THE QUARTIC INTEGRABILITY AND LONG TIME EXISTENCE OF STEEP WATER WAVES IN 2D

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ABSTRACT. It is known since the work of Dyachenko & Zakharov [52] that for the weakly nonlinear 2d infinite depth water waves, there are no 3-wave interactions and all of the 4-wave interaction coefficients vanish on the resonant manifold. In this paper we study this partial integrability from a different point of view. We construct a sequence of energy functionals $\mathfrak{E}_j(t)$, directly in the physical space, that involves material derivatives of order j of the solutions for the 2d water wave equation, so that $\frac{d}{dt}\mathfrak{E}_j(t)$ is quintic or higher order. We show that if some scaling invariant norm, and a norm involving one spacial derivative above the scaling of the initial data are of size no more than ε, then the lifespan of the solution for the 2d water wave equation is at least of order $O(\varepsilon^{-3})$, and the solution remains as regular as the initial data during this time. If only the scaling invariant norm of the data is of size ε, then the lifespan of the solution is at least of order $O(\varepsilon^{-5/2})$. Our long time existence results do not impose size restrictions on the slope of the initial interface and the magnitude of the initial velocity, they allow the interface to have arbitrary large steepnesses and initial velocities to have arbitrary large magnitudes.

1. INTRODUCTION

A class of water wave problems concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\partial\Omega(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathbf{k} - \nabla P & \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{on } \Omega(t),\ t \geq 0, \\ P = 0, & \text{on } \partial\Omega(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \partial\Omega(t)), \end{cases} \tag{1.1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. There is an important condition for these problems:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0 \tag{1.2}$$

pointwise on the interface, where $\mathbf{n}$ is the outward unit normal to the fluid interface $\partial\Omega(t)$ [41]; it is well known that when surface tension is neglected and the Taylor sign condition (1.2) fails, the water wave motion can be subject to the Taylor instability [41, 13, 10, 23].

The study of water waves dates back centuries to Newton [34], Stokes [40], Levi-Civita [31], and G.I. Taylor [41]. Nalimov [33], Yosihara [50] and Craig [19] obtained early local in time existence and uniqueness results for the 2d water wave equation (1.1) for small and smooth initial data. In [44, 45], the author showed that for dimensions $n \geq 2$, the strong Taylor sign condition

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{1.3}$$

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always holds for the infinite depth water wave problem (1.1), as long as the interface is in $C^{1+\gamma}$, $\gamma > 0$; and the initial value problem of equation (1.1) is locally well-posed in Sobolev spaces H^s, $s \geq 4$ for arbitrary given data. Since then, local wellposedness for water waves with additional effects such as the surface tension, bottom and non-zero vorticity, under the assumption (1.3),[1] were obtained, c.f. [9, 14, 15, 27, 30, 32, 35, 37, 54]; local wellposedness of (1.1) in low regularity Sobolev spaces, cf. [6, 7, 25, 1], and in a regime allowing for non-C^1 interfaces, cf. [29, 49, 3, 4, 5] were proved. Moreover the author [46, 47], Germain, Masmoudi & Shatah [24], Ionescu & Pusateri [28] and Alazard & Delort [8] obtained almost global and global existence for two and three dimensional water wave equation (1.1) for small, smooth and localized data; see [25, 26, 22, 42, 43, 12, 38, 39, 55, 2] for some additional results.

The study of the 2d water wave equation (1.1) in the Hamiltonian point of view began in [51], where Zakharov discovered that the 2d equation (1.1) can be written as a Hamiltonian system. In [52] Dyachenko & Zakharov showed that there are no three-wave interactions in the Hamiltonian 2d water wave equation (1.1) and all of the four-wave interaction coefficients vanish on the resonant manifold. Dyachenko & Zakharov [52] and Craig & Wolfolk [20] derived a formal [2] symplectic transformation that maps the Hamiltonian system of the 2d water waves to its Birkhoff normal form of order 4; and mapping properties of the transformation were studied in [21]. Building on [52, 20, 21], Berti, Feola & Pusateri [11] gave a rigorous construction of a bounded and invertible (non-symplectic) transformation in a neighborhood of the origin in phase space, mapping the 2d water wave equation (1.1) to its Birkhoff normal form up to order 4, and showed that for sufficiently small and smooth periodic initial data of size ε, the 2d water wave equation (1.1) is solvable for time of order $O(\varepsilon^{-3})$. The global and almost global existence results in [46, 47, 24, 28, 8, 25, 26, 42, 2] are consequences of the order 3 integrability[3] of the water wave system and the time decay properties of the solutions with localized initial data, the assumption that the data is sufficiently localized is crucial for these global existence results. [11] is the first rigorous long time existence result taking full advantage of the quartic integrability of the 2d water wave equation. The transformation in [11] is constructed via composing several paradifferential flow conjugations. The final resonant Poincaré-Birkhoff normal form system constructed in [11] is not a priori explicit, and an important step in [11] is a normal form uniqueness argument that allows the authors to identify the Poincaré-Birkhoff normal form system of [11] with the Birkhoff normal form system constructed in [52, 20], up to degree 4 of homogeneity.

In this paper, we study the quartic integrability of the 2d water wave equation (1.1) from a different point of view. We construct, directly in the physical space, explicit energy functionals that exhibit cancellations up to order 4 as well as important algebraic structures that allow us to prove appropriate boundedness properties. A consequence is that if some scaling invariant norm and a norm involving one order spacial derivative above the scaling of the initial data are of size no more than ε, then the lifespan of the solution for the 2d water wave equation (1.1) is at least of order $O(\varepsilon^{-3})$; if only the scaling invariant norm of the initial data is of size ε, then the solution will exist up to order $O(\varepsilon^{-5/2})$.[4] Our long time existence results do not impose size restrictions on the slope of the initial interface and the magnitude of the initial velocity, the initial interface can have arbitrary large steepness, and the magnitude of the initial velocity can be arbitrarily

[1] When there is surface tension, or bottom, or vorticity, (1.3) does not always hold, it needs to be assumed.

[2] That is, it is unbounded and non-invertible.

[3] Here we say an equation is integrable of order n if all the nonlinearities in the equation of order less than n can be removed by some type of normal form transformation.

[4] No dispersive properties of equation (1.1) is used in the proof of our long time existence results, Theorem 3.1.

large. To the best of our knowledge, this is the first long time existence result assuming smallness only on a scale invariant quantity at the initial time, and it is the first allowing steep initial interfaces. [5]

The work here is a continuation of our work in [46, 47, 49], and we expect that the results in Section 2 on the structure of the 2d water wave equation (1.1) will have other consequences. We give rigorous statements of our main results in §2 and §3.

1.1. **Outline of the paper.** We present our results on the algebraic aspect of the water wave equation, namely the quartic integrability, and their proofs in §2. In §3 we state our long time existence Theorem, its proof is given in §4. While notations and conventions will be introduced throughout the paper, a complete list can be found in Appendix A. Some of the basic equations and formulas derived in our earlier works [44, 46, 49], as well as some additional identities that are used in the derivations in this paper are collected in Appendix B. Appendix C contains the inequalities that are used for our proofs. In Appendix D we summarize the estimates obtained in §4.3.1 and §4.3.2 for easy referencing.

1.2. **Conventions.** We consider solutions of 2d the water wave equation (1.1) in the setting where the fluid domain $\Omega(t)$ is simply connected in $\mathbb{R}^2$, with the free interface $\partial\Omega(t) := \partial\Omega(t)$ being a non-self-intersecting curve,

$$\mathbf{v}(z,t) \to 0, \qquad \text{as } |z| \to \infty$$

and the interface $\partial\Omega(t)$ tending to horizontal lines at infinity. We will primarily use the Riemann mapping variable in this paper.

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2. The main results on the structure of the water wave equation

In this section we construct a sequence of energy functionals $\mathfrak{E}_j(t)$ which involves material derivatives of order j of the solutions, such that $\frac{d}{dt}\mathfrak{E}_j(t)$ is quintic.[6] The construction is based on a series of observations, given in Lemma 2.1 through Proposition 2.6, and equations (2.20)-(2.21), (2.31)-(2.32). One of the consequences, presented in §3, of the results in this section is the long time existence of solutions for the 2d water wave equation (1.1).

Let $z = x+iy = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be the interface $\partial\Omega(t)$ in Lagrangian coordinate α, so $z_t(\alpha,t) = \mathbf{v}(z(\alpha,t),t)$ is the velocity of the fluid particle on the interface, and $z_{tt}(\alpha,t) = (\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v})(z(\alpha,t),t)$ is the acceleration. Let $\Phi : \Omega(t) \to \mathscr{P}_-$ be the Riemann mapping from the fluid domain $\Omega(t)$ to the lower half plane $\mathscr{P}_-$, satisfying $\lim_{z\to\infty}\Phi_z(z) = 1$ and $\Phi(z(0,t),t) = 0$; let $\mathsf{h}(\alpha,t) := \Phi(z(\alpha,t),t)$, $b(\alpha',t) := \mathsf{h}_t \circ \mathsf{h}^{-1}(\alpha',t)$. And let $Z(\alpha',t) = X(\alpha',t) + iY(\alpha',t) := z(\mathsf{h}^{-1}(\alpha',t),t)$, $Z_t(\alpha',t) := z_t(\mathsf{h}^{-1}(\alpha',t),t)$ and $Z_{tt}(\alpha',t) := z_{tt}(\mathsf{h}^{-1}(\alpha',t),t)$ be the position, velocity and acceleration of the interface in the Riemann mapping variable α'. We know

[5] In all these works [46, 47, 24, 28, 8, 25, 1, 26, 42, 11, 55, 2], norms involving derivatives both above and below scaling are assumed small. Such smallness can not be preserved after rescaling. Also in all works [46, 47, 24, 28, 8, 25, 1, 26, 42, 11, 55, 2], the slope of the initial interface is assumed small.

[6] The "quintic" in §2 means it is a finite sum of terms homogeneous of degree 5 or higher of the unknown functions Z_t, $\frac{1}{Z_{,\alpha'}} - 1$ and their derivatives. In §3, §4 we will show that after some further modifications it is in fact $O(\epsilon^5)$ with ϵ as defined in (3.1)-(4.18).

$D_t := \partial_t + b\partial_{\alpha'}$ is the material derivative; for any function $f = f(\alpha, t)$, $D_t(f \circ \mathsf{h}^{-1}) = (\partial_t f) \circ \mathsf{h}^{-1}$, so $Z_t = D_t Z$ and $Z_{tt} = D_t Z_t$. And as derived in earlier work, cf. [44, 48] or §2.2 of [49], we have

$$b = \mathrm{Re}(I - \mathbb{H}) \left(\frac{Z_t}{Z_{,\alpha'}} \right) = \mathbb{P}_H \left(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} \right) + \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \right). \tag{2.1}$$

Since the fluid is incompressible and irrotational, so $\mathbf{v} = \nabla\varphi$, where the velocity potential φ satisfies the Bernoulli equation in the fluid domain $\Omega(t)$:

$$\begin{cases} \varphi_t + \frac{1}{2}|\nabla\varphi|^2 + P + y = 0, \qquad \Delta\varphi = 0, \qquad \text{in } \Omega(t), \\ P = 0, \qquad \text{on } \partial\Omega(t). \end{cases} \tag{2.2}$$

Let $\psi(\alpha, t) := \varphi(z(\alpha, t), t)$ be the trace of the velocity potential on the interface. As was shown in Proposition 2.3 of [46], the quantity $(I - \mathfrak{H})\psi$, where $\mathfrak{H}$ is the Hilbert transform associated with the interface z, after a suitable change of coordinates, satisfies an equation with no quadratic nonlinear terms.[7] This motivates us to begin our analysis with the quantity

$$Q := (I + \mathbb{H})(\psi \circ \mathsf{h}^{-1}) = 2\,\mathbb{P}_H(\psi \circ \mathsf{h}^{-1}).$$

Lemma 2.1. *We have*[8]

$$D_t Q = i(Z - \alpha') + \mathbb{P}_A \left(|Z_t|^2 \right), \tag{2.3}$$

$$D_t \mathbb{P}_H D_t Q + i \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} Q = i\,\mathbb{P}_A \left(Z_t \left(1 - \frac{1}{Z_{,\alpha'}} \right) + \overline{Z}_t \left(\frac{1}{\overline{Z}_{,\alpha'}} - 1 \right) \right). \tag{2.4}$$

Proof. By chain rule, (2.2) and the fact that $\nabla\varphi(z, t) = z_t$ on the interface z, we have

$$\begin{cases} \psi_t = \varphi_t + \nabla\varphi \cdot z_t = -y + \frac{1}{2}|z_t|^2, \\ \psi_\alpha = \nabla\varphi \cdot z_\alpha = z_t \cdot z_\alpha. \end{cases} \tag{2.5}$$

Changing to the Riemann mapping variable α' in the second equation yields $\partial_{\alpha'}(\psi \circ \mathsf{h}^{-1}) = Z_t \cdot Z_{,\alpha'} = \mathrm{Re}(\overline{Z}_t Z_{,\alpha'})$, which in turn gives

$$\partial_{\alpha'} Q = \overline{Z}_t Z_{,\alpha'}, \tag{2.6}$$

because the holomorphic quantities $\partial_{\alpha'} Q$ and $\overline{Z}_t Z_{,\alpha'}$ have the same real parts. A change of variables in the first equation in (2.5) yields $D_t(\psi \circ \mathsf{h}^{-1}) = -Y + \frac{1}{2}|Z_t|^2$.

We compute, by first commuting D_t with $\mathbb{H}$, then applying (2.1), Proposition B.1, (A.3) and (2.6), that

$$\begin{aligned} D_t Q &= (I + \mathbb{H}) D_t (\psi \circ \mathsf{h}^{-1}) + [D_t, \mathbb{H}](\psi \circ \mathsf{h}^{-1}) \\ &= (I + \mathbb{H})(-Y + \frac{1}{2}|Z_t|^2) + \left[\frac{Z_t}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \mathbb{P}_H (\psi \circ \mathsf{h}^{-1}) + \left[\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \mathbb{P}_A (\psi \circ \mathsf{h}^{-1}) \\ &= (I + \mathbb{H})(-Y + \frac{1}{2}|Z_t|^2) + \mathbb{P}_A \left(\frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} Q \right) - \mathbb{P}_H \left(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} \overline{Q} \right) \\ &= i(Z - \alpha') + \mathbb{P}_A \left(|Z_t|^2 \right), \end{aligned}$$

where $(I + \mathbb{H})(-Y) = i(Z - \alpha')$ because both of the holomorphic quantities $(I + \mathbb{H})(-Y)$ and $i(Z - \alpha')$ have the same real parts. This gives (2.3).[9]

[7] Observe that the basic energy used in [46] (cf. Lemma 4.1 of [46]) is in fact coordinate invariant, this means the second part of the transformation in [46]: the change of coordinates is less important if we directly work on energy functionals.

[8] Applying $D_{\alpha'}$ to both sides of (2.3) gives the first equation in the interface system (B.1).

[9] Equation (2.3) was also derived in [25] using a slightly different approach.

Observe that $\mathbb{P}_H D_t Q = i(Z-\alpha')$ by (2.3). We further compute, by (2.1) and the fact that $\mathbb{P}_A Z_t = Z_t$ and $\mathbb{P}_A \overline{Z}_t = 0$, that

$$D_t\{i(Z-\alpha')\} = i(Z_t - b) = -i\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} + i\,\mathbb{P}_A\left(Z_t\left(1-\frac{1}{Z_{,\alpha'}}\right) + \overline{Z}_t\left(\frac{1}{\overline{Z}_{,\alpha'}}-1\right)\right).$$

Observe that $i\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} = i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}Q$ by (2.6). This gives (2.4). □

Let

$$\Theta^{(0)} := Q, \qquad \Theta^{(j)} := (\mathbb{P}_H D_t)^j Q, \qquad \text{and} \tag{2.7}$$

$$D_t\mathbb{P}_H D_t \Theta^{(j)} + i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta^{(j)} =: G^{(j)}. \tag{2.8}$$

We know $\mathbb{P}_H G^{(0)} = 0$ by (2.4). We would like to find a formula for $\mathbb{P}_H G^{(j)}$. To this end we derive a recursive relation, which in turn gives the formula for $\mathbb{P}_H G^{(j)}$.

Proposition 2.2. *1. Let Θ be holomorphic, i.e. $\mathbb{P}_A\Theta = 0$, and $\Theta_1 = \mathbb{P}_H D_t\Theta$. Assume that*

$$D_t\mathbb{P}_H D_t\Theta + i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta = G, \qquad \textit{and} \qquad D_t\mathbb{P}_H D_t\Theta_1 + i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta_1 = G_1. \tag{2.9}$$

Then

$$\mathbb{P}_H G_1 - \mathbb{P}_H D_t \mathbb{P}_H G = \frac{1}{2}\mathbb{P}_H\left\{\frac{1}{\overline{Z}_{,\alpha'}}\left(< \overline{Z}_t, i\frac{1}{\overline{Z}_{,\alpha'}}, D_{\alpha'}\Theta > + < -i\frac{1}{Z_{,\alpha'}}, Z_t, D_{\alpha'}\Theta >\right)\right\}, \tag{2.10}$$

where the cubic-form $< \cdot,\cdot,\cdot >$ is defined by

$$< f, g, h > := \frac{1}{\pi i}\int \frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))(h(\alpha')-h(\beta'))}{(\alpha'-\beta')^2}\,d\beta'.$$

2. We have

$$\mathbb{P}_H(G^{(j)}) = \sum_{l=0}^{j-1}(\mathbb{P}_H D_t)^l\left(\mathbb{P}_H(G^{(j-l)}) - \mathbb{P}_H D_t \mathbb{P}_H(G^{(j-l-1)})\right), \tag{2.11}$$

where $\mathbb{P}_H(G^{(j-l)}) - \mathbb{P}_H D_t\mathbb{P}_H(G^{(j-l-1)})$ is given by (2.10) with $\Theta = \Theta^{(j-l-1)}$.

Proof. Let $F = \mathbb{P}_A D_t\Theta$ and $F_1 = \mathbb{P}_A D_t \Theta_1$. We know by (2.1) and the identity (A.4) that

$$F = \mathbb{P}_A D_t\Theta = \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}\Theta\right). \tag{2.12}$$

It is easy to check, by the definitions and the decomposition identity (A.3) that

$$D_t\mathbb{P}_H G - G_1 = -D_t\mathbb{P}_A G + D_t F_1 + i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}F + \left[D_t, i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right]\Theta;$$

using the definitions again yields

$$\mathbb{P}_A G = F_1 + \mathbb{P}_A\left(i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right),$$

so

$$D_t\mathbb{P}_H G - G_1 = -D_t\mathbb{P}_A\left(i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right) + i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}F + \left[D_t, i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right]\Theta;$$

this gives, by (2.1), (A.4), (2.12) and (B.20), that

$$\begin{aligned}\mathbb{P}_H(D_t\mathbb{P}_H G - G_1) = {}& -\mathbb{P}_H\left(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_A\left(i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right)\right)\\ &+ \mathbb{P}_H\left(i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}\Theta\right)\right) + i\,\mathbb{P}_H\left(\frac{b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right).\end{aligned} \tag{2.13}$$

Now by identity (A.4) and the definition of $D_{\alpha'}\Theta$ we can rewrite the first two terms in (2.13) as

$$-\mathbb{P}_H\left(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_A\left(i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right)\right)=-\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\mathbb{P}_H\left(\overline{Z}_t\partial_{\alpha'}\mathbb{P}_A\left(i\,\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\Theta\right)\right)\right),$$
$$\mathbb{P}_H\left(i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}\Theta\right)\right)=\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\mathbb{P}_H\left(i\,\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_A\left(Z_tD_{\alpha'}\Theta\right)\right)\right).$$

We use (B.11) and the fact that $\overline{Z}_t$ and $\frac{1}{Z_{,\alpha'}}$ are holomorphic to further rewrite

$$-\mathbb{P}_H\left(\overline{Z}_t\partial_{\alpha'}\mathbb{P}_A\left(i\,\frac{1}{\overline{Z}_{,\alpha'}}D_{\alpha'}\Theta\right)\right)=\frac{1}{2}\mathbb{P}_H\left[\overline{Z}_t,i\,\frac{1}{\overline{Z}_{,\alpha'}};D_{\alpha'}\Theta\right]$$

and

$$\mathbb{P}_H\left(i\,\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_A\left(Z_tD_{\alpha'}\Theta\right)\right)=\frac{1}{2}\mathbb{P}_H\left[-i\,\frac{1}{Z_{,\alpha'}},Z_t;D_{\alpha'}\Theta\right].$$

Replacing $b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t$ in (2.13) by (B.7) and using part 2. of Proposition B.1 gives (2.10).

It is easy to check (2.11) using the fact that $\mathbb{P}_HG^{(0)}=0$. □

We next present a new energy identity. The basic energy form, such as $\int|\partial_t\theta|^2+\mathfrak{a}\nabla_n\theta\,\overline{\theta}\,d\alpha$, is commonly used in the proof of the local well-posedness of the water wave equations. In [46] we found that the energy form

$$\int\frac{1}{\mathfrak{a}}|\partial_t\theta|^2+i\partial_\alpha\theta\,\overline{\theta}\,d\alpha,$$

with the coefficient $\mathfrak{a}$ moved to the first term and the Dirichlet-Neumann operator ∇_n replaced by $i\partial_\alpha$ was advantageous in the study of the quadratic and cubic cancellations in the energy functionals and proving long time existence, see Lemma 4.1 of [46]. In the following Proposition we introduce yet another basic energy form. We will see that this new energy form makes it easier for us to find the quartic correcting functionals to cancel out the quartic terms in the time derivatives of the basic energies.

Proposition 2.3. *Let Θ_1, Θ_2 be holomorphic, i.e. $\mathbb{P}_A\Theta_1=\mathbb{P}_A\Theta_2=0$, smooth and decay fast at infinity. Define*

$$E(t)=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta_2\overline{D_t\Theta_1}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta_1\overline{D_t\Theta_2}\,d\alpha'\right).\tag{2.14}$$

Then

$$E'(t)=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta_2\overline{(\mathbb{P}_HG_1)}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta_1\overline{(\mathbb{P}_HG_2)}\,d\alpha'\right).\tag{2.15}$$

where $G_k:=D_t\mathbb{P}_HD_t\Theta_k+i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta_k$, for $k=1,2$.

Proof. We use (B.12) to compute $E'(t)$. We have, after applying (B.18) and cancelling out equal terms,[10]

$$\begin{aligned}\frac{d}{dt}E(t)&=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta_2\overline{D_t^2\Theta_1}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta_1\overline{D_t^2\Theta_2}\,d\alpha'\right)\\&=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta_2\overline{(D_t^2\Theta_1+i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta_1)}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta_1\overline{(D_t^2\Theta_2+i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta_2)\,d\alpha'}\right);\end{aligned}\tag{2.16}$$

it is clear that the terms inserted in the second step above sum up to zero. We now want to show

$$\operatorname{Re}\int i\,\partial_{\alpha'}\Theta_2\overline{D_t\mathbb{P}_AD_t\Theta_1}\,d\alpha'=\operatorname{Re}\int i\,\partial_{\alpha'}\Theta_1\overline{D_t\mathbb{P}_AD_t\Theta_2}\,d\alpha'.\tag{2.17}$$

[10]Observe that (2.16) holds without the assumption that Θ_1, Θ_2 are holomorphic.

We begin with the term on the left hand side. Using Cauchy integral formula to insert a $\mathbb{P}_H$ gives

$$\int i\,\partial_{\alpha'}\Theta_2\overline{D_t\mathbb{P}_AD_t\Theta_1}\,d\alpha'=\int i\,\partial_{\alpha'}\Theta_2\overline{\mathbb{P}_HD_t\mathbb{P}_AD_t\Theta_1}\,d\alpha'.$$

By (2.1) and (A.3), $\mathbb{P}_HD_t\mathbb{P}_AD_t\Theta_1=\mathbb{P}_H(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_AD_t\Theta_1)$, and using Cauchy integral formula again to remove the $\mathbb{P}_H$ and insert a $\mathbb{P}_A$, and then use (2.12), we get

$$\int i\,\partial_{\alpha'}\Theta_2\overline{\mathbb{P}_HD_t\mathbb{P}_AD_t\Theta_1}\,d\alpha'=\int i\,\partial_{\alpha'}\Theta_2\overline{\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\mathbb{P}_AD_t\Theta_1}\,d\alpha'=i\int\mathbb{P}_AD_t\Theta_2\partial_{\alpha'}\overline{\mathbb{P}_AD_t\Theta_1}\,d\alpha'. \tag{2.18}$$

Exchanging roles between Θ_1 and Θ_2 in (2.18) gives us a similar identity for the right hand term in (2.17), a further integration by parts then yields (2.17). From (2.16) and (2.17), and using Cauchy integral formula again to insert a $\mathbb{P}_H$ gives (2.15). □

Let $\Theta_1=\Theta^{(j)}$, $\Theta_2=\Theta^{(j+1)}$, and define

$$E_j(t)=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\Theta^{(j+1)}}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\Theta^{(j+2)}}\,d\alpha'\right). \tag{2.19}$$

From (2.15) we get,

$$E_j'(t)=\operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\mathbb{P}_HG^{(j)}}\,d\alpha'-\int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_HG^{(j+1)}}\,d\alpha'\right);$$

compute further by using (B.12), (2.11), (2.4) and Proposition B.4, we obtain

$$\begin{aligned}\frac{d}{dt}E_j(t)&=\frac{d}{dt}\operatorname{Re}\int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_HG^{(j)}}\,d\alpha'\\&\quad+2\operatorname{Re}\sum_{l=0}^{j-1}\left\{\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_HD_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_HG^{(j-1-l)}\right)\,d\alpha'\right\}\\&\quad+\operatorname{Re}\left\{\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H\left(G^{(j+1)}-D_t\mathbb{P}_HG^{(j)}\right)\,d\alpha'\right\}.\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}&\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_HG^{(j-1-l)}\right)\\&=\frac12\mathbb{P}_H\left\{\frac{1}{\overline{Z}_{,\alpha'}}\left(<\overline{Z}_t,i\,\frac{1}{\overline{Z}_{,\alpha'}},D_{\alpha'}\Theta^{(j-l-1)}>+<-i\,\frac{1}{Z_{,\alpha'}},Z_t,D_{\alpha'}\Theta^{(j-l-1)}>\right)\right\},\end{aligned} \tag{2.21}$$

for $l=-1,0,\dots,j-1$, by Proposition 2.2.

Observe that $\frac{d}{dt}E_j(t)$ is quartic with a few desirable features: 1. the integrands depend only on the spatial derivatives of the quantities $\overline{Z}_t$, $\frac{1}{Z_{,\alpha'}}$ and $\Theta^{(j)}$; 2. it has remarkable symmetries, with the order of derivatives evenly distributed among the factors: we note that $D_{\alpha'}\Theta^{(0)}=\overline{Z}_t$. This should allow us to derive optimal low regularity estimates. However our main interest here is to find a quartic correcting functional for $E_j(t)$, so that the time derivative of the corrected energy functional is quintic and can be controlled by the scaling invariant norm $\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|_{L^2}$ and norms involving only derivatives of orders higher than the scaling and prove long time existence of the solutions. The feature of $\frac{d}{dt}E_j(t)$ allows us to do so, thanks to a few additional observations on the structure of the water wave equation.

We note that by (B.1), $-i\,\frac{1}{Z_{,\alpha'}}+i=\overline{Z}_{tt}+\textit{quadratic}$, and by (2.6), $D_{\alpha'}\Theta^{(k)}=D_t^k\overline{Z}_t+\textit{quadratic}$. Let[11]

$$\mathfrak{D}_t:=\partial_t+b(\alpha',t)\partial_{\alpha'}+b(\beta',t)\partial_{\beta'}, \tag{2.22}$$

[11] Observe that for any function $f=f(\alpha',t)$, $\mathfrak{D}_tf=D_tf$ and $\mathcal{P}f=(D_t^2+i\,\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'})f=:\mathfrak{P}f$, where $\mathfrak{P}$ is defined in (B.3).

$$\mathcal{P} := \mathfrak{D}_t^2 + i\frac{A_1(\alpha',t)}{|Z_{,\alpha'}|^2}\partial_{\alpha'} + i\frac{A_1(\beta',t)}{|Z_{,\beta'}|^2}\partial_{\beta'}. \tag{2.23}$$

And let

$$\theta := \overline{Z}_t(\alpha',t) - \overline{Z}_t(\beta',t), \tag{2.24}$$

observe that

$$\mathfrak{D}_t^j\theta = D_t^j\overline{Z}_t(\alpha',t) - D_t^j\overline{Z}_t(\beta',t), \qquad \text{for } j \geq 0. \tag{2.25}$$

So the quartic terms in

$$2\,\mathrm{Re}\sum_{l=0}^{j-1}\left\{\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)} - D_t\mathbb{P}_H G^{(j-1-l)}\right)d\alpha'\right\}$$

consists of

$$I_{1,j} := \frac{1}{\pi}\,\mathrm{Re}\sum_{l=0}^{j-1}\iint D_t^j Z_t(\alpha',t)\frac{\mathfrak{D}_t^{l+1}\left\{\mathfrak{D}_t\left(\theta(\alpha',\beta',t)\overline{\theta(\alpha',\beta',t)}\right)\mathfrak{D}_t^{j-l-1}\theta(\alpha',\beta',t)\right\}}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'. \tag{2.26}$$

Similarly the quartic terms in $\mathrm{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H\left(G^{(j+1)} - D_t\mathbb{P}_H G^{(j)}\right)d\alpha'$ are

$$I_{2,j} := \frac{1}{2\pi}\,\mathrm{Re}\iint D_t^j Z_t(\alpha',t)\frac{\mathfrak{D}_t\left(\theta(\alpha',\beta',t)\overline{\theta(\alpha',\beta',t)}\right)\mathfrak{D}_t^j\theta(\alpha',\beta',t)}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'. \tag{2.27}$$

We know it is possible to find quartic functionals $C_{i,j}$, $i=1,2$, so that $\frac{d}{dt}C_{i,j}(t) = I_{i,j} + \textit{quintic}$, based on the following observations.

Lemma 2.4. *Assume that f and $\mathcal{G}$ are smooth and decay fast at infinity. We have*

$$\begin{aligned}
&\frac{d}{dt}\iint\frac{\overline{f}\,\mathfrak{D}_t\mathcal{G} - \mathfrak{D}_t\overline{f}\,\mathcal{G}}{(\alpha'-\beta')^2}\,d\alpha'd\beta' = \iint\frac{\overline{f}\mathcal{P}\mathcal{G}}{(\alpha'-\beta')^2}\,d\alpha'd\beta' \\
&+\iint\left(-\frac{\overline{\mathcal{P}f}\mathcal{G}}{(\alpha'-\beta')^2} + \left(b_{\alpha'} + b_{\beta'} - 2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{\overline{f}\,\mathfrak{D}_t\mathcal{G} - \mathfrak{D}_t\overline{f}\,\mathcal{G}}{(\alpha'-\beta')^2}\right)d\alpha'd\beta' \\
&+i\iint\left(\partial_{\alpha'}\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + \partial_{\beta'}\frac{A_1(\beta')}{|Z_{,\beta'}|^2} - 2\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} - \frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha'-\beta'}\right)\frac{\overline{f}\,\mathcal{G}}{(\alpha'-\beta')^2}\,d\alpha'd\beta'.
\end{aligned} \tag{2.28}$$

(2.28) follows easily from (B.13) and integration by parts. We omit the details.

Lemma 2.5. *Let $\mathcal{G} = g\overline{h}q$. Then*

$$\mathcal{P}\mathcal{G} = 2\mathfrak{D}_t\left(g\,\overline{h}\right)\mathfrak{D}_t q + 2\mathfrak{D}_t\left(g\,\mathfrak{D}_t\overline{h}\right)q + \overline{h}\,q\,\mathcal{P}g + g\,\overline{h}\,\mathcal{P}q - g\,q\,\overline{\mathcal{P}h}. \tag{2.29}$$

The verification of (2.29) is straightfoward. We omit the details.

To find the quartic functionals $C_{i,j}$, we will use Lemma 2.4 with $\mathcal{G} = g\overline{h}q$. From (2.28), (2.29), we have

$$\frac{d}{dt}\iint\frac{\overline{f}\,\mathfrak{D}_t(g\,\overline{h}\,q) - (\mathfrak{D}_t\overline{f})\,g\,\overline{h}\,q}{(\alpha'-\beta')^2}\,d\alpha'd\beta' = 2\iint\frac{\overline{f}\,\mathfrak{D}_t\left(g\,\overline{h}\right)\mathfrak{D}_t q + \overline{f}\,\mathfrak{D}_t\left(g\,\mathfrak{D}_t\overline{h}\right)q}{(\alpha'-\beta')^2}\,d\alpha'd\beta' + \textit{quintic}, \tag{2.30}$$

provided $\mathcal{P}f$, $\mathcal{P}g$, $\mathcal{P}h$ and $\mathcal{P}q$ are quadratic. Observe that on the right hand side of (2.30) the operator $\mathfrak{D}_t$ acting on q in the first term is moved to the second term, acting on $\overline{h}$. And we know $\mathcal{P}\mathfrak{D}_t^j\theta = \textit{quadratic}$ by (B.4). Instead of going through the straightforward but tedious process of finding the correcting functionals

$C_{i,j}$, we will directly give the results. Before doing so, we use Lemmas 2.4 and 2.5 to write the following equation. We have, for $j,k,i,l,m\geq 0$, and $D_t^jZ_t=D_t^jZ_t(\alpha',t)$,

$$\begin{aligned}&\frac{d}{dt}\iint \frac{\left(D_t^jZ_t\,\mathfrak{D}_t-\mathfrak{D}_t(D_t^jZ_t)\right)\mathfrak{D}_t^m\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\ &=2\iint\frac{D_t^jZ_t\,\mathfrak{D}_t^m\left\{\mathfrak{D}_t\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\right)\mathfrak{D}_t^{k+1}\theta+\mathfrak{D}_t\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^{i+1}\overline{\theta}\right)\mathfrak{D}_t^k\theta\right\}}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'+R^{(m)}_{\bar{j};l,\bar{i},k},\end{aligned}\tag{2.31}$$

where

$$\begin{aligned}R^{(m)}_{\bar{j};l,\bar{i},k}=&\iint\frac{D_t^jZ_t\,[\mathcal{P},\mathfrak{D}_t^m]\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\ &+\iint\frac{D_t^jZ_t\,\mathfrak{D}_t^m\left\{(\mathcal{P}\mathfrak{D}_t^l\theta)\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta-\mathfrak{D}_t^l\theta\,\overline{(\mathcal{P}\mathfrak{D}_t^i\theta)}\,\mathfrak{D}_t^k\theta+\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,(\mathcal{P}\mathfrak{D}_t^k\theta)\right\}}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\ &-\iint\frac{\overline{(\mathcal{P}D_t^j\overline{Z}_t)}\,\mathfrak{D}_t^m\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\ &+\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{\left(D_t^jZ_t\mathfrak{D}_t-\mathfrak{D}_t(D_t^jZ_t)\right)\mathfrak{D}_t^m\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\ &+i\iint\left(\partial_{\alpha'}\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2}+\partial_{\beta'}\frac{A_1(\beta')}{|Z_{,\beta'}|^2}-2\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2}-\frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha'-\beta'}\right)\frac{D_t^jZ_t\,\mathfrak{D}_t^m\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'.\end{aligned}\tag{2.32}$$

It is clear that $R^{(m)}_{\bar{j};l,\bar{i},k}$ is quintic.

We now give the correcting functionals. Let[12]

$$\begin{aligned}C_{1,j}=&\frac{1}{2\pi}\sum_{l=0}^{j-1}\sum_{k=0}^{l}\iint\frac{\left(D_t^jZ_t\mathfrak{D}_t-D_t^{j+1}Z_t\right)\mathfrak{D}_t^{l-k}\left(\mathfrak{D}_t^k\theta\,\overline{\theta}\,\mathfrak{D}_t^{j-l-1}\theta\right)}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\ &+\frac{1}{4\pi}\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^k\iint\frac{\left(D_t^jZ_t\mathfrak{D}_t-D_t^{j+1}Z_t\right)\mathfrak{D}_t^{1+l}\theta\,\mathfrak{D}_t^k\overline{\theta}\,\mathfrak{D}_t^{j-l-2-k}\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\ &-\frac{1}{8\pi}\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^k\iint\frac{(\theta\mathfrak{D}_t-\mathfrak{D}_t\theta)\,\mathfrak{D}_t^{j-l-1}\overline{\theta}\,\mathfrak{D}_t^{j-k-1}\theta\,\mathfrak{D}_t^{k+l+1}\overline{\theta}}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\ &+\frac{1}{2\pi}\sum_{l=0}^{j-1}\iint D_t^jZ_t\,\frac{\mathfrak{D}_t^{j-l-1}\theta\,\overline{\theta}\,\mathfrak{D}_t^{1+l}\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha',\end{aligned}\tag{2.33}$$

and

$$\begin{aligned}C_{2,j}=&\frac{1}{4\pi}\sum_{k=0}^{j-1}(-1)^k\iint\frac{\left(D_t^jZ_t\mathfrak{D}_t-D_t^{j+1}Z_t\right)\theta\,\mathfrak{D}_t^k\overline{\theta}\,\mathfrak{D}_t^{j-k-1}\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\ &+\frac{1}{4\pi}(-1)^j\iint D_t^jZ_t\frac{\theta\,\mathfrak{D}_t^j\overline{\theta}\,\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'.\end{aligned}\tag{2.34}$$

Proposition 2.6. *Let $j\geq 0$. We have*

$$I_{1,j}+I_{2,j}-\frac{d}{dt}\operatorname{Re}\left(C_{1,j}+C_{2,j}\right)=R_{IC,j}\tag{2.35}$$

[12]We define $\sum_{l=0}^{-1}=\sum_{l=0}^{-2}=0$. $D_t^jZ_t=D_t^jZ_t(\alpha',t)$, $D_t^{j+1}Z_t=D_t^{j+1}Z_t(\alpha',t)$ in (2.33) and (2.34).

where

$$
\begin{aligned}
R_{IC,j} = &-\frac{1}{2\pi}\sum_{l=0}^{j-1}\mathrm{Re}\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{D_t^jZ_t\,\mathfrak{D}_t^{j-l-1}\theta\,\overline{\theta}\,\mathfrak{D}_t^{l+1}\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\
&-\frac{1}{4\pi}\mathrm{Re}\left(2\sum_{l=0}^{j-1}\sum_{k=0}^{l}R^{(l-k)}_{\overline{j};k,\overline{0},j-l-1}+\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^k\left(R^{(0)}_{\overline{j};1+l,\overline{k},j-l-2-k}-\overline{R^{(0)}_{\overline{0};j-l-1,\overline{j-k-1},l+k+1}}\right)\right)\\
&-(-1)^j\frac{1}{4\pi}\mathrm{Re}\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{D_t^jZ_t\,\theta\,\mathfrak{D}_t^j\overline{\theta}\,\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\
&-\frac{1}{4\pi}\mathrm{Re}\sum_{k=0}^{j-1}(-1)^kR^{(0)}_{\overline{j};0,\overline{k},j-k-1}.
\end{aligned}
\tag{2.36}
$$

Observe that $\mathfrak{D}_tD_t^jZ_t = D_t^{j+1}Z_t$. It is easy to prove Proposition 2.6 by (2.28)-(2.29) or (2.31)-(2.32),[13] (B.13) and the symmetry,[14] we omit the details.

Remark 2.7. It is clear from (2.36) that the remainder $R_{IC,j}$ is quintic.

We now sum up (2.20)-(2.21)-(2.26)-(2.27) and Proposition 2.6, and present it in the following. Let

$$
\mathfrak{E}_j(t) = E_j(t) - \mathrm{Re}\left(\int i\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_HG^{(j)}}\,d\alpha' + C_{1,j}(t)+C_{2,j}(t)\right). \tag{2.38}
$$

Theorem 2.8. *We have*

$$
\frac{d}{dt}\mathfrak{E}_j(t) = \mathfrak{R}_j(t), \tag{2.39}
$$

where

$$
\begin{aligned}
\mathfrak{R}_j = &\left(2\,\mathrm{Re}\sum_{l=0}^{j-1}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_HD_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_HG^{(j-1-l)}\right)d\alpha' - I_{1,j}\right)\\
&+\left(\mathrm{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H\left(G^{(j+1)}-D_t\mathbb{P}_HG^{(j)}\right)d\alpha' - I_{2,j}\right)+R_{IC,j}
\end{aligned}
\tag{2.40}
$$

is quintic.

In particular for $j=0$, equations (2.38)-(2.40) gives

$$
\mathfrak{E}_0(t) = \int\left(i\,\partial_{\alpha'}(Z-\alpha')\overline{(Z-\alpha')}+|Z_t|^2\right)d\alpha' - \frac{1}{8\pi}\iint\frac{|Z_t(\alpha')-Z_t(\beta')|^4}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta' \tag{2.41}
$$

[13] When computating $\frac{d}{dt}C_{1,j}$, use the identity

$$
\mathfrak{D}_t^{l-k}\left(\mathfrak{D}_t(\mathfrak{D}_t^k\theta\,\overline{\theta})\mathfrak{D}_t^{j-l}\theta+\mathfrak{D}_t(\mathfrak{D}_t^k\theta\,\mathfrak{D}_t\overline{\theta})\mathfrak{D}_t^{j-l-1}\theta\right)=\mathfrak{D}_t^{l-k+1}\left(\mathfrak{D}_t(\mathfrak{D}_t^k\theta\,\overline{\theta})\mathfrak{D}_t^{j-l-1}\theta\right)-\mathfrak{D}_t^{l-k}\left(\mathfrak{D}_t(\mathfrak{D}_t^{k+1}\theta\,\overline{\theta})\mathfrak{D}_t^{j-l-1}\theta\right).
$$

Also, when using (2.28)-(2.29) or (2.31)-(2.32) to compute, apply $\mathfrak{D}_t$ to the factors in the exact order given in the right hand side of (2.29) or (2.31) to facilitate cancelations.

[14] By interchanging α' with β' and use symmetry we have

$$
\begin{aligned}
&\iint\frac{f(\alpha')(g(\alpha')-g(\beta'))(h(\alpha')-h(\beta'))(q(\alpha')-q(\beta'))}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\
&=\frac{1}{2}\iint\frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))(h(\alpha')-h(\beta'))(q(\alpha')-q(\beta'))}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'.
\end{aligned}
\tag{2.37}
$$

and

$$\begin{aligned}\frac{d}{dt}\mathfrak{E}_0(t) &= \frac{1}{2}\int i\, Z_t \left(< \overline{Z}_t, i\,\frac{1-A_1}{\overline{Z}_{\alpha'}}, \overline{Z}_t > + < -i\,\frac{1-A_1}{Z_{\alpha'}}, Z_t, \overline{Z}_t > \right) d\alpha' \\ &\quad - \frac{1}{8\pi}\iint \left(b_{\alpha'} + b_{\beta'} - 2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'} \right) \frac{|Z_t(\alpha')-Z_t(\beta')|^4}{(\alpha'-\beta')^2}\, d\alpha'\, d\beta'.\end{aligned} \tag{2.42}$$

In our proof for the long time existence result, Theorem 3.1, we will only use (2.39)-(2.40) for $1 \le j \le 4$. Similar to (2.20)-(2.21), equations (2.39)-(2.40) can be used to derive optimal low regularity results. However our focus in this paper is on the long time existence of solutions to the water wave equation.

2.1. **Scaling.** The solutions for the 2d water wave equation (B.1)-(B.2) obey the following scaling law: If $\big(\overline{Z}_t, Z\big)$ is a solution of (B.1)-(B.2), then

$$\left(\overline{Z}_t^{\lambda}, Z^{\lambda}\right) := \left(\lambda^{-1/2}\overline{Z}_t(\lambda\alpha', \lambda^{1/2}t), \lambda^{-1}Z(\lambda\alpha', \lambda^{1/2}t)\right) \tag{2.43}$$

is also a solution of the equation (B.1)-(B.2). The following norms are scaling invariant for the water wave equation (B.1)-(B.2): $\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}(\mathbb{R})}$, $\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2(\mathbb{R})}$, and $\left\|\frac{1}{Z_{,\alpha'}} - 1\right\|_{L^\infty(\mathbb{R})}$.

3. The main result on the long time existence of solutions

We are now ready to present our main result on the long time existence of solutions to the 2d water wave equation (1.1).

Let

$$L(t) = \left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_{\dot H^{1/2}(\mathbb{R})} + \left\|\overline{Z}_{t,\alpha'}(t)\right\|_{L^2(\mathbb{R})} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|_{\dot H^{1/2}(\mathbb{R})} + \left\|\partial_{\alpha'}^2\overline{Z}_t(t)\right\|_{L^2(\mathbb{R})}. \tag{3.1}$$

Theorem 3.1. *1. Let $J \ge 2$. Assume that the initial data $\left(\overline{Z}_t(0), \frac{1}{Z_{,\alpha'}}(0) - 1\right) \in \cap_{\frac12 \le s \le J} \dot H^s(\mathbb{R}) \times \dot H^{s-\frac12}(\mathbb{R})$. Then there are constants $m_0 > 0$, and $0 < \varepsilon_0 \le 1$, such that for all $0 < \varepsilon \le \varepsilon_0$, if the data satisfies*

$$L(0) \le \varepsilon, \qquad \left\|\frac{1}{Z_{,\alpha'}}(0) - 1\right\|_{L^\infty} < 1, \qquad \text{and} \qquad E_1(0)E_3(0) \le m_0^2, \tag{3.2}$$

there is a constant $\mathcal{T}_0 > 0$, depending only on m_0, so that the initial value problem for the water wave equation (B.1)-(B.2) or equivalently (1.1) has a unique classical solution for the time period $[0, \frac{\mathcal{T}_0}{\varepsilon^3}]$. During this time, the solution is as regular as the initial data, $L(t) \lesssim \varepsilon$, $\left\|\frac{1}{Z_{,\alpha'}}(t) - 1\right\|_{L^\infty} < 1$ and $E_1(t)E_3(t) \lesssim m_0^2$.

2. If instead of (3.2) the data satisfies

$$\left\|\frac{1}{Z_{,\alpha'}}(0)\right\|_{\dot H^{1/2}(\mathbb{R})} + \left\|\overline{Z}_{t,\alpha'}(0)\right\|_{L^2(\mathbb{R})} \le \varepsilon, \qquad \left\|\frac{1}{Z_{,\alpha'}}(0) - 1\right\|_{L^\infty} < 1, \quad \text{and} \quad E_1(0)E_3(0) \le m_0^2, \tag{3.3}$$

then there is a constant $\mathcal{T}_1 > 0$, depending on m_0, $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(0)\right\|_{\dot H^{1/2}}$ and $\left\|\partial_{\alpha'}^2\overline{Z}_t(0)\right\|_{L^2}$, so that the initial value problem for the water wave equation (B.1)-(B.2) or equivalently (1.1) has a unique classical solution for the time period $[0, \frac{\mathcal{T}_1}{\varepsilon^{5/2}}]$. During this time, the solution is as regular as the initial data.

Remark 3.2. Observe that $E_1(t)E_3(t)$ is scaling invariant. The sole reason for the assumption on $E_1(0)E_3(0)$ is to control the evolution of the norm $\left\|\frac{1}{Z_{,\alpha'}}(t) - 1\right\|_{L^\infty}$. We will show that for ε small enough, $E_1(t)$ controls $\left\|\frac{1}{Z_{,\alpha'}}(t) - 1\right\|_{L^2}^2 + \left\|\Theta^{(2)}(t)\right\|_{\dot H^{1/2}}^2$, and $E_3(t)$ controls $\frac{1}{4}\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}^2}(t)\right\|_{L^2}^2 + \left\|\Theta^{(4)}(t)\right\|_{\dot H^{1/2}}^2$, see Proposition 4.4;

and the quantities $\Theta^{(2)}(t)$, $\Theta^{(4)}(t)$ are mainly related to the velocity Z_t, see Step 1, Step 3 in §4.3.2. We know by Lemma 4.2 that, assuming $\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\le 1$, there is a constant $c>0$, such that

$$\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{L^\infty(\mathbb{R})}^2\le c\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{L^2(\mathbb{R})}\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}^2}\right\|_{L^2(\mathbb{R})},\tag{3.4}$$

so

$$\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty}^4\le 4c^2E_1(t)E_3(t).\tag{3.5}$$

We will show that the growth of $E_1(t)E_3(t)$ can be controlled for time of order $O(\varepsilon^{-3})$, which in turn gives control of $\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty}$ for the same time period. The constants $m_0>0$ is chosen so that $\left\|\frac{1}{Z_{,\alpha'}}(0)-1\right\|_{L^\infty}<1$.[15] Since $\left\|\frac{1}{Z_{,\alpha'}}(0)-1\right\|_{L^\infty}$ can be arbitrarily close to 1, the slope of the (initial) interface can be arbitrary large. Observe also that no assumption is imposed on the magnitude of Z_t. So the magnitude of the (initial) velocity can be arbitrary large.

Remark 3.3. Part 2 of Theorem 3.1 is a direct consequence of part 1. Observe that the rescaled data $(\overline{Z}_t^\varepsilon(0),Z^\varepsilon(0))$ satisfies

$$\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}^\varepsilon}(0)\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}^2\overline{Z}_t^\varepsilon(0)\right\|_{L^2}=\varepsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(0)\right\|_{\dot H^{1/2}}+\varepsilon\left\|\partial_{\alpha'}^2\overline{Z}_t(0)\right\|_{L^2}.$$

So by part 1, the assumption (3.3) implies that the lifespan of the rescaled solution $(\overline{Z}_t^\varepsilon,Z^\varepsilon)$ is of order $O(\varepsilon^{-3})$, therefore the solution $(\overline{Z}_t,Z)$ has a lifespan of order $O(\varepsilon^{-5/2})$. In fact, the same scaling argument shows that for data satisfying (3.3) and $(Z_t(0),\frac{1}{Z_{,\alpha'}}(0)-1)\in\cap_{\frac12\le s\le J}\dot H^s\times\dot H^{s-1/2}$, the life span of the solution is at least of order $O(\varepsilon^{-3+\frac{1}{2J-2}})$, and the solution remains as smooth as the initial data during this time.

By Remark 3.3, it suffices to prove part 1 of Theorem 3.1.

Remark 3.4. In part 1 of Theorem 3.1, the smallness assumption is only imposed on the scaling invariant quantity $\left\|\frac{1}{Z_{,\alpha'}}(0)\right\|_{\dot H^{1/2}(\mathbb{R})}+\left\|\overline{Z}_{t,\alpha'}(0)\right\|_{L^2(\mathbb{R})}$ and the quantity $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(0)\right\|_{\dot H^{1/2}(\mathbb{R})}+\left\|\partial_{\alpha'}^2\overline{Z}_t(0)\right\|_{L^2(\mathbb{R})}$ which involves one higher order derivative. In this paper we do not try to lower this higher than scaling derivative to avoid technicality. We note that in all earlier works on long time existence [46, 47, 24, 8, 28, 22, 25, 26, 42, 1, 11], smallness is assumed on norms involving derivatives both above and below the scaling, so the smallness of the quantities can not be preserved after rescaling. And in all these earlier results, smallness is assumed on the slope of the initial interface.

Remark 3.5. In the regime where $\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty}<1$, the interface $Z=Z(\alpha',t)$, $\alpha'\in\mathbb{R}$ is a graph.

4. The proof of Theorem 3.1

4.1. **Some additional quintic correcting functionals.** To prove Theorem 3.1, we need some additional quintic correcting functionals to the energies $\mathfrak{E}_j(t)$, for $j\ge 2$. The goal is to have no loss of derivatives when controlling the higher order energy growth, to not involve $\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{L^\infty(\mathbb{R})}$ and any norms of order lower than scaling when bounding the cubic growth rates for the lower order energies, and to have proper control on the growth of $E_1(t)E_3(t)$.

[15] Assuming c is the optimal constant so that (3.4) holds, we choose m_0 such that $m_0^2<\frac{1}{4c^2}$.

Let $j \geq 2$. Observe that the factors $D_t^{j+1} Z_t$ and $\overline{\mathcal{P} D_t^j \overline{Z}_t}$ in $R^{(m)}_{\bar{j},l,\bar{\imath},k}$ of (2.32) have more derivatives than are controlled by $\mathfrak{E}_j(t)$; we perform an "integration by parts" in the time variable, moving one D_t from the factors $D_t^{j+1} Z_t$ and $\overline{\mathcal{P} D_t^j \overline{Z}_t}$ onto others, resulting in the following quintic correcting functional.

Let

$$F^{(m)}_{\bar{j};l,\bar{\imath},k}(t) = \iint \frac{\overline{\mathcal{P} D_t^{j-1} \overline{Z}_t} \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' + \iint \left(b_{\alpha'} + b_{\beta'} - 2 \frac{b(\alpha') - b(\beta')}{\alpha' - \beta'} \right) \frac{D_t^j Z_t \, \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \tag{4.1}$$

Computing using (B.13) and (B.18) yields

$$\begin{aligned}
\frac{d}{dt} F^{(m)}_{\bar{j};l,\bar{\imath},k}(t) &= \iint \frac{\left(\overline{\mathcal{P} D_t^j \overline{Z}_t + [D_t, \mathcal{P}] \, D_t^{j-1} \overline{Z}_t} \right) \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \frac{\overline{\mathcal{P} D_t^{j-1} \overline{Z}_t} \mathfrak{D}_t^{m+1} (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \left(b_{\alpha'} + b_{\beta'} - 2 \frac{b(\alpha') - b(\beta')}{\alpha' - \beta'} \right) \frac{\overline{\mathcal{P} D_t^{j-1} \overline{Z}_t} \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \left(b_{\alpha'} + b_{\beta'} - 2 \frac{b(\alpha') - b(\beta')}{\alpha' - \beta'} \right) \frac{D_t^{j+1} Z_t \, \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \left(b_{\alpha'} + b_{\beta'} - 2 \frac{b(\alpha') - b(\beta')}{\alpha' - \beta'} \right) \frac{D_t^j Z_t \, \mathfrak{D}_t^{m+1} (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \left(\partial_{\alpha'} D_t b + \partial_{\beta'} D_t b - 2 \frac{D_t b(\alpha') - D_t b(\beta')}{\alpha' - \beta'} \right) \frac{D_t^j Z_t \, \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta' \\
&+ \iint \left(6 \frac{(b(\alpha') - b(\beta'))^2}{(\alpha' - \beta')^2} - 4 (b_{\alpha'} + b_{\beta'}) \frac{(b(\alpha') - b(\beta'))}{(\alpha' - \beta')} + 2 b_{\alpha'} b_{\beta'} \right) \frac{D_t^j Z_t \, \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta';
\end{aligned} \tag{4.2}$$

in the sum $R^{(m)}_{\bar{j};l,\bar{\imath},k}(t) + \frac{d}{dt} F^{(m)}_{\bar{j};l,\bar{\imath},k}(t)$ those terms containing the factors $D_t^{j+1} Z_t$ and $\mathcal{P} D_t^j \overline{Z}_t$ are all cancelled out.

We also need the following correcting term for $\frac{1}{4\pi} \overline{R^{(0)}_{\bar{0};1,\bar{1},1}}$ in $R_{IC,2}$, cf. (2.36)-(2.32). Define

$$D_2(t) = \frac{1}{4\pi} \iint (\mathbb{H} b_{\alpha'}) \frac{\overline{\theta} \, \mathfrak{D}_t \theta \mathfrak{D}_t \overline{\theta} \mathfrak{D}_t \theta}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta', \qquad D_3(t) = D_4(t) = 0; \tag{4.3}$$

we have by (B.13), (B.18),

$$\begin{aligned}
\frac{d}{dt} D_2(t) &= \frac{1}{4\pi} \iint \partial_{\alpha'} D_t \mathbb{H} b \frac{\overline{\theta} \, \mathfrak{D}_t \theta \mathfrak{D}_t \overline{\theta} \mathfrak{D}_t \theta}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta' + \frac{1}{4\pi} \iint \mathbb{H} b_{\alpha'} \frac{\mathfrak{D}_t \left(\overline{\theta} \, \mathfrak{D}_t \theta \mathfrak{D}_t \overline{\theta} \mathfrak{D}_t \theta \right)}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta' \\
&+ \frac{1}{4\pi} \iint \left(b_{\beta'} - 2 \frac{b(\alpha') - b(\beta')}{\alpha' - \beta'} \right) \mathbb{H} b_{\alpha'} \frac{\overline{\theta} \, \mathfrak{D}_t \theta \mathfrak{D}_t \overline{\theta} \mathfrak{D}_t \theta}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta'.
\end{aligned} \tag{4.4}$$

Now define

$$\lambda^j := D_{\alpha'} \Theta^{(j)}(\alpha', t) - D_{\beta'} \Theta^{(j)}(\beta', t). \tag{4.5}$$

We next introduce

$$H_j(t) := \frac{1}{4\pi} \operatorname{Re} \left(\iint \frac{\overline{\lambda^j} \, \lambda^j \, \theta \, \overline{\theta} + \overline{\lambda^j} \, \overline{\lambda^j} \, \theta \, \theta}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta' - \iint \frac{\mathfrak{D}_t^j \overline{\theta} \, \mathfrak{D}_t^j \theta \, \theta \, \overline{\theta} + \mathfrak{D}_t^j \overline{\theta} \, \mathfrak{D}_t^j \overline{\theta} \, \theta \, \theta}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta' \right) \tag{4.6}$$

to deal with the derivative loss in the term

$$2\,\mathrm{Re}\sum_{l=0}^{j-1}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_H G^{(j-1-l)}\right)\,d\alpha'-I_{1,j},$$

and to ensure the control of $E_1(t)E_3(t)$. We compute by (B.13) and using the symmetry (2.37) to get

$$\begin{aligned}\frac{d}{dt}H_j=\frac{1}{\pi}\,\mathrm{Re}\left(\iint\overline{D_{\alpha'}\Theta^{(j)}}\frac{\mathfrak{D}_t\lambda^j\,\theta\,\overline{\theta}+\mathfrak{D}_t\overline{\lambda^j}\,\theta\,\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'-\iint D_t^jZ_t\frac{\mathfrak{D}_t^{j+1}\theta\,\theta\,\overline{\theta}+\mathfrak{D}_t^{j+1}\overline{\theta}\,\theta\,\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\right)\\+R_{H,j}\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}R_{H,j}&=\frac{1}{2\pi}\,\mathrm{Re}\iint\frac{\left(|\lambda^j|^2-|\mathfrak{D}_t^j\theta|^2\right)\,\theta\,\mathfrak{D}_t\overline{\theta}+\left((\overline{\lambda^j})^2-(\mathfrak{D}_t^j\overline{\theta})^2\right)\,\theta\,\mathfrak{D}_t\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'\\&+\frac{1}{4\pi}\,\mathrm{Re}\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{\left(|\lambda^j|^2-|\mathfrak{D}_t^j\theta|^2\right)\,|\theta|^2+\left((\overline{\lambda^j})^2-(\mathfrak{D}_t^j\overline{\theta})^2\right)\,\theta^2}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'.\end{aligned}\tag{4.8}$$

Observe that in the sum $I_{1,j}+\frac{d}{dt}H_j$, the term

$$\frac{1}{\pi}\,\mathrm{Re}\iint D_t^jZ_t\frac{\mathfrak{D}_t^{j+1}\theta\,\theta\,\overline{\theta}+\mathfrak{D}_t^{j+1}\overline{\theta}\,\theta\,\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'$$

in $I_{1,j}$ is replaced with

$$\frac{1}{\pi}\,\mathrm{Re}\iint\overline{D_{\alpha'}\Theta^{(j)}}\frac{\mathfrak{D}_t\lambda^j\,\theta\,\overline{\theta}+\mathfrak{D}_t\overline{\lambda^j}\,\theta\,\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta',$$

which matches better with the corresponding term in $2\,\mathrm{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_HD_t)^j\mathbb{P}_H(G^{(1)})\,d\alpha'$, and the harmless terms in $R_{H,j}$.

Let

$$F_j(t):=\frac{1}{2\pi}\sum_{l=0}^{j-1}\sum_{k=0}^{l}F^{(l-k)}_{\overline{j};k,\overline{0},j-l-1}+\frac{1}{4\pi}\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^kF^{(0)}_{\overline{j};1+l,\overline{k},j-l-2-k}+\frac{1}{4\pi}\sum_{k=0}^{j-1}(-1)^kF^{(0)}_{\overline{j};0,\overline{k},j-k-1},\tag{4.9}$$

$$\mathcal{R}^{(m)}_{\overline{j};l,\overline{i},k}(t):=R^{(m)}_{\overline{j};l,\overline{i},k}(t)+\frac{d}{dt}F^{(m)}_{\overline{j};l,\overline{i},k}(t);\tag{4.10}$$

and define

$$\begin{aligned}\mathcal{E}_j(t):&=\mathfrak{E}_j(t)-\mathrm{Re}\,F_j(t)+\mathrm{Re}\,D_j(t)-H_j(t)\\&=E_j(t)-\mathrm{Re}\left(\int i\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_HG^{(j)}}\,d\alpha'+C_{1,j}(t)+C_{2,j}(t)+F_j(t)-D_j(t)+H_j(t)\right).\end{aligned}\tag{4.11}$$

By Theorem 2.8 and (4.10), we have

$$\frac{d}{dt}\mathcal{E}_j(t)=\mathcal{R}_j(t)\tag{4.12}$$

where

$$\begin{aligned}\mathcal{R}_j(t)=&\left(2\,\mathrm{Re}\sum_{l=0}^{j-1}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_HD_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_HG^{(j-1-l)}\right)\,d\alpha'-I_{1,j}-\frac{d}{dt}H_j\right)\\&+\left(\mathrm{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H\left(G^{(j+1)}-D_t\mathbb{P}_HG^{(j)}\right)\,d\alpha'-I_{2,j}\right)+\mathcal{R}_{IC,j}+\mathrm{Re}\,\frac{d}{dt}D_j(t),\end{aligned}\tag{4.13}$$

and

$$\mathcal{R}_{IC,j}:=R_{IC,j}-\frac{d}{dt}\,\mathrm{Re}\,F_j(t)\tag{4.14}$$

is as given in (2.36), except that the quantities $R^{(l-k)}_{\overline{j};k,\overline{0},j-l-1}$, $R^{(0)}_{\overline{j};1+l,\overline{k},j-l-2-k}$ and $R^{(0)}_{\overline{j};0,\overline{k},j-k-1}$ are replaced by $\mathcal{R}^{(l-k)}_{\overline{j};k,\overline{0},j-l-1}$, $\mathcal{R}^{(0)}_{\overline{j};1+l,\overline{k},j-l-2-k}$ and $\mathcal{R}^{(0)}_{\overline{j};0,\overline{k},j-k-1}$ respectively.

4.2. **The main components in the proof of Theorem 3.1.** We are now ready to prove Theorem 3.1. We begin with the following lemmas.

Lemma 4.1. *Let $0<\delta<1$. Assume that*

$$\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{L^\infty(\mathbb{R})}\le 1-\delta. \tag{4.15}$$

Then for any holomorphic function f, i.e. $\mathbb{P}_H f = f$, we have

$$\delta\left\|f\right\|_{\dot H^{1/2}(\mathbb{R})}\le\left\|\mathbb{P}_H\left(\frac{f}{\overline{Z}_{,\alpha'}}\right)\right\|_{\dot H^{1/2}(\mathbb{R})}. \tag{4.16}$$

Proof. We know

$$f=\mathbb{P}_H f=\mathbb{P}_H\left(\frac{f}{\overline{Z}_{,\alpha'}}\right)+\mathbb{P}_H\left\{f\left(1-\frac{1}{\overline{Z}_{,\alpha'}}\right)\right\}$$

By (C.38),

$$\left\|\mathbb{P}_H\left\{f\left(1-\frac{1}{\overline{Z}_{,\alpha'}}\right)\right\}\right\|^2_{\dot H^{1/2}}=\frac{1}{2\pi}\iint\frac{|f(\alpha')-f(\beta')|^2}{(\alpha'-\beta')^2}\left|1-\frac{1}{Z_{,\alpha'}}\right|^2 d\alpha'\,d\beta'\le(1-\delta)^2\left\|f\right\|^2_{\dot H^{1/2}}.$$

This gives (4.16). □

Lemma 4.2 (Sobolev)**.** *Assume that $\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\le 1$, and $\frac{1}{Z_{,\alpha'}}$ is sufficiently smooth. Then*

$$\left\|\frac{1}{Z_{,\alpha'}}-1\right\|^2_{L^\infty(\mathbb{R})}\le 18\left\|\frac{1}{Z_{,\alpha'}}-1\right\|_{L^2(\mathbb{R})}\left\|D_{\alpha'}\frac{1}{Z^2_{,\alpha'}}\right\|_{L^2(\mathbb{R})}. \tag{4.17}$$

Proof. (4.17) is essentially a Sobolev inequality. By Fundamental Theorem of Calculus and Cauchy-Schwarz inequality, we know for any smooth complex valued function f tending to 1 at infinity,

$$\left|\frac14\left(f^4-1\right)-\frac13\left(f^3-1\right)\right|\le\int\left|\left(f^3-f^2\right)\partial_{\alpha'}f\right|\,d\alpha'\le\left\|f-1\right\|_{L^2}\left\|f^2\partial_{\alpha'}f\right\|_{L^2}.$$

On the other hand,

$$\frac14\left(f^4-1\right)-\frac13\left(f^3-1\right)=(f-1)^2\left(\frac14 f^2+\frac16 f+\frac{1}{12}\right).$$

It is easy to check that if $|1-f|\le 1$, then $\mathrm{Re}\left(\frac14 f^2+\frac16 f+\frac1{12}\right)\ge\frac1{36}$, which implies

$$\left|\frac14\left(f^4-1\right)-\frac13\left(f^3-1\right)\right|\ge\frac{1}{36}\left|f-1\right|^2.$$

Replacing f by $\frac{1}{Z_{,\alpha'}}$ yields (4.17). □

Part 1 of Theorem 3.1 follows from the following Propositions.

Proposition 4.3. *Let $0<\delta<1$, and $T_0>0$. Assume that*

$$\sup_{t\in[0,T_0]}\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty}\le 1-\delta,\qquad\sup_{t\in[0,T_0]}L(t)\le 2\epsilon. \tag{4.18}$$

There is a constant $0<\epsilon_0(\delta)\le 1$, such that for all $0<\epsilon\le\epsilon_0(\delta)$, we have 1.

$$c_2(\delta)L(t)^2\le\sum_{j=2}^4\mathcal{E}_j(t)\le c_1L(t)^2,\qquad\forall\ t\in[0,T_0] \tag{4.19}$$

where c_1 and $c_2(\delta)$ are some positive constants. 2.

$$\sum_{j=2}^{4}\mathcal{E}_j(t)\le\sum_{j=2}^{4}\mathcal{E}_j(0)+\epsilon^5 t,\qquad \forall\ t\in[0,T_0] \tag{4.20}$$

Proposition 4.4. *Assume that the assumption of Proposition 4.3 holds. There is a constant $0<\epsilon_0(\delta)\le 1$, such that for all $0<\epsilon\le\epsilon_0(\delta)$, we have 1.*

$$(1+\delta)^{-1}E_1(t)\le\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^2}^2+\left\|\Theta^{(2)}(t)\right\|_{\dot H^{1/2}}^2\le(1+\delta)E_1(t),\qquad\forall\ t\in[0,T_0]; \tag{4.21}$$

$$(1+\delta)^{-1}E_3(t)\le\frac14\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}^2}(t)\right\|_{L^2}^2+\left\|\Theta^{(4)}(t)\right\|_{\dot H^{1/2}}^2\le(1+\delta)E_3(t),\qquad\forall\ t\in[0,T_0]; \tag{4.22}$$

$$(1+\delta)^{-1}E_1(t)E_3(t)\le\mathfrak{E}_1(t)\mathcal{E}_3(t)\le(1+\delta)E_1(t)E_3(t),\qquad\forall\ t\in[0,T_0]. \tag{4.23}$$

2. There is a constant $c_3(\delta^{-1})>0$, such that

$$\mathfrak{E}_1(t)\mathcal{E}_3(t)\le\mathfrak{E}_1(0)\mathcal{E}_3(0)e^{c_3(\delta^{-1})\epsilon^3 t},\qquad\forall\ t\in[0,T_0]. \tag{4.24}$$

Let $J\ge 2$ and assume that the initial data satisfies $\left(\overline{Z}_t(0),\frac{1}{Z_{,\alpha'}}(0)-1\right)\in\cap_{\frac12\le s\le J}\dot H^s(\mathbb{R})\times\dot H^{s-\frac12}(\mathbb{R})$. By the local existence results [44, 6, 25, 4], we know there is a unique classical solution for the Cauchy problem of the water wave equation (B.1)-(B.2) for some positive time period $[0,T]$, and the solution exists and is as regular as the initial data for as long as $L(t)$ and $\|Z_{,\alpha'}(t)\|_{L^\infty}$ remain finite.

Now assume that the initial data satisfies (3.2). Let $\delta>0$ be fixed, such that $\left\|1-\frac{1}{Z_{,\alpha'}}(0)\right\|_{L^\infty}\le 1-2\delta$, $\epsilon:=\max\left\{1,\sqrt{\frac{c_1}{c_2(\delta)}}\right\}\varepsilon$, so $L(0)\le\min\left\{1,\sqrt{\frac{c_2(\delta)}{c_1}}\right\}\epsilon$. Let $T_0>0$ be the maximum time such that the solution satisfies

$$\sup_{t\in[0,T_0]}\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty}\le 1-\delta,\qquad\sup_{t\in[0,T_0]}L(t)\le 2\epsilon. \tag{4.25}$$

By Proposition 4.4, there is a $\epsilon_0=\epsilon_0(\delta)>0$, such that for all $\epsilon\le\epsilon_0$,

$$E_1(t)E_3(t)\le(1+\delta)^2E_1(0)E_3(0)e^{c_3(\delta^{-1})\epsilon^3 t},\qquad\forall\ 0\le t\le T_0 \tag{4.26}$$

so

$$E_1(t)E_3(t)\le(1+\delta)^2m_0^2(1+\delta),\qquad\forall\ 0\le t\le T_1:=\min\{T_0,\frac{\delta}{2c_3(\delta^{-1})\epsilon^3}\}. \tag{4.27}$$

By Lemma 4.2 and (4.21), (4.22), there is a constant c_0 such that

$$\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty(\mathbb{R})}^4\le c_0(1+\delta)^2E_1(t)E_3(t),\qquad\forall\ t\in[0,T_0]. \tag{4.28}$$

Let $m_0^2=\frac{(1-2\delta)^4}{c_0(1+\delta)}$. Then we have

$$\left\|\frac{1}{Z_{,\alpha'}}(t)-1\right\|_{L^\infty(\mathbb{R})}^4\le c_0(1+\delta)^5m_0^2=(1+\delta)^4(1-2\delta)^4=(1-\delta-2\delta^2)^4,\qquad\forall\ t\in[0,T_1]. \tag{4.29}$$

By Proposition 4.3, there is $\epsilon_0=\epsilon_0(\delta)$, such that for all $\epsilon\le\epsilon_0$,

$$L(t)^2\le\frac{1}{c_2(\delta)}\left(\sum_{j=2}^{4}\mathcal{E}_j(0)+\epsilon^5 t\right)\le\epsilon^2+\frac{1}{c_2(\delta)}\epsilon^5 t\le 2\epsilon^2,\quad\text{for all } t\in[0,T_2], \tag{4.30}$$

where $T_2 = \min\{T_0, \frac{c_2(\delta)}{\epsilon^3}\}$. By the maximally of T_0 we must have $\min\{T_1, T_2\} < T_0$, so there is $\mathcal{T}_0 = \mathcal{T}_0(\delta) > 0$, such that $T_0 \ge \frac{\mathcal{T}_0}{\epsilon^3}$. This proves part 1 of Theorem 3.1.

In the remainder of this paper we prove Propositions 4.3 and 4.4.[16] We assume that (4.18) holds, namely for some $0 < \delta < 1$, $0 < \epsilon \le 1$ and $T_0 > 0$,

$$\sup_{t\in[0,T_0]} \left\| \frac{1}{Z_{,\alpha'}}(t) - 1 \right\|_{L^\infty} \le 1 - \delta, \qquad \sup_{t\in[0,T_0]} L(t) \le 2\epsilon. \tag{4.31}$$

We assume $t \in [0, T_0]$ in what follows.

4.3. **The proof of Propositions 4.3 and 4.4.** From definition (2.7), (2.8), and equations (2.6), (2.3) we know

$$\Theta^{(0)} = Q, \quad \Theta^{(1)} = i(Z - \alpha'), \quad \Theta^{(2)} = -i\mathbb{P}_H b = -i\mathbb{P}_H \frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}, \tag{4.32}$$

$$D_{\alpha'}\Theta^{(0)} = \overline{Z}_t, \qquad D_{\alpha'}\Theta^{(1)} = i\left(1 - \frac{1}{Z_{,\alpha'}}\right); \qquad \text{and} \tag{4.33}$$

$$\Theta^{(j+2)} = -i\mathbb{P}_H\Big(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta^{(j)}\Big) + \mathbb{P}_H(G^{(j)}), \qquad \text{for } j \ge 1. \tag{4.34}$$

Therefore

$$\begin{aligned} E_j(t) &= \operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\Theta^{(j+1)}}\,d\alpha' - \int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\Theta^{(j+2)}}\,d\alpha'\right) \\ &= \operatorname{Re}\left(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\Theta^{(j+1)}}\,d\alpha' + \int |D_{\alpha'}\Theta^{(j)}|^2\,d\alpha' + \int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H(G^{(j)})\,d\alpha'\right). \end{aligned} \tag{4.35}$$

4.3.1. *Quantities controlled by $L(t)$, $\left\|1 - \frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_t\|_{\dot H^{1/2}}$, and $\|Z_{t,\alpha'}\|_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$.* We begin with deriving some basic estimates for the quantities involved in the proof of Propositions 4.3, 4.4. First note that by assumption (4.31),

$$\sup_{t\in[0,T_0]} \left\| \frac{1}{Z_{,\alpha'}}(t) \right\|_{L^\infty} \le 2, \qquad \sup_{t\in[0,T_0]} \|Z_{,\alpha'}(t)\|_{L^\infty} \le \delta^{-1}; \tag{4.36}$$

and by interpolation and Sobolev embedding (C.6),

$$\|Z_{t,\alpha'}\|_{L^\infty} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim L(t) \le \epsilon. \tag{4.37}$$

We estimate, by (B.1),

$$\overline{Z}_{tt} = -i\frac{A_1 - 1}{Z_{,\alpha'}} + i\left(1 - \frac{1}{Z_{,\alpha'}}\right),$$

(B.2) and (C.17), that

$$\|A_1 - 1\|_{L^2} \lesssim \|Z_t\|_{\dot H^{1/2}} \|Z_{t,\alpha'}\|_{L^2} \lesssim \epsilon \|Z_t\|_{\dot H^{1/2}}, \tag{4.38}$$

$$\|Z_{tt}\|_{L^2} \lesssim \left\|1 - \frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|A_1 - 1\|_{L^2} \lesssim \left\|1 - \frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon \|Z_t\|_{\dot H^{1/2}}; \tag{4.39}$$

by (C.18), (C.40), (B.2) that

$$A_1 \ge 1, \quad \|A_1 - 1\|_{L^\infty} + \|A_1\|_{\dot H^{1/2}} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \lesssim \epsilon^2; \tag{4.40}$$

[16] We will be brief on straightforward but tedious details for the sake of a concise presentation. We suggest the reader get familiar with the basic tools in Appendices B, C before continuing.

by (C.3), (B.1) and (4.40),

$$\|Z_{tt}\|_{\dot H^{1/2}}\lesssim \|Z_{t,\alpha'}\|_{L^2}^2+\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim \epsilon,\quad \|Z_{tt}\|_{L^\infty}\lesssim 1;\tag{4.41}$$

by (B.6), (B.7), (C.24) and (C.23) that

$$\|D_tA_1\|_{L^2}\lesssim \|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|_{L^2}^2\,\|b_{\alpha'}\|_{L^2}\,,\tag{4.42}$$

$$\|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{L^2}\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim \epsilon^2;\tag{4.43}$$

this implies that

$$\|b_{\alpha'}\|_{L^2}\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|_{L^2}\lesssim \epsilon,\quad\text{and}\quad \|D_tA_1\|_{L^2}\lesssim\epsilon^2;\tag{4.44}$$

and by (4.32) and (B.4) that

$$\left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{L^2}\lesssim\epsilon,\quad \left\|\mathcal{P}\overline{Z}_t\right\|_{L^2}\lesssim\epsilon^2,\quad \left\|D_t^2Z_t\right\|_{L^2}\lesssim \|Z_{t,\alpha'}\|_{L^2}+\epsilon^2\lesssim\epsilon.\tag{4.45}$$

In what follows we pay particular attention to the bounds $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ and $\|Z_{t,\alpha'}\|_{\dot H^{1/2}}$ in the estimates for the sake of proving Proposition 4.4. We estimate, by (B.7), and (C.40), (C.18),

$$\|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{\dot H^{1/2}}+\|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{L^\infty}\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2},\tag{4.46}$$

so by (C.4), (B.8),

$$\|b_{\alpha'}\|_{\dot H^{1/2}}+\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}};\tag{4.47}$$

this gives, by using (4.32) and (C.4), that

$$\left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{\dot H^{1/2}}+\left\|D_{\alpha'}\Theta^{(2)}\right\|_{\dot H^{1/2}}\lesssim \|b_{\alpha'}\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|b_{\alpha'}\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}};\tag{4.48}$$

and as a consequence of (4.46) and (4.37), we also have

$$\|b_{\alpha'}\|_{L^\infty}\lesssim \|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^\infty}\lesssim\epsilon;\tag{4.49}$$

and we record, by (B.8),

$$\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\lesssim\epsilon.\tag{4.50}$$

Now by (B.2), (C.17), (C.19),

$$\|\partial_{\alpha'}A_1\|_{L^2}\lesssim \|Z_{t,\alpha'}\|_{L^2}\,\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\epsilon\,\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\,;\tag{4.51}$$

and by (B.1),

$$\|Z_{tt,\alpha'}\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\,\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\,\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\,.\tag{4.52}$$

We write

$$\left[Z_t,b;\overline{Z}_{t,\alpha'}\right]=-<Z_t,\,b,\,\overline{Z}_{t,\alpha'}>+\overline{Z}_{t,\alpha'}\,[Z_t,b;1]\,,\tag{4.53}$$

estimating $\left\|<Z_t,\,b,\,\overline{Z}_{t,\alpha'}>\right\|_{\dot H^{1/2}}$ by (C.31), and the $\dot H^{1/2}$ norm of the second term by (C.4) and (C.23), (C.24) yield

$$\left\|\left[Z_t,b;\overline{Z}_{t,\alpha'}\right]\right\|_{\dot H^{1/2}}\lesssim \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\,\|b_{\alpha'}\|_{L^2}\,\|Z_{t,\alpha'}\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\,\|Z_{t,\alpha'}\|_{L^2}\,\|b_{\alpha'}\|_{\dot H^{1/2}}\,;\tag{4.54}$$

from (B.6), further using (C.40) gives

$$\|D_tA_1\|_{\dot H^{1/2}}\lesssim \|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}+\left\|[Z_t,b;\overline{Z}_{t,\alpha'}]\right\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}};\tag{4.55}$$

by (B.4) and (C.3), (C.4), this gives

$$\|Z_{ttt}\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\tag{4.56}$$

We also record, by (B.4), (4.46), (4.55), (4.51) and other relevant earlier estimates that

$$\left\|\mathcal{P}\overline{Z}_t\right\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\tag{4.57}$$

By (B.6), (C.18), (C.27), (C.42),

$$\|D_tA_1\|_{L^\infty}\lesssim\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}+\|Z_{t,\alpha'}\|_{L^4}^2\|b_{\alpha'}\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}},\tag{4.58}$$

therefore by (B.4), we also have

$$\left\|\mathcal{P}\overline{Z}_t\right\|_{L^\infty}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}},\tag{4.59}$$

and

$$\|Z_{ttt}\|_{L^\infty}\lesssim\|Z_{t,\alpha'}\|_{L^\infty}+\epsilon^2\lesssim\epsilon.\tag{4.60}$$

Now by (B.7), (B.34), and (C.17), (C.19), (C.23), (4.47),

$$\begin{aligned}\|D_t(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2}&\lesssim\|Z_{tt}\|_{\dot H^{1/2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\\&+\|Z_{t,\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}},\end{aligned}\tag{4.61}$$

this gives, by (4.52), (B.17), (B.18), (C.42) and earlier relevant estimates in this section that

$$\|\partial_{\alpha'}D_tb\|_{L^2}\lesssim\|D_t(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2}+\|D_tD_{\alpha'}Z_t\|_{L^2}+\|b_{\alpha'}\|_{L^4}^2\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\tag{4.62}$$

We estimate similarly, by (B.6), (B.34), (B.31), and (C.17), (C.19), (C.23), (C.27), (C.42) and the estimates above in this section to obtain

$$\left\|D_t^2A_1\right\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}},\tag{4.63}$$

this in turn gives, by (B.4), (B.21), because $\mathcal{P}\overline{Z}_{tt}=D_t\mathcal{P}\overline{Z}_t+[\mathcal{P},D_t]\,\overline{Z}_t$, that

$$\left\|\mathcal{P}\overline{Z}_{tt}\right\|_{L^2}+\left\|[\mathcal{P},D_t]\,\overline{Z}_t\right\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}},\tag{4.64}$$

$$\left\|D_t^3Z_t\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\tag{4.65}$$

We compute $D_t^2\frac{1}{Z_{,\alpha'}}$ starting from (B.8) and use (B.20), (C.42) and the earlier estimates in this section and get

$$\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\tag{4.66}$$

From (4.109) we estimate by (2.10), (C.29), (C.32) to get

$$\left\|\partial_{\alpha'}\Theta^{(3)}\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}^2\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2};\tag{4.67}$$

we compute by (B.17) and the definition (2.7):

$$D_t D_{\alpha'}\Theta^{(2)} = D_{\alpha'}\Theta^{(3)} + D_{\alpha'}\left[\mathbb{P}_A, b\right]\partial_{\alpha'}\Theta^{(2)} - D_{\alpha'}Z_t D_{\alpha'}\Theta^{(2)}, \tag{4.68}$$

and using (C.17), (C.20), (C.42) and the earlier estimates we obtain

$$\begin{aligned}\left\|D_t D_{\alpha'}\Theta^{(2)}\right\|_{L^2} &\lesssim \left\|\partial_{\alpha'}\Theta^{(3)}\right\|_{L^2} + \|b_{\alpha'}\|_{\dot H^{1/2}}\left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{L^4}\left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{L^4}\\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\end{aligned} \tag{4.69}$$

Now by (B.7), (C.19), (C.20),

$$\|\partial_{\alpha'}(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\|_{L^2} \lesssim \left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}} + \|Z_{t,\alpha'}\|_{L^\infty}\right)\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim \epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.70}$$

therefore

$$\|\partial_{\alpha'}b_{\alpha'}\|_{L^2} \lesssim \|\partial_{\alpha'}(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\|_{L^2} + \|Z_{t,\alpha'\alpha'}\|_{L^2} + \|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.71}$$

and we compute $\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}$ by (B.8), and use (4.70), (4.46), (4.37) to obtain

$$\left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \lesssim \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}. \tag{4.72}$$

Now by (B.2), and (C.40),

$$\|\partial_{\alpha'}A_1\|_{\dot H^{1/2}} \lesssim \|Z_{t,\alpha'}\|_{L^2}\|Z_{t,\alpha'\alpha'}\|_{L^2} \lesssim \epsilon\|Z_{t,\alpha'\alpha'}\|_{L^2}; \tag{4.73}$$

therefore by (B.1), (C.4),

$$\begin{aligned}\|Z_{tt,\alpha'}\|_{\dot H^{1/2}} &\lesssim \|\partial_{\alpha'}A_1\|_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|\partial_{\alpha'}A_1\|_{L^2} + \|A_1\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \epsilon\|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.\end{aligned} \tag{4.74}$$

Starting from (B.7), we use (B.34) to expand, and use the Sobolev inequality (C.6) to estimate the $\dot H^{1/2}$ norm of $\left[\frac{1}{Z_{,\alpha'}}, b; Z_{t,\alpha'}\right]$ and $\left[Z_t, b; \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right]$, we have, by (C.40), (C.23), (C.24), (C.25), (C.21) and (C.6),

$$\begin{aligned}\|D_t(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\|_{\dot H^{1/2}} &\lesssim \left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\\ &\quad + \epsilon^2\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \|Z_{t,\alpha'\alpha'}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)\\ &\lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \|Z_{t,\alpha'\alpha'}\|_{L^2}\right) + \epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2};\end{aligned} \tag{4.75}$$

therefore by (B.17), (B.18), (C.3), (C.4),

$$\begin{aligned}\|\partial_{\alpha'}D_t b\|_{\dot H^{1/2}} &\lesssim \|D_t(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\|_{\dot H^{1/2}} + \left\|(b_{\alpha'})^2\right\|_{\dot H^{1/2}} + \|D_{\alpha'}Z_{tt}\|_{\dot H^{1/2}} + \left\|(D_{\alpha'}Z_t)^2\right\|_{\dot H^{1/2}}\\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \epsilon\|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}};\end{aligned} \tag{4.76}$$

we also have by (C.18), (C.26) that

$$\begin{aligned}\|D_t(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\|_{L^\infty} &\lesssim \left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\\ &\quad + \|b_{\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2} \lesssim \epsilon^2.\end{aligned} \tag{4.77}$$

Compute $D_t^2\frac{1}{Z_{,\alpha'}}$ starting from (B.8), and use (B.20), (C.3), (C.4), (4.46), (4.47), (4.52), (4.61), (4.74) and (4.75), and Sobolev embedding (C.6) to find

$$\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\epsilon\left\|Z_{t,\alpha'\alpha'}\right\|_{L^2}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}. \tag{4.78}$$

Now from (B.6), using (C.19), (C.20), and (C.24), (C.21), (C.25), we have

$$\begin{aligned}\|\partial_{\alpha'}D_tA_1\|_{L^2}&\lesssim\|Z_{tt,\alpha'}\|_{L^2}\left(\|Z_{t,\alpha'}\|_{L^\infty}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right)\\&\quad+\|b_{\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{\dot H^{1/2}}+\|b_{\alpha'}\|_{\dot H^{1/2}}\|Z_{t,\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^\infty}\\&\lesssim\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right);\end{aligned}\tag{4.79}$$

therefore by (B.4),

$$\|Z_{ttt,\alpha'}\|_{L^2}\lesssim\|Z_{t,\alpha'\alpha'}\|_{L^2}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}. \tag{4.80}$$

We begin with (B.7), and use (B.32) to expand. By (C.17), (C.19), (C.20), (C.23), (C.27), (C.42), (C.25), and (C.21), we have

$$\begin{aligned}&\left\|D_t^2(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\right\|_{L^2}\lesssim\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left(\|Z_{t,\alpha'}\|_{L^\infty}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right)\\&+\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\|Z_{t,\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^\infty}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|\partial_{\alpha'}D_tb\|_{L^2}\\&+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^4}^2+\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\|Z_{tt,\alpha'}\|_{L^2}\\&+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\|b_{\alpha'}\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{ttt}\|_{\dot H^{1/2}}\\&\lesssim\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\|Z_{t,\alpha'\alpha'}\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right);\end{aligned}\tag{4.81}$$

therefore by (B.18), (B.17),

$$\left\|\partial_{\alpha'}D_t^2b\right\|_{L^2}\lesssim\|Z_{t,\alpha'\alpha'}\|_{L^2}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}. \tag{4.82}$$

Now begin with (B.8), and use (B.17), we get

$$\begin{aligned}&\left\|D_t^3\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{L^\infty}\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\left\|D_t^2(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\right\|_{L^2}\\&+\|D_t(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2}\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}+\|D_{\alpha'}Z_t\|_{L^\infty}\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\left\|D_t^2D_{\alpha'}Z_t\right\|_{L^2}\\&+\|D_tD_{\alpha'}Z_t\|_{L^2}\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}\lesssim\|Z_{t,\alpha'\alpha'}\|_{L^2}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon\|Z_{t,\alpha'}\|_{\dot H^{1/2}}.\end{aligned}\tag{4.83}$$

We compute $D_t^2A_1$ by (B.6), (B.34), (B.31), estimating the $\dot H^{1/2}$ norms of $\left[Z_{tt},b;\overline{Z}_{t,\alpha'}\right]$, $\left[Z_t,b;\overline{Z}_{tt,\alpha'}\right]$ and $D_t\left[Z_t,b;\overline{Z}_{t,\alpha'}\right]$ by the Sobolev inequality (C.6), and estimating the $\dot H^{1/2}$ norms of the remaining commutators in the expansion of $D_t^2A_1$ by (C.40). We get, by further using (C.23), (C.24), (C.25), (C.21), (C.27),

$$\left\|D_t^2A_1\right\|_{\dot H^{1/2}}\lesssim\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\|Z_{t,\alpha'\alpha'}\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right); \tag{4.84}$$

this gives, by (B.1) and the estimates above in this section,

$$\left\|D_t^2 Z_{tt}\right\|_{\dot H^{1/2}} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \epsilon\left(\|Z_{t,\alpha'\alpha'}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right). \tag{4.85}$$

We also have, by (C.18) and (C.27) that

$$\left\|D_t^2 A_1\right\|_{L^\infty} \lesssim \epsilon^2. \tag{4.86}$$

We compute $D_t^3 A_1$ by (B.6), (B.32), (B.31), and obtain, by using the inequalities in Appendix C,

$$\left\|D_t^3 A_1\right\|_{L^2} \lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \|Z_{t,\alpha'\alpha'}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right); \tag{4.87}$$

therefore by (B.4),

$$\left\|D_t^3 Z_{tt}\right\|_{L^2} \lesssim \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right). \tag{4.88}$$

We note that (4.81) also gives

$$\left\|D_t^2(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t)\right\|_{L^2} \lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right); \tag{4.89}$$

(4.89) is needed for the sake of estimating $\frac{d}{dt}\mathcal{E}_3(t)$ and proving (4.24). We claim that a similar estimate also holds for $\left\|D_t^3 A_1\right\|_{L^2}$, that is

$$\left\|D_t^3 A_1\right\|_{L^2} \lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right); \tag{4.90}$$

expanding $D_t^3 A_1$ via (B.6), (B.32), (B.31), (4.90) can be checked straightforwardly using the inequalities in Appenddix C, the only term in the expansion that needs some extra explanation is the term $\left[Z_t, D_t^2 b; \overline{Z}_{t,\alpha'}\right]$. Observe that by (B.17), (B.18),

$$\begin{aligned}\partial_{\alpha'} D_t^2 b &= D_t^2\left(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t\right) + 3 b_{\alpha'}\partial_{\alpha'} D_t b - 2(b_{\alpha'})^3 \\ &\quad - 2\operatorname{Re}\left(3 D_{\alpha'}Z_t D_{\alpha'}Z_{tt} - 2(D_{\alpha'}Z_t)^3\right) - 2\operatorname{Re}\left(Z_{ttt}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + 2\operatorname{Re}\partial_{\alpha'}\frac{Z_{ttt}}{Z_{,\alpha'}},\end{aligned} \tag{4.91}$$

so we have, by (4.89), (4.62), (4.49), (4.47), (4.44), (C.42), (4.37), (4.52), (4.60),

$$\left\|\partial_{\alpha'}\left(D_t^2 b - 2\operatorname{Re}\frac{Z_{ttt}}{Z_{,\alpha'}}\right)\right\|_{L^2} \lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right). \tag{4.92}$$

We write

$$\left[Z_t, D_t^2 b; \overline{Z}_{t,\alpha'}\right] = \left[Z_t, D_t^2 b - 2\operatorname{Re}\frac{Z_{ttt}}{Z_{,\alpha'}}; \overline{Z}_{t,\alpha'}\right] + \left[Z_t, 2\operatorname{Re}\frac{Z_{ttt}}{Z_{,\alpha'}}; \overline{Z}_{t,\alpha'}\right] \tag{4.93}$$

and apply (C.23) to the first term and (C.24) to the second term, we get, by further using (C.4),

$$\left\|\left[Z_t, D_t^2 b; \overline{Z}_{t,\alpha'}\right]\right\|_{L^2} \lesssim \epsilon^2\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right). \tag{4.94}$$

This proves (4.90). By (B.4), (B.21), (B.24), (4.89), (4.90) and the earlier estimates in this section, we conclude

$$\left\|\mathcal{P} D_t^2 \overline{Z}_t\right\|_{L^2} + \left\|\left[D_t, \mathcal{P}\right]\overline{Z}_{tt}\right\|_{L^2} + \left\|\left[D_t^2, \mathcal{P}\right]\overline{Z}_t\right\|_{L^2} \lesssim \epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right). \tag{4.95}$$

Finally we consider $\mathcal{P} D_t^3 \overline{Z}_t$. We write it as

$$\mathcal{P} D_t^3 \overline{Z}_t = D_t^3 \mathcal{P}\overline{Z}_t - \left[D_t, \mathcal{P}\right] D_t^2 \overline{Z}_t - D_t\left[D_t, \mathcal{P}\right] D_t \overline{Z}_t - D_t^2\left[D_t, \mathcal{P}\right]\overline{Z}_t$$

and use (B.4), (B.21) to compute. By the earlier estimates in this section, we have

$$\left\|[D_t,\mathcal{P}]\,\overline{Z}_{ttt}\right\|_{L^2}+\left\|D_t\,[D_t,\mathcal{P}]\,\overline{Z}_{tt}\right\|_{L^2}+\left\|D_t^2\,[D_t,\mathcal{P}]\,\overline{Z}_t\right\|_{L^2}\lesssim\epsilon^2; \tag{4.96}$$

we compute $D_t^3(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)$ from (B.7), expanding by (B.32), (B.31). We have, by (C.17), (C.19), (C.20), (C.23), (C.24), (C.22), (C.21), (C.27),

$$\left\|D_t^3(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\right\|_{L^2}\lesssim\epsilon^2. \tag{4.97}$$

we compute by (B.18),

$$\partial_{\alpha'}D_t^3b=D_t\partial_{\alpha'}D_t^2b+b_{\alpha'}\partial_{\alpha'}D_t^2b \tag{4.98}$$

and use (B.17), (B.18), and (4.91) to expand, by (C.42) and earlier estimates in this section, we have

$$\left\|\partial_{\alpha'}\left(D_t^3b-2\operatorname{Re}\frac{D_t^3Z_t}{Z_{,\alpha'}}\right)\right\|_{L^2}\lesssim\epsilon^2; \tag{4.99}$$

now we compute $D_t^4A_1$ from (B.6), expanding by (B.32), (B.31). Using a similar argument as that of (4.93) for the term $\left[Z_t,D_t^3b;\overline{Z}_{t,\alpha'}\right]$ and use the inequalities in Appendix C to estimate the remaining terms in $D_t^4A_1$ we get

$$\left\|D_t^4A_1\right\|_{L^2}\lesssim\epsilon^2. \tag{4.100}$$

This gives that

$$\left\|\mathcal{P}D_t^3\overline{Z}_t\right\|_{L^2}\lesssim\epsilon^2. \tag{4.101}$$

4.3.2. *The estimates for $E_1(t)$, $\mathfrak{E}_1(t)$ and $E_j(t)$, $\mathcal{E}_j(t)$, $j\ge2$.* In this section we prove the inequalities (4.19) and (4.21), (4.22), (4.23) of Propositions 4.3, 4.4.

Step 1. We begin with $E_1(t)$. From (4.35), (4.33),

$$E_1(t)=\left\|\Theta^{(2)}\right\|_{\dot H^{1/2}}^2+\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\int i\,\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_H(G^{(1)})\,d\alpha',$$

and by Proposition 2.2, and (C.29),

$$\left|\int i\,\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_H(G^{(1)})\,d\alpha'\right|\lesssim\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\|Z_{t,\alpha'}\|_{L^2}^2\lesssim\epsilon^2\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2, \tag{4.102}$$

so there is $\epsilon_0=\epsilon_0(\delta)>0$, such that for all $0<\epsilon\le\epsilon_0$, (4.21) holds. We note that by Lemma 4.1,

$$\delta\,\|Z_t\|_{\dot H^{1/2}}\le\left\|\Theta^{(2)}\right\|_{\dot H^{1/2}}. \tag{4.103}$$

Now we consider $\mathfrak{E}_1(t)$. We estimate, by (C.29), (C.30) and (4.39), (4.45), (4.103) that

$$\begin{aligned}|C_{1,1}(t)|+|C_{2,1}(t)|&\lesssim\|Z_{tt}\|_{L^2}^2\|Z_{t,\alpha'}\|_{L^2}^2+\|Z_{ttt}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|Z_t\|_{\dot H^{1/2}}^2\\&\lesssim\frac{\epsilon^2}{\delta^2}\left(\left\|\Theta^{(2)}\right\|_{\dot H^{1/2}}^2+\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\end{aligned}$$

so there is $\epsilon_0=\epsilon_0(\delta)>0$, such that for all $0<\epsilon\le\epsilon_0$,

$$(1+\delta)^{-1/2}E_1(t)\le\mathfrak{E}_1(t)\le(1+\delta)^{1/2}E_1(t). \tag{4.104}$$

Step 2. We next consider $E_2(t)$ and $\mathcal{E}_2(t)$. By (4.35),

$$E_2(t)=\left\|\Theta^{(3)}\right\|_{\dot H^{1/2}}^2+\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}^2+\int i\,\partial_{\alpha'}\overline{\Theta^{(2)}}\mathbb{P}_H(G^{(2)})\,d\alpha'. \tag{4.105}$$

First we have by Lemma 4.1, and (C.1), (C.3)

$$\delta\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\le\left\|\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)\right\|_{\dot H^{1/2}}\lesssim\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}; \tag{4.106}$$

and from the identity

$$\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}=\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\left[\mathbb{P}_H,\overline{Z}_t\right]\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}\right)+\left[\mathbb{P}_A,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'},$$

and (4.32) (C.17), (C.19), that

$$\left\|\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}-i\partial_{\alpha'}\Theta^{(2)}\right\|_{L^2}\lesssim\left\|\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{\dot H^{1/2}}\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\lesssim\epsilon\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}, \tag{4.107}$$

so by (4.36), there is $\epsilon_0=\epsilon_0(\delta)>0$, such that for $0<\epsilon\le\epsilon_0$,

$$\delta\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{L^2}\lesssim\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2},\qquad\delta^2\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\lesssim\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}\lesssim\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}. \tag{4.108}$$

Now by (4.35), (4.32),

$$\Theta^{(3)}=\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)+\mathbb{P}_H(G^{(1)}); \tag{4.109}$$

we estimate, by (2.11), (4.36), (4.106) and (C.3), (C.31), (C.29) to get

$$\left\|\mathbb{P}_H(G^{(1)})\right\|_{\dot H^{1/2}}\lesssim\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\left\|Z_{t,\alpha'}\right\|_{L^2}^2\lesssim\frac{\epsilon^2}{\delta}\left\|\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)\right\|_{\dot H^{1/2}}, \tag{4.110}$$

and by (2.11)-(2.21), (B.25), (C.15), (B.28), (C.29), (C.30), (C.27), (4.36) to yield

$$\begin{aligned}\left\|\mathbb{P}_H(G^{(2)})\right\|_{L^2}&\lesssim\left\|b_{\alpha'}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}^2+\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\left(\left\|Z_{tt}\right\|_{\dot H^{1/2}}+\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\right)\\&+\left\|Z_{t,\alpha'}\right\|_{L^2}^2\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\frac{\epsilon^2}{\delta^2}\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2},\end{aligned} \tag{4.111}$$

so there is $\epsilon_0=\epsilon_0(\delta)>0$, such that for $0<\epsilon\le\epsilon_0$,

$$(1+\delta)^{-1}E_2(t)\le\left\|\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)\right\|_{\dot H^{1/2}}^2+\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}^2\le(1+\delta)E_2(t). \tag{4.112}$$

Now we consider the terms $C_{1,2}+C_{2,2}+F_2+H_2$ in $\mathcal{E}_2(t)$. Observe that those terms with the factor $D_t^3Z_t$ in $C_{1,2}$, $C_{2,2}$ can be combined with those in F_2 with the factor $\overline{\mathcal{P}D_t\overline{Z}_t}$, and we know

$$-D_t^3Z_t+\overline{\mathcal{P}D_t\overline{Z}_t}=-i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}Z_{tt}. \tag{4.113}$$

So by (C.29), (C.30), (C.31), (C.3), (C.27), (C.37), we have

$$\begin{aligned}|C_{1,2}+C_{2,2}+F_2+H_2|&\lesssim\left\|Z_{ttt}\right\|_{L^2}^2\left\|Z_{t,\alpha'}\right\|_{L^2}^2+\left\|Z_{ttt}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|Z_{tt}\right\|_{\dot H^{1/2}}^2\\&+\left\|Z_{t,\alpha'}\right\|_{L^2}^2\left\|Z_{tt}\right\|_{\dot H^{1/2}}\left(\left\|Z_{tt}\right\|_{\dot H^{1/2}}+\left\|\frac{A_1}{|Z_{,\alpha'}|^2}\right\|_{\dot H^{1/2}}\right)+\left\|b_{\alpha'}\right\|_{L^2}\left\|Z_{ttt}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}^2\\&+\left\|Z_{tt}\right\|_{\dot H^{1/2}}^4+\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}^2\left\|Z_{t,\alpha'}\right\|_{L^2}^2\\&\lesssim\frac{\epsilon^2}{\delta^4}\left(\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}^2+\left\|\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)\right\|_{\dot H^{1/2}}^2\right);\end{aligned} \tag{4.114}$$

we also have by (C.36),

$$|D_2(t)|\lesssim\left\|b_{\alpha'}\right\|_{L^2}\left\|Z_{t,\alpha'}\right\|_{L^2}\left\|Z_{tt}\right\|_{\dot H^{1/2}}^2\lesssim\frac{\epsilon^2}{\delta^4}\left(\left\|D_{\alpha'}\Theta^{(2)}\right\|_{L^2}^2+\left\|\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\left(1-\frac{1}{Z_{,\alpha'}}\right)\right)\right\|_{\dot H^{1/2}}^2\right). \tag{4.115}$$

By (4.106), (4.108), there is $\epsilon_0 = \epsilon_0(\delta) > 0$, such that for $0 < \epsilon \le \epsilon_0$,

$$c_2(\delta)\left(\left\|1-\frac{1}{Z_{,\alpha'}}\right\|^2_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|^2_{L^2}\right)\le \mathcal{E}_2(t)\le c_1\left(\left\|1-\frac{1}{Z_{,\alpha'}}\right\|^2_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|^2_{L^2}\right). \tag{4.116}$$

for some constants $c_1 > 0$ and $c_2(\delta) > 0$.

Step 3. We now study $E_3(t)$ and $\mathcal{E}_3(t)$. By (4.32), (4.34), (4.35), we know

$$E_3(t) = \left\|\Theta^{(4)}\right\|^2_{\dot H^{1/2}} + \left\|D_{\alpha'}\Theta^{(3)}\right\|^2_{L^2} + \int i\,\partial_{\alpha'}\overline{\Theta}^{(3)}\mathbb{P}_H(G^{(3)})\,d\alpha', \tag{4.117}$$

where

$$\Theta^{(4)} = -\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)+\mathbb{P}_H(G^{(2)}),\quad D_{\alpha'}\Theta^{(3)} = -D_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\right)+D_{\alpha'}\mathbb{P}_H(G^{(1)}).$$

We compute

$$\begin{aligned}&D_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\right)-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} =\\ &=\left[\frac{1}{Z_{,\alpha'}},\mathbb{P}_H\right]\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|^2}+\left[\mathbb{P}_H,\frac{1}{Z^2_{,\alpha'}}\right]\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}-\left[\mathbb{P}_A,\frac{1}{|Z_{,\alpha'}|^2}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},\end{aligned} \tag{4.118}$$

so by (C.17), (C.3), and (4.36),

$$\left\|D_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\right)-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\frac{\epsilon}{\delta^2}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}. \tag{4.119}$$

We estimate by (2.10), (C.32) and (C.29), and get

$$\left\|\partial_{\alpha'}\mathbb{P}_H(G^{(1)})\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|^2_{L^2}\lesssim\frac{\epsilon^2}{\delta^2}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}. \tag{4.120}$$

Now

$$\mathbb{P}_H\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} = \partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\left[\mathbb{P}_H,\overline{Z}_t\right]\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}, \tag{4.121}$$

and by Lemma 4.1, (C.40) and (C.5),

$$\begin{aligned}\delta\|Z_{t,\alpha'}\|_{\dot H^{1/2}}&\le\left\|\mathbb{P}_H\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right\|_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\\ &\lesssim\frac{1}{\delta}\left(\left\|D_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right\|_{L^2}\right)+\|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2};\end{aligned} \tag{4.122}$$

using Lemma 4.1 again gives

$$\delta\left\|D_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right\|_{\dot H^{1/2}}\le\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)\right\|_{\dot H^{1/2}}, \tag{4.123}$$

therefore

$$\delta^3\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)\right\|_{\dot H^{1/2}}+\epsilon\delta\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.124}$$

on the other hand, by (C.4), (4.45) ,(4.48) we also have

$$\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)\right\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}. \tag{4.125}$$

We next estimate $\left\|\mathbb{P}_H\left(G^{(2)}\right)\right\|_{\dot{H}^{1/2}}$. By (2.11)-(2.21), (C.3), (C.31), and (C.29),

$$\left\|\mathbb{P}_H\left(G^{(2)}-D_t\mathbb{P}_H(G^{(1)})\right)\right\|_{\dot{H}^{1/2}}\lesssim\|Z_{t,\alpha'}\|_{L^2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\lesssim\epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2},\tag{4.126}$$

by (B.25), (B.28), (C.40), (C.32), (C.29), (C.4), (C.30), (C.27),

$$\begin{aligned}&\left\|\mathbb{P}_H D_t\mathbb{P}_H(G^{(1)})\right\|_{\dot{H}^{1/2}}\lesssim\|b_{\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\|Z_{t,\alpha'}\|_{L^2}^2\\&+\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot{H}^{1/2}}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\\&+\left\|D_t<\overline{Z}_t,i\frac{1}{\overline{Z}_{,\alpha'}},\overline{Z}_t>\right\|_{\dot{H}^{1/2}}+\left\|D_t<Z_t,-i\frac{1}{Z_{,\alpha'}},\overline{Z}_t>\right\|_{\dot{H}^{1/2}}.\end{aligned}\tag{4.127}$$

To estimate $\left\|D_t<\overline{Z}_t,i\frac{1}{\overline{Z}_{,\alpha'}},\overline{Z}_t>\right\|_{\dot{H}^{1/2}}$ and $\left\|D_t<Z_t,-i\frac{1}{Z_{,\alpha'}},\overline{Z}_t>\right\|_{\dot{H}^{1/2}}$, we use (B.28) to expand. We use Sobolev inequality (C.6) to estimate the following $\dot{H}^{1/2}$ norm. By (C.27), (C.42), we have

$$\begin{aligned}&\left\|\int\partial_{\beta'}\mathfrak{D}_t\left(\frac{1}{\alpha'-\beta'}\right)\theta\theta\left(\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{\overline{Z}_{,\beta'}}\right)d\beta'\right\|_{\dot{H}^{1/2}}^2\\&\lesssim\|b_{\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}^2\|b_{\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^4}^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\epsilon^5\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\end{aligned}\tag{4.128}$$

For the $\dot{H}^{1/2}$ norm of the remaining term in $D_t<\overline{Z}_t,i\frac{1}{\overline{Z}_{,\alpha'}},\overline{Z}_t>$, we use (C.31). We get

$$\begin{aligned}\left\|D_t<\overline{Z}_t,i\frac{1}{\overline{Z}_{,\alpha'}},\overline{Z}_t>\right\|_{\dot{H}^{1/2}}&\lesssim\epsilon^2\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}^{1/2}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^{1/2}\\&+\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot{H}^{1/2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}^2\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}}.\end{aligned}\tag{4.129}$$

Therefore

$$\left\|\mathbb{P}_H\left(G^{(2)}\right)\right\|_{\dot{H}^{1/2}}\lesssim\epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon^2\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}.\tag{4.130}$$

We now consider the term $\int i\,\partial_{\alpha'}\overline{\Theta}^{(3)}\mathbb{P}_H(G^{(3)})\,d\alpha'$ in (4.117). The estimate for $\left\|\mathbb{P}_H(G^{(3)})\right\|_{L^2}$ is as usual, by using the formula (2.11)-(2.21), the expansions (B.25) and (B.29), (B.34), and the inequalities (C.29), (C.30), (C.19), (C.20), (C.27) and (C.42); observe that by (C.19), the estimates (4.126)-(4.129) for (4.130) can be used to treat the second term in the expansion by (B.25). We have

$$\begin{aligned}\left\|\mathbb{P}_H(G^{(3)})\right\|_{L^2}&\lesssim\epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon^2\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}+\epsilon^2\|Z_{tt,\alpha'}\|_{L^2}+\epsilon^2\left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}}\\&+\epsilon^2\left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon^2\|b_{\alpha'}\|_{\dot{H}^{1/2}}+\epsilon^2\|\partial_{\alpha'}D_tb\|_{L^2}+\epsilon^2\|Z_{ttt}\|_{\dot{H}^{1/2}}\\&\lesssim\epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\epsilon^2\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}};\end{aligned}\tag{4.131}$$

so there is a $\epsilon_0=\epsilon(\delta)>0$, such that for all $0<\epsilon\le\epsilon_0$, the inequality (4.22) in Proposition 4.4 holds. Finally we can estimate the correcting terms $C_{1,3}+C_{2,3}+F_3+H_3$ as in Step 2 by combining the terms in $C_{1,3}$ and $C_{2,3}$ with the factor $D_t^4Z_t$ with the terms in F_3 with the factor $\overline{\mathcal{P}D_t^2\overline{Z}_t}$ and use (C.29), (C.30), (C.27), (C.31), (C.4), and (C.42) to obtain

$$
\begin{aligned}
&|C_{1,3}+C_{2,3}+F_3+H_3| \lesssim \left\|D_t^2 Z_t\right\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|Z_{ttt}\|_{\dot H^{1/2}}^2+\left\|D_t^2 Z_t\right\|_{L^2}^2\|Z_{tt,\alpha'}\|_{L^2}^2\\
&+\left\|D_t^3 Z_t\right\|_{L^2}\left(\|Z_{t,\alpha'}\|_{L^2}^2\left\|D_t^3 Z_t\right\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\left\|D_t^2 Z_t\right\|_{L^2}+\|Z_{tt,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot H^{1/2}}^2\right)\\
&+\left\|D_t^3 Z_t\right\|_{L^2}\left(\|b_{\alpha'}\|_{L^4}\left\|D_t^2 Z_t\right\|_{L^4}\|Z_{t,\alpha'}\|_{L^2}^2+\|b_{\alpha'}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\|Z_{tt}\|_{L^\infty}\right)\\
&+\|Z_{t,\alpha'}\|_{L^2}^2\left\|D_{\alpha'}\Theta^{(3)}\right\|_{L^2}^2+\|Z_{ttt}\|_{\dot H^{1/2}}\left(\|Z_{t,\alpha'}\|_{L^2}^2\|Z_{ttt}\|_{\dot H^{1/2}}+\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot H^{1/2}}\right)\\
&+\|Z_{ttt}\|_{\dot H^{1/2}}\left\|\partial_{\alpha'}\frac{A_1}{|Z_{,\alpha'}|^2}\right\|_{L^2}\left(\|Z_{t,\alpha'}\|_{L^2}^2\|Z_{ttt}\|_{L^2}+\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{\dot H^{1/2}}^2\right)\\
&\lesssim \epsilon^2\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|D_{\alpha'}\Theta^{(3)}\right\|_{L^2}^2\right).
\end{aligned}
\tag{4.132}
$$

So there is $\epsilon_0=\epsilon(\delta)>0$, such that for all $0<\epsilon\le\epsilon_0$,

$$(1+\delta)^{-1/2}E_3(t)\le\mathcal{E}_3(t)\le(1+\delta)^{1/2}E_3(t). \tag{4.133}$$

This together with (4.104) proves the inequality (4.23) in Proposition 4.4. Moreover, there are constants $c_1>0$ and $c_2(\delta)>0$, such that for all $0<\epsilon\le\epsilon_0(\delta)$,

$$c_2(\delta)\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2\right)\le\mathcal{E}_3(t)\le c_1\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2\right). \tag{4.134}$$

We also note that by (4.125), (4.130),

$$\left\|\Theta^{(4)}\right\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\epsilon. \tag{4.135}$$

Because

$$D_t\Theta^{(3)}=\Theta^{(4)}+\mathbb{P}_A D_t\Theta^{(3)}=\Theta^{(4)}+[\mathbb{P}_A,b]\,\partial_{\alpha'}\Theta^{(3)}, \tag{4.136}$$

$$D_t^2\Theta^{(2)}=\Theta^{(4)}+[\mathbb{P}_A,b]\,\partial_{\alpha'}\Theta^{(3)}+D_t\,[\mathbb{P}_A,b]\,\partial_{\alpha'}\Theta^{(2)}, \tag{4.137}$$

using the identity

$$\left[b,b;\partial_{\alpha'}\Theta^{(2)}\right]=-<b,b,\partial_{\alpha'}\Theta^{(2)}>+\partial_{\alpha'}\Theta^{(2)}\,[b,b;1]\,, \tag{4.138}$$

we have, by (C.40), (B.34), (C.31), (C.4), (C.24), (C.23),

$$\left\|D_t\Theta^{(3)}\right\|_{\dot H^{1/2}}+\left\|D_t^2\Theta^{(2)}\right\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\epsilon. \tag{4.139}$$

Step 4. We now consider $E_4(t)$ and $\mathcal{E}_4(t)$. By (4.35),

$$E_4(t)=\left\|\Theta^{(5)}\right\|_{\dot H^{1/2}}^2+\left\|D_{\alpha'}\Theta^{(4)}\right\|_{L^2}^2+\int i\,\partial_{\alpha'}\overline{\Theta}^{(4)}\mathbb{P}_H(G^{(4)})\,d\alpha', \tag{4.140}$$

and by (4.34), (4.109),

$$
\begin{aligned}
\Theta^{(5)}&=-i\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta^{(3)}\right)+\mathbb{P}_H(G^{(3)})\\
&=i\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{1}{|Z_{,\alpha'}|^2}\right)-i\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H(G^{(1)})\right)+\mathbb{P}_H(G^{(3)}),
\end{aligned}
\tag{4.141}
$$

$$D_{\alpha'}\Theta^{(4)}=-D_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)+D_{\alpha'}\mathbb{P}_H(G^{(2)}). \tag{4.142}$$

Since

$$\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{1}{|Z_{,\alpha'}|^2}\right)-\frac{1}{|Z_{,\alpha'}|^2\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \tag{4.143}$$
$$=-\left[\mathbb{P}_H,\frac{1}{|Z_{,\alpha'}|^2}\right]\partial_{\alpha'}\mathbb{P}_A\frac{1}{|Z_{,\alpha'}|^2}+\left[\mathbb{P}_H,\frac{1}{|Z_{,\alpha'}|^2Z_{,\alpha'}}\right]\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}-\left[\mathbb{P}_A,\frac{1}{|Z_{,\alpha'}|^2\overline{Z}_{,\alpha'}}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}=:I,$$

and by (C.40),

$$\|I\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.144}$$

by (C.4), (C.5),

$$\left\|\frac{1}{|Z_{,\alpha'}|^2\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2, \tag{4.145}$$

$$\delta^3\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}\lesssim\left\|\frac{1}{|Z_{,\alpha'}|^2\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2; \tag{4.146}$$

therefore

$$\delta^3\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}-\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{1}{|Z_{,\alpha'}|^2}\right)\right\|_{\dot H^{1/2}} \tag{4.147}$$
$$\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.$$

Also, we know $\mathbb{P}_H b=\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}=i\Theta^{(2)}$, and

$$\partial_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}} \tag{4.148}$$
$$=\mathbb{P}_H\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)-\left[\mathbb{P}_A,\frac{1}{|Z_{,\alpha'}|^2}\right]\partial_{\alpha'}^2\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}},$$

applying (C.14) on the second term, we get

$$\left\|\partial_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right\|_{L^2}\lesssim\|b_{\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.149}$$

now since

$$\partial_{\alpha'}^2\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\frac{\overline{Z}_{t,\alpha'\alpha'}}{\overline{Z}_{,\alpha'}}=-\left[\mathbb{P}_A,\frac{1}{\overline{Z}_{,\alpha'}}\right]\partial_{\alpha'}\overline{Z}_{t,\alpha'}+2\left[\mathbb{P}_H,\overline{Z}_{t,\alpha'}\right]\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}+\left[\mathbb{P}_H,\overline{Z}_t\right]\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}, \tag{4.150}$$

applying (C.17), (C.19), (C.20) yields

$$\left\|\partial_{\alpha'}^2\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\frac{\overline{Z}_{t,\alpha'\alpha'}}{\overline{Z}_{,\alpha'}}\right\|_{L^2}\lesssim\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\lesssim\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.151}$$

therefore

$$\delta^4\|Z_{t,\alpha'\alpha'}\|_{L^2}-\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\left\|D_{\alpha'}\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}\right)\right\|_{L^2} \tag{4.152}$$
$$\lesssim\|Z_{t,\alpha'\alpha'}\|_{L^2}+\epsilon\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.$$

We next consider $\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H(G^{(1)})\right)\right\|_{\dot{H}^{1/2}}$. By (2.10), and applying (C.4), (4.120), (2.11), (C.27), (C.32), (C.31) gives

$$\begin{aligned}
&\left\|\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_H(G^{(1)})\right)\right\|_{\dot{H}^{1/2}} \lesssim \left\|\partial_{\alpha'}\mathbb{P}_H(G^{(1)})\right\|_{\dot{H}^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \|Z_{t,\alpha'}\|_{L^2}^2 \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} \|Z_{t,\alpha'}\|_{L^2}^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \|Z_{t,\alpha'}\|_{L^2}^2 + \|Z_{t,\alpha'}\|_{\dot{H}^{1/2}} \|Z_{t,\alpha'}\|_{L^2} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \\
&\lesssim \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.
\end{aligned} \tag{4.153}$$

We now estimate $\left\|\partial_{\alpha'}\mathbb{P}_H\left(G^{(2)}\right)\right\|_{L^2}$. By (2.11)-(2.21), using (B.25), (B.28) to expand, then applying (C.29), (C.32), (C.17), (C.20), (C.27), we get

$$\begin{aligned}
&\left\|\partial_{\alpha'}\mathbb{P}_H\left(G^{(2)}\right)\right\|_{L^2} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \|Z_{t,\alpha'}\|_{L^2} + \|b_{\alpha'}\|_{\dot{H}^{1/2}} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 \\
&+ \left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 + \left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 \\
&+ \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2} \|Z_{tt,\alpha'}\|_{L^2} + \|b_{\alpha'}\|_{L^\infty} \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 \\
&\lesssim \epsilon^2 \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}.
\end{aligned} \tag{4.154}$$

And we estimate $\left\|\mathbb{P}_H(G^{(3)})\right\|_{\dot{H}^{1/2}}$ by (2.11)-(2.21), expanding with (B.25), (B.29) and (B.34). Again the estimate is straightforward but tedious. We use (C.31) to estimate $\left\|\mathbb{P}_H\left(G^{(3)} - D_t\mathbb{P}_H G^{(2)}\right)\right\|_{\dot{H}^{1/2}}$ and the $\dot{H}^{1/2}$ norm of the term

$$\frac{1}{\pi}\int \frac{\mathfrak{D}_t^2\left((\overline{Z}_t(\alpha') - \overline{Z}_t(\beta'))^2\left(\frac{1}{\overline{Z}_{,\alpha'}} - \frac{1}{\overline{Z}_{,\beta'}}\right)\right)}{(\alpha'-\beta')^2}\,d\beta'$$

in the expansion of $D_t^2 < \overline{Z}_t, i\frac{1}{\overline{Z}_{,\alpha'}}, \overline{Z}_t >$; we use Sobolev inequality (C.6) to estimate the $\dot{H}^{1/2}$ norms of the remaining terms in the expansion of $D_t^2 < \overline{Z}_t, i\frac{1}{\overline{Z}_{,\alpha'}}, \overline{Z}_t >$, as well as $\left\|D_t < \overline{Z}_t, i\frac{1}{\overline{Z}_{,\alpha'}}, i\frac{1}{Z_{,\alpha'}} >\right\|_{\dot{H}^{1/2}}$ and $\left\|\left[b, b, \partial_{\alpha'}\left\{\frac{1}{\overline{Z}_{,\alpha'}} < \overline{Z}_t, i\frac{1}{\overline{Z}_{,\alpha'}}, \overline{Z}_t >\right\}\right]\right\|_{\dot{H}^{1/2}}$. Observe that we have (4.129), and the estimate in (4.154) is useful for the estimate of the second term in the expansion of $\mathbb{P}_H D_t \mathbb{P}_H G^{(2)}$ by (B.25). We have

$$\begin{aligned}
&\left\|\mathbb{P}_H(G^{(3)})\right\|_{\dot{H}^{1/2}} \lesssim \epsilon^2 \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} + \epsilon^2 \left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon^2 \left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} \\
&+ \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon^2 \|Z_{t,\alpha'}\|_{\dot{H}^{1/2}} \\
&\lesssim \epsilon^2 \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon^2 \|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}.
\end{aligned} \tag{4.155}$$

We estimate $\left\|\mathbb{P}_H(G^{(4)})\right\|_{L^2}$ by (2.11)-(2.21), expanding with (B.25)-(B.26), (B.29), (B.32). The estimate is routine, using the inequalities in Appendix C and the estimates in §4.3.1. Going through the terms carefully, we get

$$\left\|\mathbb{P}_H(G^{(4)})\right\|_{L^2} \lesssim \epsilon^2 \|Z_{t,\alpha'\alpha'}\|_{L^2} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{1/2}} + \epsilon^2 \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \epsilon^2 \|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}. \tag{4.156}$$

Finally the correcting terms $C_{1,4}+C_{2,4}+F_4+H_4$ can be estimated similarly as in Step 2 by combining the terms in $C_{i,4}$ with the factor $D_t^5 Z_t$ with those terms in F_4 with the factor $\overline{\mathcal{P}D_t^3\overline{Z}_t}$ and use the equation

$$-D_t^5 Z_t+\overline{\mathcal{P}D_t^3\overline{Z}_t}=-i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_t^3 Z_t \tag{4.157}$$

and we have, by applying the inequalities in Appendix C,

$$\begin{aligned}&|C_{1,4}+C_{2,4}+F_4+H_4|\\ &\lesssim \epsilon^2\left(\|Z_{t,\alpha'\alpha'}\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}+\left\|D_{\alpha'}\Theta^{(4)}\right\|_{L^2}\right)^2.\end{aligned} \tag{4.158}$$

This together with (4.134) shows that there is a $\epsilon_0=\epsilon_0(\delta)>0$, such that for all of $0<\epsilon\le\epsilon_0$,

$$\mathcal{E}_3(t)+\mathcal{E}_4(t)\lesssim\|Z_{t,\alpha'\alpha'}\|_{L^2}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2\le c_2(\delta)^{-1}(\mathcal{E}_3(t)+\mathcal{E}_4(t)), \tag{4.159}$$

for some constant $c_2(\delta)>0$. Combining with (4.37), (4.116) proves the inequality (4.19) in Proposition 4.3. We also note that by (4.142), (4.149), (4.151), (4.154), (4.141), (4.147), (4.153), (4.155), (4.109),

$$\left\|\partial_{\alpha'}\Theta^{(4)}\right\|_{L^2}+\left\|\Theta^{(5)}\right\|_{\dot H^{1/2}}+\left\|\partial_{\alpha'}\Theta^{(3)}\right\|_{\dot H^{1/2}}\lesssim\epsilon. \tag{4.160}$$

Because

$$\begin{aligned}\partial_{\alpha'}D_t\Theta^{(3)}&=\partial_{\alpha'}\Theta^{(4)}+\partial_{\alpha'}\left[\mathbb{P}_A,b\right]\partial_{\alpha'}\Theta^{(3)},\\ D_t^2\Theta^{(3)}&=\Theta^{(5)}+\left[\mathbb{P}_A,b\right]\partial_{\alpha'}\Theta^{(4)}+D_t\left[\mathbb{P}_A,b\right]\partial_{\alpha'}\Theta^{(3)},\\ D_tD_{\alpha'}\Theta^{(3)}&=D_{\alpha'}D_t\Theta^{(3)}-D_{\alpha'}Z_tD_{\alpha'}\Theta^{(3)},\end{aligned} \tag{4.161}$$

using the identity

$$\left[b,b;\partial_{\alpha'}\Theta^{(3)}\right]=-<b,b,\partial_{\alpha'}\Theta^{(3)}>+\partial_{\alpha'}\Theta^{(3)}\left[b,b;1\right], \tag{4.162}$$

we have by (C.17), (C.20), (C.40), (B.34), (C.31), (C.4), (C.24), (C.23),

$$\left\|\partial_{\alpha'}D_t\Theta^{(3)}\right\|_{L^2}+\left\|D_t^2\Theta^{(3)}\right\|_{\dot H^{1/2}}+\left\|D_tD_{\alpha'}\Theta^{(3)}\right\|_{L^2}\lesssim\epsilon. \tag{4.163}$$

4.3.3. *The estimates for $\frac{d}{dt}\mathfrak{E}_1(t)$, $\frac{d}{dt}\mathcal{E}_j(t)$, $j\ge 2$.* In this section we prove the inequalities (4.20) and (4.24). This requires us to estimate $\frac{d}{dt}\mathfrak{E}_1(t)$, $\frac{d}{dt}\mathcal{E}_j(t)$, $2\le j\le 4$. We will use Theorem 2.8 for $\frac{d}{dt}\mathfrak{E}_1(t)$, and (4.12)-(4.13)-(4.14) for $\frac{d}{dt}\mathcal{E}_j(t)$, $2\le j\le 4$. We refer the reader to Appendix D for a list of the quantities controlled by ϵ.

Step 5. In this step we want to use (2.39)-(2.40) to show that for $0<\epsilon\le\epsilon_0(\delta)$, where $\epsilon_0(\delta)$ is as in Steps 1-4,

$$\frac{d}{dt}\mathfrak{E}_1(t)\lesssim\frac{\epsilon^3}{\delta^3}\mathfrak{E}_1(t),\qquad \text{for } t\in[0,T_0]. \tag{4.164}$$

The prove for (4.164) is quite straightforward. Observe that $D_{\alpha'}\Theta^{(1)}=i\left(1-\frac{1}{Z_{,\alpha'}}\right)$ by (4.33), and by (B.1),

$$\overline{Z}_{tt}=i\left(1-\frac{1}{Z_{,\alpha'}}\right)-i\,\frac{A_1-1}{Z_{,\alpha'}}=D_{\alpha'}\Theta^{(1)}-i\,\frac{A_1-1}{Z_{,\alpha'}}; \tag{4.165}$$

recall the notation (4.5), (2.24), (2.27), we write

$$\begin{aligned}
&\operatorname{Re}\int i\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_H\left(G^{(2)}-D_t\mathbb{P}_HG^{(1)}\right)d\alpha'-I_{2,1}\\
&=\frac{1}{2\pi}\operatorname{Re}\iint(\overline{D_{\alpha'}\Theta^{(1)}}(\alpha',t)-Z_{tt}(\alpha',t))\frac{\mathfrak{D}_t(\theta\overline{\theta})\mathfrak{D}_t\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\
&+\frac{1}{2\pi}\operatorname{Re}\iint\overline{D_{\alpha'}\Theta^{(1)}}(\alpha',t)\frac{\mathfrak{D}_t(\theta\overline{\theta})(\lambda^1-\mathfrak{D}_t\theta)}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\
&+\frac{1}{2}\operatorname{Re}\int i\,\overline{D_{\alpha'}\Theta^{(1)}}(\alpha',t)\left(<\overline{Z}_t,i\frac{1-A_1}{\overline{Z}_{,\alpha'}},-i\frac{1}{Z_{,\alpha'}}>+<i\frac{A_1-1}{Z_{,\alpha'}},Z_t,-i\frac{1}{Z_{,\alpha'}}>\right)d\alpha',
\end{aligned}\tag{4.166}$$

so by (C.29), and the estimates in §4.3.1,

$$\begin{aligned}
&\left|\operatorname{Re}\int i\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_H\left(G^{(2)}-D_t\mathbb{P}_HG^{(1)}\right)d\alpha'-I_{2,1}\right|\lesssim\|A_1-1\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt,\alpha'}\|_{L^2}\|Z_{tt}\|_{L^2}\\
&+\|A_1-1\|_{L^2}\|Z_{t,\alpha'}\|_{L^2}\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\left(\|Z_{tt,\alpha'}\|_{L^2}+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)\\
&\lesssim\epsilon^3\left(\|Z_t\|_{\dot H^{1/2}}+\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)^2;
\end{aligned}\tag{4.167}$$

observe that $D_{\alpha'}\Theta^{(0)}=\overline{Z}_t$, we write

$$\begin{aligned}
&2\operatorname{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_HD_t\mathbb{P}_H\left(G^{(1)}-D_t\mathbb{P}_HG^{(0)}\right)d\alpha'-I_{1,1}\\
&=\frac{1}{\pi}\operatorname{Re}\int Z_{tt}\left(D_t\int\frac{\mathfrak{D}_t(\theta\overline{\theta})\,\theta}{(\alpha'-\beta')^2}\,d\beta'-\int\frac{\mathfrak{D}_t\left(\mathfrak{D}_t(\theta\overline{\theta})\,\theta\right)}{(\alpha'-\beta')^2}\,d\beta'\right)d\alpha'\\
&+\frac{1}{\pi}\operatorname{Re}\int\partial_{\alpha'}\overline{\Theta^{(1)}}\left(\mathbb{P}_HD_t\mathbb{P}_H\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{\overline{Z}_{,\alpha'}}D_t\right)\int\frac{\mathfrak{D}_t(\theta\overline{\theta})\,\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\
&+\frac{1}{\pi}\operatorname{Re}\int\left(\overline{D_{\alpha'}\Theta^{(1)}}-Z_{tt}\right)D_t\int\frac{\mathfrak{D}_t(\theta\overline{\theta})\,\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\\
&+\operatorname{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_HD_t\mathbb{P}_H\left\{\frac{1}{\overline{Z}_{,\alpha'}}\left(<\overline{Z}_t,i\frac{1-A_1}{\overline{Z}_{,\alpha'}},\overline{Z}_t>+<i\frac{A_1-1}{Z_{,\alpha'}},Z_t,\overline{Z}_t>\right)\right\}d\alpha';
\end{aligned}\tag{4.168}$$

now

$$\mathbb{P}_HD_t\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}f\right)-\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}D_tf\right)=\mathbb{P}_H\left\{D_t\left(\frac{1}{\overline{Z}_{,\alpha'}}\right)f\right\}-\mathbb{P}_H\left[b,\mathbb{P}_A\right]\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}f\right),\tag{4.169}$$

so by (B.28), (B.25), (C.27), (C.29), (C.30), (C.3),

$$\left|2\operatorname{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(1)}}\mathbb{P}_HD_t\mathbb{P}_H\left(G^{(1)}-D_t\mathbb{P}_HG^{(0)}\right)d\alpha'-I_{1,1}\right|\lesssim\frac{\epsilon^3}{\delta}\left(\|Z_t\|_{\dot H^{1/2}}^2+\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right).\tag{4.170}$$

Now the estimates for all the terms in $R_{IC,1}$ are straightforward, using (C.27), (C.29), (C.30), (C.35), (C.36), (C.33), (C.46), (C.48) and the estimates in §4.3.1, we have

$$|R_{IC,1}|\lesssim\epsilon^3\left(\|Z_t\|_{\dot H^{1/2}}+\left\|1-\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\right)^2.\tag{4.171}$$

This together with (4.167), (4.170), (4.103), (4.104), (4.21) proves (4.164).

In the remainder of this paper we will show that for $0<\epsilon\le\epsilon_0(\delta)$, where $\epsilon_0(\delta)$ is as in Steps 2-4,

$$\frac{d}{dt}\mathcal{E}_2(t) \lesssim \epsilon^5, \quad \frac{d}{dt}\mathcal{E}_3(t) \lesssim \epsilon^3 \left(\|Z_{t,\alpha'}\|^2_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|^2_{L^2} \right), \quad \frac{d}{dt}\mathcal{E}_4(t) \lesssim \epsilon^5. \tag{4.172}$$

this together with (4.164), (4.134) proves the inequalities (4.20), (4.24) and finishes the proof for Propositions 4.3, 4.4 as well as Theorem 3.1.

Step 6. We begin with those terms in the remainder $\mathcal{R}_{IC,j} + \mathrm{Re}\,\frac{d}{dt}D_j(t)$, $j \geq 2$, (cf. (4.13)-(4.14)) for which the desired estimates can be directly derived from the inequalities in Appendix C, keep in mind that lower order norms such as $\left\|1 - \frac{1}{Z_{,\alpha'}}\right\|_{L^2}$ or $\|Z_t\|_{\dot H^{1/2}}$ are NOT allowed, as they are not controlled by $L(t)$; and since $\left\|\frac{1}{Z_{,\alpha'}} - 1\right\|_{L^\infty} \leq 1$, $\|Z_{tt}\|_{L^\infty} \lesssim 1$, we can involve $\left\|\frac{1}{Z_{,\alpha'}} - 1\right\|_{L^\infty}$ and $\|Z_{tt}\|_{L^\infty}$ only in sextic or higher order terms, at most once.

We first consider the following type of terms

$$M_{1,j} := \iint \left(b_{\alpha'} + b_{\beta'} - 2\frac{b(\alpha') - b(\beta')}{\alpha' - \beta'}\right) \frac{D_t^j Z_t\, \mathfrak{D}_t^l \theta\, \mathfrak{D}_t^i \overline{\theta}\, \mathfrak{D}_t^k \theta}{(\alpha' - \beta')^2}\, d\alpha' d\beta' \tag{4.173}$$

where $l + i + k = j$. This type of terms appears in (4.2), (2.36) and (2.32), we use (C.35), (C.36) and the estimates in §4.3.1 to handle $M_{1,j}$.

For $j = 2$, $(l, i, k) = (0, 0, 2), (0, 1, 1)$ and their permutations, so

$$|M_{1,2}| \lesssim \left\|D_t^2 Z_t\right\|_{L^2} \|b_{\alpha'}\|_{L^\infty} \left(\|Z_{t,\alpha'}\|^2_{L^2} \left\|D_t^2 Z_t\right\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \|Z_{tt}\|^2_{\dot H^{1/2}} \right) \lesssim \epsilon^5. \tag{4.174}$$

For $j = 3$, $(l, i, k) = (0, 0, 3), (0, 1, 2), (1, 1, 1)$ and their permutations, so

$$\begin{aligned} |M_{1,3}| &\lesssim \left\|D_t^3 Z_t\right\|_{L^2} \|b_{\alpha'}\|_{L^\infty} \left(\|Z_{t,\alpha'}\|^2_{L^2} \left\|D_t^3 Z_t\right\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \|Z_{tt,\alpha'}\|_{L^2} \left\|D_t^2 Z_t\right\|_{L^2} \right) \\ &+ \left\|D_t^3 Z_t\right\|_{L^2} \|b_{\alpha'}\|_{L^\infty} \|Z_{tt,\alpha'}\|_{L^2} \|Z_{tt}\|^2_{\dot H^{1/2}} \lesssim \epsilon^3 \left(\|Z_{t,\alpha'}\|^2_{\dot H^{1/2}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|^2_{L^2} \right). \end{aligned} \tag{4.175}$$

For $j = 4$, $(l, i, k) = (0, 0, 4), (0, 1, 3), (0, 2, 2), (1, 1, 2)$ and their permutations, so a similar argument also gives

$$|M_{1,4}| \lesssim \epsilon^5. \tag{4.176}$$

We next consider terms of the type

$$M_{2,j} := \iint \frac{\overline{\mathcal{P} D_t^m \overline{Z}_t} \mathfrak{D}_t^l \theta\, \mathfrak{D}_t^i \overline{\theta}\, \mathfrak{D}_t^k \theta}{(\alpha' - \beta')^2}\, d\alpha' d\beta' \tag{4.177}$$

where $m + l + i + k = 2j - 1$. We discuss the following cases:

1. $m = j - 1$, $l + i + k = j$, as appeared in (4.2);
2. $i = j$, $m + l + k = j - 1$, as appeared in (2.32)- (2.36);
3. $i = 0$, $m + l + k = 2j - 1$, $0 \leq m, l, k \leq j - 1$, as appeared in (2.32)-(2.36);
4. $m = 0$, $l + i + k = 2j - 1$, $0 \leq l, i, k \leq j - 1$, as appeared in (2.32)-(2.36).

Observe that we can use the symmetry (2.37) to rewrite the corresponding terms in (2.32) as (4.177), and vice versa. We use (C.29), (C.30) and the estimates in §4.3.1 to obtain the inequalities (4.178), (4.179), and (4.180) below.

For $j = 2$, we have the following cases: $m = 1$, $(l, i, k) = (0, 0, 2), (0, 1, 1)$ and permutations; $i = 2$, $(m, l, k) = (0, 0, 1)$ and permutations; $i = 0$, $(m, l, k) = (1, 1, 1)$; $m = 0$, $(l, i, k) = (1, 1, 1)$. It is clear that by

(C.29), (C.30) and the estimates in §4.3.1 we have

$$|M_{2,2}| \lesssim \epsilon^5. \tag{4.178}$$

For $j = 3$, we have the following cases: $m = 2$, $(l, i, k) = (0, 0, 3), (0, 1, 2), (1, 1, 1)$ and permutations; $i = 3$, $(m, l, k) = (0, 0, 2), (0, 1, 1)$ and permutations; $i = 0$, $(m, l, k) = (1, 2, 2)$ and permutations; $m = 0$, $(l, i, k) = (1, 2, 2)$ and permutations. Observe that for the case where $i = 3$, we need to use the symmetry (2.37) to rewrite $M_{2,3}$ as in (2.32). We have by (C.29), (C.30) and the estimates in §4.3.1,

$$|M_{2,3}| \lesssim \epsilon^3 \left(\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}}^2 + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \right). \tag{4.179}$$

For $j = 4$, we have the following cases: $m = 3$, $(l, i, k) = (0, 0, 4), (0, 1, 3), (0, 2, 2), (1, 1, 2)$ and permutations; $i = 4$, $(m, l, k) = (0, 0, 3), (0, 1, 2), (1, 1, 1)$ and permutations; $i = 0$, $(m, l, k) = (3, 3, 1), (3, 2, 2)$ and permutations; $m = 0$, $(l, i, k) = (3, 3, 1), (3, 2, 2)$ and permutations. Again we use the form in (2.32) for the case where $i = 4$. We have quite straightforwardly that

$$|M_{2,4}| \lesssim \epsilon^5. \tag{4.180}$$

Observe that those terms

$$\iint \frac{D_t^j Z_t \, \mathfrak{D}_t^m \left\{ (\mathcal{P}\mathfrak{D}_t^l \theta) \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta - \mathfrak{D}_t^l \theta \, \overline{(\mathcal{P}\mathfrak{D}_t^i \theta)} \, \mathfrak{D}_t^k \theta + \mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, (\mathcal{P}\mathfrak{D}_t^k \theta) \right\}}{(\alpha' - \beta')^2} \, d\alpha' \, d\beta'$$

in (2.32), with $m + l + i + k = j - 1$ can be treated exactly in the same way as $M_{2,j}$, we do not specifically go over these terms.

Now we consider the terms $\iint \frac{\overline{[D_t, \mathcal{P}] D_t^{j-1} \overline{Z}_t} \mathfrak{D}_t^m (\mathfrak{D}_t^l \theta \, \mathfrak{D}_t^i \overline{\theta} \, \mathfrak{D}_t^k \theta)}{(\alpha' - \beta')^2} \, d\alpha' d\beta'$ in (4.2). We use product rules and complex conjugate to reduce it to the following form

$$M_{3,j} := \iint \frac{[D_t, \mathcal{P}] \, D_t^{j-1} \overline{Z}_t \, \mathfrak{D}_t^l \overline{\theta} \, \mathfrak{D}_t^i \theta \, \mathfrak{D}_t^k \overline{\theta}}{(\alpha' - \beta')^2} \, d\alpha' d\beta', \tag{4.181}$$

where $l + i + k = j - 1$. We know by (B.21),

$$[D_t, \mathcal{P}] \, D_t^{j-1} \overline{Z}_t = i \left(\frac{D_t A_1}{A_1} + b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t \right) \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_t^{j-1} \overline{Z}_t, \tag{4.182}$$

so $M_{3,j}$ is sextic; and we have, by (C.3) and the estimates in §4.3.1,

$$\begin{aligned} |M_{3,j}| &\lesssim \left\| D_t^{j-1} Z_t \right\|_{\dot{H}^{1/2}} \left\| \left(\frac{D_t A_1}{A_1} + b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t \right) \frac{A_1}{|Z_{,\alpha'}|^2} < D_t^l Z_t, D_t^i \overline{Z}_t, D_t^k Z_t > \right\|_{\dot{H}^{1/2}} \\ &\lesssim \left\| D_t^{j-1} Z_t \right\|_{\dot{H}^{1/2}} (\|D_t A_1\|_{L^\infty} + \|b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t\|_{L^\infty}) \left\| < D_t^l Z_t, D_t^i \overline{Z}_t, D_t^k Z_t > \right\|_{\dot{H}^{1/2}} \\ &+ \left\| D_t^{j-1} Z_t \right\|_{\dot{H}^{1/2}} \left(\left\| \frac{D_t A_1}{|Z_{,\alpha'}|^2} \right\|_{\dot{H}^{1/2}} + \left\| \frac{(b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t) A_1}{|Z_{,\alpha'}|^2} \right\|_{\dot{H}^{1/2}} \right) \left\| < D_t^l Z_t, D_t^i \overline{Z}_t, D_t^k Z_t > \right\|_{L^\infty}; \end{aligned} \tag{4.183}$$

we can also use (C.4) to get

$$\begin{aligned} |M_{3,j}| &\lesssim \left\| D_t^{j-1} Z_t \right\|_{\dot{H}^{1/2}} (\|D_t A_1\|_{L^\infty} + \|b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t\|_{L^\infty}) \left\| < D_t^l Z_t, D_t^i \overline{Z}_t, D_t^k Z_t > \right\|_{\dot{H}^{1/2}} \\ &+ \left\| D_t^{j-1} Z_t \right\|_{\dot{H}^{1/2}} \left(\left\| \frac{D_t A_1}{|Z_{,\alpha'}|^2} \right\|_{\dot{H}^1} + \left\| \frac{(b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t) A_1}{|Z_{,\alpha'}|^2} \right\|_{\dot{H}^1} \right) \left\| < D_t^l Z_t, D_t^i \overline{Z}_t, D_t^k Z_t > \right\|_{L^2}. \end{aligned} \tag{4.184}$$

For $j = 2$, we have $(l, i, k) = (0, 0, 1)$ and permutations; for $j = 3$, we have $(l, i, k) = (0, 0, 2), (0, 1, 1)$ and permutations; for $j = 4$, we have $(l, i, k) = (0, 0, 3), (0, 1, 2), (1, 1, 1)$ and permutations. We use (4.183) for

$j=2,3$ and (4.184) for $j=4$. By (C.3), (C.31), (C.29), (C.30) and the estimates in §4.3.1 we get

$$|M_{3,2}|\lesssim\epsilon^5,\quad |M_{3,3}|\lesssim\epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\quad |M_{3,4}|\lesssim\epsilon^6. \tag{4.185}$$

Next we look at the terms of type

$$M_{4,j}:=\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{\overline{\mathcal{P}D_t^{j-1}\overline{Z}_t}\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha'-\beta')^2}\,d\alpha'd\beta' \tag{4.186}$$

and

$$M_{5,j}:=\iint\left(6\frac{(b(\alpha')-b(\beta'))^2}{(\alpha'-\beta')^2}-4(b_{\alpha'}+b_{\beta'})\frac{(b(\alpha')-b(\beta'))}{(\alpha'-\beta')}+2b_{\alpha'}b_{\beta'}\right)\frac{D_t^jZ_t\,\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha'-\beta')^2}\,d\alpha'd\beta' \tag{4.187}$$

in (4.2), where $l+i+k=j-1$. Observe that both terms are sextic, and we used product rules to reduce the terms in (4.2) to the forms of (4.186), (4.187). We take advantage of the fact that these terms are sextic and use (C.27), (C.42) to deduce for $m=4,5$,

$$|M_{m,2}|\lesssim\epsilon^5,\quad |M_{m,3}|\lesssim\epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\quad |M_{m,4}|\lesssim\epsilon^5. \tag{4.188}$$

Now we consider the term

$$M_{6,j}:=\iint\frac{D_t^jZ_t\,[\mathcal{P},\mathfrak{D}_t^m]\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'=\iint\frac{[\mathfrak{P},D_t^m]\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)\,\mathfrak{D}_t^j\overline{\theta}}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta' \tag{4.189}$$

in (2.32), where $m+l+i+k=j-1$. Here we used the symmetry (2.37) to get the second equality above. We expand $[\mathfrak{P},D_t^m]$ by (B.21), (B.24), and use (C.29) and the estimates in §4.3.1 to obtain

$$|M_{6,2}|\lesssim\epsilon^6,\quad |M_{6,3}|\lesssim\epsilon^4\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\quad |M_{6,4}|\lesssim\epsilon^6. \tag{4.190}$$

Now we treat the terms

$$M_{7,j}:=i\iint\left(\partial_{\beta'}\frac{A_1(\beta')}{|Z_{,\beta'}|^2}-2\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2}-\frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha'-\beta'}\right)\frac{D_t^jZ_t(\alpha')\,\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta' \tag{4.191}$$

and

$$M_{8,j}:=\iint\left(\partial_{\beta'}D_tb(\beta')-2\frac{D_tb(\alpha')-D_tb(\beta')}{\alpha'-\beta'}\right)\frac{D_t^jZ_t(\alpha')\,\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha'-\beta')^2}\,d\alpha'd\beta' \tag{4.192}$$

where $l+i+k=j-1$ in (2.32)-(2.36) and (4.2). Here we used the product rules to convert the terms in (2.32) and (4.2) to the forms in (4.191), (4.192). We use (C.46) and (C.48) and the estimates in §4.3.1 to obtain for $m=7,8$,

$$|M_{m,2}|\lesssim\epsilon^5,\quad |M_{m,3}|\lesssim\epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\quad |M_{m,4}|\lesssim\epsilon^5. \tag{4.193}$$

We also have the following terms from $R^{(0)}_{0;l,\bar{i},k}$, where $l+i+k=2j-1$, $0\le l,i,k\le j-1$, in (2.32)-(2.36),

$$\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{(Z_t\mathfrak{D}_t-Z_{tt})\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'; \tag{4.194}$$

we use the symmetry (2.37) to rewrite it as

$$M_{9,j}:=\frac12\iint\left(b_{\alpha'}+b_{\beta'}-2\frac{b(\alpha')-b(\beta')}{\alpha'-\beta'}\right)\frac{\left(\overline{\theta}\mathfrak{D}_t-\mathfrak{D}_t\overline{\theta}\right)\left(\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta\right)}{(\alpha'-\beta')^2}\,d\alpha'\,d\beta'; \tag{4.195}$$

and use product rules to expand. By (C.35), (C.36) and the estimates in §4.3.1, we have

$$\begin{aligned}|M_{9,3}| &\lesssim \|b_{\alpha'}\|_{L^\infty} \|Z_{t,\alpha'}\|_{L^2} \left(\left\|D_t^3 Z_t\right\|_{L^2} \|Z_{tt,\alpha'}\|_{L^2} \|Z_{ttt}\|_{L^2} + \|Z_{ttt}\|_{\dot H^{1/2}}^2 \|Z_{ttt}\|_{L^2}\right)\\ &+ \|b_{\alpha'}\|_{L^\infty} \|Z_{tt,\alpha'}\|_{L^2}^2 \|Z_{ttt}\|_{L^2}^2 \lesssim \epsilon^3 \left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2 + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right),\end{aligned} \tag{4.196}$$

and

$$|M_{9,4}| \lesssim \epsilon^5. \tag{4.197}$$

For $j = 2$, $(l, i, k) = (1, 1, 1)$, we have, by (C.36),

$$\begin{aligned}&\left|\iint \left(b_{\alpha'} + b_{\beta'} - 2\frac{b(\alpha') - b(\beta')}{\alpha' - \beta'}\right) \frac{\overline{\theta}\mathfrak{D}_t\left(\mathfrak{D}_t\theta\,\mathfrak{D}_t\overline{\theta}\,\mathfrak{D}_t\theta\right)}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'\right|\\ &\lesssim \|Z_{ttt}\|_{L^2} \|Z_{t,\alpha'}\|_{L^2} \|Z_{tt}\|_{\dot H^{1/2}}^2 \|b_{\alpha'}\|_{L^\infty} \lesssim \epsilon^5;\end{aligned} \tag{4.198}$$

and by (C.37),

$$\left|\iint \left(b_{\alpha'} + b_{\beta'} - 2\frac{b(\alpha') - b(\beta')}{\alpha' - \beta'}\right) \frac{\mathfrak{D}_t\overline{\theta}\mathfrak{D}_t\theta\,\mathfrak{D}_t\overline{\theta}\,\mathfrak{D}_t\theta}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'\right| \lesssim \|b_{\alpha'}\|_{L^\infty} \|Z_{tt}\|_{\dot H^{1/2}}^4 \lesssim \epsilon^5; \tag{4.199}$$

so

$$|M_{9,2}| \lesssim \epsilon^5. \tag{4.200}$$

We can estimate the second and third term on the right hand side of (4.4) similarly and obtain

$$\begin{aligned}&\left|\iint \mathbb{H}b_{\alpha'} \frac{\mathfrak{D}_t\left(\overline{\theta}\,\mathfrak{D}_t\theta\mathfrak{D}_t\overline{\theta}\mathfrak{D}_t\theta\right)}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'\right| \lesssim \|\mathbb{H}b_{\alpha'}\|_{L^\infty} \|Z_{tt}\|_{\dot H^{1/2}}^4\\ &+ \|\mathbb{H}b_{\alpha'}\|_{L^2} \|Z_{ttt}\|_{L^\infty} \|Z_{tt}\|_{\dot H^{1/2}}^2 \|Z_{t,\alpha'}\|_{L^2} \lesssim \epsilon^5;\end{aligned} \tag{4.201}$$

$$\begin{aligned}&\left|\iint \left(b_{\beta'} - 2\frac{b(\alpha') - b(\beta')}{\alpha' - \beta'}\right) \mathbb{H}b_{\alpha'} \frac{\overline{\theta}\,\mathfrak{D}_t\theta\mathfrak{D}_t\overline{\theta}\mathfrak{D}_t\theta}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'\right|\\ &\lesssim \|\mathbb{H}b_{\alpha'}\|_{L^2} \|b_{\alpha'}\|_{L^\infty} \|Z_{tt}\|_{\dot H^{1/2}}^2 \|Z_{t,\alpha'}\|_{L^2} \|Z_{tt}\|_{L^\infty} \lesssim \epsilon^5.\end{aligned} \tag{4.202}$$

Observe that the first two (non-zero) terms in $R^{(0)}_{0;l,\bar{i},k}$, $l+i+k = 2j-1$, $0 \le l, i, k \le j-1$, cf. (2.32)-(2.36), have been covered in $M_{2,j}$, there is one more term left, which is

$$i\iint \left(\partial_{\alpha'} \frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + \partial_{\beta'} \frac{A_1(\beta')}{|Z_{,\beta'}|^2} - 2\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} - \frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha' - \beta'}\right) \frac{Z_t\,\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'. \tag{4.203}$$

We use symmetry (2.37) to rewrite it as

$$M_{10,j} := \frac{1}{2}i\iint \left(\partial_{\alpha'} \frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + \partial_{\beta'} \frac{A_1(\beta')}{|Z_{,\beta'}|^2} - 2\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} - \frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha' - \beta'}\right) \frac{\overline{\theta}\,\mathfrak{D}_t^l\theta\,\mathfrak{D}_t^i\overline{\theta}\,\mathfrak{D}_t^k\theta}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'. \tag{4.204}$$

By (C.27), (C.42) and the estimates in §4.3.1, we have

$$|M_{10,3}| \lesssim \epsilon^3 \left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2 + \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right), \qquad |M_{10,4}| \lesssim \epsilon^5. \tag{4.205}$$

For $j = 2$ and $(l, i, k) = (1, 1, 1)$, we have by (C.37) and Hölder's inequality,

$$\left|\iint \left(\frac{\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} - \frac{A_1(\beta')}{|Z_{,\beta'}|^2}}{\alpha' - \beta'}\right) \frac{\overline{\theta}\,\mathfrak{D}_t\theta\,\mathfrak{D}_t\overline{\theta}\,\mathfrak{D}_t\theta}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta'\right| \lesssim \|Z_{tt}\|_{\dot H^{1/2}}^3 \left\|\frac{A_1}{|Z_{,\alpha'}|^2}\right\|_{\dot H^{1/2}} \|Z_{t,\alpha'}\|_{L^\infty} \lesssim \epsilon^5. \tag{4.206}$$

Step 7. We have treated all the terms in $\mathcal{R}_{IC,j} + \mathrm{Re}\,\frac{d}{dt} D_j(t)$ but the following two: the first is

$$M_{11,j} := \iint \partial_{\alpha'} \left(i\,\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + D_t b(\alpha') \right) \frac{D_t^j Z_t(\alpha')\,\mathfrak{D}_t^l \theta\,\mathfrak{D}_t^i \overline{\theta}\,\mathfrak{D}_t^k \theta}{(\alpha'-\beta')^2}\, d\alpha'\, d\beta' \tag{4.207}$$

where $l+i+k = j-1$, $2 \le j \le 4$; the second is for $j=2$ only, from $\mathrm{Re}\left(\frac{1}{4\pi} R^{(0)}_{\overline{0};1,\overline{1},1} + \frac{d}{dt} D_2(t)\right)$:

$$M_{12,2} := \iint \partial_{\alpha'} \left(i\,\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + D_t \mathbb{H} b(\alpha') \right) \frac{\overline{\theta}\,\mathfrak{D}_t \theta\,\mathfrak{D}_t \overline{\theta}\,\mathfrak{D}_t \theta}{(\alpha'-\beta')^2}\, d\alpha'\, d\beta', \tag{4.208}$$

here we used the symmetry (2.37) to rewrite the term from (2.32), and used product rules to arrive at the term in (4.207).

By (B.39),

$$\mathbb{P}_H \left(D_t b + i \frac{A_1}{|Z_{,\alpha'}|^2} - i \right) = i \mathbb{P}_H \left(\frac{A_1 - 1}{|Z_{,\alpha'}|^2} \right) + i \mathbb{P}_H (G^{(1)}) + i\,[b, \mathbb{P}_H]\, \partial_{\alpha'} \overline{\Theta^{(2)}}; \tag{4.209}$$

using (B.22), (B.38) gives

$$\begin{aligned} D_t \mathbb{H} b + i \frac{A_1}{|Z_{,\alpha'}|^2} - i &= [D_t, \mathbb{H}]\, b + 2i\, \mathrm{Im}\, \mathbb{P}_H \left(D_t b + i \frac{A_1}{|Z_{,\alpha'}|^2} - i \right) \\ &= i\,\frac{A_1 - 1}{|Z_{,\alpha'}|^2} + 2i\, \mathrm{Re}\, \mathbb{P}_H (G^{(1)}) + \frac{1}{2}\, [b, \mathbb{H}]\, \partial_{\alpha'} b; \end{aligned} \tag{4.210}$$

and we know for $j \ge 2$

$$\mathbb{P}_A D_t^j \overline{Z}_t = -i \left[\mathbb{P}_A, \frac{A_1}{|Z_{,\alpha'}|^2} \right] \partial_{\alpha'} D_t^{j-2} \overline{Z}_t + \mathbb{P}_A \mathcal{P} D_t^{j-2} \overline{Z}_t; \tag{4.211}$$

so by (C.17), (C.19), (C.20), (4.120) and the estimates in §4.3.1, we have for $2 \le j \le 4$,

$$\left\| \partial_{\alpha'} \left(D_t \mathbb{H} b + i \frac{A_1}{|Z_{,\alpha'}|^2} \right) \right\|_{L^2} + \left\| \partial_{\alpha'} \mathbb{P}_H \left(D_t b + i \frac{A_1}{|Z_{,\alpha'}|^2} \right) \right\|_{L^2} + \left\| \mathbb{P}_A D_t^j \overline{Z}_t \right\|_{L^2} \lesssim \epsilon^2, \tag{4.212}$$

in particular for $j = 3$,

$$\left\| \partial_{\alpha'} \mathbb{P}_H \left(D_t b + i \frac{A_1}{|Z_{,\alpha'}|^2} \right) \right\|_{L^2} + \left\| \mathbb{P}_A D_t^3 \overline{Z}_t \right\|_{L^2} \lesssim \epsilon \left(\|Z_{t,\alpha'}\|_{\dot{H}^{1/2}} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right); \tag{4.213}$$

by (C.40), (C.4), (4.153), and the estimates in §4.3.1, the following also holds,

$$\left\| \partial_{\alpha'} \mathbb{P}_H \left(D_t b + i \frac{A_1}{|Z_{,\alpha'}|^2} \right) \right\|_{\dot{H}^{1/2}} \lesssim \epsilon^2. \tag{4.214}$$

Observe that $M_{12,2}$ is sextic. Applying (C.36) gives

$$|M_{12,2}| \lesssim \left\| \partial_{\alpha'} \left(D_t \mathbb{H} b + i \frac{A_1}{|Z_{,\alpha'}|^2} \right) \right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}\, \|Z_{tt}\|^2_{\dot{H}^{1/2}}\, \|Z_{tt}\|_{L^\infty} \lesssim \epsilon^5. \tag{4.215}$$

We rewrite $M_{11,j}$ as

$$\begin{aligned} M_{11,j} &= \iint \partial_{\alpha'} \mathbb{P}_H \left(i\,\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + D_t b(\alpha') \right) \frac{D_t^j Z_t(\alpha')\,\mathfrak{D}_t^l \theta\,\mathfrak{D}_t^i \overline{\theta}\,\mathfrak{D}_t^k \theta}{(\alpha'-\beta')^2}\, d\alpha'\, d\beta' \\ &+ \iint \partial_{\alpha'} \mathbb{P}_A \left(i\,\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + D_t b(\alpha') \right) \frac{\mathbb{P}_H \left(D_t^j Z_t \right)(\alpha')\,\mathfrak{D}_t^l \theta\,\mathfrak{D}_t^i \overline{\theta}\,\mathfrak{D}_t^k \theta}{(\alpha'-\beta')^2}\, d\alpha'\, d\beta' \\ &+ \int \partial_{\alpha'} \mathbb{P}_A \left(i\,\frac{A_1(\alpha')}{|Z_{,\alpha'}|^2} + D_t b(\alpha') \right) \left[\mathbb{P}_H, < D_t^l \overline{Z}_t, D_t^i Z_t, D_t^k \overline{Z}_t > \right] \mathbb{P}_A \left(D_t^j Z_t \right)(\alpha')\, d\alpha' \\ &= I_j + II_j + III_j, \end{aligned} \tag{4.216}$$

where in the last term we used the Cauchy integral formula to rewrite it as a commutator. We have for $j = 2$, $(l,i,k) = (0,0,1)$ and permutations; $j = 3$, $(l,i,k) = (0,0,2), (0,1,1)$ and permutations; $j = 4$,

$(l,i,k)=(0,0,3),(0,1,2),(1,1,1)$ and permutations. Observe that the first two terms in (4.216), I_j, II_j, are sextic; we apply (C.27), (C.42) to obtain

$$|I_2|+|II_2|+|I_4|+|II_4|\lesssim \epsilon^5,\quad |I_3|+|II_3|\lesssim \epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right); \tag{4.217}$$

applying (C.17), (C.31) on III_j yields

$$|III_2|+|III_4|\lesssim \epsilon^5,\quad |III_3|\lesssim \epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right), \tag{4.218}$$

therefore

$$|M_{11,2}|+|M_{11,4}|\lesssim \epsilon^5,\quad |M_{11,3}|\lesssim \epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right).$$

Sum up the estimates in Step 6 and Step 7 we have

$$\left|\mathcal{R}_{IC,2}+\operatorname{Re}\frac{d}{dt}D_2(t)\right|+|\mathcal{R}_{IC,4}|\lesssim \epsilon^5,\quad |\mathcal{R}_{IC,3}|\lesssim \epsilon^3\left(\|Z_{t,\alpha'}\|_{\dot H^{1/2}}^2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2\right). \tag{4.219}$$

Step 8. In this step we treat the remaining terms in (4.13), namely

$$2\operatorname{Re}\sum_{l=0}^{j-1}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_H G^{(j-1-l)}\right)d\alpha'-I_{1,j}-\frac{d}{dt}H_j, \tag{4.220}$$

$$\operatorname{Re}\int i\,\partial_{\alpha'}\overline{\Theta^{(j)}}\mathbb{P}_H\left(G^{(j+1)}-D_t\mathbb{P}_H G^{(j)}\right)d\alpha'-I_{2,j}. \tag{4.221}$$

Let

$$J_{l,j}=\frac{1}{2\pi}\iint D_t^j Z_t(\alpha',t)\frac{\mathfrak{D}_t^{l+1}\left\{\mathfrak{D}_t\left(\theta(\alpha',\beta',t)\overline{\theta(\alpha',\beta',t)}\right)\mathfrak{D}_t^{j-l-1}\theta(\alpha',\beta',t)\right\}}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha', \tag{4.222}$$

By (2.26), (2.27),

$$I_{1,j}=2\operatorname{Re}\sum_{l=0}^{j-1}J_{l,j},\qquad I_{2,j}=\operatorname{Re}J_{-1,j}.$$

Because

$$D_{\alpha'}\Theta^{(1)}=i(1-\frac{1}{Z_{,\alpha'}})$$

by (2.10), (4.33) and the notations (2.24) and (4.5)[17] we can write

$$\mathbb{P}_H\left(G^{(j-l)}-D_t\mathbb{P}_H G^{(j-1-l)}\right)=\frac{1}{2\pi i}\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\,d\beta'\right), \tag{4.223}$$

[17]Observe that $\lambda^0=\theta$.

and

$$
\begin{aligned}
N_{l,j} &:= 2\pi \left(\int i\,\partial_{\alpha'} \overline{\Theta^{(j)}}\, (\mathbb{P}_H D_t)^{l+1}\, \mathbb{P}_H \left(G^{(j-l)} - D_t \mathbb{P}_H G^{(j-1-l)} \right) d\alpha' - J_{l,j} \right) \\
&= \int \partial_{\alpha'} \overline{\Theta^{(j)}} \left\{ (\mathbb{P}_H D_t)^{l+1}\, \mathbb{P}_H \frac{1}{\overline{Z}_{,\alpha'}} - \frac{1}{\overline{Z}_{,\alpha'}} D_t^{l+1} \right\} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&+ \int \overline{D_{\alpha'}\Theta^{(j)}} \left\{ D_t^{l+1} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\, d\beta' - \int \frac{\mathfrak{D}_t^{l+1} \left((\theta \overline{\lambda^1} + \overline{\theta}\lambda^1)\lambda^{j-l-1} \right)}{(\alpha'-\beta')^2}\, d\beta' \right\} d\alpha' \\
&+ \int \overline{D_{\alpha'}\Theta^{(j)}} \int \frac{\mathfrak{D}_t^{l+1} \left(\theta(\overline{\lambda^1} - \mathfrak{D}_t \overline{\theta})\lambda^{j-l-1} + \overline{\theta}(\lambda^1 - \mathfrak{D}_t\theta)\lambda^{j-l-1} \right)}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&+ \int \overline{D_{\alpha'}\Theta^{(j)}} \int \frac{\mathfrak{D}_t^{l+1} \left(\mathfrak{D}_t(\theta\overline{\theta})(\lambda^{j-l-1} - \mathfrak{D}_t^{j-l-1}\theta) \right)}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&+ \int \left(\overline{D_{\alpha'}\Theta^{(j)}} - D_t^j Z_t \right) \int \frac{\mathfrak{D}_t^{l+1} \left(\mathfrak{D}_t(\theta\overline{\theta}) \mathfrak{D}_t^{j-l-1}\theta \right)}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&= N_{l,j,1} + N_{l,j,2} + N_{l,j,3} + N_{l,j,4} + N_{l,j,5}.
\end{aligned} \tag{4.224}
$$

Observe that there is a derivative loss in $N_{j-1,j,3}$ and $N_{j-1,j,5}$.[18] Because $\lambda^0 = \theta$, so $N_{j-1,j,4} = 0$. We combine $N_{j-1,j,3} + N_{j-1,j,5}$ with $-\pi \frac{d}{dt} H_j$, cf. (4.7), and write

$$
\begin{aligned}
&\operatorname{Re}(N_{j-1,j,3} + N_{j-1,j,5}) - \pi \frac{d}{dt} H_j = \tilde{N}_{j,3} + \tilde{N}_{j,4} + \tilde{N}_{j,5} - \pi R_{H,j} \\
&:= \operatorname{Re} \int \overline{D_{\alpha'}\Theta^{(j)}} \int \frac{\overline{\theta}\theta \left(\mathfrak{D}_t^j \lambda^1 - \mathfrak{D}_t \lambda^j \right) + \theta\theta \left(\mathfrak{D}_t^j \overline{\lambda^1} - \mathfrak{D}_t \overline{\lambda^j} \right)}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&+ \operatorname{Re} \iint D_t^j Z_t \frac{\mathfrak{D}_t^j \left(\theta(\overline{\lambda^1} - \mathfrak{D}_t\overline{\theta})\theta + \overline{\theta}(\lambda^1 - \mathfrak{D}_t\theta)\theta \right) - \overline{\theta}\theta \mathfrak{D}_t^j(\lambda^1 - \mathfrak{D}_t\theta) - \theta\theta \mathfrak{D}_t^j(\overline{\lambda^1} - \mathfrak{D}_t\overline{\theta})}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' \\
&+ \operatorname{Re} \int \left(\overline{D_{\alpha'}\Theta^{(j)}} - D_t^j Z_t \right) \int \frac{\mathfrak{D}_t^j \left((\theta\,\overline{\lambda^1} + \overline{\theta}\lambda^1)\theta \right) - \overline{\theta}\,\theta\,\mathfrak{D}_t^j \lambda^1 - \theta\,\theta\,\mathfrak{D}_t^j \overline{\lambda^1}}{(\alpha'-\beta')^2}\, d\beta'\, d\alpha' - \pi R_{H,j};
\end{aligned} \tag{4.225}
$$

this cancels out the derivative lossing terms for $j = 4$, and enables us to get the desired estimates for $j = 3$.

We sum up the above decomposition:

$$
\begin{aligned}
&\pi \left(2 \operatorname{Re} \sum_{l=0}^{j-1} \int i\,\partial_{\alpha'} \overline{\Theta^{(j)}} (\mathbb{P}_H D_t)^{l+1} \mathbb{P}_H \left(G^{(j-l)} - D_t \mathbb{P}_H G^{(j-1-l)} \right) d\alpha' - I_{1,j} - \frac{d}{dt} H_j(t) \right) \\
&= \operatorname{Re} \left(\sum_{l=0}^{j-2} \sum_{k=1}^{5} N_{l,j,k} + \sum_{k=1}^{2} N_{j-1,j,k} \right) + \sum_{k=3}^{5} \tilde{N}_{j,k} - \pi R_{H,j},
\end{aligned} \tag{4.226}
$$

and

$$
2\pi \left(\operatorname{Re} \int i\,\partial_{\alpha'} \overline{\Theta^{(j)}} \mathbb{P}_H \left(G^{(j+1)} - D_t \mathbb{P}_H G^{(j)} \right) d\alpha' - I_{2,j} \right) = \operatorname{Re} \sum_{k=1}^{5} N_{-1,j,k}. \tag{4.227}
$$

Observe that all the terms in (4.224), (4.225) are quintic.

[18] Namely $N_{j-1,j,3}$ and $N_{j-1,j,5}$ contain factors that cannot be controlled by $\mathcal{E}_j(t)$.

Now we use (B.26) to write

$$\begin{aligned}&\mathbb{P}_H\left((\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{\overline{Z}_{,\alpha'}}D_t^{l+1}\right)=\left((\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H-\mathbb{P}_H D_t^{l+1}\right)\frac{1}{\overline{Z}_{,\alpha'}}+\mathbb{P}_H\left[D_t^{l+1},\frac{1}{\overline{Z}_{,\alpha'}}\right]\\ &=-\sum_{k=0}^{l}(\mathbb{P}_H D_t)^k\mathbb{P}_H\left[b,\mathbb{P}_A\right]\partial_{\alpha'}D_t^{l-k}\frac{1}{\overline{Z}_{,\alpha'}}+\mathbb{P}_H\sum_{k=0}^{l}\binom{l+1}{k+1}\left(D_t^{k+1}\frac{1}{\overline{Z}_{,\alpha'}}\right)D_t^{l-k};\end{aligned}\tag{4.228}$$

because

$$D_{\alpha'}\Theta^{(1)}-\overline{Z}_{tt}=i\,\frac{A_1-1}{Z_{,\alpha'}},\tag{4.229}$$

by (B.41) we have

$$\begin{aligned}&D_{\alpha'}\Theta^{(l+1)}-D_t^{l+1}\overline{Z}_t=D_{\alpha'}\Theta^{(l+1)}-D_t^lD_{\alpha'}\Theta^{(1)}+D_t^l(D_{\alpha'}\Theta^{(1)}-\overline{Z}_{tt})\\ &=\sum_{k=0}^{l-1}(\mathbb{P}_H D_t)^k\left[\frac{1}{Z_{,\alpha'}},\mathbb{P}_H\right]\partial_{\alpha'}D_t\Theta^{(l-k)}+\sum_{k=0}^{l-1}(\mathbb{P}_H D_t)^k\left[\mathbb{P}_H,\frac{1}{Z_{,\alpha'}}D_{\alpha'}\Theta^{(l-k)}\right]Z_{t,\alpha'}\\ &+i\sum_{k=0}^{l-1}D_t^k[\mathbb{P}_A,b]\partial_{\alpha'}(\mathbb{P}_H D_t)^{l-1-k}\frac{1}{Z_{,\alpha'}}+i\,D_t^l\left(\frac{A_1-1}{Z_{,\alpha'}}\right),\end{aligned}\tag{4.230}$$

and

$$\begin{aligned}&D_tD_{\alpha'}\Theta^{(j)}-D_t^jD_{\alpha'}\Theta^{(1)}=\sum_{k=0}^{j-2}D_t(\mathbb{P}_H D_t)^k\left[\frac{1}{Z_{,\alpha'}},\mathbb{P}_H\right]\partial_{\alpha'}D_t\Theta^{(j-1-k)}\\ &+\sum_{k=0}^{j-2}D_t(\mathbb{P}_H D_t)^k\left[\mathbb{P}_H,\frac{1}{Z_{,\alpha'}}D_{\alpha'}\Theta^{(j-1-k)}\right]Z_{t,\alpha'}+i\sum_{k=0}^{j-2}D_t^{k+1}[\mathbb{P}_A,b]\partial_{\alpha'}(\mathbb{P}_H D_t)^{j-2-k}\frac{1}{Z_{,\alpha'}}.\end{aligned}\tag{4.231}$$

We are now ready to do the estimates. We begin with $N_{l,j,1}$. Observe that by the Cauchy integral formula, we can insert a $\mathbb{P}_H$ to write it as

$$N_{l,j,1}=\int\partial_{\alpha'}\overline{\Theta^{(j)}}\,\mathbb{P}_H\left\{(\mathbb{P}_H D_t)^{l+1}\,\mathbb{P}_H\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{\overline{Z}_{,\alpha'}}D_t^{l+1}\right\}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha'\tag{4.232}$$

and it is clear that $N_{-1,j,1}=0$. We estimate $|N_{l,j,1}|$ for $0\le l\le j-1$, $2\le j\le 4$. By (4.228) we need to estimate, for $0\le k\le l$,

$$A_{k,l,j}:=\left\|(\mathbb{P}_H D_t)^k\mathbb{P}_H\left[b,\mathbb{P}_A\right]\partial_{\alpha'}D_t^{l-k}\left(\frac{1}{\overline{Z}_{,\alpha'}}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\,d\beta'\right)\right\|_{L^2},\tag{4.233}$$

$$B_{k,l,j}:=\left\|\mathbb{P}_H\;\left\{\left(D_t^{k+1}\frac{1}{\overline{Z}_{,\alpha'}}\right)D_t^{l-k}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\,d\beta'\right\}\right\|_{L^2}.\tag{4.234}$$

By (B.29), (C.29), (C.30), (C.27), we have, for $k<l$ or $l<j-1$ and $2\le j\le 4$,

$$\left\|D_t^{l-k}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^{j-l-1}}{(\alpha'-\beta')^2}\,d\beta'\right\|_{L^2}\lesssim\epsilon^3,\tag{4.235}$$

and by (4.129) and (C.31), (C.32),

$$\left\|\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^1}{(\alpha'-\beta')^2}\,d\beta'\right\|_{\dot H^{1/2}}+\left\|D_t\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\|_{\dot H^{1/2}}\lesssim\epsilon^2\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right),\tag{4.236}$$

$$\left\|\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\|_{\dot H^{1/2}}\lesssim\epsilon^3,\quad\left\|\partial_{\alpha'}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\|_{L^2}\lesssim\epsilon^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2},\tag{4.237}$$

therefore by (B.25), (B.32), (C.19), (C.20), (C.23), (C.21), (C.27), we have, except for the cases where $l = k = j-1$ for $j = 3, 4$,

$$A_{k,l,2} \lesssim \epsilon^4, \quad A_{k,l,3} \lesssim \epsilon^3 \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}} \right), \quad A_{k,l,4} \lesssim \epsilon^4. \tag{4.238}$$

We also have, by (B.29), (C.29), (C.27), (C.33), (C.42), that except for the cases where $l = k = j-1$ for $2 \le j \le 4$,

$$B_{k,l,2} \lesssim \epsilon^4, \quad B_{k,l,3} \lesssim \epsilon^3 \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}} \right), \quad B_{k,l,4} \lesssim \epsilon^4. \tag{4.239}$$

Now all the terms in $A_{j-1,j-1,j}$, $j = 3, 4$, after expanding by (B.25), (B.32), can be handled similarly, except for one, namely

$$\mathbb{P}_H \left[D_t^{j-1} b, \mathbb{P}_A \right] \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta} \lambda^1) \lambda^0}{(\alpha' - \beta')^2} \, d\beta' \right); \tag{4.240}$$

for this term, we write

$$D_t^{j-1} b = D_t^{j-1} b - 2 \operatorname{Re} \frac{D_t^{j-1} Z_t}{Z_{,\alpha'}} + 2 \operatorname{Re} \frac{D_t^{j-1} Z_t}{Z_{,\alpha'}} \tag{4.241}$$

and use (4.92) and (4.99) and (C.19) to estimate the first term and (C.17) to estimate the second term,

$$\begin{aligned} &\left\| \mathbb{P}_H \left[D_t^{j-1} b, \mathbb{P}_A \right] \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta} \lambda^1) \lambda^0}{(\alpha' - \beta')^2} \, d\beta' \right) \right\|_{L^2} \\ &\lesssim \left\| \partial_{\alpha'} \left(D_t^{j-1} b - 2 \operatorname{Re} \frac{D_t^{j-1} Z_t}{Z_{,\alpha'}} \right) \right\|_{L^2} \left\| \frac{1}{\overline{Z}_{,\alpha'}} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta} \lambda^1) \lambda^0}{(\alpha' - \beta')^2} \, d\beta' \right\|_{\dot H^{1/2}} \\ &+ \left\| \frac{D_t^{j-1} Z_t}{Z_{,\alpha'}} \right\|_{\dot H^{1/2}} \left\| \partial_{\alpha'} \left(\frac{1}{\overline{Z}_{,\alpha'}} \int \frac{(\theta \overline{\lambda^1} + \overline{\theta} \lambda^1) \lambda^0}{(\alpha' - \beta')^2} \, d\beta' \right) \right\|_{L^2}, \end{aligned} \tag{4.242}$$

and we get

$$A_{2,2,3} \lesssim \epsilon^3 \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}} \right), \qquad A_{3,3,4} \lesssim \epsilon^4. \tag{4.243}$$

To estimate $B_{j-1,j-1,j}$ for $2 \le j \le 4$, we first compute using (B.8),

$$\begin{aligned} D_t^j \frac{1}{\overline{Z}_{,\alpha'}} &= D_t^{j-1} \left(\frac{1}{\overline{Z}_{,\alpha'}} (b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t) \right) + \left[D_t^{j-1}, \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right] Z_t \\ &+ \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathbb{P}_H D_t^{j-1} Z_t + \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathbb{P}_A D_t^{j-1} Z_t, \end{aligned} \tag{4.244}$$

and by (B.20), we write

$$\left[D_t^{j-1}, \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right] Z_t = \sum_{k=0}^{j-2} D_t^k \left(\frac{b_{\alpha'} - 2 \operatorname{Re} D_{\alpha'} Z_t}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_t^{j-2-k} Z_t \right), \tag{4.245}$$

and using the fact $\mathbb{P}_H Z_t = 0$ we write

$$\mathbb{P}_H D_t^{j-1} Z_t = \sum_{k=0}^{j-2} D_t^k \left[\mathbb{P}_H, D_t \right] D_t^{j-2-k} Z_t = \sum_{k=0}^{j-2} D_t^k \left[\mathbb{P}_H, b \right] \partial_{\alpha'} D_t^{j-2-k} Z_t; \tag{4.246}$$

using the expansions in (4.244), (4.245) and (4.246) and the estimates in §4.3.1, we have,

$$\left\| D_t^j \frac{1}{\overline{Z}_{,\alpha'}} - \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathbb{P}_A D_t^{j-1} Z_t \right\|_{L^2} \lesssim \epsilon^2, \qquad j = 2, 4, \tag{4.247}$$

$$\left\|D_t^3\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_A D_t^2 Z_t\right\|_{L^2}\lesssim\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right); \tag{4.248}$$

now we rewrite

$$\begin{aligned}&\mathbb{P}_H\left\{\left(D_t^j\frac{1}{\overline{Z}_{,\alpha'}}\right)\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\}\\ &=\mathbb{P}_H\left\{\left(D_t^j\frac{1}{\overline{Z}_{,\alpha'}}-\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathbb{P}_A D_t^{j-1}Z_t\right)\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\}\\ &\quad+\left[\mathbb{P}_H,\frac{1}{|Z_{,\alpha'}|^2}\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right]\partial_{\alpha'}\mathbb{P}_A D_t^{j-1}Z_t,\end{aligned} \tag{4.249}$$

by (C.29), (C.32),

$$\left\|\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\|_{L^\infty}\lesssim\|Z_{t,\alpha'}\|_{L^2}^2, \tag{4.250}$$

$$\left\|\int\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta'\right\|_{\dot H^1}\lesssim\|Z_{t,\alpha'}\|_{L^2}^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}; \tag{4.251}$$

therefore by (C.19), (4.247), (4.248), we have

$$B_{2,2,3}\lesssim\epsilon^3\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right),\qquad B_{1,1,2}+B_{3,3,4}\lesssim\epsilon^4. \tag{4.252}$$

Sum up (4.232)-(4.252) we conclude

$$\begin{aligned}&|N_{l,j,1}|\lesssim\epsilon^5,\qquad -1\le l\le j-1,\ j=2,4;\\ &|N_{l,3,1}|\lesssim\epsilon^3\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right)^2,\quad -1\le l\le 2.\end{aligned} \tag{4.253}$$

Now we handle $N_{l,j,2}$. Observe that $N_{-1,j,2}=0$, so we work on the cases where $0\le l\le j-1$. We use (B.29) to expand. The estimates are routine for all the terms after expansion, using (C.46), (C.48), (C.27), (C.42), (C.35), (C.49), except that when $l=j-1$ for $j=3,4$, we need to again decompose $D_t^{j-1}b$ by (4.241), and treat all the terms as usual, except

$$\int\partial_{\beta'}\operatorname{Re}\frac{D_t^{j-1}Z_t(\beta')}{Z_{,\beta'}}\frac{(\theta\overline{\lambda^1}+\overline{\theta}\lambda^1)\lambda^0}{(\alpha'-\beta')^2}\,d\beta', \tag{4.254}$$

for which we first perform integration by parts, then apply (C.46), (C.48) for $\mathsf{b}=\operatorname{Re}\frac{D_t^{j-1}Z_t}{Z_{,\alpha'}}$. We have

$$\begin{aligned}&|N_{l,j,2}|\lesssim\epsilon^5,\qquad -1\le l\le j-1,\ j=2,4;\\ &|N_{l,3,2}|\lesssim\epsilon^3\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right)^2,\quad -1\le l\le 2.\end{aligned} \tag{4.255}$$

We consider the remaining terms in (4.226)-(4.227). Using (4.230) and the estimates in §4.3.1, §4.3.2, (C.17), (C.20), (C.19), (C.23), (C.27) we have

$$\left\|D_{\alpha'}\Theta^{(1)}-D_t\overline{Z}_t\right\|_{\dot H^{1/2}}\lesssim\epsilon^2, \tag{4.256}$$

$$\left\|D_{\alpha'}\Theta^{(1)}-D_t\overline{Z}_t\right\|_{\dot H^1}\lesssim\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right), \tag{4.257}$$

$$\left\|D_t^l\left(D_{\alpha'}\Theta^{(j-l)}-D_t^{j-l}\overline{Z}_t\right)\right\|_{L^2}\lesssim\epsilon^2,\quad j=2,4,\ 0\le l\le j, \tag{4.258}$$

$$\left\| D_t^l \left(D_{\alpha'} \Theta^{(3-l)} - D_t^{3-l} \overline{Z}_t \right) \right\|_{L^2} \lesssim \epsilon \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \| Z_{t,\alpha'} \|_{\dot H^{1/2}} \right), \quad 0 \le l \le 3; \tag{4.259}$$

and using (4.231) we have[19]

$$\begin{aligned} &\left\| D_t D_{\alpha'} \Theta^{(j)} - D_t^j D_{\alpha'} \Theta^{(1)} \right\|_{L^2} \lesssim \epsilon \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \| Z_{t,\alpha'} \|_{\dot H^{1/2}} \right), \quad \text{for } j = 2, 3, \\ &\left\| D_t D_{\alpha'} \Theta^{(4)} - D_t^4 D_{\alpha'} \Theta^{(1)} \right\|_{L^2} \lesssim \epsilon^2; \end{aligned} \tag{4.260}$$

this gives, by (C.29), (C.30),

$$\begin{aligned} &\sum_{l=-1}^{j-2} \sum_{k=3}^{5} |N_{l,j,k}| + \sum_{k=3}^{5} \left| \tilde N_{j,k} \right| + |R_{H,j}| \lesssim \epsilon^5, \qquad \text{for } j = 2, 4; \\ &\sum_{l=-1}^{j-2} \sum_{k=3}^{5} |N_{l,j,k}| + \sum_{k=3}^{5} \left| \tilde N_{j,k} \right| + |R_{H,j}| \lesssim \epsilon^3 \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \| Z_{t,\alpha'} \|_{\dot H^{1/2}} \right)^2, \quad \text{for } j = 3. \end{aligned} \tag{4.261}$$

Sum up the results in Steps 6-8, we get

$$\frac{d}{dt} \left(\mathcal{E}_2(t) + \mathcal{E}_3(t) + \mathcal{E}_4(t) \right) \lesssim \epsilon^5, \qquad \frac{d}{dt} \mathcal{E}_3(t) \lesssim \epsilon^3 \left(\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} + \| Z_{t,\alpha'} \|_{\dot H^{1/2}} \right)^2. \tag{4.262}$$

This together with (4.164), (4.134) gives the inequalities (4.20) and (4.24), and finishes the proof for Propositions 4.3, 4.4 and Theorem 3.1.

Appendix A. Notations and conventions

We use the following notations and conventions throughout the paper: compositions are always in terms of the spatial variables and we write for $f = f(\cdot, t)$, $g = g(\cdot, t)$, $f(g(\cdot, t), t) := f \circ g(\cdot, t) := U_g f(\cdot, t)$. We identify (x, y) with the complex number $x + iy$; $\operatorname{Re} z$, $\operatorname{Im} z$ are the real and imaginary parts of z; $\overline{z} = \operatorname{Re} z - i \operatorname{Im} z$ is the complex conjugate of z. $\overline{\Omega}$ is the closure of the domain Ω, $\partial\Omega$ is the boundary of Ω, $\mathscr{P}_- := \{ z \in \mathbb{C} : \operatorname{Im} z < 0 \}$ is the lower half plane. $[A, B] := AB - BA$ is the commutator of operators A and B.

We use $z = x + iy = z(\alpha, t)$, $z_t = z_t(\alpha, t)$ and $z_{tt}(\alpha, t)$ to denote the position, velocity and acceleration of the interface in Lagrangian coordinate α; $Z = X + iY = Z(\alpha', t)$, $Z_t = Z_t(\alpha', t)$ and $Z_{tt}(\alpha', t)$ denote the position, velocity and acceleration of the interface in the Riemann mapping variable α'; $\mathsf{h}(\alpha, t) = \alpha'$ is the coordinate change from the Lagrangian variable α to the Riemann mapping variable α';[20] $b = \mathsf{h}_t \circ \mathsf{h}^{-1}$, and the material derivative is $D_t = \partial_t + b \partial_{\alpha'}$. We write

$$Z_{,\alpha'} = \partial_{\alpha'} Z(\alpha', t), \qquad Z_{t,\alpha'} = \partial_{\alpha'} Z_t, \qquad Z_{tt,\alpha'} = \partial_{\alpha'} Z_{tt}, \qquad \text{etc.}$$

Let $\mathbb{H}$ be the Hilbert transform associated with the lower half plane $\mathscr{P}_-$:

$$\mathbb{H} f(\alpha') = \frac{1}{\pi i} \text{pv.} \int \frac{1}{\alpha' - \beta'} f(\beta') \, d\beta'. \tag{A.1}$$

We know $\mathbb{H}^2 = I$, and a function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, is the boundary value of a holomorphic function in $\mathscr{P}_-$ if and only if $f = \mathbb{H} f$. We define the projections to the space of holomorphic, and respectively, anti-holomorphic functions in the lower half plane by

$$\mathbb{P}_H := \frac{1}{2}(I + \mathbb{H}), \qquad \text{and} \qquad \mathbb{P}_A := \frac{1}{2}(I - \mathbb{H}). \tag{A.2}$$

[19] Again we use the decomposition (4.241) to treat the term $\left[\mathbb{P}_A, D_t^{j-1} b \right] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}$.

[20] $\mathsf{h}(\alpha, t) = \Phi(z(\alpha, t); t)$, where $\Phi(\cdot, t) : \Omega \to \mathscr{P}_-$ is the Riemann mapping satisfying $\Phi(z(0, t); t) = 0$ and $\lim_{z \to \infty} \Phi_z(z, t) = 1$.

It is clear that the decomposition identity

$$\mathbb{P}_H + \mathbb{P}_A = I \tag{A.3}$$

and the projection identity

$$\mathbb{P}_H \mathbb{P}_A = \mathbb{P}_A \mathbb{P}_H = 0 \tag{A.4}$$

hold. We will often call a function $f \in L^p$, $1 \le p < \infty$, that is the boundary value of a holomorphic function in $\mathscr{P}_-$ simply by "holomorphic".

We define

$$D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}, \tag{A.5}$$

$$[f, g; h] := \frac{1}{\pi i} \int \frac{(f(x) - f(y))(g(x) - g(y))}{(x - y)^2} h(y)\, dy, \tag{A.6}$$

and

$$< f, g, h >:= \frac{1}{\pi i} \int \frac{(f(x) - f(y))(g(x) - g(y))(h(x) - h(y))}{(x - y)^2}\, dy. \tag{A.7}$$

We use the following notations for functional spaces: $H^s = H^s(\mathbb{R})$ is the Sobolev space with norm $\|f\|_{H^s} := (\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi)^{1/2}$, $\dot{H}^s = \dot{H}^s(\mathbb{R})$ is the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^s} = c(\int |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi)^{1/2}$, and we define

$$\|f\|^2_{\dot{H}^{1/2}} = \|f\|^2_{\dot{H}^{1/2}(\mathbb{R})} := \int i\mathbb{H}\partial_x f(x) \overline{f}(x)\, dx = \frac{1}{2\pi} \iint \frac{|f(x) - f(y)|^2}{(x - y)^2}\, dx\, dy. \tag{A.8}$$

$L^p = L^p(\mathbb{R})$ is the L^p space with $\|f\|_{L^p} := (\int |f(x)|^p\, dx)^{1/p}$ for $1 \le p < \infty$, and $f \in L^\infty$ if $\|f\|_{L^\infty} := \operatorname{ess\,sup} |f(x)| < \infty$. When not specified, all the norms $\|f\|_{H^s}$, $\|f\|_{\dot{H}^s}$, $\|f\|_{L^p}$, $1 \le p \le \infty$ are in terms of the spatial variable only, and $\|f\|_{H^s(\mathbb{R})}$, $\|f\|_{\dot{H}^s(\mathbb{R})}$, $\|f\|_{L^p(\mathbb{R})}$, $1 \le p \le \infty$ are in terms of the spatial variable. $C^j(X)$ is the space of j-times continuously differentiable functions on the set X; $C^j_0(\mathbb{R})$ is the space of j-times continuously differentiable functions that decays at the infinity.

We use c, C to denote universal constants. $c(a_1, \dots)$, $C(a_1, \dots)$, $M(a_1, \dots)$ are constants depending on $a_1, \dots$; constants appearing in different contexts need not be the same. We write $f \lesssim g$ if there is a universal constant c, such that $f \le cg$.

The following are some additional notations used in this paper:

$Q := (I + \mathbb{H})\psi \circ \mathsf{h}^{-1}$, where $\psi \circ \mathsf{h}^{-1}$ is the trace of the velocity potential on the interface; $\Theta^{(0)} := Q$, $\Theta^{(j)} := (\mathbb{P}_H D_t)^j Q$, and $G^{(j)} := D_t \mathbb{P}_H D_t \Theta^{(j)} + i \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \Theta^{(j)}$.

We define $\theta := \overline{Z}_t(\alpha', t) - \overline{Z}_t(\beta', t)$; $\lambda^j = D_{\alpha'} \Theta^{(j)}(\alpha', t) - D_{\beta'} \Theta^{(j)}(\beta', t)$; $\mathfrak{D}_t := \partial_t + b(\alpha', t)\partial_{\alpha'} + b(\beta', t)\partial_{\beta'}$, and $\mathcal{P} := \mathfrak{D}_t^2 + i \frac{A_1(\alpha', t)}{|Z_{,\alpha'}|^2} \partial_{\alpha'} + i \frac{A_1(\beta', t)}{|Z_{,\beta'}|^2} \partial_{\beta'}$. Observe that when acting on a function independent of β', i.e. $f = f(\alpha', t)$, $\mathcal{P} f = (D_t^2 + i \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}) f$.

We denote by $M(f)$ the Hardy-Littlewood maximum function of f.

Appendix B. Equations and Identities

Here we give some basic equations and identities that will be used in this paper. First we recall some of the equations and formulas derived in our earlier work, see [44, 48] or §2.2, §2.3 and §2.7 of [49].

B.1. **Interface equations.** We know that the interface equations for the 2d water waves is given by

$$\begin{cases} \overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}} \\ \overline{Z}_t = \mathbb{H}\overline{Z}_t, \quad \frac{1}{Z_{,\alpha'}} - 1 = \mathbb{H}\left(\frac{1}{Z_{,\alpha'}} - 1\right), \end{cases} \tag{B.1}$$

where the quantities A_1 and b satisfy

$$A_1 = 1 - \operatorname{Im}\left[Z_t, \mathbb{H}\right]\overline{Z}_{t,\alpha'} = 1 - \frac{1}{2}\operatorname{Im}[Z_t, \overline{Z}_t; 1], \qquad b = \operatorname{Re}(I - \mathbb{H})\left(\frac{Z_t}{Z_{,\alpha'}}\right). \tag{B.2}$$

Let

$$\mathfrak{P} := D_t^2 + i\,\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}. \tag{B.3}$$

The quasi-linear equation for the water waves is

$$\mathcal{P}\overline{Z}_t = \mathfrak{P}\overline{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ \mathsf{h}^{-1}(\overline{Z}_{tt} - i) \tag{B.4}$$

where

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ \mathsf{h}^{-1} = \frac{D_t A_1}{A_1} + b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t, \tag{B.5}$$

with

$$D_t A_1 = -\operatorname{Im}\left([Z_{tt}, \mathbb{H}]\,\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\overline{Z}_{tt} - [Z_t, b; \overline{Z}_{t,\alpha'}]\right) = -\operatorname{Im}\left(\left[\overline{Z}_t, Z_{tt}; 1\right] - \left[Z_t, b; \overline{Z}_{t,\alpha'}\right]\right), \tag{B.6}$$

and[21]

$$b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t = \operatorname{Re}\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'} + [Z_t, \mathbb{H}]\,\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \frac{1}{2}\left[\frac{1}{Z_{,\alpha'}}, Z_t; 1\right] - \frac{1}{2}\left[\overline{Z}_t, \frac{1}{\overline{Z}_{,\alpha'}}; 1\right]. \tag{B.7}$$

We also have

$$D_t\left(\frac{1}{Z_{,\alpha'}}\right) = \frac{1}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'} Z_t) = \frac{1}{Z_{,\alpha'}}(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t) + \frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^2}. \tag{B.8}$$

B.2. **Basic identities.** We give some basic identities that will be used in our calculations. We begin with Proposition B.1, which is a consequence of the fact that the product of holomorphic functions is holomorphic and (A.4).

Proposition B.1. *Assume that* $f,\ g \in L^2(\mathbb{R})$.

1. Assume either both f, g *are holomorphic:* $f = \mathbb{H}f$, $g = \mathbb{H}g$, *or both are anti-holomorphic:* $f = -\mathbb{H}f$, $g = -\mathbb{H}g$. *Then*

$$[f, \mathbb{H}]g = 0. \tag{B.9}$$

2. If $\mathbb{P}_A f = \mathbb{P}_A g = 0$, *then* $\mathbb{P}_A(fg) = 0$; *and if* $\mathbb{P}_H f = \mathbb{P}_H g = 0$, *then* $\mathbb{P}_H(fg) = 0$.

Proposition B.2. *1. We have*

$$[f, g; h] = [f, \mathbb{H}]\partial_{\alpha'}(gh) + [g, \mathbb{H}]\partial_{\alpha'}(fh) - [fg, \mathbb{H}]\partial_{\alpha'} h. \tag{B.10}$$

2. If h *is holomorphic, i.e.* $\mathbb{P}_A h = 0$, *then*

$$\mathbb{P}_H[f, g; h] = -2\mathbb{P}_H(f\partial_{\alpha'}\mathbb{P}_A(gh)) - 2\mathbb{P}_H(g\partial_{\alpha'}\mathbb{P}_A(fh)). \tag{B.11}$$

[21] The second equality in (B.6) and (B.7) are obtained by integration by parts.

(B.10) is obtained by integration by parts. (B.11) follows from (B.10), (A.4) and (A.3) .

We often use the following equalities to calculate the time and spatial derivatives to our energy functionals, and the following version of the Cauchy integral formula to derive our equations and formulas.

Proposition B.3. *For f, g smooth and decay fast at infinity,*

$$\frac{d}{dt}\int f(\alpha',t)\,d\alpha' = \int (D_t + b_{\alpha'}) f(\alpha',t)\,d\alpha'; \tag{B.12}$$

$$\frac{d}{dt}\iint g(\alpha',\beta',t)\,d\alpha' d\beta' = \iint (\mathfrak{D}_t + b_{\alpha'} + b_{\beta'}) g(\alpha',\beta',t)\,d\alpha' d\beta'; \tag{B.13}$$

$$D_t \int g(\alpha',\beta',t)\,d\beta' = \int (\mathfrak{D}_t + b_{\beta'}) g(\alpha',\beta',t)\,d\beta'; \tag{B.14}$$

$$\partial_{\alpha'} \int g(\alpha',\beta',t)\,d\beta' = \int (\partial_{\alpha'} + \partial_{\beta'}) g(\alpha',\beta',t)\,d\beta'. \tag{B.15}$$

(B.12), (B.13), (B.14), (B.15) follow from the simple fact that $\int (b\partial_{\alpha'} + b_{\alpha'}) f\,d\alpha' = \int \partial_{\alpha'}(bf)\,d\alpha' = 0$, $\int (b\partial_{\beta'} + b_{\beta'}) g\,d\beta' = \int \partial_{\beta'}(bg)\,d\beta' = 0$, and $\int \partial_{\beta'} g\,d\beta' = 0$.

Proposition B.4 (Cauchy integral formula)**.** *For any $\Theta \in L^1(\mathbb{R})$, satisfying $\mathbb{P}_A\Theta = 0$ or $\mathbb{P}_H\Theta = 0$,*

$$\int \Theta(\alpha')\,d\alpha' = 0. \tag{B.16}$$

B.3. **Commutator identities.** We include here various commutator identities that are necessary for our proofs. Some have already appeared in Appendix B.5 of [29] and Appendix B of [49]. We have

$$[D_t, D_{\alpha'}] = -(D_{\alpha'} Z_t) D_{\alpha'}, \tag{B.17}$$

$$[D_t, \partial_{\alpha'}] = -b_{\alpha'}\partial_{\alpha'}. \tag{B.18}$$

By product rules,

$$[D_t, \frac{1}{Z_{,\alpha'}}] f = D_t\left(\frac{1}{Z_{,\alpha'}}\right) f. \tag{B.19}$$

From (B.8) and (B.18) it implies that

$$\left[D_t, \frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right] f = \frac{b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t}{|Z_{,\alpha'}|^2}\partial_{\alpha'} f, \tag{B.20}$$

$$[D_t, \mathfrak{P}]\, f = \left[D_t, i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right] f = i\left(\frac{D_t A_1}{A_1} + b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t\right)\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} f. \tag{B.21}$$

We also have

$$[D_t, \mathbb{H}] = 2[D_t, \mathbb{P}_H] = [b, \mathbb{H}]\partial_{\alpha'} = \left[\frac{Z_t}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\mathbb{P}_H + \left[\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\mathbb{P}_A, \tag{B.22}$$

where the last equality in (B.22) is a consequence of (2.1), (A.3) and (B.9). In general, for operators A, B and C,

$$[A, BC] = [A, B]C + B[A, C]. \tag{B.23}$$

We use the following identities in our computations. We have

$$[\mathcal{P}, \mathfrak{D}_t^m] = \sum_{k=0}^{m-1} \mathfrak{D}_t^k\,[\mathcal{P}, \mathfrak{D}_t]\,\mathfrak{D}_t^{m-k-1}, \tag{B.24}$$

$$\mathbb{P}_H D_t \mathbb{P}_H = \mathbb{P}_H D_t - \mathbb{P}_H D_t \mathbb{P}_A = \mathbb{P}_H D_t - \mathbb{P}_H\,[b, \mathbb{P}_A]\,\partial_{\alpha'}, \tag{B.25}$$

and we compute

$$(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H = \mathbb{P}_H D_t^{l+1} + \sum_{k=0}^{l}\left((\mathbb{P}_H D_t)^{k+1}\mathbb{P}_H D_t^{l-k} - (\mathbb{P}_H D_t)^k \mathbb{P}_H D_t^{l+1-k}\right)$$
$$= \mathbb{P}_H D_t^{l+1} - \sum_{k=0}^{l}(\mathbb{P}_H D_t)^k \mathbb{P}_H D_t \mathbb{P}_A D_t^{l-k} = \mathbb{P}_H D_t^{l+1} - \sum_{k=0}^{l}(\mathbb{P}_H D_t)^k \mathbb{P}_H \,[b, \mathbb{P}_A]\, \partial_{\alpha'} D_t^{l-k},$$

so

$$(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H = \mathbb{P}_H D_t^{l+1} - \sum_{k=0}^{l}(\mathbb{P}_H D_t)^k \mathbb{P}_H \,[b, \mathbb{P}_A]\, \partial_{\alpha'} D_t^{l-k}. \tag{B.26}$$

A similar computation also gives

$$(\mathbb{P}_H D_t)^{l+1}\mathbb{P}_H = D_t^{l+1}\mathbb{P}_H - \sum_{k=0}^{l} D_t^k \,[\mathbb{P}_A, b]\, \partial_{\alpha'} (\mathbb{P}_H D_t)^{l-k}\mathbb{P}_H. \tag{B.27}$$

Let $\mathfrak{f} := f(\alpha') - f(\beta')$, $\mathfrak{g} := g(\alpha') - g(\beta')$ and $\mathfrak{h} := h(\alpha') - h(\beta')$. We use (B.14) and (B.18) to get

$$D_t < f,\, g,\, h >= \frac{1}{\pi i}\int \frac{\mathfrak{D}_t(\mathfrak{f}\,\mathfrak{g}\,\mathfrak{h})}{(\alpha'-\beta')^2}\,d\beta' + \frac{1}{\pi i}\int \partial_{\beta'}\mathfrak{D}_t\left(\frac{1}{\alpha'-\beta'}\right)\mathfrak{f}\,\mathfrak{g}\,\mathfrak{h}\,d\beta'; \tag{B.28}$$

and by induction,

$$D_t^n < f,\, g,\, h >= \frac{1}{\pi i}\sum_{k=0}^{n}\binom{n}{k}\int \partial_{\beta'}\mathfrak{D}_t^k\left(\frac{1}{\alpha'-\beta'}\right)\mathfrak{D}_t^{n-k}(\mathfrak{f}\,\mathfrak{g}\,\mathfrak{h})\,d\beta', \tag{B.29}$$

Similarly,

$$D_t^n[f,\, g;\, h] = \frac{1}{\pi i}\sum_{k=0}^{n}\binom{n}{k}\int \partial_{\beta'}\mathfrak{D}_t^k\left(\frac{1}{\alpha'-\beta'}\right)\mathfrak{D}_t^{n-k}\left((f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))h(\beta')\right)\,d\beta'; \tag{B.30}$$

$$D_t^n[f,\, g;\, \partial_{\alpha'}h] = \frac{1}{\pi i}\sum_{k=0}^{n}\binom{n}{k}\int \mathfrak{D}_t^k\left(\frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))}{(\alpha'-\beta')^2}\right)\partial_{\beta'}D_t^{n-k}h(\beta')\,d\beta'; \tag{B.31}$$

$$D_t^n[f, \mathbb{H}]\partial_{\alpha'}g = \frac{1}{\pi i}\sum_{k=0}^{n}\binom{n}{k}\int \mathfrak{D}_t^k\left(\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right)\partial_{\beta'}D_t^{n-k}g(\beta')\,d\beta'; \tag{B.32}$$

where

$$\mathfrak{D}_t\left(\frac{1}{\alpha'-\beta'}\right) = -\frac{b(\alpha')-b(\beta')}{(\alpha'-\beta')^2}, \quad \mathfrak{D}_t^2\left(\frac{1}{\alpha'-\beta'}\right) = -\frac{D_t b(\alpha')-D_t b(\beta')}{(\alpha'-\beta')^2} + 2\frac{(b(\alpha')-b(\beta'))^2}{(\alpha'-\beta')^3}, \text{ etc;} \tag{B.33}$$

in particular,

$$D_t[f, \mathbb{H}]\partial_{\alpha'}g = [D_t f, \mathbb{H}]\partial_{\alpha'}g + [f, \mathbb{H}]\partial_{\alpha'}D_t g - [f, b; \partial_{\alpha'}g]. \tag{B.34}$$

B.4. **Some additional equations.** We know by definition (2.7), (2.8) and equation (2.3) that

$$\Theta^{(0)} = Q, \quad \Theta^{(1)} = i(Z-\alpha'), \quad \Theta^{(2)} = -i\mathbb{P}_H b = -i\mathbb{P}_H\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}, \tag{B.35}$$

$$\Theta^{(j+2)} = -i\mathbb{P}_H\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta^{(j)}\right) + \mathbb{P}_H(G^{(j)}), \qquad j \ge 1. \tag{B.36}$$

This implies

$$D_{\alpha'}\Theta^{(0)} = \overline{Z}_t, \qquad D_{\alpha'}\Theta^{(1)} = i\left(1 - \frac{1}{Z_{,\alpha'}}\right); \tag{B.37}$$

$$b = i\,\Theta^{(2)} - i\,\overline{\Theta^{(2)}}, \qquad D_t b = i\,D_t\Theta^{(2)} - i\,\overline{D_t\Theta^{(2)}}, \qquad \text{and} \tag{B.38}$$

$$\mathbb{P}_H\left(D_t b + i\frac{A_1}{|Z_{,\alpha'}|^2} - i\right) = i\mathbb{P}_H\left(\frac{A_1-1}{|Z_{,\alpha'}|^2}\right) + i\mathbb{P}_H(G^{(1)}) + i\,[b,\mathbb{P}_H]\,\partial_{\alpha'}\overline{\Theta^{(2)}}. \tag{B.39}$$

We compute, by (B.24), (B.21) that

$$\begin{aligned}\left[\mathcal{P}, D_t^l\right]\overline{Z}_t &= \sum_{k=0}^{l-1} D_t^{l-1-k}\left[\mathcal{P}, D_t\right] D_t^k\overline{Z}_t\\ &= -i\sum_{k=0}^{l-1} D_t^{l-1-k}\left\{\left(\frac{D_tA_1}{A_1} + b_{\alpha'} - 2\operatorname{Re} D_{\alpha'}Z_t\right)\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_t^k\overline{Z}_t\right\}.\end{aligned} \tag{B.40}$$

and we have, by (B.23), (B.17) and (B.27), that

$$\begin{aligned}D_{\alpha'}\Theta^{(l+1)} - D_t^l D_{\alpha'}\Theta^{(1)} &= \sum_{k=0}^{l-1}(\mathbb{P}_H D_t)^k\left[\frac{1}{Z_{,\alpha'}}, \mathbb{P}_H\right]\partial_{\alpha'}D_t\Theta^{(l-k)}\\ &+ \sum_{k=0}^{l-1}(\mathbb{P}_H D_t)^k\mathbb{P}_H(D_{\alpha'}Z_t D_{\alpha'}\Theta^{(l-k)}) - \sum_{k=0}^{l-1} D_t^k[\mathbb{P}_A, b]\partial_{\alpha'}(\mathbb{P}_H D_t)^{l-1-k}D_{\alpha'}\Theta^{(1)}.\end{aligned} \tag{B.41}$$

Appendix C. Basic inequalities

We will use the following equalities or inequalities in this paper. The first set: Lemma C.1 through Proposition C.14 are either classical results, or simple consequences of definitions and classical results, and some have already appeared in Appendix A of [49]. Lemma C.15 and Proposition C.16 are from Section 5.1 of [49].

Lemma C.1. *For any function $f \in \dot{H}^{1/2}(\mathbb{R})$,*

$$\|f\|_{\dot{H}^{1/2}}^2 = \|\mathbb{P}_H f\|_{\dot{H}^{1/2}}^2 + \|\mathbb{P}_A f\|_{\dot{H}^{1/2}}^2\,; \tag{C.1}$$

$$\int i\partial_{\alpha'} f\,\overline{f}\,d\alpha' = \|\mathbb{P}_H f\|_{\dot{H}^{1/2}}^2 - \|\mathbb{P}_A f\|_{\dot{H}^{1/2}}^2\,. \tag{C.2}$$

Proposition C.2. *Let $f,\ g \in C^1(\mathbb{R})$. Then*

$$\|fg\|_{\dot{H}^{1/2}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{H}^{1/2}} + \|g\|_{L^\infty}\|f\|_{\dot{H}^{1/2}}; \tag{C.3}$$

$$\|fg\|_{\dot{H}^{1/2}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{H}^{1/2}} + \|f'\|_{L^2}\|g\|_{L^2}; \tag{C.4}$$

$$\|g\|_{\dot{H}^{1/2}} \lesssim \|f^{-1}\|_{L^\infty}(\|fg\|_{\dot{H}^{1/2}} + \|f'\|_{L^2}\|g\|_{L^2}). \tag{C.5}$$

(C.4) is straightforward from the definition of $\dot{H}^{1/2}$ and Hardy's inequality. The remaining two are from Appendix A of [49].

Proposition C.3 (Sobolev inequality)**.** *Let $f \in C_0^1(\mathbb{R})$. Then*

$$\|f\|_{L^\infty}^2 \le 2\|f\|_{L^2}\|f'\|_{L^2}, \qquad \|f\|_{\dot{H}^{1/2}}^2 \le \|f\|_{L^2}\|f'\|_{L^2}. \tag{C.6}$$

Proposition C.4 (Maximum inequality)**.** *Let $1 < p \le \infty$, Then for all $f \in L^p$,*

$$\|M(f)\|_{L^p} \lesssim \|f\|_{L^p}\,. \tag{C.7}$$

Proposition C.5 (Hardy's inequalities)**.** *Let $1 < p < \infty$, $f \in C^1(\mathbb{R})$, with $f' \in L^p(\mathbb{R})$. Then*

$$\sup_{x\in\mathbb{R}}\int\frac{|f(x)-f(y)|^p}{|x-y|^p}dy \lesssim \|f'\|_{L^p}^p\,; \tag{C.8}$$

and

$$\iint \frac{|f(x)-f(y)|^{2p}}{|x-y|^{2p}}\,dxdy \lesssim \|f'\|_{L^p}^{2p}. \tag{C.9}$$

Let $H\in C^1(\mathbb{R};\mathbb{R}^d)$, $A_i\in C^1(\mathbb{R})$, $i=1,\dots m$, and $F\in C^\infty(\mathbb{R})$. Define

$$C_1(A_1,\dots,A_m,f)(x)=\text{pv.}\int F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^{m+1}}f(y)\,dy. \tag{C.10}$$

Proposition C.6. *There exist constants $c_1=c_1(F,\|H'\|_{L^\infty})$, $c_2=c_2(F,\|H'\|_{L^\infty})$, such that*

1. For any $f\in L^2,\ A_i'\in L^\infty,\ 1\le i\le m$,

$$\|C_1(A_1,\dots,A_m,f)\|_{L^2}\le c_1\|A_1'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^2}. \tag{C.11}$$

2. For any $f\in L^\infty,\ A_i'\in L^\infty,\ 2\le i\le m,\ A_1'\in L^2$,

$$\|C_1(A_1,\dots,A_m,f)\|_{L^2}\le c_2\|A_1'\|_{L^2}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^\infty}. \tag{C.12}$$

(C.11) is a result of Coifman, McIntosh and Meyer [18]. (C.12) is a consequence of the Tb Theorem, a proof is given in [46].

Let H, A_i F satisfy the same assumptions as in (C.10). Define

$$C_2(A,f)(x)=\int F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^{m}}\partial_y f(y)\,dy. \tag{C.13}$$

The following are consequences of Proposition C.6 and integration by parts.

Proposition C.7. *There exist constants c_3, c_4 and c_5, depending on F and $\|H'\|_{L^\infty}$, such that*

1. For any $f\in L^2,\ A_i'\in L^\infty,\ 1\le i\le m$,

$$\|C_2(A,f)\|_{L^2}\le c_3\|A_1'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^2}. \tag{C.14}$$

2. For any $f\in L^\infty,\ A_i'\in L^\infty,\ 2\le i\le m,\ A_1'\in L^2$,

$$\|C_2(A,f)\|_{L^2}\le c_4\|A_1'\|_{L^2}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^\infty}. \tag{C.15}$$

3. For any $f'\in L^2,\ A_1\in L^\infty,\ \ A_i'\in L^\infty,\ 2\le i\le m$,

$$\|C_2(A,f)\|_{L^2}\le c_5\|A_1\|_{L^\infty}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f'\|_{L^2}. \tag{C.16}$$

Proposition C.8. *Assume that f,g,h are smooth and decay fast at infinity. Then*

$$\|[f,\mathbb{H}]g\|_{L^2}\lesssim\|f\|_{\dot H^{1/2}}\|g\|_{L^2}; \tag{C.17}$$

$$\|[f,\mathbb{H}]g\|_{L^\infty}\lesssim\|f'\|_{L^2}\|g\|_{L^2}; \tag{C.18}$$

$$\|[f,\mathbb{H}]\partial_{\alpha'}g\|_{L^2}\lesssim\|f'\|_{L^2}\|g\|_{\dot H^{1/2}}; \tag{C.19}$$

$$\|[f,\mathbb{H}]\partial_{\alpha'}g\|_{L^2}\lesssim\|f'\|_{\dot H^{1/2}}\|g\|_{L^2}; \tag{C.20}$$

$$\|[f,h;\partial_{\alpha'}g]\|_{L^2}\lesssim\|f'\|_{L^2}\|h'\|_{L^\infty}\|g\|_{\dot H^{1/2}}, \tag{C.21}$$

$$\|[f,h;\partial_{\alpha'}g]\|_{L^2}\lesssim\|f'\|_{L^2}\|h'\|_{\dot H^{1/2}}\|g\|_{L^\infty}+\|f'\|_{\dot H^{1/2}}\|h'\|_{L^2}\|g\|_{L^\infty}. \tag{C.22}$$

(C.20) follows from integration by parts, Cauchy-Schwarz inequality and the definition (A.8), and (C.22) follows from integration by parts, Cauchy-Schwarz inequality, Hardy's inequality and the definition (A.8). The remaining inequalities are from Appendix A of [49].

Proposition C.9. *For any f, g, h smooth and decay fast at spatial infinity, we have*

$$\|[f,g;h]\|_{L^p} \lesssim \|f'\|_{L^2}\|g'\|_{L^2}\|h\|_{L^p}, \qquad 1\le p\le\infty; \tag{C.23}$$

$$\|[f,g;h]\|_{L^2} \lesssim \|f'\|_{L^2}\|g\|_{\dot H^{1/2}}\|h\|_{L^\infty}; \tag{C.24}$$

$$\|[f,g;h]\|_{L^2} \lesssim \|f\|_{\dot H^{1/2}}\|g'\|_{L^\infty}\|h\|_{L^2}; \tag{C.25}$$

$$\|[f,g;h]\|_{L^\infty} \lesssim \|f'\|_{L^2}\|g'\|_{L^\infty}\|h\|_{L^2}. \tag{C.26}$$

(C.23) directly follows from Hölder's inequality and Hardy's inequality. (C.24) and (C.25) are direct consequences of Cauchy-Schwarz inequality and the definition (A.8). (C.26) is from Appendix A of [49].

Proposition C.10. *Assume that $f_i\in C^1(\mathbb{R})$, with $f_i'\in L^{p_i}$, $g\in L^q$, where*

$$\sum_{i=1}^n \frac{1}{p_i}+\frac{1}{q}=\frac{1}{p}+1, \qquad 1<p_i\le\infty,\ 1\le q\le\infty,\ p>0.$$

Then

$$\left\|\int \frac{\Pi_{i=1}^n(f_i(x)-f_i(y))}{(x-y)^n}g(y)\,dy\right\|_{L^p} \lesssim \Pi_{i=1}^n\|f_i'\|_{L^{p_i}}\|g\|_{L^q}. \tag{C.27}$$

Observe that $\left|\frac{f(x)-f(y)}{x-y}\right| \le \min\{M(f')(x), M(f')(y)\}$. (C.27) is a direct consequence of the Maximum inequality (C.7) and Hölder's inequality.

We also have the following inequalities for the cubic form $<\cdot,\cdot,\cdot>$.

Proposition C.11. *For any f, g, h smooth and decay fast at spatial infinity, we have*

$$\|<f,g,h>\|_{L^2} \lesssim \|f'\|_{L^2}\|g\|_{L^\infty}\|h\|_{\dot H^{1/2}}; \tag{C.28}$$

$$\|<f,g,h>\|_{L^p} \lesssim \|f'\|_{L^2}\|g'\|_{L^2}\|h\|_{L^p}, \qquad 1\le p\le\infty; \tag{C.29}$$

$$\|<f,g,h>\|_{L^2} \lesssim \|f'\|_{L^2}\|g\|_{\dot H^{1/2}}\|h\|_{\dot H^{1/2}}; \tag{C.30}$$

$$\|<f,g,h>\|_{\dot H^{1/2}} \lesssim \|f'\|_{L^2}\|g'\|_{L^2}\|h\|_{\dot H^{1/2}}; \tag{C.31}$$

$$\|<f,g,h>\|_{\dot H^1} \lesssim \|f'\|_{L^2}\|g'\|_{L^2}\|h'\|_{L^2}; \tag{C.32}$$

$$\|<f,g,h>\|_{L^\infty} \lesssim \|f'\|_{L^2}(\|g'\|_{L^2}\|h\|_{\dot H^{1/2}}+\|h'\|_{L^2}\|g\|_{\dot H^{1/2}}). \tag{C.33}$$

(C.28), (C.29), and (C.32) are easy consequences of Hölder's inequality, Hardy's inequality, and the definition (A.8). (C.30) and (C.31) follow from interpolation; (C.33) follows from the inequality

$$\|\partial_{\alpha'}<f,g,h>\|_{L^1} \le \|f'\|_{L^2}(\|g'\|_{L^2}\|h\|_{\dot H^{1/2}}+\|h'\|_{L^2}\|g\|_{\dot H^{1/2}}),$$

which in turn follows from Cauchy-Schwarz inequality and Hardy's inequality.

Proposition C.12. *Assume that f, g, h are holomorphic, i.e. $f=\mathbb{P}_H f$, $g=\mathbb{P}_H g$ and $h=\mathbb{P}_H h$. Then*

$$\|<f,g,h>\|_{L^\infty} \lesssim \|f'\|_{L^2}\|g'\|_{L^2}\|h\|_{\dot H^{1/2}}. \tag{C.34}$$

Observe that for f, g holomorphic, $[f,g;1]=0$. The same argument for (C.33) gives (C.34).

Proposition C.13. *Assume that f, g, h are smooth and decay fast at infinity,* $\mathrm{b}=\mathrm{b}(x,y)$ *is bounded. Then*

$$\left\|\int \mathrm{b}(x,y)\frac{(f(x)-f(y))(g(x)-g(y))(h(x)-h(y))}{(x-y)^2}\,dy\right\|_{L^2} \lesssim \|\mathrm{b}\|_{L^\infty(\mathbb{R}^2)}\|f'\|_{L^2}\|g'\|_{L^2}\|h\|_{L^2}; \tag{C.35}$$

$$\left\|\int \mathrm{b}(x,y)\frac{(f(x)-f(y))(g(x)-g(y))(h(x)-h(y))}{(x-y)^2}\,dy\right\|_{L^2} \lesssim \|\mathrm{b}\|_{L^\infty(\mathbb{R}^2)}\|f'\|_{L^2}\|g\|_{\dot H^{1/2}}\|h\|_{\dot H^{1/2}}. \tag{C.36}$$

(C.35) follows from Cauchy-Schwarz inequality and Hardy's inequality; (C.36) follows from interpolation.

Proposition C.14. *Assume that* $f_k \in \dot{H}^{1/2}(\mathbb{R})$, $k = 1, 2, 3, 4$. *Then*

$$\left|\int \frac{\Pi_{k=1}^4 (f_k(x) - f_k(y))}{(x-y)^2}\, dx\, dy\right| \lesssim \Pi_{k=1}^4 \|f_k\|_{\dot{H}^{1/2}}. \tag{C.37}$$

Proof. Observe that by symmetry, we can write

$$\int \frac{\Pi_{k=1}^4 (f_k(x) - f_k(y))}{(x-y)^2}\, dx\, dy = 2\int f_1(x) \int \frac{\Pi_{k=2}^4 (f_k(x) - f_k(y))}{(x-y)^2}\, dy\, dx.$$

By (C.30), we have, for $f_k \in H^1(\mathbb{R})$, $k = 1, 2$,

$$\left|\int \frac{\Pi_{k=1}^4 (f_k(x) - f_k(y))}{(x-y)^2}\, dx\, dy\right| \lesssim \|f_1\|_{L^2} \|f_2'\|_{L^2} \|f_3\|_{\dot{H}^{1/2}} \|f_4\|_{\dot{H}^{1/2}};$$

similarly we also have

$$\left|\int \frac{\Pi_{k=1}^4 (f_k(x) - f_k(y))}{(x-y)^2}\, dx\, dy\right| \lesssim \|f_1'\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{\dot{H}^{1/2}} \|f_4\|_{\dot{H}^{1/2}}.$$

(C.37) then follows by interpolation. □

The equality and inequalities in Lemma C.15 and Proposition C.16 are from Section 5.1 and Lemma 6.3 of [49].

Lemma C.15 (cf. Lemma 5.4, [49])**.** *Assume that* f, g, f_1, $g_1 \in H^1(\mathbb{R})$ *are the boundary values of some holomorphic functions on* $\mathscr{P}_-$. *Then*

$$\int \partial_{\alpha'} \mathbb{P}_A(\overline{f} g)(\alpha') f_1(\alpha') \overline{g}_1(\alpha')\, d\alpha' = -\frac{1}{2\pi i} \iint \frac{(\overline{f}(\alpha') - \overline{f}(\beta'))(f_1(\alpha') - f_1(\beta'))}{(\alpha' - \beta')^2} g(\beta') \overline{g}_1(\alpha')\, d\alpha' d\beta'. \tag{C.38}$$

Proposition C.16 (cf. Proposition 5.6, Lemma 6.3, [49])**.** *Assume that* f, $g \in H^1(\mathbb{R})$. *We have*

$$\|[f, \mathbb{H}]\, g\|_{\dot{H}^{1/2}} \lesssim \|f\|_{\dot{H}^{1/2}} (\|g\|_{L^\infty} + \|\mathbb{H} g\|_{L^\infty}); \tag{C.39}$$

$$\|[f, \mathbb{H}]\, g\|_{\dot{H}^{1/2}} \lesssim \|\partial_{\alpha'} f\|_{L^2} \|g\|_{L^2}; \tag{C.40}$$

$$\|[f, \mathbb{H}]\, \partial_{\alpha'} g\|_{\dot{H}^{1/2}} \lesssim \|g\|_{\dot{H}^{1/2}} (\|\partial_{\alpha'} f\|_{L^\infty} + \|\partial_{\alpha'} \mathbb{H} f\|_{L^\infty}); \tag{C.41}$$

$$\|f\|_{L^4}^2 \lesssim \|f\|_{L^2} \|f\|_{\dot{H}^{1/2}}. \tag{C.42}$$

Finally, we need the following inequalities for our proof. Let $\fint_x^y \mathsf{b} = \frac{\int_y^x \mathsf{b}(\alpha)\, d\alpha}{x - y}$. We have

Proposition C.17. *Let* $\mathsf{b} \in BMO(\mathbb{R})$, $f_1, \dots, f_n \in C^1(\mathbb{R})$ *such that* $f_1', \dots, f_n' \in L^2(\mathbb{R})$, *and* $g \in L^2(\mathbb{R})$. *Then*

$$\left\|\int \left(\mathsf{b}(y) - \fint_x^y \mathsf{b}\right) \frac{\Pi_{i=1}^2 (f_i(x) - f_i(y))}{(x-y)^2}\, dy\right\|_{L^\infty} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2}; \tag{C.43}$$

$$\left\|\int \left(\mathsf{b}(y) - \fint_x^y \mathsf{b}\right)^2 \frac{\Pi_{i=1}^2 (f_i(x) - f_i(y))}{(x-y)^2}\, dy\right\|_{L^\infty} \lesssim \|\mathsf{b}\|_{BMO}^2 \|f_1'\|_{L^2} \|f_2'\|_{L^2}; \tag{C.44}$$

$$\left\|\int \left(\mathsf{b}(x) + \mathsf{b}(y) - 2\fint_x^y \mathsf{b}\right) \frac{\Pi_{i=1}^2 (f_i(x) - f_i(y))}{(x-y)^2} g(y)\, dy\right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2} \|g\|_{L^2}; \tag{C.45}$$

$$\left\|\int \frac{(\mathsf{b}(x) - \mathsf{b}(y)) \Pi_{i=1}^2 (f_i(x) - f_i(y))}{(x-y)^2} g(y)\, dy\right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2} \|g\|_{L^2}; \tag{C.46}$$

$$\left\| \int \left(\mathsf{b}(x) + \mathsf{b}(y) - 2\fint_x^y \mathsf{b} \right) \frac{\Pi_{i=1}^3 (f_i(x) - f_i(y))}{(x-y)^3} \, dy \right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2} \|f_3'\|_{L^2} ; \tag{C.47}$$

$$\left\| \int \frac{(\mathsf{b}(x) - \mathsf{b}(y)) \Pi_{i=1}^3 (f_i(x) - f_i(y))}{(x-y)^3} \, dy \right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2} \|f_3'\|_{L^2} ; \tag{C.48}$$

and for $n \geq 3$,

$$\left\| \int \frac{(\mathsf{b}(x) - \mathsf{b}(y)) \Pi_{i=1}^n (f_i(x) - f_i(y))}{(x-y)^n} g(y) \, dy \right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f_1'\|_{L^2} \|f_2'\|_{L^2} \|g \Pi_{i=3}^n M(f_i')\|_{L^2} , \tag{C.49}$$

where $M(f)$ *is the Hardy-Littlewood maximum function of* f.

Remark C.18. Observe that $\dot{H}^{1/2}(\mathbb{R}) \subset BMO(\mathbb{R})$, and $\|\mathsf{b}\|_{BMO} \lesssim \|\mathsf{b}\|_{\dot{H}^{1/2}}$ for all $\mathsf{b} \in \dot{H}^{1/2}$. We will primarily use Proposition C.17 for $\mathsf{b} \in \dot{H}^{1/2}(\mathbb{R})$. We sometimes also use Proposition C.17 for $\mathsf{b} \in L^\infty(\mathbb{R})$, with the inequality $\|\mathsf{b}\|_{BMO} \lesssim \|\mathsf{b}\|_{L^\infty}$.

Proof. It suffices to prove for the case that $f_1 = f_2$. We begin with the following lemmas.

Lemma C.19 (Schur test). *Let*

$$Tg(x) = \int e^{\mathsf{b}(x)} K(x,y) e^{\mathsf{b}(y)} g(y) \, dy.$$

where b *is a real valued measurable function,* K *is measurable on* $\mathbb{R}^2$*. Assume that*

$$\max\{ \sup_x \int e^{2\mathsf{b}(y)} |K(x,y)| \, dy, \sup_y \int e^{2\mathsf{b}(x)} |K(x,y)| \, dx \} := M < \infty.$$

Then for any $g \in L^2$,

$$\|Tg\|_{L^2} \leq M \|g\|_{L^2} .$$

Let $\mathsf{b} \in BMO(\mathbb{R})$ be real valued, and $h(x) = \int_0^x e^{\mathsf{b}(y)} \, dy$. We know there is a constant $\gamma_0 > 0$, such that for any $\mathsf{b} \in BMO(\mathbb{R})$ satisfying $\|\mathsf{b}\|_{BMO} \leq \gamma_0$, e^{b} is a A_2 weight and $U_{h^{-1}}$ is a bounded map from $BMO(\mathbb{R})$ to $BMO(\mathbb{R})$, with $\|f \circ h^{-1}\|_{BMO} \leq 2 \|f\|_{BMO}$ for all $f \in BMO(\mathbb{R})$.[22] And for any real valued BMO function b with $\|\mathsf{b}\|_{BMO} \leq \gamma < 1$, $z(x) = \int_0^x e^{i\mathsf{b}(\alpha)} \, d\alpha$ defines a chord-arc curve, cf. [16, 17], with

$$(1-\gamma)|x-y| \leq |z(x) - z(y)| \leq |x-y|, \qquad \text{for all } x, y \in \mathbb{R}. \tag{C.50}$$

We have

Lemma C.20. *Let* $\mathsf{b} \in BMO(\mathbb{R})$ *be real valued with* $\|\mathsf{b}\|_{BMO} \leq \gamma_0/2$*, and* $h'(x) = e^{\mathsf{b}(x)}$*. For any* f *such that* $f' \in L^2(\mathbb{R})$*, there is a constant* $c(\gamma_0) > 0$*, such that*

$$\sup_x \int e^{2\mathsf{b}(y)} \left| \frac{f(x) - f(y)}{h(x) - h(y)} \right|^2 dy \leq c(\gamma_0) \|f'\|_{L^2}^2 . \tag{C.51}$$

Proof. We know

$$f(x) - f(y) = \int_y^x f'(\alpha) \, d\alpha = \int_{h(y)}^{h(x)} f' \circ h^{-1}(\beta) (h^{-1})'(\beta) \, d\beta,$$

hence

$$\sup_x \left| \frac{f(x) - f(y)}{h(x) - h(y)} \right| \leq M \left(f' \circ h^{-1} (h^{-1})' \right) (h(y))$$

[22] These are consequences of John-Nirenberg's inequality and the theory of A_p weights, cf. [16, 17].

where $M(g)$ is the Hardy-Littlewood Maximal function of g. Because $e^{\mathsf{b}\circ h^{-1}}$ is a A_2 weight, we have

$$\begin{aligned}\sup_x \int e^{2\mathsf{b}(y)} \left|\frac{f(x)-f(y)}{h(x)-h(y)}\right|^2 dy &\le \int e^{2\mathsf{b}(y)} M^2\left(f'\circ h^{-1}(h^{-1})'\right)(h(y))\,dy \\ = \int e^{\mathsf{b}\circ h^{-1}(y)} M^2\left(f'\circ h^{-1}(h^{-1})'\right)(y)\,dy &\lesssim \int e^{\mathsf{b}\circ h^{-1}(y)} \left|f'\circ h^{-1}(h^{-1})'\right|^2(y)\,dy = \|f'\|_{L^2}^2 .\end{aligned} \tag{C.52}$$

This proves Lemma C.20. □

Now let b be a BMO function satisfying $\|\mathsf{b}\|_{BMO} \le 1$, and z be a complex number in $D := \left\{|z| < \min\{\frac14, \frac{\gamma_0}{2}\}\right\}$. Let $h_z(x) = \int_0^x e^{z\mathsf{b}(\alpha)}\,d\alpha$ and

$$T_1(f, \mathsf{b}; z)(x) : = \int e^{2z\mathsf{b}(y)} \left(\frac{f(x)-f(y)}{h_z(x)-h_z(y)}\right)^2 dy, \tag{C.53}$$

$$T_2(f, g, \mathsf{b}; z)(x) : = \int e^{z\mathsf{b}(x)} e^{z\mathsf{b}(y)} \left(\frac{f(x)-f(y)}{h_z(x)-h_z(y)}\right)^2 g(y)\,dy. \tag{C.54}$$

By Lemma C.20, and (C.50), we have for all f satisfying $f' \in L^2$,

$$\|T_1(f, \mathsf{b}; z)\|_{L^\infty} \le c(\gamma_0)\|f'\|_{L^2}^2, \tag{C.55}$$

and by Lemmas C.19, C.20, and (C.50), we have for all f, g, satisfying $f',\ g \in L^2$,

$$\|T(f, g, \mathsf{b}; z)\|_{L^2(\mathbb{R})} \le c(\delta_o)\|f'\|_{L^2}^2\|g\|_{L^2}, \tag{C.56}$$

where $c(\gamma_0) > 0$ is a constant depending only on γ_0. Let $q_1 \in L^1(\mathbb{R})$, $q_2 \in L^2(\mathbb{R})$, and

$$F_1(z) := \int q_1(x) T_1(f, \mathsf{b}; z)(x)\,dx, \tag{C.57}$$

$$F_2(z) := \int q_2(x) T_2(f, g, \mathsf{b}; z)(x)\,dx. \tag{C.58}$$

Then F_1, F_2 are holomorphic functions in the domain D, satisfying

$$|F_1(z)| \le c(\gamma_0)\|q_1\|_{L^1}\|f'\|_{L^2}^2, \quad |F_2(z)| \le c(\gamma_0)\|q_2\|_{L^2}\|f'\|_{L^2}^2\|g\|_{L^2}, \qquad \text{for all } z \in D. \tag{C.59}$$

And by Cauchy integral theorem, we have

$$|F_1'(0)| \lesssim c(\gamma_0)\|q_1\|_{L^1}\|f'\|_{L^2}^2, \quad |F_1''(0)| \lesssim c(\gamma_0)\|q_1\|_{L^1}\|f'\|_{L^2}^2, \tag{C.60}$$

$$\text{and} \quad |F_2'(0)| \lesssim c(\gamma_0)\|q_2\|_{L^2}\|f'\|_{L^2}^2\|g\|_{L^2}. \tag{C.61}$$

We compute $F_1'(0)$, $F_1''(0)$, and obtain

$$\begin{aligned} F_1'(0) &= 2\iint q_1(x)\left(\mathsf{b}(y) - \fint_x^y \mathsf{b}\right)\frac{(f(x)-f(y))^2}{(x-y)^2}\,dy\,dx \\ F_1''(0) &= \iint \frac{q_1(x)(f(x)-f(y))^2}{(x-y)^2}\left(4\left(\mathsf{b}(y) - \fint_x^y \mathsf{b}\right)^2 - 2\fint_x^y (\mathsf{b}(\alpha) - \fint_x^y \mathsf{b})^2\,d\alpha\right)dxdy. \end{aligned} \tag{C.62}$$

Because $\sup_{x,y} \fint_x^y \left|\mathsf{b}(\alpha) - \fint_x^y \mathsf{b}\right|^2 d\alpha \lesssim \|\mathsf{b}\|_{BMO}^2 \le 1$ and by Hardy's inequality,

$$\iint |q_1(x)| \left|\frac{(f(x)-f(y))^2}{(x-y)^2}\right| dx\,dy \lesssim \|q_1\|_{L^1}\|f'\|_{L^2}^2,$$

so

$$\left|\iint \frac{q_1(x)(f(x)-f(y))^2}{(x-y)^2}\left(\fint_x^y (\mathsf{b}(\alpha) - \fint_x^y \mathsf{b})^2\,d\alpha\right)dxdy\right| \lesssim \|q_1\|_{L^1}\|f'\|_{L^2}^2,$$

(C.43) and (C.44) then follow from (C.60) and (C.62). We compute

$$F_2'(0) = \iint q_2(x) \left(\mathsf{b}(x) + \mathsf{b}(y) - 2 \fint_x^y \mathsf{b} \right) \frac{(f(x) - f(y))^2}{(x-y)^2} g(y)\, dy\, dx. \tag{C.63}$$

By (C.61) this gives us (C.45).

Now the Schur test, (C.44) and Hardy's inequality yields

$$\left\| \int \left(\mathsf{b}(y) - \fint_x^y \mathsf{b} \right) \frac{(f(x) - f(y))^2}{(x-y)^2} g(y)\, dy \right\|_{L^2} \lesssim \|\mathsf{b}\|_{BMO} \|f'\|_{L^2}^2 \|g\|_{L^2}. \tag{C.64}$$

This together with (C.45) gives (C.46).

(C.47), (C.48), and (C.49) can be proved similarly by considering

$$T_3(f, f_3, \mathsf{b}; z) := \int e^{z\mathsf{b}(x)} e^{z\mathsf{b}(y)} \left(\frac{f(x) - f(y)}{h_z(x) - h_z(y)} \right)^2 \frac{f_3(x) - f_3(y)}{x - y}\, dy, \tag{C.65}$$

$$T_4(f, f_3, \dots f_n, g, \mathsf{b}; z) := \int e^{z\mathsf{b}(x)} e^{z\mathsf{b}(y)} \left(\frac{f(x) - f(y)}{h_z(x) - h_z(y)} \right)^2 \frac{\Pi_{i=3}^n (f_i(x) - f_i(y))}{(x-y)^{n-2}} g(y)\, dy, \tag{C.66}$$

and using the fact that

$$\left| \frac{f_i(x) - f_i(y)}{x - y} \right| \le M(f_i')(y), \qquad 3 \le i \le n.$$

We omit the details.

□

Appendix D. Main quantities controlled by $L(t)$ and $\left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2} + \|Z_{t,\alpha'}\|_{\dot H^{1/2}}$

Assume that

$$L(t) = \|Z_{t,\alpha'}\|_{L^2} + \left\| \frac{1}{Z_{,\alpha'}} \right\|_{\dot H^{1/2}} + \|Z_{t,\alpha'\alpha'}\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\dot H^{1/2}} \le 2\epsilon, \quad \left\| \frac{1}{Z_{,\alpha'}} - 1 \right\|_{L^\infty} \le 1 - \delta < 1.$$

We have shown in §4.3.1 and §4.3.2 that the following quantities are controlled by ϵ:

$$\begin{aligned}
&\|Z_{t,\alpha'}\|_{L^\infty},\ \|Z_{t,\alpha'}\|_{\dot H^{1/2}},\ \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_{L^2},\ \|Z_{tt}\|_{\dot H^{1/2}},\ \|b_{\alpha'}\|_{L^2},\ \left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{L^2},\ \left\|D_t^2 Z_t\right\|_{L^2},\ \|b_{\alpha'}\|_{L^\infty},\\
&\left\|D_t \frac{1}{Z_{,\alpha'}}\right\|_{L^\infty},\ \|Z_{ttt}\|_{L^\infty},\ \|\partial_{\alpha'} b_{\alpha'}\|_{L^2},\ \left\|\partial_{\alpha'} D_t \frac{1}{Z_{,\alpha'}}\right\|_{L^2},\ \|Z_{tt,\alpha'}\|_{\dot H^{1/2}},\ \|\partial_{\alpha'} D_t b\|_{\dot H^{1/2}},\\
&\left\|D_t^2 \frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}},\ \|Z_{ttt,\alpha'}\|_{L^2},\ \|\partial_{\alpha'} D_t^2 b\|_{L^2},\ \left\|D_t^3 \frac{1}{Z_{,\alpha'}}\right\|_{L^2},\ \left\|D_t^2 Z_{tt}\right\|_{\dot H^{1/2}},\ \left\|D_t^3 Z_{tt}\right\|_{L^2},\\
&\left\|\partial_{\alpha'}\Theta^{(4)}\right\|_{L^2},\ \left\|\Theta^{(5)}\right\|_{\dot H^{1/2}},\ \left\|\partial_{\alpha'}\Theta^{(3)}\right\|_{\dot H^{1/2}},\ \left\|\partial_{\alpha'} D_t \Theta^{(3)}\right\|_{L^2},\ \left\|D_t^2\Theta^{(3)}\right\|_{\dot H^{1/2}},\ \left\|D_t D_{\alpha'}\Theta^{(3)}\right\|_{L^2};
\end{aligned} \tag{D.1}$$

the following quantities are controlled by ϵ^2:

$$\begin{aligned}
&\|A_1 - 1\|_{L^\infty},\ \|A_1\|_{\dot H^{1/2}},\ \|D_t A_1\|_{L^2},\ \|b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t\|_{L^2},\ \left\|\mathcal{P}\overline{Z}_t\right\|_{L^2},\\
&\|\partial_{\alpha'} A_1\|_{\dot H^{1/2}},\ \|D_t(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t)\|_{\dot H^{1/2}},\ \|D_t(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t)\|_{L^\infty},\\
&\left\|D_t^2 A_1\right\|_{\dot H^{1/2}},\ \left\|D_t^2 A_1\right\|_{L^\infty},\ \left\|[D_t, \mathcal{P}]\,\overline{Z}_{ttt}\right\|_{L^2},\ \left\|D_t\,[D_t, \mathcal{P}]\,\overline{Z}_{tt}\right\|_{L^2},\ \left\|D_t^2\,[D_t, \mathcal{P}]\,\overline{Z}_t\right\|_{L^2},\\
&\left\|D_t^3(b_{\alpha'} - 2\operatorname{Re} D_{\alpha'} Z_t)\right\|_{L^2},\ \left\|\partial_{\alpha'}\left(D_t^3 b - 2\operatorname{Re}\frac{D_t^3 Z_t}{Z_{,\alpha'}}\right)\right\|_{L^2},\ \left\|D_t^4 A_1\right\|_{L^2},\ \left\|\mathcal{P} D_t^3 \overline{Z}_t\right\|_{L^2};
\end{aligned} \tag{D.2}$$

the following quantities are controlled by $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}$:

$$\begin{aligned}&\|b_{\alpha'}\|_{\dot H^{1/2}},\ \left\|D_t\frac{1}{Z_{,\alpha'}}\right\|_{\dot H^{1/2}},\ \left\|\partial_{\alpha'}\Theta^{(2)}\right\|_{\dot H^{1/2}},\ \left\|D_{\alpha'}\Theta^{(2)}\right\|_{\dot H^{1/2}},\ \|Z_{tt,\alpha'}\|_{L^2},\ \|Z_{ttt}\|_{\dot H^{1/2}},\\ &\|\partial_{\alpha'}D_tb\|_{L^2},\ \|D_tD_{\alpha'}Z_t\|_{L^2},\ \left\|D_t^3Z_t\right\|_{L^2},\ \left\|D_t^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2},\\ &\left\|\partial_{\alpha'}\Theta^{(3)}\right\|_{L^2},\ \left\|D_tD_{\alpha'}\Theta^{(2)}\right\|_{L^2},\ \left\|\Theta^{(4)}\right\|_{\dot H^{1/2}},\ \left\|D_t\Theta^{(3)}\right\|_{\dot H^{1/2}},\ \left\|D_t^2\Theta^{(2)}\right\|_{\dot H^{1/2}};\end{aligned}\tag{D.3}$$

the following quantities are controlled by $\epsilon\left(\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}+\|Z_{t,\alpha'}\|_{\dot H^{1/2}}\right)$:

$$\begin{aligned}&\|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{\dot H^{1/2}},\ \|b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t\|_{L^\infty},\ \|\partial_{\alpha'}A_1\|_{L^2},\ \|D_tA_1\|_{\dot H^{1/2}},\ \left\|\mathcal{P}\overline{Z}_t\right\|_{\dot H^{1/2}},\\ &\|D_tA_1\|_{L^\infty},\ \left\|\mathcal{P}\overline{Z}_t\right\|_{L^\infty},\ \|D_t(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2},\ \left\|D_t^2A_1\right\|_{L^2},\ \left\|\mathcal{P}\overline{Z}_{tt}\right\|_{L^2},\ \left\|[\mathcal{P},D_t]\,\overline{Z}_t\right\|_{L^2},\\ &\left\|D_t^2(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\right\|_{L^2},\ \left\|D_t^3A_1\right\|_{L^2},\ \left\|\partial_{\alpha'}\left(D_t^2b-2\operatorname{Re}\frac{Z_{ttt}}{Z_{,\alpha'}}\right)\right\|_{L^2},\ \left\|\mathcal{P}D_t^2\overline{Z}_t\right\|_{L^2},\\ &\left\|[D_t,\mathcal{P}]\,\overline{Z}_{tt}\right\|_{L^2},\ \left\|\left[D_t^2,\mathcal{P}\right]\overline{Z}_t\right\|_{L^2},\ \|\partial_{\alpha'}(b_{\alpha'}-2\operatorname{Re}D_{\alpha'}Z_t)\|_{L^2},\ \|\partial_{\alpha'}D_tA_1\|_{L^2}.\end{aligned}\tag{D.4}$$

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