial_\beta \lambda \partial_\alpha \chi + \partial_\alpha \lambda \partial_\beta \chi + \text{cubic and higher order terms}$$

The quadratic term $\partial_\beta \lambda \partial_\alpha \chi - \partial_\alpha \lambda \partial_\beta \chi$ is new in 3d. Observe that this is one of the null forms studied in [26]. It is also null for our equation and can be written as the factor $1/t$ times a quadratic expression involving some "invariant vector fields" for $\partial_t^2 - e_2 \partial_\alpha + e_1 \partial_\beta$, see [42]. Therefore this term is cubic in nature and equation (3.25) is of the type "linear + cubic and higher order perturbations".

We refer the reader to [42] for the proof of Propositions 3.4 and 3.5.

We remark that the 3d transformations is recently used in [37] to give a rigorous justification of the modulation approximation for the 3d water wave equation (1.1). Besides the 3d transforms, [37] uses the method of normal form to handle the quadratic term $\partial_\beta \lambda \partial_\alpha \chi - \partial_\alpha \lambda \partial_\beta \chi$ since this term is truly quadratic in the modulation regime.

3.3. Global in time behavior of solutions for the 2d and 3d water waves. In [41, 42] equations (3.8) and (3.25) together with the coordinate changes (3.9) and (3.26) are used to prove the almost global wellposedness of the 2d and global wellposedness of the 3d water wave equations (1.1) for small, smooth and localized initial data. The basic idea is what we illustrated with the model equation (3.4); it is made rigorous by the method of invariant vector fields. This involves constructing invariant vector fields for the operator $\partial_t^2 - e_2 \partial_\alpha + e_1 \partial_\beta$ (the invariant vector fields for $\partial_t^2 - i \partial_\alpha$ for the 2d case is available due to the well studied Schrödinger operator $i \partial_t - \partial_x^2$), proving generalized Sobolev inequalities that give $L^2 \rightarrow L^\infty$ estimates with the decay rate $1/t^{1/2}$ for the 2d and $1/t$ for the 3d water waves, using equations (3.8)-(3.9) and (3.25)-(3.26) to show that properly constructed energies that involve invariant vector fields remain bounded for the time period $[0, e^{c/\epsilon}]$ for the 2d and for all time for the 3d water waves for data of size $O(\epsilon)$. The projection $\frac{1}{2}(I - \mathfrak{F})$ is used in various ways to project away "quadratic noises" in the course of deriving the energy estimates. We remark that it is more natural to treat $D_t^2 - iA\partial_\alpha$ and $D_t^2 - A\mathcal{N} \times \nabla$ as the main operators for the 2d and 3d water wave equations than treating them as the perturbations of the linear operators $\partial_t^2 - i \partial_\alpha$ and $\partial_t^2 - e_2 \partial_\alpha + e_1 \partial_\beta$. The almost global well-posedness for the 2d and global well-posedness for the 3d water wave equations follow from the local well-posedness results, the uniform boundedness of the energies and continuity arguments. For details of the proofs see [41, 42]. We state the results.

Let $|D| = \sqrt{-\Delta}$, $H^s(\mathbb{R}^{n-1}) = \{f \mid (I + |D|)^s f \in L^2(\mathbb{R}^{n-1})\}$, with $\|f\|_{H^s(\mathbb{R}^{n-1})} = \|(I + |D|)^s f\|_{L^2(\mathbb{R}^{n-1})}$.

2d water waves. Let $s \geq 12$, $\max\{\lfloor \frac{s}{2} \rfloor + 3, 11\} \leq l \leq s - 1$. Assume

$$\begin{aligned} z(\alpha, 0) &= (\alpha, y(\alpha)), & z_t(\alpha, 0) &= \mathbf{u}(\alpha), & z_{tt}(\alpha, 0) &= \mathbf{w}(\alpha) & \alpha \in \mathbb{R}, \\ \mathbf{v}(z, 0) &= \mathbf{g}(z), & z &\in \Omega(0) \end{aligned} \quad (3.31)$$

and the data in (3.31) satisfies the 2d water wave system (2.4). In particular $\bar{\mathbf{g}}$ is a holomorphic function in the initial fluid domain $\Omega(0)$ and $\mathbf{g}(z(\alpha, 0)) = \mathbf{u}(\alpha)$. Let $\Gamma = \partial_\alpha, \alpha \partial_\alpha$. Assume that

$$\sum_{|j| \leq s-1} (\|\Gamma^j y_\alpha\|_{H^{1/2}(\mathbb{R})} + \|\Gamma^j \mathbf{u}\|_{H^{3/2}(\mathbb{R})} + \|\Gamma^j \mathbf{w}\|_{H^1(\mathbb{R})}) < \infty.$$

Let

$$\epsilon = \sum_{|j| \leq l} (\|\Gamma^j y\|_{H^1(\mathbb{R})} + \|\Gamma^j \mathbf{u}\|_{H^1(\mathbb{R})}) + \sum_{j \leq l-2} \|(z \partial_z)^j \bar{\mathbf{g}}\|_{L^2(\Omega(0))}.$$

Theorem 3.6 (2d Theorem, c.f.[41]). *There exist ϵ_0 and $c > 0$, such that for $\epsilon < \epsilon_0$, the initial value problem (2.4)-(3.31) has a unique classical solution for the time period $[0, e^{c/\epsilon}]$. During this time, the interface is a graph, the solution is as regular as the initial data and remains small. Moreover the L^∞ norm of the steepness $\partial_\alpha(z - \bar{z})$, the velocity z_t and acceleration z_{tt} decay at rate $1/t^{1/2}$.*

3d water waves. Let $s \geq 27$, $\max\{\lfloor \frac{s}{2} \rfloor + 1, 17\} \leq l \leq s - 10$. Assume that initially

$$\xi(\alpha, \beta, 0) = (\alpha, \beta, y^0(\alpha, \beta)), \quad \xi_t(\alpha, \beta, 0) = \mathbf{u}^0(\alpha, \beta), \quad \xi_{tt}(\alpha, \beta, 0) = \mathbf{w}^0(\alpha, \beta), \quad (3.32)$$

and the data in (3.32) satisfies the 3d water wave system (2.38). Let $\Gamma = \partial_\alpha, \partial_\beta, \alpha \partial_\alpha + \beta \partial_\beta, \alpha \partial_\beta - \beta \partial_\alpha$. Assume that

$$\sum_{\substack{|j| \leq s-1 \\ \partial = \partial_\alpha, \partial_\beta}} \|\Gamma^j |D|^{1/2} y^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j \partial y^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j \mathbf{u}^0\|_{H^{3/2}(\mathbb{R}^2)} + \|\Gamma^j \mathbf{w}^0\|_{H^1(\mathbb{R}^2)} < \infty \quad (3.33)$$

Let

$$\epsilon = \sum_{\substack{|j| \leq l+3 \\ \partial = \partial_\alpha, \partial_\beta}} \|\Gamma^j |D|^{1/2} y^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j \partial y^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j \mathbf{u}^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j \mathbf{w}^0\|_{L^2(\mathbb{R}^2)}. \quad (3.34)$$

Theorem 3.7 (3d Theorem, c.f. [42]). *There exists $\epsilon_0 > 0$, such that for $0 < \epsilon \leq \epsilon_0$, the initial value problem (2.38)-(3.32) has a unique classical solution globally in time. For each time $0 \leq t < \infty$, the interface is a graph, the solution has the same regularity as the initial data and remains small. Moreover the L^∞ norm of the steepness and the acceleration on the interface, the derivative of the velocity on the interface decay at rate $\frac{1}{t}$.*

Remark 3.8. 1. The existence time in Theorem 3.6 is extended to global in [22, 3, 20] by further understanding the nature of the cubic nonlinearities in equation (1.1).

2. In [41] a quick dispersive estimate was proved by the vector field method. An in-depth analysis on the dispersion of the linear water wave operator $\partial_t^2 + |D|$ is performed in [6]; a threshold is found so that when the amount of small frequency waves in the initial data is below the threshold, the solution of the linear water wave equation (3.2) decays with rate $1/\sqrt{t}$, while above the threshold, there is a growth factor in the linear solution. Consequences in the nonlinear setting remain to be understood.

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APPENDIX A. BASIC ANALYSIS PREPARATIONS

In this section we present some inequalities and identities on \mathbb{R} that are used to guide the derivation of the quasilinear structure of the 2d water waves. Corresponding inequalities and identities are available in all dimensions \mathbb{R}^d , we refer the reader to [40, 42] for those.

Let $H \in C^1(\mathbb{R}; \mathbb{R}^l)$, $A_i \in C^1(\mathbb{R})$, $i = 1, \dots, m$, $F \in C^\infty(\mathbb{R}^l)$. Define

$$C_1(H, A, f)(x) = p.v. \int F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^{m+1}} f(y) dy. \quad (\text{A.1})$$

Proposition A.1. *There exist constants $c_1 = c_1(F, \|H'\|_{L^\infty})$, $c_2 = c_2(F, \|H'\|_{L^\infty})$, such that*

1. For any $f \in L^2$, $A'_i \in L^\infty$, $1 \leq i \leq m$,

$$\|C_1(H, A, f)\|_{L^2} \leq c_1 \|A'_1\|_{L^\infty} \dots \|A'_m\|_{L^\infty} \|f\|_{L^2}. \quad (\text{A.2})$$

2. For any $f \in L^\infty$, $A'_i \in L^\infty$, $2 \leq i \leq m$, $A'_1 \in L^2$,

$$\|C_1(H, A, f)\|_{L^2} \leq c_2 \|A'_1\|_{L^2} \|A'_2\|_{L^\infty} \dots \|A'_m\|_{L^\infty} \|f\|_{L^\infty}. \quad (\text{A.3})$$

(A.2) is a result of Coifman, McIntosh and Meyer [12]. (A.3) is a consequence of the Tb Theorem, a proof is given in [41].

Let H, A_i, F satisfy the same assumptions as in (A.1). Define

$$C_2(H, A, f)(x) = \int F\left(\frac{H(x) - H(y)}{x - y}\right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^m} \partial_y f(y) dy. \quad (\text{A.4})$$

We have the following inequalities.

Proposition A.2. *There exist constants $c_3 = c_3(F, \|H'\|_{L^\infty})$, $c_4 = c_4(F, \|H'\|_{L^\infty})$, such that*

1. For any $f \in L^2$, $A'_i \in L^\infty$, $1 \leq i \leq m$,

$$\|C_2(H, A, f)\|_{L^2} \leq c_3 \|A'_1\|_{L^\infty} \dots \|A'_m\|_{L^\infty} \|f\|_{L^2}. \quad (\text{A.5})$$

2. For any $f \in L^\infty$, $A'_i \in L^\infty$, $2 \leq i \leq m$, $A'_1 \in L^2$,

$$\|C_2(H, A, f)\|_{L^2} \leq c_4 \|A'_1\|_{L^2} \|A'_2\|_{L^\infty} \dots \|A'_m\|_{L^\infty} \|f\|_{L^\infty}. \quad (\text{A.6})$$

Using integration by parts, the operator $C_2(H, A, f)$ can be easily converted into a sum of operators of the form $C_1(H, A, f)$. (A.5) and (A.6) follow from (A.2) and (A.3).

The following identities are useful to compute the derivatives of the integral operators.

Let

$$\mathbf{K}f(x, t) = p.v. \int K(x, y; t) f(y, t) dy$$

where either K or $(x - y)K(x, y; t)$ is continuous and bounded, and K is smooth away from the diagonal $\Delta = \{(x, y) \mid x = y\}$. We have for $f \in C^1(\mathbb{R}^{1+1})$ vanishing as $|x| \rightarrow \infty$,

$$\begin{aligned} [\partial_t, \mathbf{K}]f(x, t) &= \int \partial_t K(x, y; t) f(y, t) dy \\ [\partial_x, \mathbf{K}]f(x, t) &= \int (\partial_x + \partial_y) K(x, y; t) f(y, t) dy \end{aligned} \quad (\text{A.7})$$

The first identity in (A.7) is straightforward, the second is obtained by integration by parts.

Using (A.7), $\partial_{\alpha'}^s [f, \mathbb{H}] \partial_{\alpha'} g$ equals to the sum of $[\partial_{\alpha'}^s f, \mathbb{H}] \partial_{\alpha'} g$,²⁶ $[f, \mathbb{H}] \partial_{\alpha'}^{s+1} g$ and some intermediate terms. By Propositions A.1 and A.2 and the Sobolev embedding, $[f, \mathbb{H}] \partial_{\alpha'} g$ has the same regularity as f and g .

²⁶Expand $[\partial_{\alpha'}^s f, \mathbb{H}] \partial_{\alpha'} g = \partial_{\alpha'}^s f \mathbb{H}(\partial_{\alpha'} g) - \mathbb{H}(\partial_{\alpha'}^s f \partial_{\alpha'} g)$ and estimate term by term.

When the Riemann mapping is not used, typically we work in the regime where the interface is chord-arc, that is there are constants $\mu_1 > 0$ and $\mu_2 > 0$, such that

$$\mu_1|\alpha - \beta| \leq |z(\alpha, t) - z(\beta, t)| \leq \mu_2|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}.$$

By (A.7), derivatives of $[f, \mathfrak{H}] \frac{\partial_{\alpha} g}{z_{\alpha}}$ are of types (A.1) and (A.4), where in (A.1) or (A.4) $F \in C_0^{\infty}(\mathbb{C})$ is chosen to be $F(z) = \frac{1}{z^m}$ for $\mu_1 \leq |z| \leq \mu_2$ and $H(\alpha) = z(\alpha, t)$. An application of Proposition A.1 and A.2 and Sobolev embedding shows that $[f, \mathfrak{H}] \frac{\partial_{\alpha} g}{z_{\alpha}}$ has the same regularity as f , g and z .

APPENDIX B. COMMUTATOR IDENTITIES

We need the following identities in our derivation of the quasilinear structure (2.27) and Propositions 3.1 and 3.2. Let $z = z(\cdot, t)$ define a non-selfintersecting curve, and \mathfrak{H} be the Hilbert transform as defined in (2.2).

Lemma B.1 (Lemma 2.1 of [41]). *Assume that $z_t, z_{\alpha} - 1 \in C^1([0, T], H^1(\mathbb{R}))$, $f \in C^1(\mathbb{R} \times (0, T))$ satisfies $f_{\alpha}(\alpha, t) \rightarrow 0$, as $|\alpha| \rightarrow \infty$. We have*

$$\begin{aligned} [\partial_t, \mathfrak{H}]f &= [z_t, \mathfrak{H}] \frac{f_{\alpha}}{z_{\alpha}} \\ [\partial_t^2, \mathfrak{H}]f &= [z_{tt}, \mathfrak{H}] \frac{f_{\alpha}}{z_{\alpha}} + 2[z_t, \mathfrak{H}] \frac{f_{t\alpha}}{z_{\alpha}} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 f_{\beta} d\beta \\ [\mathfrak{a}\partial_{\alpha}, \mathfrak{H}]f &= [\mathfrak{a}z_{\alpha}, \mathfrak{H}] \frac{f_{\alpha}}{z_{\alpha}}, \quad \partial_{\alpha} \mathfrak{H}f = z_{\alpha} \mathfrak{H} \frac{f_{\alpha}}{z_{\alpha}} \\ [\partial_t^2 - i\mathfrak{a}\partial_{\alpha}, \mathfrak{H}]f &= 2[z_t, \mathfrak{H}] \frac{f_{t\alpha}}{z_{\alpha}} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 f_{\beta} d\beta \\ (I - \mathfrak{H})(-i\mathfrak{a}t\bar{z}_{\alpha}) &= 2[z_{tt}, \mathfrak{H}] \frac{\bar{z}_{t\alpha}}{z_{\alpha}} + 2[z_t, \mathfrak{H}] \frac{\bar{z}_{tt\alpha}}{z_{\alpha}} - \frac{1}{\pi i} \int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 \bar{z}_{t\beta} d\beta \end{aligned} \tag{B.1}$$

The proof of Lemma B.1 is straightforward by integration by parts. We omit the proof.

Let $\Omega \subset \mathbb{C}$ be a domain with boundary $\Sigma : z = z(\alpha)$, $\alpha \in I$ oriented clockwise. Let \mathfrak{H} be defined by (2.2).

Lemma B.2 (Lemma 2.2 of [41]). *1. If $f = \mathfrak{H}f$, $g = \mathfrak{H}g$, then $[f, \mathfrak{H}]g = 0$.*

2. For any $f, g \in L^2$, we have $[f, \mathfrak{H}]\mathfrak{H}g = -[\mathfrak{H}f, \mathfrak{H}]g$.

The first statement follows from the fact that the product of holomorphic functions is holomorphic. Observe that the first statement also holds for $f = -\mathfrak{H}f$ and $g = -\mathfrak{H}g$, the second statement follows from the first by applying the first identity to $(I \pm \mathfrak{H})f$ and $(I \pm \mathfrak{H})g$.

We do not give the commutator identities for 3d, but refer the reader to [40, 42] for details.

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