A BLOW-UP CRITERIA AND THE EXISTENCE OF 2D GRAVITY WATER WAVES WITH ANGLED CRESTS

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ABSTRACT. We consider the two dimensional gravity water wave equation in the regime that includes free surfaces with angled crests. We assume that the fluid is inviscid, incompressible and irrotational, the air density is zero, and we neglect the surface tension. In [21] it was shown that in this regime, only a degenerate Taylor inequality $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds, with degeneracy at the singularities; an energy functional $\mathfrak{E}$ was constructed and an aprori estimate was proved. In this paper we show that a (generalized) solution of the water wave equation with smooth data will remain smooth so long as $\mathfrak{E}(t)$ remains finite; and for any data satisfying $\mathfrak{E}(0) < \infty$, the equation is solvable locally in time, for a period depending only on $\mathfrak{E}(0)$.

1. Introduction

A class of water wave problems concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{k} - \nabla P \qquad \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, \qquad \text{on } \Omega(t),\ t \geq 0, \\ P = 0, \qquad \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases} \tag{1.1}$$

euler

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. There is an important condition for these problems:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0 \tag{1.2}$$

taylor

pointwise on the interface, where $\mathbf{n}$ is the outward unit normal to the interface $\Sigma(t)$ [29]. It is well known that when surface tension is neglected and the Taylor sign condition (1.2) fails, the water wave motion can be subject to the Taylor instability [29, 6, 5]. In [30, 31], we showed that for dimensions $n \geq 2$, the strong Taylor stability criterion

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{1.3}$$

taylor-s

always holds for the infinite depth water wave problem (1.1), as long as the interface is non-self-intersecting and smooth; and the initial value problem of the water wave system (1.1) is uniquely solvable locally in time in Sobolev spaces H^s, $s \geq 4$ for arbitrary given data. Earlier work include Nalimov [25], Yosihara [36] and Craig [12] on local existence and uniqueness for small and smooth data for the 2d water wave equation (1.1). There have been much work recently, local wellposedness for water waves with additional effects such as surface tension, bottom and vorticity have been proved, c.f. [4, 8, 9, 17, 22, 24, 26, 27, 37];

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local wellposedness of (1.1) in low regularity Sobolev spaces where the interfaces are in $C^{3/2}$ has been obtained, c.f. [1, 2]. In all of these work, the strong Taylor stability criterion (1.3) is assumed.[1] In addition, in the last few years, almost global and global wellposedness for the water wave equation (1.1) in both two and three dimensional spaces for small, smooth and localized initial data have been proved, c.f. [32, 33, 14, 18, 3].

In [21], we studied the 2d water wave equation (1.1) in the regime that includes free interfaces with angled crests. We constructed an energy functional $\mathfrak{E}(t)$ in this framework and proved an a priori estimate. In this paper we introduce a notion of generalized solutions of (1.1) – a generalized solution is classical provided the interface is non-self-intersecting;[2] we prove a blow-up criteria that states that for smooth initial data, a unique generalized solution of the 2d water wave equation exists and remains smooth so long as $\mathfrak{E}(t)$ remains finite; and we show that for data satisfying $\mathfrak{E}(0) < \infty$, a generalized solution of the 2d water wave equation (1.1) exists for a time period depending only on $\mathfrak{E}(0)$; if in addition the initial interface is chord-arc,[3] there is a $T > 0$, depending only on $\mathfrak{E}(0)$ and the chord-arc constant, so that the interface remains chord-arc and a classical solution of the 2d water wave equation (1.1) exists for time $t \in [0, T]$. The (generalized) solution is constructed by mollifying the initial data and by showing that the sequence of (generalized) solutions for the mollified data converges to a (generalized) solution for the given data.

The rest of the paper is organized as follows: in section 2, we state and refine the earlier results this paper is built upon, this includes the local wellposedness result for Sobolev data in [30], and the energy functional $\mathfrak{E}$ constructed and the a priori estimate proved in [21], in the context of generalized solutions; the notion of generalized solutions will be introduced in §2.2 and §2.3. In section 3 we present the main results: a blow-up criteria via the energy functional $\mathfrak{E}$ and the local existence of water waves with angled crests. We prove the blow-up criteria in sections 4 and the local existence in section 5. The majority of the notation are introduced in §2.1, with the rest throughout the paper. Some basic preparatory results in analysis are given in Appendix A; various identities that are useful for the paper are derived in Appendix B. Finally in Appendix C, we list the quantities which have been shown in [21] are controlled by $\mathfrak{E}$.

2. Preliminaries

notation prelim

2.1. Notation and convention. We consider solutions of the water wave equation (1.1) in the setting where the fluid domain $\Omega(t)$ is simply connected, with the free interface $\Sigma(t) := \partial\Omega(t)$ a Jordan curve,[4]

$$\mathbf{v}(z,t) \to 0, \qquad \text{as } |z| \to \infty$$

and the interface $\Sigma(t)$ tending to horizontal lines at infinity.[5]

We use the following notations and conventions: $[A, B] := AB - BA$ is the commutator of operators A and B. $H^s(\mathbb{R})$ is the Sobolev space with norm $\|f\|_{H^s} := (\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi)^{1/2}$, $\dot{H}^{1/2}$ is the Sobolev space with norm $\|f\|_{\dot{H}^{1/2}} := (\int |\xi| |\hat{f}(\xi)|^2 \, d\xi)^{1/2}$, $L^p = L^p(\mathbb{R})$ is the L^p space with $\|f\|_{L^p} := (\int |f(x)|^p \, dx)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_{L^\infty} :=$ ess sup $|f(x)|$. We write $f(t) := f(\cdot, t)$, with $\|f(t)\|_{H^s}$ being the Sobolev norm, $\|f(t)\|_{L^p}$ being the L^p norm of $f(t)$ in the spatial variable. When not specified, all the H^s and L^p norms are in terms of the spatial variables. Compositions are always in terms of the spatial variables and we write for $f = f(\cdot, t)$, $g = g(\cdot, t)$, $f(g(\cdot, t), t) := f \circ g(\cdot, t) := U_g f(\cdot, t)$. We

[1] When there is surface tension, or vorticity, or a bottom, (1.3) doesn't always hold.

[2] By non-self-intersecting we mean it is a Jordan curve.

[3] A curve is chord-arc if the arc-length and the chord length between any two points on the curve are comparable.

[4] That is, $\Sigma(t)$ is homeomorphic to the line $\mathbb{R}$.

[5] The problem with velocity $\mathbf{v}(z, t) \to (c, 0)$ as $|z| \to \infty$ can be reduced to the one with $\mathbf{v} \to 0$ at infinity by studying the solutions in a moving frame. $\Sigma(t)$ may tend to two different lines at $+\infty$ and $-\infty$.

identify (x, y) with the complex number $x+iy$; $\operatorname{Re} z$, $\operatorname{Im} z$ are the real and imaginary parts of z; $\overline{z} = \operatorname{Re} z - i\operatorname{Im} z$ is the complex conjugate of z. $\overline{\Omega}$ is the closure of the domain Ω, $\partial\Omega$ is the boundary of Ω, $P_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ is the lower half plane. We write

$$[f, g; h] := \frac{1}{\pi i}\int \frac{(f(x)-f(y))(g(x)-g(y))}{(x-y)^2} h(y)\, dy. \tag{2.1}$$

eq:comm

We use c, C to denote universal constants and $c(a, b)$, $C(a)$, $M(a)$ etc. to denote constants that depends on a, b and respectively a etc. Constants appearing in different contexts need not be the same. We write $f \lesssim g$ if there is a universal constant c, such that $f \le cg$. RHS, LHS are the short codes for the "right hand side" and the "left hand side".

surface-equation

2.2. **The equation for the free surface in Lagrangian and Riemann mapping variables.** Let the free interface $\Sigma(t) : z = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be given by Lagrangian parameter α, so $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t); t)$ is the velocity of the fluid particles on the interface, $z_{tt}(\alpha, t) = \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}(z(\alpha, t); t)$ is the acceleration; notice that $P = 0$ on $\Sigma(t)$ implies that ∇P is normal to $\Sigma(t)$, therefore $\nabla P = -i\mathfrak{a} z_\alpha$, where

$$\mathfrak{a} = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}}; \tag{2.2}$$

frak-a

and the first and third equation of (1.1) gives

$$z_{tt} + i = i\mathfrak{a} z_\alpha. \tag{2.3}$$

interface-l

The second equation of (1.1): $\operatorname{div} \mathbf{v} = \operatorname{curl} \mathbf{v} = 0$ implies that $\overline{\mathbf{v}}$ is holomorphic in the fluid domain $\Omega(t)$, hence $\overline{z}_t$ is the boundary value of a holomorphic function in $\Omega(t)$. By Proposition A.1 the second equation of (1.1) is equivalent to $\overline{z}_t = \mathfrak{H}\overline{z}_t$, where $\mathfrak{H}$ is the Hilbert transform associated with the fluid domain $\Omega(t)$. So the motion of the fluid interface $\Sigma(t) : z = z(\alpha, t)$ is given by

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \overline{z}_t = \mathfrak{H}\overline{z}_t. \end{cases} \tag{2.4}$$

interface-e

(2.4) is a fully nonlinear equation. In [30], Riemann mapping was introduced to analyze the quasi-linear structure of (2.4).

Let $\Phi(\cdot, t) : \overline{\Omega(t)} \to \overline{P}_-$ be the Riemann mapping taking $\overline{\Omega(t)}$ to the closure of the lower half plane $\overline{P}_-$, satisfying $\lim_{z\to\infty} \Phi_z(z, t) = 1$. Let

$$h(\alpha, t) := \Phi(z(\alpha, t), t), \tag{2.5}$$

h

so $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Let h^{-1} be defined by

$$h(h^{-1}(\alpha', t), t) = \alpha', \quad \alpha' \in \mathbb{R};$$

and

$$Z(\alpha', t) := z \circ h^{-1}(\alpha', t), \quad Z_t(\alpha', t) := z_t \circ h^{-1}(\alpha', t), \quad Z_{tt}(\alpha', t) := z_{tt} \circ h^{-1}(\alpha', t) \tag{2.6}$$

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be the reparametrization of the position, velocity and acceleration of the interface in the Riemann mapping variable α'. Let

$$Z_{,\alpha'}(\alpha', t) := \partial_{\alpha'} Z(\alpha', t), \quad Z_{t,\alpha'}(\alpha', t) := \partial_{\alpha'} Z_t(\alpha', t), \quad Z_{tt,\alpha'}(\alpha', t) := \partial_{\alpha'} Z_{tt}(\alpha', t), \tag{2.7}$$

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etc. We note that $\Phi^{-1}(\alpha', t) = Z(\alpha', t)$, so $(\Phi^{-1})_{z'}(\alpha', t) = Z_{,\alpha'}(\alpha', t)$, and by Proposition A.1,

$$Z_{,\alpha'} - 1 = \mathbb{H}(Z_{,\alpha'} - 1), \qquad \frac{1}{Z_{,\alpha'}} - 1 = \mathbb{H}(\frac{1}{Z_{,\alpha'}} - 1). \tag{2.8}$$

interface-holo

Observe that $\overline{\mathbf{v}}\circ\Phi^{-1} : P_- \to \mathbb{C}$ is holomorphic in the lower half plane P_- with $\overline{\mathbf{v}}\circ\Phi^{-1}(\alpha', t) = \overline{Z}_t(\alpha', t)$. Precomposing (2.3) with h^{-1} and applying Proposition A.1 to $\overline{\mathbf{v}}\circ\Phi^{-1}$ on P_- gives the free surface equation in the Riemann mapping variable:

$$\begin{cases} Z_{tt} + i = i\mathcal{A}Z_{,\alpha'} \\ \overline{Z}_t = \mathbb{H}\overline{Z}_t \end{cases} \tag{2.9}$$

interface-r

where $\mathcal{A}\circ h = \mathfrak{a}h_\alpha$ and $\mathbb{H}$ is the Hilbert transform associated with the lower half plane P_-:

$$\mathbb{H}f(\alpha') = \frac{1}{\pi i}\text{pv.}\int \frac{1}{\alpha' - \beta'} f(\beta')\,d\beta'.$$

From the chain rule, we know for $f = f(\cdot, t)$, $U_h^{-1}\partial_t U_h f = (\partial_t + b\partial_{\alpha'})f$ where

$$b := h_t \circ h^{-1};$$

so $Z_{tt} = (\partial_t + b\partial_{\alpha'})Z_t = (\partial_t + b\partial_{\alpha'})^2 Z$. Let $A_1 := \mathcal{A}|Z_{,\alpha'}|^2$. Multiply $\overline{Z}_{,\alpha'}$ to the first equation of (2.9) yields

$$\overline{Z}_{,\alpha'}(Z_{tt} + i) = iA_1. \tag{2.10}$$

interface-a1

In [30], it was shown that systems (1.1), (2.4) and (2.9)-(2.8) with b, A_1 given by (2.18), (2.19) are equivalent in the regime of nonself-intersecting interfaces $z = z(\cdot, t)$.[6]

However the system (2.9)-(2.8) is well defined even if $Z = Z(\cdot, t)$ is self-intersecting. In constructing the approximating sequence of solutions from the mollified data, it is convenient to allow self-intersecting solutions of (2.9)-(2.8). In this context, Z and z, Z_t, z_t etc. are related via (2.6) and (2.7) through a homeomorphism $h = h(\cdot, t) : \mathbb{R} \to \mathbb{R}$, and from (2.9)-(2.8) we can show that h, A_1 satisfy (2.18)-(2.19), see Appendix B.1. For not necessarily non-self-intersecting solutions Z of (2.9)-(2.8) we will abuse terminologies by continue saying Z, Z_t etc. are in the Riemann mapping variable, z, z_t etc. are in the Lagrangian coordinates, Z, Z_t, Z_{tt} are the interface, velocity and acceleration.

Let's consider the solution of (2.9)-(2.8) in the "fluid domain".[7]

general-soln

2.3. **Generalized solutions of the water wave equation.** [8]

Let $Z = Z(\cdot, t)$ be a solution of (2.9)-(2.8), let $F(\cdot, t) : P_- \to \mathbb{C}$, $\Psi(\cdot, t) : P_- \to \mathbb{C}$ be holomorphic functions, continuous on $\overline{P}_-$, such that

$$F(\alpha', t) = \overline{Z}_t(\alpha', t), \qquad \Psi(\alpha', t) = Z(\alpha', t), \qquad \Psi_{z'}(\alpha', t) = Z_{,\alpha'}(\alpha', t). \tag{2.11}$$

eq:270

By (B.4) of Appendix B.1 and (2.11),

$$h_t \circ h^{-1} = \frac{Z_t}{Z_{,\alpha'}} - \frac{\Psi_t}{\Psi_{z'}} = \frac{\overline{F}}{\Psi_{z'}} - \frac{\Psi_t}{\Psi_{z'}}. \tag{2.12}$$

eq:271

Now $\overline{z}_t(\alpha, t) = \overline{Z}_t(h(\alpha, t), t) = F(h(\alpha, t), t)$, so

$$\overline{z}_{tt} = F_t \circ h + F_{z'} \circ h h_t = U_h\{F_t - \frac{\Psi_t}{\Psi_{z'}}F_{z'} + \frac{\overline{F}}{\Psi_{z'}}F_{z'}\}$$

therefore $\overline{Z}_{tt}$ is the trace of the function $F_t - \frac{\Psi_t}{\Psi_{z'}}F_{z'} + \frac{\overline{F}}{\Psi_{z'}}F_{z'}$ on ∂P_-; $Z_{,\alpha'}(\overline{Z}_{tt} - i)$ is then the trace of the function $\Psi_{z'}F_t - \Psi_t F_{z'} + \overline{F}F_{z'} - i\Psi_{z'}$ on ∂P_-. By (2.10),

$$\Psi_{z'}F_t - \Psi_t F_{z'} + \overline{F}F_{z'} - i\Psi_{z'} = iA_1, \qquad \text{on } \partial P_-. \tag{2.13}$$

eq:272

[6] When $\Sigma(t) : Z = Z(\cdot, t)$ becomes self-intersecting, it is not physical to assume $P \equiv 0$ on $\Sigma(t)$. So in general we do not consider beyond the regime of non-self-intersecting interfaces.

[7] It makes sense to talk about fluid domain only when $Z = Z(\cdot, t)$ is non-self-intersecting. Here we just abuse the terminology.

[8] Here and in §2.2 we give a generic discussion. The statements are rigorous if the quantities involved are sufficiently regular.

On the left hand side of (2.13), $\Psi_{z'}F_t - \Psi_t F_{z'} - i\Psi_{z'}$ is holomorphic on P_-,while $\overline{F}F_{z'} = \partial_{z'}(\overline{F}F)$; we recall from complex analysis, $\partial_{z'} = \frac{1}{2}(\partial_{x'} - i\partial_{y'})$. So there is a real valued function $\mathfrak{P} : \mathbb{P}_- \to \mathbb{R}$, such that

$$\Psi_{z'}F_t - \Psi_t F_{z'} + \overline{F}F_{z'} - i\Psi_{z'} = -(\partial_{x'} - i\partial_{y'})\mathfrak{P}, \qquad \text{on } P_- \tag{2.14}$$ eq:273

moreover by (2.13), because iA_1 is purely imaginary,

$$\mathfrak{P} = 0, \qquad \text{on } \partial P_-. \tag{2.15}$$ eq:274

We note that by applying $\partial_{x'} + i\partial_{y'}$ to both sides of (2.14), $\mathfrak{P}$ satisfies

$$\Delta\mathfrak{P} = -2|F_{z'}|^2 \qquad \text{on } P_-. \tag{2.16}$$ eq:275

If in addition $\Sigma(t) = \{Z = Z(\alpha', t) := \Psi(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is a Jordan curve with

$$\lim_{|\alpha'| \to \infty} Z_{,\alpha'}(\alpha', t) = 1,$$

let $\Omega(t)$ be the domain bounded by $Z = Z(\cdot, t)$ from the above, then $Z = Z(\alpha', t)$, $\alpha' \in \mathbb{R}$ winds the boundary of $\Omega(t)$ exactly once. By the argument principle, $\Psi : \overline{P}_- \to \overline{\Omega}(t)$ is one-to-one and onto, $\Psi^{-1} : \Omega(t) \to P_-$ exists and is a holomorphic function. In this case, it is easy to check by the chain rule that equation (2.14) is equivalent to

$$(F \circ \Psi^{-1})_t + \overline{F} \circ \Psi^{-1}(F \circ \Psi^{-1})_z + (\partial_x - i\partial_y)(\mathfrak{P} \circ \Psi^{-1}) = i, \qquad \text{on } \Omega(t) \tag{2.17}$$ eq;276

This is the Euler equation, i.e. the first equation of (1.1) in complex form. Therefore $\overline{\mathbf{v}} = F \circ \Psi^{-1}$, $P = \mathfrak{P} \circ \Psi^{-1}$ is a solution of the water wave equation (1.1), with $\Sigma(t) : Z = Z(\cdot, t)$ the boundary of the fluid domain $\Omega(t)$.

In what follows we give the local wellposedness result of [30] and the a priori estimate of [21] for solutions of (2.9)-(2.8).

2.4. **Local wellposedness in Sobolev spaces.** In [30] we derived a quasi-linearization of (2.9)-(2.8), the system (4.6)-(4.7) of [30] by taking one derivative to t to equation (2.3) and analyzed the quantities b and A_1;[9] and via A_1, we showed that the strong Taylor inequality (1.3) always holds for smooth nonself-intersecting interfaces. In addition, we proved that the Cauchy problem of the system (4.6)-(4.7) of [30] is locally well-posed in Sobolev spaces.

prop:a1 **Proposition 2.1** (Lemma 3.1 and (4.7) of [30], Proposition 2.2 and (2.18) of [35]). *We have*
1.

$$b := h_t \circ h^{-1} = \operatorname{Re}\big([Z_t, \mathbb{H}](\frac{1}{Z_{,\alpha'}} - 1)\big) + 2\operatorname{Re} Z_t. \tag{2.18}$$ b

2.

$$A_1 = 1 - \operatorname{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi}\int \frac{|Z_t(\alpha', t) - Z_t(\beta', t)|^2}{(\alpha' - \beta')^2}\, d\beta' \ge 1. \tag{2.19}$$ a1

3.

$$-\frac{\partial P}{\partial \mathbf{n}}\Big|_{Z = Z(\cdot, t)} = \frac{A_1}{|Z_{,\alpha'}|}; \tag{2.20}$$ taylor-formula

in particular if the interface $\Sigma(t) \in C^{1,\gamma}$ for some $\gamma > 0$, then the strong Taylor sign condition (1.3) *holds.*

Remark 2.2. By (2.20), the Taylor sign condition (1.2) always holds. Assume $\Sigma(t)$ is non-self-intersecting with angled crests, assume the interior angle at a crest is ν. Around the crest, we know the Riemann mapping Φ^{-1} (we move the singular point to the origin) behaves like

$$\Phi^{-1}(z') \approx (z')^r, \qquad \text{with } \nu = r\pi$$

[9] [35] has a slightly different and shorter derivation. [21] has the derivation in a periodic setting. The reader may want to consult [21, 35] for the derivations. The identities in Appendix B.1 provide yet another derivation of the quasi-linearization and (2.18), (2.19) from (2.9)-(2.8), without assuming $Z = Z(\cdot, t)$ being non-self-intersecting. We note that (2.20) only makes sense for non-self-intersecting interfaces.

so $Z_{,\alpha'} \approx (\alpha')^{r-1}$. From (2.10) and the fact $A_1 \ge 1$, the interior angle at the crest must be $\le \pi$ if the acceleration $|Z_{tt}| \ne \infty$, and $-\frac{\partial P}{\partial \mathbf{n}} = 0$ at the singularities where the interior angles are $< \pi$,[10] cf. [21], §3.

Let $h(\alpha, 0) = \alpha$ for $\alpha \in \mathbb{R}$; let the initial interface $Z(\cdot, 0) := Z(0)$, the initial velocity $Z_t(\cdot, 0) := Z_t(0)$ be given such that $Z(0)$ satisfy (2.8) and $Z_t(0)$ satisfy $\overline{Z}_t(0) = \mathbb{H}\overline{Z}_t(0)$; let A_1 be given by (2.19), the initial acceleration $Z_{tt}(0)$ satisfy (2.10), and $a_0 = \frac{A_1(\cdot,0)}{|Z_{,\alpha'}(\cdot,0)|^2}$. By Theorem 5.11 of [30] and a refinement of the argument in §6 of [30], the following local existence result holds.

prop:local-s **Proposition 2.3** (local existence in Sobolev spaces, cf. Theorem 5.11, §6 of [30])**.** *Let $s \ge 4$. Assume that $Z_t(0) \in H^{s+1/2}(\mathbb{R})$, $Z_{tt}(0) \in H^s(\mathbb{R})$ and $a_0 \ge c_0 > 0$ for some constant $c_0 > 0$.*[11] *Then there is $T > 0$,*[12] *such that on $[0, T]$, the initial value problem of* (2.9)-(2.8)-(2.18)-(2.19) *has a unique solution $Z = Z(\cdot, t)$, satisfying $(Z_{tt}, Z_t) \in C^l([0, T], H^{s-l}(\mathbb{R}) \times H^{s+1/2-l}(\mathbb{R}))$, and $Z_{,\alpha'} - 1 \in C^l([0, T], H^{s-l}(\mathbb{R}))$, for $l = 0, 1$.*

Moreover if T^* is the supremum over all such times T, then either $T^* = \infty$, or $T^* < \infty$, but

$$\sup_{[0,T^*)} (\|Z_{tt}(t)\|_{H^3} + \|Z_t(t)\|_{H^{3+1/2}}) = \infty. \tag{2.21}$$ eq:1

Proof. Notice that the system (4.6)-(4.7) of [30] is a system for the horizontal velocity $w = \operatorname{Re} Z_t$ and horizontal acceleration $u = \operatorname{Re} Z_{tt}$, the interface doesn't appear explicitly; it is well-defined even if the interface $Z = Z(\cdot, t)$ is self-intersecting. The first part of Proposition 2.3 follows from Theorem 5.11, and the argument from the second half of page 70 to the first half of page 71 of §6 of [30].

Now assume $T^* < \infty$, and

$$\sup_{[0,T^*)} (\|Z_{tt}(t)\|_{H^3} + \|Z_t(t)\|_{H^{3+1/2}}) := M_0 < \infty. \tag{2.22}$$ eq:10

We want to show that the solution Z of the system (2.9)-(2.8)-(2.18)-(2.19) can be extended beyond T^* by a time $T' > 0$ that depends only on M_0, c_0, $\|Z_t(0)\|_{H^s}$ and $\|Z_{tt}(0)\|_{H^s}$, contradicting with the maximality of T^*.

Let $T < T^*$ be arbitrary chosen. Let $a = a(\cdot, t)$, $b = b(\cdot, t)$ be given by (4.7) of [30], and let $h = h(\cdot, t)$ satisfy

$$\begin{cases} \frac{dh}{dt} = b(h, t) \\ h(\alpha, 0) = \alpha. \end{cases} \tag{2.23}$$ eq:3

By Theorem 5.11 of [30] and the argument in §6 of [30], we know $b \in C([0, T], H^{s+1/2}(\mathbb{R}))$ with $\|b(t)\|_{H^{s+1/2}} \le c(\|Z_t(t)\|_{H^{s+1/2}}, \|Z_{tt}(t)\|_{H^s})$, and $h(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with $h(\alpha, t) - \alpha \in C([0, T], H^{s+1/2})$. Moreover $Z(\alpha', t) := z \circ h^{-1}(\alpha', t)$ satisfies (2.10), and for $t \in [0, T]$,

$$\|Z_{tt}(t)\|_{H^s} + \|Z_t(t)\|_{H^{s+1/2}} \le d_0 e^{Kt}(\|Z_{tt}(0)\|_{H^s} + \|Z_t(0)\|_{H^{s+1/2}}), \tag{2.24}$$ eq:2

where $K = K(M(T), \mathfrak{a}(T), s)$, $d_0 = d(M(T), \mathfrak{a}(T), s)$ are constants depending on

$$\mathfrak{a}(T) := \inf_{\mathbb{R}\times[0,T]} a(\alpha', t), \qquad M(T) = \sup_{[0,T]} (\|Z_{tt}(t)\|_{H^3} + \|Z_t(t)\|_{H^{3+1/2}}),$$

and $K(M(T), \mathfrak{a}(T), s) \to \infty$, $d(M(T), \mathfrak{a}(T), s) \to \infty$ as $M(T) \to \infty$, $\mathfrak{a}(T) \to 0$. We want to show that $\mathfrak{a}(T) \ge \frac{1}{C(M_0, c_0)}$ for some constant $C(M_0, c_0) > 0$.

[10] $\mathcal{A} := \frac{A_1}{|Z_{\alpha'}|^2}$ equals zero alongside $-\frac{\partial P}{\partial \mathbf{n}}$.

[11] Let $s \ge 4$. as a consequence of $(Z_t(0), Z_{tt}(0)) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ and $a_0 \ge c_0 > 0$, $Z_{,\alpha'}(0) - 1 \in H^s(\mathbb{R})$. In general by (2.10), $(Z_t, Z_{tt}) \in H^{s+1/2} \times H^s$ implies $\frac{1}{Z_{,\alpha'}} - 1 \in H^s$, and $(Z_t, \frac{1}{Z_{,\alpha'}} - 1) \in H^{s+1/2} \times H^s$ implies $Z_{tt} \in H^s$.

[12] T depends only on c_0, $\|Z_t(0)\|_{H^{s+1/2}}$ and $\|Z_{tt}(0)\|_{H^s}$.

By (2.10),

$$a(\alpha', t) := \frac{|Z_{tt}(t) + i|^2}{A_1(t)} = \frac{A_1(t)}{|Z_{,\alpha'}(t)|^2},$$

so it suffices to show that there is a constant $c(M_0, c_0)$, such that $\|Z_{,\alpha'}(t)\|_{L^\infty} \le c(M_0, c_0)$ for all $t \in [0, T]$. From the assumption $a_0 = \frac{A_1(\cdot,0)}{|Z_{,\alpha'}(\cdot,0)|^2} \ge c_0$, $|Z_{,\alpha'}(\cdot, 0)|^2 \le \frac{A_1(\cdot,0)}{c_0}$. Applying the Hardy's inequality Proposion A.3 and Cauchy-Schwarz on (2.19) yields

$$\|Z_{,\alpha'}(0)\|_{L^\infty}^2 \le \frac{\|A_1(0)\|_{L^\infty}}{c_0} \lesssim \frac{1 + \|Z_{t,\alpha'}(0)\|_{L^2}^2}{c_0}.$$

We calculate $\|Z_{,\alpha'}(t)\|_{L^\infty}$ by the fundamental theorem of calculus.

Differentiating (2.23) gives

$$\begin{cases} \frac{dh_\alpha}{dt} = b_{\alpha'}(h, t)h_\alpha \\ h_\alpha(\alpha, 0) = 1. \end{cases} \tag{2.25}$$

eq:4

So on $[0, T]$,

$$e^{-\int_0^t \|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})}\, d\tau} \le h_\alpha(\alpha, t) \le e^{\int_0^t \|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})}\, d\tau};$$

and by Sobolev embedding, $\|b_{\alpha'}(\tau)\|_{L^\infty(\mathbb{R})} \lesssim \|b(\tau)\|_{H^2(\mathbb{R})} \le c(\|Z_t(\tau)\|_{H^2}, \|Z_{tt}(\tau)\|_{H^2})$. Because $h(\alpha, 0) = \alpha$, $z(\alpha, 0) = Z(\alpha, 0)$ and

$$z_\alpha(\alpha, t) = Z_\alpha(\alpha, 0) + \int_0^t z_{t\alpha}(\alpha, \tau)\, d\tau.$$

By the chain rule $z_{t\alpha} = Z_{t,\alpha'} \circ hh_\alpha$, $z_\alpha = Z_{,\alpha'} \circ hh_\alpha$; so for $t \in [0, T]$,

$$\|Z_{,\alpha'}(t)\|_{L^\infty} \le (\|Z_{,\alpha'}(0)\|_{L^\infty} + \int_0^t \|Z_{t,\alpha'}(\tau)\|_{L^\infty} \|h_\alpha(\tau)\|_{L^\infty}\, d\tau) \|\frac{1}{h_\alpha(t)}\|_{L^\infty} \le C(M_0, c_0)$$

for some constant $C(M_0, c_0)$ depending on M_0, c_0 and T^*, and

$$\mathfrak{a}(T) = \inf_{\mathbb{R}\times[0,T]} a(\alpha', t) = \inf_{\mathbb{R}\times[0,T]} \frac{A_1}{|Z_{,\alpha'}|^2} \ge \frac{1}{C(M_0, c_0)}. \tag{2.26}$$

eq:5

Now (2.22), (2.24) and (2.26) gives

$$\|Z_{tt}(T)\|_{H^s} + \|Z_t(T)\|_{H^{s+1/2}} \le c(M_0, c_0, \|Z_{tt}(0)\|_{H^s}, \|Z_t(0)\|_{H^{s+1/2}}),$$
$$\text{and} \quad a(\cdot, T) \ge \frac{1}{C(M_0, c_0)} > 0.$$

So by the first part of Proposition 2.3, the solution Z can be extended onto $[T, T + T']$, for some $T' > 0$ depending only on M_0, c_0 and $\|Z_{tt}(0)\|_{H^s}, \|Z_t(0)\|_{H^{s+1/2}}$. This contradicts with the definition of T^*, so either $T^* = \infty$ or (2.21) holds.

□

Let $D_\alpha := \frac{1}{z_\alpha}\partial_\alpha$ and $D_{\alpha'} := \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}$. By (2.8) and the basic fact that product of holomorphic functions is holomorphic, if g is the boundary value of a holomorphic function on P_-, then $D_{\alpha'}g$ is also the boundary value of a holomorphic function on P_-. Notice that for any function f,

$$(D_\alpha f) \circ h^{-1} = D_{\alpha'}(f \circ h^{-1}).$$

a priori

2.5. **An a priori estimate for water waves with angled crests.** In [21], we studied the water wave equation (1.1) in the regime that includes interfaces with angled crests in a symmetric periodic setting, we constructed an energy functional for this regime and proved an a priori estimate. The same analysis applies to the whole line setting. The main difference is that in the whole line case, we do not need to consider the means of the various quantities; and in the proof of the a priori estimate, the argument in the footnote 21 of [21] works, so we do not need the Peter-Paul trick. Hence in the whole line case, that part of the proof is simpler. Additionally, with the minor modifications given in Appendix B.1, the argument in [21] applies more generally to solutions of (2.9)-(2.8)-(2.18)-(2.19), without any non-self-intersecting assumptions, and the characterization of the energy given in §10 of [21] also holds. In this subsection we present the results of [21] in the whole line setting for solutions of (2.9)-(2.8)-(2.18)-(2.19), we will only show how to handle the differences between the symmetric periodic and the whole line cases.

Let

$$E_a(t)=\|(\partial_t D_\alpha^2\bar z_t)\circ h^{-1}(t)\|_{L^2(1/A_1)}^2+\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\overline{Z}_t(t)\|_{\dot H^{1/2}}^2+\|D_{\alpha'}^2\overline{Z}_t(t)\|_{L^2(1/A_1)}^2. \tag{2.27}$$ eq:ea

and

$$E_b(t)=\|\partial_t D_\alpha\bar z_t(t)\|_{L^2(\frac{1}{\mathfrak{a}})}^2+\|D_{\alpha'}\overline{Z}_t(t)\|_{\dot H^{1/2}}^2+\|\overline{Z}_{t,\alpha'}(t)\|_{L^2}. \tag{2.28}$$ eq:eb

Let

$$\mathfrak{E}(t)=E_a(t)+E_b(t)+\|\overline{Z}_{tt}(t)-i\|_{L^\infty} \tag{2.29}$$ energy

Notice that we replaced the third term $|\bar z_{tt}(\alpha_0,t)-i|$ in the energy of [21] by $\|\overline{Z}_{tt}(t)-i\|_{L^\infty}$. In §10 of [21], we showed that the regime $\mathfrak{E}<\infty$ includes interfaces with angled crests with interior angles $<\frac{\pi}{2}$, in particular, the self-similar solutions constructed in [34] has finite energy $\mathfrak{E}$.

prop:a priori

Theorem 2.4 (cf. Theorem 2 of [21])**.** *Let $Z=Z(\cdot,t)$, $t\in[0,T']$ be a solution of the system* (2.9)-(2.8)-(2.18)-(2.19)*, satisfying $(Z_{tt,\alpha'},Z_{t,\alpha'})\in C^l([0,T'],H^{s-l}(\mathbb{R})\times H^{s+1/2-l}(\mathbb{R}))$, $l=0,1$ for some $s\geq 3$ and $Z_{tt}\in C([0,T'],L^\infty(\mathbb{R}))$. Then there are $T:=T(\mathfrak{E}(0))>0$, $C=C(\mathfrak{E}(0))>0$, depending only on $\mathfrak{E}(0)$,*[13] *such that*

$$\sup_{[0,\min\{T,T'\}]}\mathfrak{E}(t)\leq C(\mathfrak{E}(0))<\infty. \tag{2.30}$$ a priori-e

Proof. Let $h(\alpha,0)=\alpha$, $\alpha\in\mathbb{R}$,

$$\mathfrak{e}(t)=E_a(t)+E_b(t)$$

We only need to show how to handle the term $\|\overline{Z}_{tt}(t)-i\|_{L^\infty}$. The argument in §4.4.3 of [21] shows that for each given $\alpha\in\mathbb{R}$,

$$\frac{d}{dt}|\bar z_{tt}(\alpha,t)-i|\leq(\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\|_{L^\infty}+\|D_\alpha\bar z_t\|_{L^\infty})|\bar z_{tt}(\alpha,t)-i|. \tag{2.31}$$

Notice from the estimate for $\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\|_{L^\infty}$ in [21] and Sobolev embedding that in fact

$$\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\|_{L^\infty}+\|D_\alpha\bar z_t\|_{L^\infty}\leq c(\mathfrak{e}),$$

where c is a polynomial with nonnegative universal coefficients. Therefore

$$\frac{d}{dt}|\bar z_{tt}(\alpha,t)-i|\leq c(\mathfrak{e})|\bar z_{tt}(\alpha,t)-i|.$$

By Gronwall, $|\bar z_{tt}(\alpha,t)-i|\leq|\bar z_{tt}(\alpha,0)-i|e^{\int_0^t c(\mathfrak{e}(\tau))\,d\tau}$ hence

$$\|\bar z_{tt}(t)-i\|_{L^\infty}\leq\|\bar z_{tt}(0)-i\|_{L^\infty}e^{\int_0^t c(\mathfrak{e}(\tau))\,d\tau}.$$

[13] $T(e)$ is decreasing with respect to e, and $C(e)$ is increasing with respect to e.

Now let

$$\mathfrak{E}_1(t) = \mathfrak{e}(t) + \|\overline{z}_{tt}(0) - i\|_{L^\infty} e^{\int_0^t c(\mathfrak{e}(\tau))\, d\tau}$$

so $\mathfrak{E}(t) \le \mathfrak{E}_1(t)$, and $\mathfrak{E}(0) = \mathfrak{E}_1(0)$. By the whole line counterpart of Theorem 2 of [21], $\frac{d}{dt}\mathfrak{e}(t) \le p(\mathfrak{E}(t))$ for some polynomial p with nonnegative universal coefficients, therefore

$$\frac{d}{dt}\mathfrak{E}_1(t) \le p(\mathfrak{E}_1(t)) + C(\mathfrak{E}_1(t)). \tag{2.32}$$

Applying Gronwall again yields the conclusion of Theorem 2.4.

□

Let

$$\begin{aligned}\mathcal{E}(t) = \|\overline{Z}_{t,\alpha'}\|_{L^2}^2 + \|D_{\alpha'}^2 \overline{Z}_t\|_{L^2}^2 + \|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 + \|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\|_{L^2}^2 \\ + \|\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \overline{Z}_t\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'} \overline{Z}_t\|_{\dot{H}^{1/2}}^2 + \|\frac{1}{Z_{,\alpha'}}\|_{L^\infty}^2.\end{aligned} \tag{2.33}$$

energy1

As was shown in §10 of [21], we have the following characterization of the energy $\mathfrak{E}$.

prop:energy-eq

Proposition 2.5 (A characterization of $\mathfrak{E}$ via $\mathcal{E}$, cf. §10 of [21])**.** *There are polynomials C_1 and C_2, with nonnegative universal coefficients, such that for solutions Z of* (2.9)-(2.8),

$$\mathcal{E}(t) \le C_1(\mathfrak{E}(t)), \qquad \textit{and} \quad \mathfrak{E}(t) \le C_2(\mathcal{E}(t)). \tag{2.34}$$

energy-equiv

2.6. **A description of the class $\mathcal{E} < \infty$ in the fluid domain.** We give here an equivalent description of the class $\mathcal{E} < \infty$ for solutions Z of (2.9)-(2.8) in the "fluid domain".

Let $1 < p \le \infty$, and

$$K_y(x) = \frac{-y}{\pi(x^2+y^2)}, \qquad y < 0 \tag{2.35}$$

poisson

be the Poisson kernel. We know for any holomorphic function G on P_-,

$$\sup_{y<0} \|G(x+iy)\|_{L^p(\mathbb{R},dx)} < \infty$$

if and only if there exists $g \in L^p(\mathbb{R})$ such that $G(x+iy) = K_y * g(x)$. In this case, $\sup_{y<0} \|G(x+iy)\|_{L^p(\mathbb{R},dx)} = \|g\|_{L^p}$. Moreover, if $g \in L^p(\mathbb{R})$, $1 < p < \infty$, $\lim_{y\to 0-} K_y * g(x) = g(x)$ in $L^p(\mathbb{R})$ and if $g \in L^\infty \cap C(\mathbb{R})$, $\lim_{y \to 0-} K_y * g(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $Z = Z(\cdot, t)$ be a solution of (2.9)-(2.8), let Ψ, F be holomorphic functions on P_-, continuous on $\overline{P}_-$, such that

$$Z(\alpha', t) = \Psi(\alpha', t), \quad \overline{Z}_t(\alpha', t) = F(\alpha', t).$$

Notice that all the quantities in (2.33) are boundary values of some holomorphic functions on P_-. Let $z' = x' + iy'$, where $x', y' \in \mathbb{R}$. $\mathcal{E}(t) < \infty$ is equivalent to[14]

$$\begin{aligned}\mathcal{E}_1(t) := \sup_{y'<0} \|F_{z'}(t)\|_{L^2(\mathbb{R},dx')}^2 + \sup_{y'<0} \|\frac{1}{\Psi_{z'}} \partial_{z'} \big(\frac{1}{\Psi_{z'}} F_{z'}\big)(t)\|_{L^2(\mathbb{R},dx')}^2 \\ + \sup_{y'<0} \|\partial_{z'} \big(\frac{1}{\Psi_{z'}}\big)(t)\|_{L^2(\mathbb{R},dx')}^2 + \sup_{y'<0} \|\frac{1}{\Psi_{z'}}(t)\|_{L^\infty(\mathbb{R},dx')}^2 \\ + \sup_{y'<0} \|\frac{1}{\{\Psi_{z'}\}^2} \partial_{z'} \big(\frac{1}{\Psi_{z'}} F_{z'}\big)(t)\|_{\dot{H}^{1/2}(\mathbb{R},dx')}^2 + \sup_{y'<0} \|\frac{1}{\Psi_{z'}} F_{z'}(t)\|_{\dot{H}^{1/2}(\mathbb{R},dx')}^2 \\ + \sup_{y'<0} \|\frac{1}{\Psi_{z'}} \partial_{z'} \left(\frac{1}{\Psi_{z'}} \partial_{z'} \big(\frac{1}{\Psi_{z'}}\big)\right)(t)\|_{L^2(\mathbb{R},dx')}^2 < \infty.\end{aligned} \tag{2.36}$$

domain-energy

[14]It is clear $\mathcal{E}(t) = \mathcal{E}_1(t)$ for smooth $Z = Z(\cdot, t)$. Otherwise this equivalence is understood at a formal level, and is made rigorous according to the circumstances.

main

3. The main results

We are now ready to state the main results of the paper. For simplicity we present and prove the results in the whole line setting. The same results hold for the symmetric periodic setting as studied in [21] and the proofs are similar, except for some minor modifications.

Let $h(\alpha, 0) = \alpha$ for $\alpha \in \mathbb{R}$; let the initial interface $Z(\cdot, 0) := Z(0)$, the initial velocity $Z_t(\cdot, 0) := Z_t(0)$ be given such that $Z(0)$ satisfy (2.8) and $Z_t(0)$ satisfy $\overline{Z}_t(0) = \mathbb{H}\overline{Z}_t(0)$; let A_1 be given by (2.19), the initial acceleration $Z_{tt}(0)$ satisfy (2.10).

blow-up **Theorem 3.1** (A blow-up criteria via $\mathfrak{E}$)**.** *Let $s \geq 4$. Assume $Z_{,\alpha'}(0) \in L^\infty(\mathbb{R})$, $Z_t(0) \in H^{s+1/2}(\mathbb{R})$ and $Z_{tt}(0) \in H^s(\mathbb{R})$. Then there is $T > 0$, such that on $[0, T]$, the initial value problem of (2.9)-(2.8) has a unique solution $Z = Z(\cdot, t)$, satisfying $(Z_{tt}, Z_t) \in C^l([0, T], H^{s-l}(\mathbb{R}) \times H^{s+1/2-l}(\mathbb{R}))$ for $l = 0, 1$, and $Z_{,\alpha'} - 1 \in C([0, T], H^s(\mathbb{R}))$.*

Moreover if T^* is the supremum over all such times T, then either $T^* = \infty$, or $T^* < \infty$, but

$$\sup_{[0,T^*)} \mathfrak{E}(t) = \infty \tag{3.1}$$ eq:30

Remark 3.2. 1. Assume $Z_{,\alpha'}(0) \in L^\infty(\mathbb{R})$. We note that by the definition $\mathcal{A} := \frac{A_1}{|Z_{,\alpha'}|^2}$, $a_0 = \frac{A_1(\cdot,0)}{|Z_{,\alpha'}(\cdot,0)|^2} \geq c_0 > 0$ for some constant $c_0 > 0$. So the first part of Theorem 3.1 is the local wellposedness in Sobolev spaces as stated in Proposition 2.3. The novelty of Theorem 3.1 is the new blow up criteria via the energy functional $\mathfrak{E}$.

2. Notice that $\sup_{[0,T^*)} \mathfrak{E}(t) < \infty$ if and only if $\sup_{[0,T^*)} \mathcal{E}(t) < \infty$, by Proposition 2.5.

By the discussion of §2.3, a solution of (2.9)-(2.8) is a solution of the water wave equation (1.1) if and only if $\Sigma(t) = \{Z = Z(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is Jordan. So we can modify the statement of Theorem 3.1 to give a blow-up criteria for the water wave equation (1.1). For the first half of the statements in Corollary 3.3, see Theorem 6.1 of [30].

blow-up1 **Corollary 3.3** (A blow-up criteria via $\mathfrak{E}$)**.** *Let $s \geq 4$. Assume in addition $Z = Z(\cdot, 0)$ is non-self-intersecting. Then there is $T > 0$, such that on $[0, T]$, the initial value problem of (1.1) has a unique solution, with the properties that the interface $Z = Z(\cdot, t)$ is nonself-intersecting and $(Z_{tt}, Z_t) \in C^l([0, T], H^{s-l}(\mathbb{R}) \times H^{s+1/2-l}(\mathbb{R}))$ for $l = 0, 1$, and $Z_{,\alpha'} - 1 \in C([0, T], H^s(\mathbb{R}))$.*

Moreover if T^* is the supremum over all such times T, then either $T^* = \infty$, or $T^* < \infty$, but

$$\sup_{[0,T^*)} \mathfrak{E}(t) = \infty, \qquad \text{or} \qquad Z = Z(\cdot, t) \text{ becomes self-intersecting at } t = T^* \tag{3.2}$$ eq:30'

id

3.1. The initial data.

[15] Let $\Omega(0)$ be the initial fluid domain, with the interface $\Sigma(0) := \partial\Omega(0)$ being a Jordan curve that tends to horizontal lines at infinity, and let $\Phi(\cdot, 0) : \Omega(0) \to P_-$ be the Riemann Mapping such that $\lim_{z\to\infty} \Phi_z(z, 0) = 1$. We know $\Phi(\cdot, 0) : \overline{\Omega(0)} \to \overline{P}_-$ is a homeomorphism. Let $\Psi(\cdot, 0) := \Phi^{-1}(\cdot, 0)$, and $Z(\alpha', 0) := \Psi(\alpha', 0)$, so $Z = Z(\alpha', 0) : \mathbb{R} \to \Sigma(0)$ is the parametrization of $\Sigma(0)$ in the Riemann Mapping variable. Let $\mathbf{v}(\cdot, 0) : \Omega(0) \to \mathbb{C}$ be the initial velocity field, and $F(z', 0) = \overline{\mathbf{v}}(\Psi(z', 0), 0)$. Assume $\overline{\mathbf{v}}(\cdot, 0)$ is holomorphic on $\Omega(0)$, so $F(\cdot, 0)$ is holomorphic on P_-. Assume $F(\cdot, 0)$, $\Psi(\cdot, 0)$ satisfy (2.36) at $t = 0$. In addition, assume [16]

$$c_0 := \sup_{y'<0} \|F(x' + iy', 0)\|_{L^2(\mathbb{R}, dx')} + \sup_{y'<0} \left\| \frac{1}{\Psi_{z'}(x' + iy', 0)} - 1 \right\|_{L^2(\mathbb{R}, dx')} < \infty. \tag{3.3}$$ iid

[15] We only need to assume that $F(\cdot, 0), \Psi(\cdot, 0)$ are given and are holomorphic on P_- and continuous on $\overline{P}_-$, satisfying $\lim_{z'\to\infty} \Psi_{z'}(z', 0) = 1$, $\Psi_{z'}(z', 0) \neq 0$ on P_-, (2.36) at $t = 0$ and (3.3). We give the initial data as is to put it in the context of the water waves (1.1).

[16] Let $Z_{tt}(0)$ be given by (2.10). Under the assumption (2.36) at $t = 0$, this is equivalent to assuming $\|Z_t(0)\|_{L^2} + \|Z_{tt}(0)\|_{L^2} < \infty$.

th:local **Theorem 3.4** (Local existence in the $\mathfrak{E} < \infty$ regime)**.** *1. There exists $T_0 > 0$, depending only on $\mathcal{E}_1(0)$, such that on $[0, T_0]$, the initial value problem of the water wave equation* (1.1) *has a generalized solution $(F, \Psi, \mathfrak{P})$ in the sense of* (2.14)-(2.15)*, with the properties that $F(\cdot, t), \Psi(\cdot, t)$ are holomorphic on P_- for each fixed $t \in [0, T_0]$, $F, \Psi, \frac{1}{\Psi_{z'}}, \mathfrak{P}$ are continuous on $\overline{P}_- \times [0, T_0]$, F, Ψ are continuous differentiable on $P_- \times [0, T_0]$, $\mathfrak{P}$ is continuous differentiable with respect to the spatial variables on $P_- \times [0, T_0]$; during this time, $\mathcal{E}_1(t) < \infty$ and*

$$\sup_{y'<0} \|F(x' + iy', t)\|_{L^2(\mathbb{R}, dx')} + \sup_{y'<0} \|\frac{1}{\Psi_{z'}(x' + iy', t)} - 1\|_{L^2(\mathbb{R}, dx')} < \infty. \tag{3.4}$$ iidt

The generalized solution gives rise to a solution $(\overline{\mathbf{v}}, P) = (F \circ \Psi^{-1}, \mathfrak{P} \circ \Psi^{-1})$ of the water wave equation (1.1) *so long as $\Sigma(t) = \{Z = \Psi(\alpha', t) \mid \alpha' \in \mathbb{R}\}$ is a Jordan curve.*

2. If in addition, the initial interface is chord-arc, that is, $Z_{,\alpha'}(\cdot, 0) \in L^1_{loc}(\mathbb{R})$ and there is $0 < \delta < 1$, such that

$$\delta \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma, 0)|\, d\gamma \le |Z(\alpha', 0) - Z(\beta', 0)| \le \int_{\alpha'}^{\beta'} |Z_{,\alpha'}(\gamma, 0)|\, d\gamma, \quad \forall -\infty < \alpha' < \beta' < \infty.$$

Then there is $T_0 > 0, T_1 > 0$, T_0, T_1 depend only on $\mathcal{E}_1(0)$, such that on $[0, \min\{T_0, \frac{\delta}{T_1}\}]$, the initial value problem of the water wave equation (1.1) *has a solution, satisfying $\mathcal{E}_1(t) < \infty$ and* (3.4)*, and the interface $Z = Z(\cdot, t)$ is chord-arc.*

proof1 4. The proof of Theorem 3.1

We only need to prove the second part, the blow-up criteria of Theorem 3.1. We assume $T^* < \infty$, for otherwise we are done.

Let $Z = Z(\cdot, t)$, $t \in [0, T^*)$ be a solution of (2.9)-(2.8):

$$Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}, \tag{4.1}$$ interface-e-1

with constraint

$$\begin{cases} \overline{Z}_t = \mathbb{H}\overline{Z}_t, \\ Z_{,\alpha'} - 1 = \mathbb{H}(Z_{,\alpha'} - 1), \qquad \frac{1}{Z_{,\alpha'}} - 1 = \mathbb{H}(\frac{1}{Z_{,\alpha'}} - 1); \end{cases} \tag{4.2}$$ interface-e-2

satisfying $(Z_{tt}, Z_t) \in C^l([0, T^*), H^{s-l}(\mathbb{R}) \times H^{s+1/2-l}(\mathbb{R}))$ for $l = 0, 1$, and $Z_{,\alpha'} - 1 \in C([0, T^*), H^s(\mathbb{R}))$. Precompose (4.1) with h gives

$$z_{tt} + i = i\mathfrak{a}z_\alpha \tag{4.3}$$ interface-e2

where $\mathfrak{a}h_\alpha := \mathcal{A} \circ h$. Differentiating (4.3) with respect to t yields

$$\overline{z}_{ttt} + i\mathfrak{a}\overline{z}_{t\alpha} = -i\mathfrak{a}_t\overline{z}_\alpha = \frac{\mathfrak{a}_t}{\mathfrak{a}}(\overline{z}_{tt} - i) \tag{4.4}$$ quasi-l

Precompose (4.4) with h^{-1}. This gives the corresponding equation in the Riemann mapping variable:

$$\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\overline{Z}_{tt} - i) \tag{4.5}$$ quasi-r

We know $\overline{Z}_{ttt} = (\partial_t + b\partial_{\alpha'})^2\overline{Z}_t$ and $\overline{Z}_{tt} = (\partial_t + b\partial_{\alpha'})\overline{Z}_t$, where $b := h_t \circ h^{-1}$. The analysis in Appendix B.1 shows that b and $A_1 := \mathcal{A}|Z_{,\alpha'}|^2$ are as given in (2.18), (2.19), and

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\operatorname{Im}(2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\partial_{\alpha'}\overline{Z}_t - [Z_t, Z_t; D_{\alpha'}\overline{Z}_t])}{A_1}. \tag{4.6}$$ at

where

$$[Z_t, Z_t; D_{\alpha'}\overline{Z}_t] := \frac{1}{\pi i} \int \frac{(Z_t(\alpha', t) - Z_t(\beta', t))^2}{(\alpha' - \beta')^2} D_{\beta'}\overline{Z}_t(\beta', t)\, d\beta'. \tag{4.7}$$ zzz

(4.4)-(4.2) or equivalently (4.5)-(4.2) with b, A_1 and $\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}$ given by (2.18), (2.19) and (4.6) is a quasilinear equation of the hyperbolic type in the regime of smooth interfaces, with the right hand side consisting of lower order terms. [17] However in the regime that includes interfaces with angled crests, since $\mathcal{A}$ and $-\frac{\partial P}{\partial \mathbf{n}}$ equal to zero at the crests where the interior angles are $< \pi$, the left hand side of (4.5) (or (4.4)) is degenerate hyperbolic.

We have the following basic energy inequality.

basic-e **Lemma 4.1** (Basic energy inequality)**.** *Assume $\theta = \theta(\alpha, t)$, $\alpha \in \mathbb{R}$, $t \in [0, T)$ is smooth, decays fast at the spatial infinity and satisfies $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$ and*

$$\partial_t^2 \theta + i\mathfrak{a}\partial_\alpha \theta = G_\theta. \tag{4.8}$$ eq:40

Let

$$E_\theta(t) := \int \frac{1}{\mathfrak{a}}|\theta_t|^2\, d\alpha + i\int \partial_\alpha \theta \overline{\theta}\, d\alpha + \int \frac{1}{\mathfrak{a}}|\theta|^2\, d\alpha \tag{4.9}$$ eq:41

Then

$$\frac{d}{dt}E_\theta(t) \le \Big(\Big\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \Big\|_{L^\infty} + 1 \Big) E_\theta(t) + 2E_\theta(t)^{1/2} \left(\int \frac{|G_\theta|^2}{\mathfrak{a}}\, d\alpha \right)^{1/2}. \tag{4.10}$$ eq:42

Remark 4.2. Since $\mathcal{A} \circ h := \mathfrak{a} h_\alpha$, upon changing to the Riemann mapping variable,

$$E_\theta(t) = \int \frac{1}{\mathcal{A}}(|\theta_t \circ h^{-1}|^2 + |\theta \circ h^{-1}|^2)\, d\alpha' + i\int \partial_{\alpha'}(\theta \circ h^{-1})\overline{\theta} \circ h^{-1}\, d\alpha'$$

By $\theta \circ h^{-1} = \mathbb{H}(\theta \circ h^{-1})$ and (A.5),

$$i\int \partial_\alpha \theta \overline{\theta}\, d\alpha = i\int \partial_{\alpha'}(\theta \circ h^{-1})\overline{\theta} \circ h^{-1}\, d\alpha' = \|\theta \circ h^{-1}\|^2_{\dot{H}^{1/2}} \ge 0.$$

Proof. We have [18]

$$\begin{aligned} \frac{d}{dt}E_\theta(t) &= 2\operatorname{Re}\int \frac{1}{\mathfrak{a}}\theta_{tt}\overline{\theta}_t\, d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\theta_t|^2\, d\alpha + i\int \partial_\alpha \theta_t \overline{\theta}\, d\alpha + i\int \partial_\alpha \theta \overline{\theta}_t\, d\alpha \\ &\quad + 2\operatorname{Re}\int \frac{1}{\mathfrak{a}}\theta_t \overline{\theta}\, d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\theta|^2\, d\alpha \\ &= 2\operatorname{Re}\int \frac{1}{\mathfrak{a}}(\theta_{tt} + i\mathfrak{a}\partial_\alpha \theta)\overline{\theta}_t\, d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}^2}(|\theta_t|^2 + |\theta|^2)\, d\alpha + 2\operatorname{Re}\int \frac{1}{\mathfrak{a}}\theta_t \overline{\theta}\, d\alpha \end{aligned} \tag{4.11}$$ eq:43

Here in the second step we used integration by parts on the third term. (4.8), Cauchy-Schwarz and the fact that $i\int \partial_\alpha \theta \overline{\theta}\, d\alpha \ge 0$ gives (4.10).

□

Apply $D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}$, $k = 2, 3$ to (4.4), then commute $D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}$ with $\partial_t^2 + i\mathfrak{a}\partial_\alpha$ yields

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}\overline{z}_t = D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}(-i\mathfrak{a}_t \overline{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}]\overline{z}_t \tag{4.12}$$ eq:44

Let

$$E_k(t) := E_{D_\alpha(\frac{\partial_\alpha}{h_\alpha})^{k-1}\overline{z}_t}(t). \tag{4.13}$$

Because $\mathcal{A} = \frac{A_1}{|Z_{,\alpha'}|^2}$ and $U_h^{-1}D_\alpha U_h = D_{\alpha'} = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}$,

$$E_k(t) = \int \frac{1}{A_1}(|\partial_{\alpha'}^k \overline{Z}_t|^2 + |Z_{,\alpha'}U_h^{-1}\partial_t U_h \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^k \overline{Z}_t|^2)\, d\alpha' + \left\| \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^k \overline{Z}_t \right\|^2_{\dot{H}^{1/2}} \tag{4.14}$$

We prove Theorem 3.1 via the following two Propositions.

step1 **Proposition 4.3.** *There exists a polynomial $p_1 = p_1(x)$ with universal coefficients such that*

$$\frac{d}{dt}E_2(t) \le p_1(\mathfrak{E}(t))E_2(t). \tag{4.15}$$

[17](4.5) is equivalent to the quasi-linear system (4.6)-(4.7) of [30]. The only difference is that (4.5) is in terms of Z_t and Z_{tt} and (4.6)-(4.7) of [30] is in terms of the real components $\operatorname{Re} Z_t$ and $\operatorname{Re} Z_{tt}$.

[18]Some variants of the proof have been given in [32] and [21]. We prove (4.10) nevertheless.

step2 **Proposition 4.4.** *There exist polynomials $p_2 = p_2(x, y)$ and $p_3 = p_3(x, y)$ with universal coefficients such that*

$$\frac{d}{dt}E_3(t) \le p_2(\mathfrak{E}(t), E_2(t))E_3(t) + p_3(\mathfrak{E}(t), E_2(t)). \tag{4.16}$$

Propositions 4.3 and 4.4 give that

$$\begin{aligned} &E_2(t) \le E_2(0)e^{\int_0^t p_1(\mathfrak{E}(s))\,ds}; \qquad \text{and} \\ &E_3(t) \le (E_3(0) + \int_0^t p_3(\mathfrak{E}(s), E_2(s))\,ds)e^{\int_0^t p_2(\mathfrak{E}(s),E_2(s))\,ds}, \end{aligned} \tag{4.17}$$ step1-2

so for $T^* < \infty$, $E_2(0)+E_3(0) < \infty$ and $\sup_{[0,T^*)} \mathfrak{E}(t) < \infty$ implies $\sup_{[0,T^*)}(E_2(t)+E_3(t)) < \infty$. In §4.1 and §4.2 we will prove Propositions 4.3 and 4.4. We will complete the proof of Theorem 3.1 in §4.3 by showing that $\sup_{[0,T^*)}(\|Z_t(t)\|_{H^{3+1/2}} + \|Z_{tt}(t)\|_{H^3})$ is controlled by $\sup_{[0,T^*)}(E_2(t) + E_3(t))$ and the initial data.

proof-prop1

4.1. The proof of Proposition 4.3.

Proof. We prove Proposition 4.3 by applying the basic energy inequality, Lemma 4.1 to $D_\alpha(\frac{\partial_\alpha}{h_\alpha})\overline{z}_t$ of (4.12), notice that $(I - \mathbb{H})(U_h^{-1}D_\alpha(\frac{\partial_\alpha}{h_\alpha})\overline{z}_t) = (I - \mathbb{H})D_{\alpha'}\overline{Z}_{t,\alpha'} = 0$. Using (B.16) (B.15) and (B.22), we expand the right hand side of (4.12):

$$\begin{aligned} G_2 &:= D_\alpha\frac{\partial_\alpha}{h_\alpha}(-i\mathfrak{a}_t\overline{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha\frac{\partial_\alpha}{h_\alpha}]\overline{z}_t \\ &= D_\alpha\frac{\partial_\alpha}{h_\alpha}(-i\mathfrak{a}_t\overline{z}_\alpha) - 2(D_\alpha z_{tt}D_\alpha\frac{\partial_\alpha}{h_\alpha}\overline{z}_t + D_\alpha z_t\partial_t D_\alpha\frac{\partial_\alpha}{h_\alpha}\overline{z}_t) \\ &\quad - D_\alpha\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}\} - D_\alpha U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{tt,\alpha'}\} - iD_\alpha U_h\{\mathcal{A}_{\alpha'}\overline{Z}_{t,\alpha'}\} \end{aligned} \tag{4.18}$$ eq:45

We can control $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty}$ by a polynomial of $\mathfrak{E}$, see Appendix C. What remains to be shown is that

$$\int\frac{|G_2|^2}{\mathfrak{a}}\,d\alpha \le C(\mathfrak{E})E_2, \tag{4.19}$$ eq:46

for some polynomial $C(\mathfrak{E})$. Changing to the Riemann mapping variables and using $\mathcal{A} = \frac{A_1}{|Z_{,\alpha'}|^2}$, $A_1 \ge 1$,

$$\int\frac{|G_2|^2}{\mathfrak{a}}\,d\alpha = \int\frac{|G_2|^2}{\mathfrak{a}h_\alpha}h_\alpha\,d\alpha = \int\frac{|Z_{,\alpha'}U_h^{-1}G_2|^2}{A_1}\,d\alpha' \le \int|Z_{,\alpha'}U_h^{-1}G_2|^2\,d\alpha'. \tag{4.20}$$ eq:54

So it suffices to show that

$$\int|Z_{,\alpha'}U_h^{-1}G_2|^2\,d\alpha' \le C(\mathfrak{E})E_2.$$

Let

$$G_{2,0} := D_\alpha\frac{\partial_\alpha}{h_\alpha}(-i\mathfrak{a}_t\overline{z}_\alpha); \tag{4.21}$$ eq:55

$$G_{2,1} := -2(D_\alpha z_{tt}D_\alpha\frac{\partial_\alpha}{h_\alpha}\overline{z}_t + D_\alpha z_t\partial_t D_\alpha\frac{\partial_\alpha}{h_\alpha}\overline{z}_t); \qquad \text{and} \tag{4.22}$$ eq:56

$$\begin{aligned} G_{2,2} := &-D_\alpha\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}\} - D_\alpha U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{tt,\alpha'}\} \\ &- iD_\alpha U_h\{\mathcal{A}_{\alpha'}\overline{Z}_{t,\alpha'}\}, \end{aligned} \tag{4.23}$$ eq:57

so $G_2 = G_{2,0}+G_{2,1}+G_{2,2}$. We know by $\overline{z}_{tt} - i = -i\mathfrak{a}\overline{z}_\alpha$ (4.3) and $U_h^{-1}D_\alpha U_h = D_{\alpha'} := \frac{\partial_{\alpha'}}{Z_{,\alpha'}}$,

$$Z_{,\alpha'}U_h^{-1}G_{2,0} = \partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt} - i)); \tag{4.24}$$ eq:551

$$Z_{,\alpha'}U_h^{-1}G_{2,1} = -2(D_{\alpha'}Z_{tt}\partial_{\alpha'}^2\overline{Z}_t + D_{\alpha'}Z_t\big(Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t\big)); \tag{4.25}$$ eq:561

$$Z_{,\alpha'}U_h^{-1}G_{2,2} = -\partial_{\alpha'}U_h^{-1}\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}\} - \partial_{\alpha'}\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{tt,\alpha'}\} \\ - i\partial_{\alpha'}\{\mathcal{A}_{\alpha'}\overline{Z}_{t,\alpha'}\}. \tag{4.26}$$ eq:571

Step 1: Quantities controlled by E_2 and a polynomial of $\mathfrak{E}$. By the definition of E_2, and the fact that $\|A_1\|_{L^\infty} \le C(\mathfrak{E})$ (cf. Appendix C), we know

$$\int \frac{|D_\alpha \frac{\partial_\alpha}{h_\alpha}\overline{z}_t|^2}{\mathfrak{a}}\,d\alpha, \quad \int \frac{|\partial_t D_\alpha \frac{\partial_\alpha}{h_\alpha}\overline{z}_t|^2}{\mathfrak{a}}\,d\alpha \le E_2 \tag{4.27}$$ eq:47

$$\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}^2, \quad \left\|Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t\right\|_{L^2}^2, \quad \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t\right\|_{\dot H^{1/2}}^2 \le C(\mathfrak{E})E_2. \tag{4.28}$$ eq:48

We commute $Z_{,\alpha'}$ with $U_h^{-1}\partial_t U_h$ in the second quantity of (4.28)

$$Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t = U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t + [Z_{,\alpha'}, U_h^{-1}\partial_t U_h]\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t \tag{4.29}$$

By (B.26) and Appendix C,

$$\left|\left\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t\right\|_{L^2} - \left\|Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\overline{Z}_t\right\|_{L^2}\right| \le C(\mathfrak{E})\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}, \tag{4.30}$$ eq:49

so

$$\left\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t\right\|_{L^2}^2 \le C(\mathfrak{E})E_2 \tag{4.31}$$ eq:50

Step 2. Controlling $G_{2,1}$. By (4.25), Appendix C and (4.28),

$$\int |Z_{,\alpha'}U_h^{-1}G_{2,1}|^2\,d\alpha \le C(\mathfrak{E})E_2. \tag{4.32}$$ eq:51

Step 3. Controlling $G_{2,2}$. We expand further the terms in $Z_{,\alpha'}U_h^{-1}G_{2,2}$ by the product rule,

$$\partial_{\alpha'}U_h^{-1}\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}\} = (h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'} \\ + \{U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'}\}\partial_{\alpha'}\overline{Z}_{t,\alpha'} + \{\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\}U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'} \\ + \{\partial_{\alpha'}U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'}\}\overline{Z}_{t,\alpha'}; \tag{4.33}$$ eq:52

$$\partial_{\alpha'}\{(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{tt,\alpha'}\} = \{\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\}\overline{Z}_{tt,\alpha'} + (h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}; \\ \partial_{\alpha'}\{\mathcal{A}_{\alpha'}\overline{Z}_{t,\alpha'}\} = (\partial_{\alpha'}\mathcal{A}_{\alpha'})\overline{Z}_{t,\alpha'} + \mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \tag{4.34}$$ eq:53

Step 3.1. The quantity $\partial_{\alpha'}^k(h_t\circ h^{-1})$. By equation (B.5) in Appendix B.1,

$$h_t\circ h^{-1}(\alpha',t) = \frac{Z_t(\alpha',t)}{Z_{,\alpha'}(\alpha',t)} + \Xi(\alpha',t). \tag{4.35}$$ b1

where $(I-\mathbb{H})\Xi(\cdot,t) = 0$. Differentiating with respect to α' yields

$$(h_t\circ h^{-1})_{\alpha'} = \frac{Z_{t,\alpha'}}{Z_{,\alpha'}} + Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + \partial_{\alpha'}\Xi. \tag{4.36}$$ eq:70

Rewrite $\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} = 2\,\mathrm{Re}\,\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} - \frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}$ and move $2\,\mathrm{Re}\,\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}$ to the left, we obtain

$$(h_t\circ h^{-1})_{\alpha'} - 2\,\mathrm{Re}\,\frac{Z_{t,\alpha'}}{Z_{,\alpha'}} = -\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} + Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + \partial_{\alpha'}\Xi; \tag{4.37}$$ eq:71

differentiating (4.36) with respect to α' and using the fact $\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} = 2\,\mathrm{Re}\,\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} - \frac{\partial_{\alpha'}^2\overline{Z}_t}{\overline{Z}_{,\alpha'}}$ gives

$$\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'} - 2\,\mathrm{Re}\,\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} = 2Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - \frac{\partial_{\alpha'}^2\overline{Z}_t}{\overline{Z}_{,\alpha'}} + Z_t\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} + \partial_{\alpha'}^2\Xi. \tag{4.38}$$ eq:72

Notice that $(I-\mathbb{H})\partial_{\alpha'}^k\Xi=0$, $k=1,2$. Apply $(I-\mathbb{H})$ to both sides of (4.37) and (4.38), then take the real parts. Rewrite the last two terms on the right hand sides as commutators via the fact that $(I-\mathbb{H})\partial_{\alpha'}^k\overline{Z}_t=0$ and $(I-\mathbb{H})\partial_{\alpha'}^k\frac{1}{Z_{,\alpha'}}=0$, $k=1,2$.[19] We get

$$(h_t\circ h^{-1})_{\alpha'}-2\operatorname{Re}\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}=\operatorname{Re}\{-[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\overline{Z}_{t,\alpha'}+[Z_t,\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\} \tag{4.39}$$ eq:73

and

$$\begin{aligned}\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}-2\operatorname{Re}\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}}&=\operatorname{Re}\{2(I-\mathbb{H})(Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\\&\quad-[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t+[Z_t,\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\}.\end{aligned} \tag{4.40}$$ eq:74

From (4.40), by Hölder's inequality, (A.8) and (A.12),

$$\left\|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}-2\operatorname{Re}\frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}}\right\|_{L^2}\lesssim\|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2} \tag{4.41}$$ eq:78

Step 3.2. The estimates for the quantities involving $\overline{Z}_t$. Commuting $\partial_{\alpha'}$ with $U_h^{-1}\partial_t U_h$ and using (B.18) gives

$$\begin{aligned}\partial_{\alpha'}U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}&=U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t+[\partial_{\alpha'},U_h^{-1}\partial_t U_h]\overline{Z}_{t,\alpha'}\\&=U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t+(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t,\end{aligned} \tag{4.42}$$

so by (4.28), (4.31) and Appendix C,

$$\|\partial_{\alpha'}U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}\|_{L^2}^2\le C(\mathfrak{E})E_2. \tag{4.43}$$ eq:60

We estimate $\|Z_{t,\alpha}\|_{L^\infty}$ by (A.3), Appendix C and (4.28),

$$\|Z_{t,\alpha'}\|_{L^\infty}^2\le 2\|Z_{t,\alpha'}\|_{L^2}\|\partial_{\alpha'}^2 Z_t\|_{L^2}\le C(\mathfrak{E})E_2^{1/2}. \tag{4.44}$$ eq:61

We compute $\partial_{\alpha'}^2\overline{Z}_{tt}$ by (B.19),

$$\begin{aligned}\partial_{\alpha'}^2\overline{Z}_{tt}-U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t&=[\partial_{\alpha'}^2,U_h^{-1}\partial_t U_h]\overline{Z}_t\\&=2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t+\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'},\end{aligned} \tag{4.45}$$ eq:75

where by (4.41), (4.28), (4.44) and Appendix C,

$$\begin{aligned}\|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}\|_{L^2}&\lesssim\|D_{\alpha'}Z_t\|_{L^\infty}\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}+\|Z_{t,\alpha'}\|_{L^\infty}^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\\&\lesssim C(\mathfrak{E})E_2^{1/2}.\end{aligned} \tag{4.46}$$ eq:99

Therefore (4.45), (4.46), (4.31), (4.28) and Appendix C gives that

$$\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}^2\lesssim C(\mathfrak{E})E_2. \tag{4.47}$$ eq:79

As a consequence of (A.3), (4.47) and Appendix C,

$$\|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^\infty}^2\le 2\|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^2}\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.48}$$ eq:80

We compute $U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}$ by commuting $U_h^{-1}\partial_t U_h$ with $\partial_{\alpha'}$ and using (B.18),

$$U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}=\partial_{\alpha'}\overline{Z}_{tt}+[U_h^{-1}\partial_t U_h,\partial_{\alpha'}]\overline{Z}_t=\overline{Z}_{tt,\alpha'}-(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{t,\alpha'}; \tag{4.49}$$ eq:83

(4.48), (4.44) and Appendix C imply that

$$\|U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}\|_{L^\infty}^2\lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.50}$$ eq:81

[19] If $(I-\mathbb{H})g=0$, then $(I-\mathbb{H})(fg)=[f,\mathbb{H}]g$.

Step 3.3. The estimate for the terms involving $\partial_{\alpha'}^k(h_t\circ h^{-1})$ in (4.33) *and* (4.34). By Steps 3.1 and 3.2, we can give the estimates for some of the terms in (4.33) and (4.34). First, because $\|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\le C(\mathfrak{E})$ (cf. Appendix C) and (4.43),

$$\|(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_h^{-1}\partial_t U_h\overline{Z}_{t,\alpha'}\|_{L^2}^2\le C(\mathfrak{E})E_2; \tag{4.51}$$

eq:86

and from (4.47),

$$\|(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\|_{L^2}^2\le C(\mathfrak{E})E_2. \tag{4.52}$$

eq:85

From (4.41), (4.28), (4.44), (4.48) and Appendix C,

$$\begin{aligned}\|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\overline{Z}_{tt,\alpha'}\|_{L^2}&\lesssim\|D_{\alpha'}Z_{tt}\|_{L^\infty}\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}\\&+\|Z_{t,\alpha'}\|_{L^\infty}\|Z_{tt,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/2};\end{aligned} \tag{4.53}$$

eq:82

additionally from (4.49),

$$\begin{aligned}\|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}U_h^{-1}\partial_tU_h\overline{Z}_{t,\alpha'}\|_{L^2}&\lesssim(\|D_{\alpha'}Z_{tt}\|_{L^\infty}+C(\mathfrak{E})\|D_{\alpha'}Z_t\|_{L^\infty})\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}\\&+(\|Z_{tt,\alpha'}\|_{L^\infty}+C(\mathfrak{E})\|Z_{t,\alpha'}\|_{L^\infty})\|Z_{t,\alpha'}\|_{L^\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\lesssim C(\mathfrak{E})E_2^{1/2}.\end{aligned} \tag{4.54}$$

eq:84

Step 3.4. The terms involving $\partial_{\alpha'}^kU_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'}$, $k=0,1$. We first consider $U_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'}$. Applying $U_h^{-1}\partial_tU_h$ to (4.39) gives

$$\begin{aligned}U_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'}&=2\operatorname{Re}U_h^{-1}\partial_tU_h\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}\\&+\operatorname{Re}\{-U_h^{-1}\partial_tU_h[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\overline{Z}_{t,\alpha'}+U_h^{-1}\partial_tU_h[Z_t,\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\};\end{aligned} \tag{4.55}$$

eq:88

we know

$$U_h^{-1}\partial_tU_h\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}=U_h^{-1}\partial_t\frac{z_{t\alpha}}{z_\alpha}=\frac{Z_{tt,\alpha'}}{Z_{,\alpha'}}-\big(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}\big)^2=D_{\alpha'}Z_{tt}-(D_{\alpha'}Z_t)^2. \tag{4.56}$$

eq:91

We compute the last two terms on the RHS of (4.55) by (B.25),

$$\begin{aligned}U_h^{-1}\partial_tU_h[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\overline{Z}_{t,\alpha'}&=[U_h^{-1}\partial_tU_h\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\overline{Z}_{t,\alpha'}\\&+[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}](\partial_{\alpha'}\overline{Z}_{tt})-[\frac{1}{\overline{Z}_{,\alpha'}},h_t\circ h^{-1};\overline{Z}_{t,\alpha'}];\end{aligned} \tag{4.57}$$

eq:87

and

$$\begin{aligned}U_h^{-1}\partial_tU_h[Z_t,\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}&=[Z_{tt},\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\\&+[Z_t,\mathbb{H}](\partial_{\alpha'}U_h^{-1}\partial_tU_h\frac{1}{Z_{,\alpha'}})-[Z_t,h_t\circ h^{-1};\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}].\end{aligned} \tag{4.58}$$

eq:89

Now by the product rule,

$$\begin{aligned}\partial_{\alpha'}U_h^{-1}\partial_tU_h\frac{1}{Z_{,\alpha'}}&=\partial_{\alpha'}\{\frac{1}{Z_{,\alpha'}}((h_t\circ h^{-1})_{\alpha'}-D_{\alpha'}Z_t)\}\\&=\big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\big)((h_t\circ h^{-1})_{\alpha'}-D_{\alpha'}Z_t)+(D_{\alpha'}(h_t\circ h^{-1})_{\alpha'}-D_{\alpha'}^2Z_t);\end{aligned} \tag{4.59}$$

eq:90

commuting $U_h^{-1}\partial_tU_h$ with $\partial_{\alpha'}$ and using (B.18) gives

$$\begin{aligned}U_h^{-1}\partial_tU_h\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}&=\partial_{\alpha'}U_h^{-1}\partial_tU_h\frac{1}{Z_{,\alpha'}}+[U_h^{-1}\partial_tU_h,\partial_{\alpha'}]\frac{1}{Z_{,\alpha'}}\\&=\partial_{\alpha'}U_h^{-1}\partial_tU_h\frac{1}{Z_{,\alpha'}}-(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}.\end{aligned} \tag{4.60}$$

eq:98

Applying Appendix C yields

$$\|\partial_{\alpha'} U_h^{-1}\partial_t U_h \frac{1}{Z_{,\alpha'}}\|_{L^2} + \|U_h^{-1}\partial_t U_h \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\|_{L^2} \le C(\mathfrak{E}); \tag{4.61}$$

eq:97

and from (4.55), by (4.56), (4.57), (4.58), (4.61) and (A.18), (A.17) and Appendix C,

$$\|U_h^{-1}\partial_t U_h (h_t\circ h^{-1})_{\alpha'}\|_{L^\infty} \lesssim C(\mathfrak{E}). \tag{4.62}$$

eq:92

We analyze $\partial_{\alpha'} U_h^{-1}\partial_t U_h (h_t\circ h^{-1})_{\alpha'}$ similarly. Commuting $\partial_{\alpha'}$ with $U_h^{-1}\partial_t U_h$ and using (B.18) gives

$$\begin{aligned}\partial_{\alpha'} U_h^{-1}\partial_t U_h (h_t\circ h^{-1})_{\alpha'} &= [\partial_{\alpha'}, U_h^{-1}\partial_t U_h](h_t\circ h^{-1})_{\alpha'} + U_h^{-1}\partial_t U_h \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\\ &= (h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'} + U_h^{-1}\partial_t U_h \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}.\end{aligned} \tag{4.63}$$

eq:93

We compute the second term on the RHS of (4.63) via (4.40):

$$\begin{aligned}U_h^{-1}\partial_t U_h \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'} - 2\operatorname{Re} U_h^{-1}\partial_t U_h \frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} &= \operatorname{Re}\{2U_h^{-1}\partial_t U_h (I-\mathbb{H})(Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\\ &- U_h^{-1}\partial_t U_h [\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}]\partial_{\alpha'}^2 \overline{Z}_t + U_h^{-1}\partial_t U_h [Z_t,\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\};\end{aligned} \tag{4.64}$$

eq:94

commuting $U_h^{-1}\partial_t U_h$ with $\frac{\partial_{\alpha'}}{Z_{,\alpha'}} := D_{\alpha'}$ and using(B.12) gives

$$\begin{aligned}U_h^{-1}\partial_t U_h \frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}} &= D_{\alpha'} U_h^{-1}\partial_t U_h Z_{t,\alpha'} + [U_h^{-1}\partial_t U_h, D_{\alpha'}]Z_{t,\alpha'}\\ &= \frac{1}{Z_{,\alpha'}}\partial_{\alpha'} U_h^{-1}\partial_t U_h Z_{t,\alpha'} - \frac{1}{Z_{,\alpha'}}(D_{\alpha'}Z_t)(\partial_{\alpha'}^2 Z_t);\end{aligned} \tag{4.65}$$

eq:95

for the first term on the RHS of (4.64), we commute $U_h^{-1}\partial_t U_h$ with $(I-\mathbb{H})$ and use (B.23),

$$\begin{aligned}&U_h^{-1}\partial_t U_h (I-\mathbb{H})(Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) = (I-\mathbb{H})U_h^{-1}\partial_t U_h (Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\\ &\quad - [U_h^{-1}\partial_t U_h, \mathbb{H}](Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\\ &\quad = (I-\mathbb{H})U_h^{-1}\partial_t U_h (Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) - [h_t\circ h^{-1}, \mathbb{H}]\partial_{\alpha'}(Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}});\end{aligned} \tag{4.66}$$

eq:96

we use product rule to expand further the terms on the RHS of (4.66). By (4.50), (4.61), (4.44), Appendix C and (A.11),

$$\left\|U_h^{-1}\partial_t U_h (I-\mathbb{H})(Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\right\|_{L^2}^2 \le C(\mathfrak{E})E_2^{1/2}. \tag{4.67}$$

We use (B.25) to compute the last two terms on the RHS of (4.64), then use (A.11), (A.12) and (4.61), (4.50), (4.44), (4.48) and Appendix C to do the estimates, we get

$$\left\|U_h^{-1}\partial_t U_h [\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}]\partial_{\alpha'}^2 \overline{Z}_t\right\|_{L^2}^2 + \left\|U_h^{-1}\partial_t U_h [Z_t,\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 \le C(\mathfrak{E})E_2^{1/2}, \tag{4.68}$$

eq:100

therefore

$$\left\|U_h^{-1}\partial_t U_h \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'} - 2\operatorname{Re} U_h^{-1}\partial_t U_h \frac{\partial_{\alpha'}^2 Z_t}{Z_{,\alpha'}}\right\|_{L^2}^2 \le C(\mathfrak{E})E_2^{1/2}. \tag{4.69}$$

eq:101

We now conclude the estimates for the two terms involving $U_h^{-1}\partial_t U_h (h_t\circ h^{-1})_{\alpha'}$ in (4.33). By (4.63), (4.41), (4.44), (4.69) and (4.28) and Appendix C,

$$\left\|\{\partial_{\alpha'} U_h^{-1}\partial_t U_h (h_t\circ h^{-1})_{\alpha'}\}\overline{Z}_{t,\alpha'}\right\|_{L^2}^2 \le C(\mathfrak{E})E_2; \tag{4.70}$$

eq:102

by (4.62) and (4.28),

$$\left\|\{U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'}\}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{L^2}^2 \le C(\mathfrak{E})E_2. \tag{4.71}$$ eq:110

Finally we estimate the L^2 norms of the two terms on the RHS of the second equation in (4.34).

Step 3.5. The L^2 norm of $\partial_{\alpha'}(\mathcal{A}_{\alpha'}\overline{Z}_{t,\alpha'})$. We begin with the first equation of (2.9) $\mathcal{A} := \frac{Z_{tt}+i}{iZ_{,\alpha'}}$. Differentiating with respect to α' gives

$$\partial_{\alpha'}\mathcal{A} = -iD_{\alpha'}Z_{tt} - i(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \tag{4.72}$$ eq:0103

By Appendix C,

$$\|\partial_{\alpha'}\mathcal{A}\|_{L^\infty} \le C(\mathfrak{E}), \tag{4.73}$$ eq:0104

therefore by (4.28),

$$\|\mathcal{A}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}^2 \le C(\mathfrak{E})E_2. \tag{4.74}$$ eq:106

We now consider the term $(\partial_{\alpha'}\mathcal{A}_{\alpha'})\overline{Z}_{t,\alpha'}$ in (4.34). We calculate $\partial_{\alpha'}^2\mathcal{A}$ by differentiating the equation $i\mathcal{A} = \frac{Z_{tt}+i}{Z_{,\alpha'}}$ (2.9) twice:

$$i\partial_{\alpha'}^2\mathcal{A} = \frac{\partial_{\alpha'}^2 Z_{tt}}{Z_{,\alpha'}} + 2\partial_{\alpha'}Z_{tt}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + (Z_{tt}+i)\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}. \tag{4.75}$$ eq:108

Applying $(I-\mathbb{H})$ then taking the imaginary parts gives

$$\partial_{\alpha'}^2\mathcal{A} = \operatorname{Im}(I-\mathbb{H})(\frac{\partial_{\alpha'}^2 Z_{tt}}{Z_{,\alpha'}}) + 2\operatorname{Im}(I-\mathbb{H})(\partial_{\alpha'}Z_{tt}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) + \operatorname{Im}(I-\mathbb{H})((Z_{tt}+i)\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}); \tag{4.76}$$ eq:109

we rewrite the first term on the right by commuting out $\frac{1}{Z_{,\alpha'}}$

$$(I-\mathbb{H})(\frac{\partial_{\alpha'}^2 Z_{tt}}{Z_{,\alpha'}}) = \frac{1}{Z_{,\alpha'}}(I-\mathbb{H})(\partial_{\alpha'}^2 Z_{tt}) + [\frac{1}{Z_{,\alpha'}},\mathbb{H}](\partial_{\alpha'}^2 Z_{tt}); \tag{4.77}$$ eq:107

using $(I-\mathbb{H})\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} = 0$ we rewrite the third term on the right of (4.76) as a commutator

$$(I-\mathbb{H})((Z_{tt}+i)\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}) = [Z_{tt},\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \tag{4.78}$$ eq:111

so

$$\begin{aligned}\partial_{\alpha'}^2\mathcal{A} = \operatorname{Im}\{&\frac{1}{Z_{,\alpha'}}(I-\mathbb{H})(\partial_{\alpha'}^2 Z_{tt}) + [\frac{1}{Z_{,\alpha'}},\mathbb{H}](\partial_{\alpha'}^2 Z_{tt})\}\\ &+ \operatorname{Im}\{2(I-\mathbb{H})(\partial_{\alpha'}Z_{tt}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) + [Z_{tt},\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\}\end{aligned} \tag{4.79}$$ eq:112

We apply (A.11), (A.12) and Hölder. This gives

$$\left\|\partial_{\alpha'}^2\mathcal{A} - \operatorname{Im}\{\frac{1}{Z_{,\alpha'}}(I-\mathbb{H})(\partial_{\alpha'}^2 Z_{tt})\}\right\|_{L^2} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|Z_{tt,\alpha'}\|_{L^\infty}, \tag{4.80}$$ eq:0112

so

$$\left\|(\partial_{\alpha'}^2\mathcal{A})\overline{Z}_{t,\alpha'}\right\|_{L^2} \lesssim \|D_{\alpha'}\overline{Z}_t\|_{L^\infty}\|\partial_{\alpha'}^2 Z_{tt}\|_{L^2} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}\|\overline{Z}_{t,\alpha'}\|_{L^\infty}\|Z_{tt,\alpha'}\|_{L^\infty}. \tag{4.81}$$ eq:114

By (4.28), (4.44), (4.48) and Appendix C,

$$\left\|(\partial_{\alpha'}^2\mathcal{A})\overline{Z}_{t,\alpha'}\right\|_{L^2}^2 \le C(\mathfrak{E})E_2.$$

This completes the proof for

$$\int |Z_{,\alpha'}U_h^{-1}G_{2,2}|^2\,d\alpha \le C(\mathfrak{E})E_2. \tag{4.82}$$ eq:115

Step 4. Controlling $\|Z_{,\alpha'}U_h^{-1}G_{2,0}\|_{L^2}$. We are left with controlling $\|Z_{,\alpha'}U_h^{-1}G_{2,0}\|_{L^2}$. By (4.24), we must show

$$\int |\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i))|^2\,d\alpha' \le C(\mathfrak{E})E_2. \tag{4.83}$$ eq:118

We expand $\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i))$ by the product rule

$$\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i)) = \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^2\overline{Z}_{tt}+2\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\overline{Z}_{tt,\alpha'}+\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})(\overline{Z}_{tt}-i) \tag{4.84}$$ eq:119

where we estimate the L^2 norm of $\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})$ by (4.6), (A.8), (A.11), (A.12) and (A.9)

$$\begin{aligned}\Big\|\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\Big\|_{L^2} \lesssim &\|Z_{t,\alpha'}\|_{L^\infty}\|Z_{tt,\alpha'}\|_{L^2}+\|Z_{t,\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\|D_{\alpha'}\overline{Z}_t\|_{L^\infty}\\ &+\|Z_{t,\alpha'}\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2}\Big\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\Big\|_{L^\infty}\end{aligned} \tag{4.85}$$ eq:120

so by Appendix C, (4.28) and (4.44), (4.48),

$$\int |\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^2\overline{Z}_{tt}+2\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\overline{Z}_{tt,\alpha'}|^2\,d\alpha' \le C(\mathfrak{E})E_2. \tag{4.86}$$ eq:121

What remains is the term $\int |\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})(\overline{Z}_{tt}-i)|^2\,d\alpha'$. We begin with (4.12), together with (4.18) and (4.21), (4.22), (4.23):

$$(\partial_t^2+i\mathfrak{a}\partial_\alpha)D_\alpha(\frac{\partial_\alpha}{h_\alpha})\overline{z}_t = D_\alpha\frac{\partial_\alpha}{h_\alpha}(-i\mathfrak{a}_t\overline{z}_\alpha)+G_{2,1}+G_{2,2}. \tag{4.87}$$ eq:116

Precomposing with h^{-1} then multiply $Z_{,\alpha'}$ gives, using $\overline{z}_{tt}-i=-i\mathfrak{a}\overline{z}_\alpha$ (4.3),

$$Z_{,\alpha'}U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)D_\alpha(\frac{\partial_\alpha}{h_\alpha})\overline{z}_t = \partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i))+Z_{,\alpha'}U_h^{-1}(G_{2,1}+G_{2,2}). \tag{4.88}$$ eq:117

By commuting $(\partial_t^2+i\mathfrak{a}\partial_\alpha)$ with $\frac{h_\alpha}{z_\alpha}$ we rewrite the left hand side as

$$U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t+Z_{,\alpha'}U_h^{-1}[(\partial_t^2+i\mathfrak{a}\partial_\alpha),\frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t; \tag{4.89}$$ eq:122

(4.88) now yields

$$U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t = \partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)(\overline{Z}_{tt}-i)+e \tag{4.90}$$ eq:125

where

$$\begin{aligned}e:=&-Z_{,\alpha'}U_h^{-1}[(\partial_t^2+i\mathfrak{a}\partial_\alpha),\frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t+Z_{,\alpha'}U_h^{-1}(G_{2,1}+G_{2,2})\\ &+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^2\overline{Z}_{tt}+2\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\overline{Z}_{tt,\alpha'}.\end{aligned} \tag{4.91}$$

Observe $(I-\mathbb{H})U_h^{-1}(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t=(I-\mathbb{H})\partial_{\alpha'}^2\overline{Z}_t=0$. We want to use the "almost holomorphicity" of the LHS of (4.90) and the fact that $\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)$ is real valued to estimate $\int|\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})(\overline{Z}_{tt}-i)|^2\,d\alpha$. We first show that the error term e is well behaved. By (B.29),

$$\begin{aligned}&Z_{,\alpha'}U_h^{-1}[(\partial_t^2+i\mathfrak{a}\partial_\alpha),\frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t = 2((h_t\circ h^{-1})_{\alpha'}-D_{\alpha'}Z_t)U_h^{-1}\partial_tU_h\partial_{\alpha'}^2\overline{Z}_t\\ &+((h_t\circ h^{-1})_{\alpha'}-D_{\alpha'}Z_t)^2\partial_{\alpha'}^2\overline{Z}_t+(U_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'}-U_h^{-1}\partial_tU_hD_{\alpha'}Z_t)\partial_{\alpha'}^2\overline{Z}_t\\ &+(Z_{tt}+i)\partial_{\alpha'}\big(\frac{1}{Z_{,\alpha'}}\big)\partial_{\alpha'}^2\overline{Z}_t;\end{aligned} \tag{4.92}$$ eq:123

from (B.12), $U_h^{-1}\partial_tU_hD_{\alpha'}Z_t=D_{\alpha'}Z_{tt}-(D_{\alpha'}Z_t)^2$, therefore by Appendix C, (4.28), (4.31) and (4.62),

$$\int\Big|Z_{,\alpha'}U_h^{-1}[(\partial_t^2+i\mathfrak{a}\partial_\alpha),\frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t\Big|^2\,d\alpha' \le C(\mathfrak{E})E_2. \tag{4.93}$$ eq:124

The estimates (4.93), (4.86), (4.82) and (4.32) give that

$$\int |e|^2\, d\alpha' \le C(\mathfrak{E})E_2. \tag{4.94}$$

eq:126

Now we apply $(I-\mathbb{H})$ to both sides of (4.90), then rewrite $(I-\mathbb{H})(\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)(\overline{Z}_{tt}-i))$ by commuting out $(\overline{Z}_{tt}-i)$:

$$\begin{aligned}(I-\mathbb{H})\big(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t\big) = (\overline{Z}_{tt}-i)(I-\mathbb{H})(\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))\\ +[\overline{Z}_{tt},\mathbb{H}](\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))+(I-\mathbb{H})e\end{aligned} \tag{4.95}$$

eq:127

Since $\mathbb{H}$ is purely imaginary, $|\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)|\le |(I-\mathbb{H})(\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))|$ hence

$$\begin{aligned}&\|(\overline{Z}_{tt}-i)\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\|_{L^2}\le \|(\overline{Z}_{tt}-i)(I-\mathbb{H})(\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))\|_{L^2}\\ &\lesssim \|(I-\mathbb{H})\big(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t\big)\|_{L^2}+\|[\overline{Z}_{tt},\mathbb{H}](\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))\|_{L^2}+\|e\|_{L^2}.\end{aligned} \tag{4.96}$$

eq:128

By (A.12) and (4.85), (4.48), (4.44) and Appendix C,

$$\|[\overline{Z}_{tt},\mathbb{H}](\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big))\|_{L^2}\lesssim \|Z_{tt,\alpha'}\|_{L^\infty}\|\partial_{\alpha'}\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\|_{L^2}\le C(\mathfrak{E})E_2^{1/2} \tag{4.97}$$

therefore

$$\|(\overline{Z}_{tt}-i)\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\|_{L^2}\lesssim \|(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t)\|_{L^2}+C(\mathfrak{E})E_2^{1/2}. \tag{4.98}$$

eq:129

In what follows we will show that

$$\|(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t)\|_{L^2}\le C(\mathfrak{E})E_2^{1/2}$$

and complete the proof for Proposition 4.3.

Step 4.1. Controlling $\|(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t)\|_{L^2}$. We introduce the following notations. We write $f_1\equiv f_2$, if $(I-\mathbb{H})(f_1-f_2)=0$. We define $\mathbb{P}_H := \frac{(I+\mathbb{H})}{2}$ and $\mathbb{P}_A := \frac{(I-\mathbb{H})}{2}$, so $\mathbb{P}_H+\mathbb{P}_A=I$, and $\mathbb{P}_H-\mathbb{P}_A=\mathbb{H}$. By Proposition A.1, $\mathbb{P}_H$ is the projection onto the space of holomorphic functions in the lower half plane P_-, and $\mathbb{P}_A$ is the projection onto the space of anti-holomorphic functions in P_-.

We want to derive an estimate of $\|(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h\|_{L^2}$ for a generic U satisfying $U=\mathbb{H}U$, i.e. $U\equiv 0$. Observe $D_{\alpha'}U\equiv 0$. By (B.8) of Proposition B.1, U satisfies

$$\begin{aligned}U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h\equiv 2\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}(U_h^{-1}\partial_tU_h-\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'})U\\ +Z_t^2D_{\alpha'}^2U+2(Z_{tt}+i)D_{\alpha'}U.\end{aligned} \tag{4.99}$$

eq:134

What we will do first is to use (4.99) to rewrite $(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h$ into a favorable form so that desired estimate will follow.

We expand on the RHS of (4.99) the term

$$Z_t^2D_{\alpha'}^2U=\big(\frac{Z_t}{Z_{,\alpha'}}\big)^2\partial_{\alpha'}^2U+\frac{Z_t^2}{Z_{,\alpha'}}\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U$$

by the product rule, and decompose $\frac{Z_t}{Z_{,\alpha'}} = \mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}}) + \mathbb{P}_H(\frac{Z_t}{Z_{,\alpha'}})$. We have, because $\partial_{\alpha'}(U_h^{-1}\partial_t U_h - \frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'})U \equiv 0$ by (B.6),

$$\begin{aligned}U_h^{-1}(\partial_t^2 + i\mathfrak{a}\partial_\alpha)U\circ h \equiv 2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}(U_h^{-1}\partial_t U_h - (\mathbb{P}_A+\mathbb{P}_H)(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'})U \\ + \big((\mathbb{P}_A+\mathbb{P}_H)(\frac{Z_t}{Z_{,\alpha'}})\big)^2\partial_{\alpha'}^2 U \\ + \frac{Z_t^2}{Z_{,\alpha'}}\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U + 2(Z_{tt}+i)D_{\alpha'}U.\end{aligned} \tag{4.100}$$

eq:135

We expand further the factor $\partial_{\alpha'}(\mathbb{P}_H(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}U)$ on the RHS by the product rule. After cancelation we obtain

$$\begin{aligned}U_h^{-1}(\partial_t^2 + i\mathfrak{a}\partial_\alpha)U\circ h \equiv 2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}(U_h^{-1}\partial_t U_h - \mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'})U \\ - 2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\{\mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}) + \mathbb{P}_H(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\}\partial_{\alpha'}U + \\ \big(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\big)^2\partial_{\alpha'}^2U + \frac{Z_t^2}{Z_{,\alpha'}}\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U + 2(Z_{tt}+i)D_{\alpha'}U.\end{aligned} \tag{4.101}$$

eq:136

Now because $\mathbb{P}_H(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})$ and $\partial_{\alpha'}U$ are holomorphic,

$$2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\mathbb{P}_H(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U \equiv 2\frac{Z_t}{Z_{,\alpha'}}\mathbb{P}_H(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U,$$

moreover

$$-2\frac{Z_t}{Z_{,\alpha'}}\mathbb{P}_H(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) + \frac{Z_t^2}{Z_{,\alpha'}}\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}}) = -\frac{Z_t}{Z_{,\alpha'}}\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}); \qquad \text{and}$$

$$\begin{aligned}-\frac{Z_t}{Z_{,\alpha'}}\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}U = -Z_t\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}}U) \\ + Z_t\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})U;\end{aligned}$$

and by straightforward expansion,

$$[Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \equiv -2Z_t\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}});$$

and

$$Z_t\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\mathbb{H}(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) = \{\mathbb{P}_H\big(Z_t\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\big)\}^2 - \{\mathbb{P}_A\big(Z_t\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\big)\}^2.$$

Therefore

$$\begin{aligned}U_h^{-1}(\partial_t^2 + i\mathfrak{a}\partial_\alpha)U\circ h \equiv 2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}(U_h^{-1}\partial_t U_h - \mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'})U \\ - 2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}})\partial_{\alpha'}U + \big(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\big)^2\partial_{\alpha'}^2U \\ + \frac12\{[Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\}\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}}U) - \{\mathbb{P}_A\big(Z_t\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}})\big)\}^2U \\ + 2\frac{(Z_{tt}+i)}{Z_{,\alpha'}}\partial_{\alpha'}U.\end{aligned} \tag{4.102}$$

eq:137

We further rewrite

$$2\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}})\partial_{\alpha'}U \equiv \mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}})(I-\mathbb{H})(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}U).$$

Apply $(I-\mathbb{H})$ to both sides of (4.102), and rewrite terms of the form $(I-\mathbb{H})(g_1g_2)$ with $g_2=\mathbb{H}g_2$ as $[g_1,\mathbb{H}]g_2$. We obtain

$$\begin{aligned}
&(I-\mathbb{H})U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h=2[\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}}),\mathbb{H}]\partial_{\alpha'}(U_h^{-1}\partial_tU_h-\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'})U\\
&-(I-\mathbb{H})\{\mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}})[\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}}),\mathbb{H}]\partial_{\alpha'}U\}+[\big(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\big)^2,\mathbb{H}]\partial_{\alpha'}^2U\\
&+[\frac12[Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},\mathbb{H}]\partial_{\alpha'}(\frac{1}{Z_{,\alpha'}}U)\\
&-(I-\mathbb{H})\{\big(\mathbb{P}_A\big(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\big)\big)^2U\}+2[\frac{Z_{tt}+i}{Z_{,\alpha'}},\mathbb{H}]\partial_{\alpha'}U.
\end{aligned}\tag{4.103}$$

eq:0137

We further use the identity[20]

$$-2[g_1,\mathbb{H}]\partial_{\alpha'}(g_1g_2)+[g_1^2,\mathbb{H}]\partial_{\alpha'}g_2=-\frac{1}{\pi i}\int\big(\frac{g_1(\alpha')-g_1(\beta')}{\alpha'-\beta'}\big)^2g_2(\beta')\,d\beta':=-[g_1,g_1;g_2]$$

to rewrite the sum of second part of the first and the third terms on the right:

$$\begin{aligned}
&-2[\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}}),\mathbb{H}]\partial_{\alpha'}\left(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\partial_{\alpha'}U\right)+[\big(\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\big)^2,\mathbb{H}]\partial_{\alpha'}^2U\\
&=-[\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}}),\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}});\partial_{\alpha'}U].
\end{aligned}\tag{4.104}$$

eq:138

We are now ready to give the estimate for $(I-\mathbb{H})U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h$. We have, by (A.11), (A.15) and Hölder's inequality,

$$\begin{aligned}
&\|(I-\mathbb{H})U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h\|_{L^2}\lesssim\|\partial_{\alpha'}\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\|_{L^\infty}\|U_h^{-1}\partial_tU_hU\|_{L^2}+\\
&\|\mathbb{P}_H(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}})\|_{L^\infty}\|\partial_{\alpha'}\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\|_{L^\infty}\|U\|_{L^2}+\|\partial_{\alpha'}\mathbb{P}_A(\frac{Z_t}{Z_{,\alpha'}})\|_{L^\infty}^2\|U\|_{L^2}\\
&+\|\partial_{\alpha'}\big([Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\big)\|_{L^2}\|\frac{1}{Z_{,\alpha'}}U\|_{\dot H^{1/2}}\\
&+\|\mathbb{P}_A\big(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\big)\|_{L^\infty}^2\|U\|_{L^2}++\|\partial_{\alpha'}(\frac{Z_{tt}+i}{Z_{,\alpha'}})\|_{L^\infty}\|U\|_{L^2}.
\end{aligned}\tag{4.105}$$

eq:139

Now by (A.19),

$$\|\partial_{\alpha'}\big([Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\big)\|_{L^2}\lesssim\|Z_{t,\alpha'}\|_{L^2}^2\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}$$

and because

$$\partial_{\alpha'}(\frac{Z_{tt}+i}{Z_{,\alpha'}})=D_{\alpha'}Z_{tt}+(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},$$

$$\|\partial_{\alpha'}(\frac{Z_{tt}+i}{Z_{,\alpha'}})\|_{L^\infty}\le\|D_{\alpha'}Z_{tt}\|_{L^\infty}+\|(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^\infty}.$$

We can conclude now by Appendix C that for any U satisfying $U=\mathbb{H}U$,

$$\|(I-\mathbb{H})U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U\circ h\|_{L^2}\le C(\mathfrak{E})(\|U\|_{L^2}+\|U_h^{-1}\partial_tU_hU\|_{L^2}+\|\frac{1}{Z_{,\alpha'}}U\|_{\dot H^{1/2}}).\tag{4.106}$$

eq:140

As a consequence of (4.106) and (4.28), (4.31),

$$\|(I-\mathbb{H})(U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)(\frac{\partial_\alpha}{h_\alpha})^2\bar z_t)\|_{L^2}\le C(\mathfrak{E})E_2^{1/2}.\tag{4.107}$$

eq:142

[20] It is an easy consequence of integration by parts.

(4.98) then gives

$$\|(\overline{Z}_{tt}-i)\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\|_{L^2}^2\lesssim C(\mathfrak{E})E_2. \tag{4.108}$$

eq:143

Sum up (4.82), (4.32), (4.86) and (4.108),

$$\int |Z_{,\alpha'}U_h^{-1}G_2|^2\,d\alpha'\le C(\mathfrak{E})E_2.$$

This finishes the proof of Proposition 4.3.

□

proof-prop2

4.2. **The proof of Proposition 4.4.**

Proof. We prove Proposition 4.4 by applying Lemma 4.1 to (4.12) for $k=3$, notice that $(I-\mathbb{H})U_h^{-1}D_\alpha(\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t=(I-\mathbb{H})D_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t=0$. For $k=3$, the right hand side of (4.12) is

$$G_3:=D_\alpha(\frac{\partial_\alpha}{h_\alpha})^2(-i\mathfrak{a}_t\overline{z}_\alpha)+[\partial_t^2+i\mathfrak{a}\partial_\alpha,D_\alpha(\frac{\partial_\alpha}{h_\alpha})^2]\overline{z}_t \tag{4.109}$$

eq:211

Similar to the proof for Proposition 4.3, we only need to show that

$$\int |Z_{,\alpha'}U_h^{-1}G_3|^2\,d\alpha'\le C(\mathfrak{E},E_2)E_3. \tag{4.110}$$

eq:201

We expand $Z_{,\alpha'}U_h^{-1}G_3$ by (B.16), (B.15), (B.22). We have

$$\begin{aligned}Z_{,\alpha'}U_h^{-1}G_3&=\partial_{\alpha'}^3\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i)\big)+Z_{,\alpha'}U_h^{-1}[\partial_t^2+i\mathfrak{a}\partial_\alpha,D_\alpha](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t\\&\quad+\partial_{\alpha'}U_h^{-1}[\partial_t^2+i\mathfrak{a}\partial_\alpha,\frac{\partial_\alpha}{h_\alpha}]\frac{\partial_\alpha\overline{z}_t}{h_\alpha}+\partial_{\alpha'}^2U_h^{-1}[\partial_t^2+i\mathfrak{a}\partial_\alpha,\frac{\partial_\alpha}{h_\alpha}]\overline{z}_t\\&:=Z_{,\alpha'}U_h^{-1}G_{3,0}+Z_{,\alpha'}U_h^{-1}G_{3,1}+Z_{,\alpha'}U_h^{-1}G_{3,2}+Z_{,\alpha'}U_h^{-1}G_{3,3}\end{aligned} \tag{4.111}$$

eq:202

where

$$\begin{aligned}Z_{,\alpha'}U_h^{-1}G_{3,0}:&=\partial_{\alpha'}^3\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i)\big)\\&=\partial_{\alpha'}^3\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)(\overline{Z}_{tt}-i)+3\partial_{\alpha'}^2\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\overline{Z}_{tt,\alpha'}\\&\quad+3\partial_{\alpha'}\big(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\big)\partial_{\alpha'}^2\overline{Z}_{tt}+\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\partial_{\alpha'}^3\overline{Z}_{tt};\end{aligned} \tag{4.112}$$

eq:206

$$\begin{aligned}Z_{,\alpha'}U_h^{-1}G_{3,1}:&=Z_{,\alpha'}U_h^{-1}[\partial_t^2+i\mathfrak{a}\partial_\alpha,D_\alpha](\frac{\partial_\alpha}{h_\alpha})^2\overline{z}_t\\&=-2D_{\alpha'}Z_{tt}\partial_{\alpha'}^3\overline{Z}_t-2(D_{\alpha'}Z_t)Z_{,\alpha'}U_h^{-1}\partial_tU_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t;\end{aligned} \tag{4.113}$$

eq:203

$$\begin{aligned}Z_{,\alpha'}U_h^{-1}G_{3,2}&:=\partial_{\alpha'}U_h^{-1}[\partial_t^2+i\mathfrak{a}\partial_\alpha,\frac{\partial_\alpha}{h_\alpha}]\frac{\partial_\alpha\overline{z}_t}{h_\alpha}\\&=-\partial_{\alpha'}U_h^{-1}\partial_tU_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t\}-\partial_{\alpha'}\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}\partial_tU_h\overline{Z}_{t,\alpha'}\}\\&\quad-i\partial_{\alpha'}\{\mathcal{A}_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t\}\\&=-(\partial_{\alpha'}U_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'})\partial_{\alpha'}^2\overline{Z}_t-(U_h^{-1}\partial_tU_h(h_t\circ h^{-1})_{\alpha'})\partial_{\alpha'}^3\overline{Z}_t\\&\quad-\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}(U_h^{-1}\partial_tU_h\partial_{\alpha'}^2\overline{Z}_t+\partial_{\alpha'}U_h^{-1}\partial_tU_h\partial_{\alpha'}\overline{Z}_t)\\&\quad-(h_t\circ h^{-1})_{\alpha'}(\partial_{\alpha'}U_h^{-1}\partial_tU_h\partial_{\alpha'}^2\overline{Z}_t+\partial_{\alpha'}^2U_h^{-1}\partial_tU_h\partial_{\alpha'}\overline{Z}_t)\\&\quad-i(\partial_{\alpha'}\mathcal{A}_{\alpha'})\partial_{\alpha'}^2\overline{Z}_t-i\mathcal{A}_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t;\end{aligned} \tag{4.114}$$

eq:204

and

$$\begin{aligned} Z_{,\alpha'}U_h^{-1}G_{3,3} &:= \partial_{\alpha'}^2 U_h^{-1}[\partial_t^2 + i\mathfrak{a}\partial_\alpha, \frac{\partial_\alpha}{h_\alpha}]\bar{z}_t \\ &= -\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_t\} - \partial_{\alpha'}^2\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt}\} \\ &\quad - i\partial_{\alpha'}^2\{\mathcal{A}_{\alpha'}\partial_{\alpha'}\overline{Z}_t\}. \end{aligned} \tag{4.115}$$ eq:205

Step 1. Quantities controlled by E_3 and a polynomial of $\mathfrak{C}$ and E_2. By the definition of E_3, and the fact that $\|A_1\|_{L^\infty} \le C(\mathfrak{C})$ (cf. Appendix C),

$$\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}^2, \quad \left\|Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t\right\|_{L^2}^2, \quad \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t\right\|_{\dot{H}^{1/2}}^2 \le C(\mathfrak{C})E_3. \tag{4.116}$$ eq:207

We commute $Z_{,\alpha'}$ with $U_h^{-1}\partial_t U_h$ of the second quantity in (4.116):

$$U_h^{-1}\partial_t U_h\partial_{\alpha'}^3\overline{Z}_t = Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t - [Z_{,\alpha'}, U_h^{-1}\partial_t U_h]\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t \tag{4.117}$$

By (B.26) and Appendix C, we have

$$\left|\left\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^3\overline{Z}_t\right\|_{L^2} - \left\|Z_{,\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t\right\|_{L^2}\right| \le C(\mathfrak{C})\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}, \tag{4.118}$$ eq:208

so

$$\left\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^3\overline{Z}_t\right\|_{L^2}^2 \le C(\mathfrak{C})E_3 \tag{4.119}$$ eq:209

By (B.18),

$$\partial_{\alpha'}U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t - U_h^{-1}\partial_t U_h\partial_{\alpha'}^3\overline{Z}_t = [\partial_{\alpha'}, U_h^{-1}\partial_t U_h]\partial_{\alpha'}^2\overline{Z}_t = (h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t,$$

so

$$\|\partial_{\alpha'}U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t\|_{L^2}^2 \le C(\mathfrak{C})E_3. \tag{4.120}$$ eq:212

As a consequence of (A.3), (4.28), (4.116), (4.31) and (4.120),

$$\|\partial_{\alpha'}^2\overline{Z}_t\|_{L^\infty}^2 \le C(\mathfrak{C}, E_2)E_3^{1/2}, \quad \|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t\|_{L^\infty}^2 \le C(\mathfrak{C}, E_2)E_3^{1/2}. \tag{4.121}$$ eq:213

By (B.18) again,

$$\begin{aligned} \partial_{\alpha'}^2 U_h^{-1}\partial_t U_h\partial_{\alpha'}\overline{Z}_t - \partial_{\alpha'}U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t &= \partial_{\alpha'}[\partial_{\alpha'}, U_h^{-1}\partial_t U_h]\partial_{\alpha'}\overline{Z}_t \\ &= \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t + (h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t, \end{aligned}$$

so by (4.120), (4.116), and (4.41), (4.121), (4.28), (4.44) and Appendix C,

$$\|\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h\partial_{\alpha'}\overline{Z}_t\|_{L^2}^2 \le C(\mathfrak{C}, E_2)E_3 + C(\mathfrak{C}, E_2); \tag{4.122}$$ eq:214

and consequently by (A.3) and (4.43),

$$\|\partial_{\alpha'}U_h^{-1}\partial_t U_h\partial_{\alpha'}\overline{Z}_t\|_{L^\infty}^2 \le C(\mathfrak{C}, E_2)E_3^{1/2} + C(\mathfrak{C}, E_2). \tag{4.123}$$ eq:215

Step 2. Controlling $G_{3,1}$. By (4.113), Appendix C and (4.116),

$$\int |Z_{,\alpha'}U_h^{-1}G_{3,1}|^2\,d\alpha \le C(\mathfrak{C})E_3. \tag{4.124}$$ eq:210

Step 3. Controlling $G_{3,2}$. By (4.63), (4.69), (4.41), (4.65), (4.43), (4.28), (4.121), (4.62), (4.116), (4.123), (4.120), (4.122), (4.73), (4.80), (4.48) and Appendix C, we can control each of the terms in (4.114). Sum up, we have

$$\int |Z_{,\alpha'}U_h^{-1}G_{3,2}|^2\,d\alpha \le C(\mathfrak{C}, E_2)E_3 + C(\mathfrak{C}, E_2). \tag{4.125}$$ eq:216

Step 4. Controlling $G_{3,3}$. Expanding $G_{3,3}$ in (4.115) by the product rule, we find that the additional types of terms that have not already appeared in (4.114) and controlled in the previous step are

$$(\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'})\partial_{\alpha'}\overline{Z}_t,\quad (\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'})U_h^{-1}\partial_t U_h\partial_{\alpha'}\overline{Z}_t,$$
$$\partial_{\alpha'}^2\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_{tt}\},\quad \text{and}\quad (\partial_{\alpha'}^3\mathcal{A})\partial_{\alpha'}\overline{Z}_t.$$

Step 4.1. Controlling $\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}$ and $\partial_{\alpha'}^3 Z_{tt}$. We begin with controlling $\partial_{\alpha'}A_1$, $\partial_{\alpha'}^2 A_1$ and $\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}$. Differentiating (2.19) gives

$$\partial_{\alpha'}A_1 = -\operatorname{Im}[Z_{t,\alpha'},\mathbb{H}]\overline{Z}_{t,\alpha'} - \operatorname{Im}[Z_t,\mathbb{H}]\partial_{\alpha'}\overline{Z}_{t,\alpha'} \tag{4.126}$$ eq:104

so by (A.18), Appendix C and (4.28),

$$\|\partial_{\alpha'}A_1\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}\|\partial_{\alpha'}^2 Z_t\|_{L^2} \lesssim C(\mathfrak{E})E_2^{1/2}. \tag{4.127}$$ eq:105

Differentiating again with respect to α' then apply (A.11), (A.13) and (A.3) gives

$$\|\partial_{\alpha'}^2 A_1\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^2 Z_t\|_{L^2} \le C(\mathfrak{E},E_2). \tag{4.128}$$ eq:219

To estimate $\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}$ we begin with (2.10):

$$-i\frac{1}{Z_{,\alpha'}} = \frac{\overline{Z}_{tt}-i}{A_1}.$$

Taking two derivatives with respect to α' gives

$$-i\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} = \frac{\partial_{\alpha'}^2\overline{Z}_{tt}}{A_1} - 2\overline{Z}_{tt,\alpha'}\frac{\partial_{\alpha'}A_1}{A_1^2} + (\overline{Z}_{tt}-i)\Big(-\frac{\partial_{\alpha'}^2 A_1}{A_1^2} + 2\frac{(\partial_{\alpha'}A_1)^2}{A_1^3}\Big); \tag{4.129}$$ eq:220

therefore

$$\begin{aligned}\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2} \lesssim &\|\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2} + \|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^2}\|\partial_{\alpha'}A_1\|_{L^\infty}\\ &+\|\frac{1}{Z_{,\alpha'}}\|_{L^\infty}(\|\partial_{\alpha'}^2 A_1\|_{L^2} + \|\partial_{\alpha'}A_1\|_{L^2}\|\partial_{\alpha'}A_1\|_{L^\infty}) \le C(\mathfrak{E},E_2),\end{aligned} \tag{4.130}$$ eq:221

and consequently by (A.3),

$$\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^\infty} \le C(\mathfrak{E},E_2). \tag{4.131}$$ eq:225

We are now ready to give the estimates for $\|\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2}$ and $\|\partial_{\alpha'}^3\overline{Z}_{tt}\|_{L^2}$. Rewriting the first term on the right of (4.40) as a commutator then differentiating yields,

$$\begin{aligned}\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'} - 2\operatorname{Re}\Big(\frac{\partial_{\alpha'}^3 Z_t}{Z_{,\alpha'}} + \partial_{\alpha'}^2 Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big) = \operatorname{Re}\{&2\partial_{\alpha'}[Z_{t,\alpha'},\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}})\\ &-\partial_{\alpha'}[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t + \partial_{\alpha'}[Z_t,\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\}.\end{aligned} \tag{4.132}$$ eq:217

Expanding the right hand side of (4.132) by the product rule. By (A.11), (A.12),

$$\begin{aligned}\|\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2} \lesssim \|&\frac{1}{Z_{,\alpha'}}\|_{L^\infty}\|\partial_{\alpha'}^3 Z_t\|_{L^2} + \|\partial_{\alpha'}^2 Z_t\|_{L^\infty}\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}\\ &+\|Z_{t,\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2} \le C(\mathfrak{E},E_2)E_3^{1/2} + C(\mathfrak{E},E_2).\end{aligned} \tag{4.133}$$ eq:218

For $\|\partial_{\alpha'}^3\overline{Z}_{tt}\|_{L^2}$, we differentiate (4.45) with respect to α':

$$\begin{aligned}\partial_{\alpha'}^3\overline{Z}_{tt} - \partial_{\alpha'}U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\overline{Z}_t = 3\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\overline{Z}_t + 2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^3\overline{Z}_t\\ +\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\overline{Z}_t,\end{aligned} \tag{4.134}$$ eq:230

therefore by (4.41), (4.116), (4.120), (4.121),

$$\|\partial_{\alpha'}^3 \overline{Z}_{tt}\|_{L^2}^2 \le C(\mathfrak{E}, E_2)E_3 + C(\mathfrak{E}, E_2), \tag{4.135}$$

eq:222

and as a consequence of (A.3),

$$\|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^\infty}^2 \le C(\mathfrak{E}, E_2)E_3^{1/2} + C(\mathfrak{E}, E_2). \tag{4.136}$$

eq:223

Step 4.2. Controlling $\partial_{\alpha'}^3 \mathcal{A}$. We differentiate (4.79) with respect to α' and use the product rule to expand. We have,

$$\begin{aligned}\|\partial_{\alpha'}^3 \mathcal{A}\|_{L^2} &\le \|\frac{1}{Z_{,\alpha'}}\|_{L^\infty}\|\partial_{\alpha'}^3 \overline{Z}_{tt}\|_{L^2} + \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^\infty}\|\partial_{\alpha'}^2 \overline{Z}_{tt}\|_{L^2}\\ &+ \|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2}\|\partial_{\alpha'}\overline{Z}_{tt}\|_{L^\infty} \le C(\mathfrak{E}, E_2)E_3^{1/2} + C(\mathfrak{E}, E_2).\end{aligned} \tag{4.137}$$

eq:224

Step 4.3. Controlling $\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'}$. By (B.16) and (B.18),

$$\begin{aligned}\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h(h_t\circ h^{-1})_{\alpha'} &= U_h^{-1}\partial_t U_h\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\\ &+ (\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'})^2 + 2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\end{aligned} \tag{4.138}$$

eq:226

where

$$\begin{aligned}&\|(\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'})^2 + 2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2}\\ &\lesssim \|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\|_{L^2}\|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty} + \|\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2}\|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\\ &\lesssim \|\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\|_{L^2}^{3/2}\|\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2}^{1/2} + \|\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2}\|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\\ &\le C(\mathfrak{E}, E_2)E_3^{1/2} + C(\mathfrak{E}, E_2).\end{aligned} \tag{4.139}$$

eq:227

For $U_h^{-1}\partial_t U_h\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}$, we differentiate (4.132) and use the product rule and (B.25) to expand the derivatives,

$$\begin{aligned}&U_h^{-1}\partial_t U_h\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'} - 2\,\mathrm{Re}(U_h^{-1}\partial_t U_h(\frac{\partial_{\alpha'}^3 Z_t}{Z_{,\alpha'}}) + U_h^{-1}\partial_t U_h(\partial_{\alpha'}^2 Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}))\\ &= \mathrm{Re}\, U_h^{-1}\partial_t U_h\{2\partial_{\alpha'}[Z_{t,\alpha'},\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}) - \partial_{\alpha'}[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}]\partial_{\alpha'}^2\overline{Z}_t + \partial_{\alpha'}[Z_t,\mathbb{H}]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\};\end{aligned} \tag{4.140}$$

eq:228

we then use (A.11), (A.12), (A.13), (A.16) and Hölder's inequality to do the estimates. We have

$$\begin{aligned}&\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'}\|_{L^2} \lesssim \|U_h^{-1}\partial_t U_h\partial_{\alpha'}^3 Z_t\|_{L^2}\|\frac{1}{Z_{,\alpha'}}\|_{L^\infty}\\ &+ \|\partial_{\alpha'}^3 Z_t\|_{L^2}\|U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\|_{L^\infty} + \|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2 Z_t\|_{L^\infty}\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}\\ &+ \|\partial_{\alpha'}^2 Z_t\|_{L^\infty}(\|U_h^{-1}\partial_t U_h\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2} + \|\partial_{\alpha'}U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}}\|_{L^2})\\ &+ \|\partial_{\alpha'}^2 Z_t\|_{L^\infty}\|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2} + \|U_h^{-1}\partial_t U_h\partial_{\alpha'}Z_t\|_{L^\infty}\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2}\\ &+ \|\partial_{\alpha'}Z_t\|_{L^\infty}\|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2} + \|Z_{tt,\alpha'}\|_{L^\infty}\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2}\\ &+ \|Z_{t,\alpha'}\|_{L^\infty}\|U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\|_{L^2}.\end{aligned} \tag{4.141}$$

eq:229

Now by (B.18), (B.16),

$$U_h^{-1}\partial_t U_h\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} = \partial_{\alpha'}^2 U_h^{-1}\partial_t U_h\frac{1}{Z_{,\alpha'}} - \partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - 2(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \tag{4.142}$$

eq:231

and

$$U_h^{-1}\partial_t U_h \frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}((h_t\circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t); \tag{4.143}$$ eq:232

$$\begin{aligned}\partial_{\alpha'}^2 U_h^{-1}\partial_t U_h \frac{1}{Z_{,\alpha'}} &= (\partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}})((h_t\circ h^{-1})_{\alpha'} - D_{\alpha'} Z_t)\\ &+ 2(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}})(\partial_{\alpha'}(h_t\circ h^{-1})_{\alpha'} - \partial_{\alpha'} D_{\alpha'} Z_t) + \frac{1}{Z_{,\alpha'}}(\partial_{\alpha'}^2(h_t\circ h^{-1})_{\alpha'} - \partial_{\alpha'}^2 D_{\alpha'} Z_t);\end{aligned} \tag{4.144}$$ eq:233

we further expand

$$\partial_{\alpha'} D_{\alpha'} Z_t = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2 Z_t + \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\partial_{\alpha'} Z_t;$$

and

$$\partial_{\alpha'}^2 D_{\alpha'} Z_t = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3 Z_t + 2\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2 Z_t + \partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} Z_{t,\alpha'}.$$

Therefore

$$\|U_h^{-1}\partial_t U_h \frac{1}{Z_{,\alpha'}}\|_{L^\infty} \le C(\mathfrak{E}), \quad \|U_h^{-1}\partial_t U_h \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\|_{L^2} \le C(\mathfrak{E}, E_2) E_3^{1/2} + C(\mathfrak{E}, E_2). \tag{4.145}$$ eq:234

By (4.141),

$$\|U_h^{-1}\partial_t U_h \partial_{\alpha'}^2 (h_t\circ h^{-1})_{\alpha'}\|_{L^2} \le C(\mathfrak{E}, E_2) E_3^{1/2} + C(\mathfrak{E}, E_2). \tag{4.146}$$ eq:235

Step 4.4. Conclusion for $G_{3,3}$. We expand $G_{3,3}$ by product rules. Sum up the estimates in Steps 4.1-4.3, we have

$$\int |Z_{,\alpha'} U_h^{-1} G_{3,3}|^2\, d\alpha' \le C(\mathfrak{E}, E_2) E_3 + C(\mathfrak{E}, E_2). \tag{4.147}$$ eq:236

Step 5. Controlling $G_{3,0}$. We estimate $\|Z_{,\alpha'} U_h^{-1} G_{3,0}\|_{L^2}$ using similar ideas as that in Step 4 for Proposition 4.3. By (4.112), we must control $\|\partial_{\alpha'}^3(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})(\overline{Z}_{tt} - i)\|_{L^2}$, $\|\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\overline{Z}_{tt,\alpha'}\|_{L^2}$, $\|\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}$ and $\|(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\partial_{\alpha'}^3\overline{Z}_{tt}\|_{L^2}$. First by (4.135) and Appendix C,

$$\|(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\partial_{\alpha'}^3\overline{Z}_{tt}\|_{L^2}^2 \le C(\mathfrak{E}, E_2) E_3 + C(\mathfrak{E}, E_2). \tag{4.148}$$ eq:237

By (4.85) and (A.3),

$$\|\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\partial_{\alpha'}^2\overline{Z}_{tt}\|_{L^2}^2 \le C(\mathfrak{E}, E_2) E_3 + C(\mathfrak{E}, E_2). \tag{4.149}$$ eq:238

By (4.6),

$$\begin{aligned}\|\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\|_{L^2} \lesssim\ & \|\partial_{\alpha'}^2 Z_t\|_{L^2}(\|Z_{tt,\alpha'}\|_{L^\infty} + \|\mathbb{H} Z_{tt,\alpha'}\|_{L^\infty}) + \|\partial_{\alpha'}^2 Z_{tt}\|_{L^2}\|Z_{t,\alpha'}\|_{L^\infty}\\ &+ (\|\partial_{\alpha'}^2 Z_t\|_{L^\infty}\|Z_{t,\alpha'}\|_{L^2} + \|\partial_{\alpha'}^2 Z_t\|_{L^2}\|Z_{t,\alpha'}\|_{L^\infty})\|D_{\alpha'} Z_t\|_{L^\infty}\\ &+ \|\partial_{\alpha'} Z_t\|_{L^\infty}^2\|\partial_{\alpha'} D_{\alpha'} Z_t\|_{L^2} + \|\partial_{\alpha'}(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\|_{L^2}\|\partial_{\alpha'} A_1\|_{L^\infty}\\ &+ \|\frac{\mathfrak{a}_t}{\mathfrak{a}}\|_{L^\infty}\|\partial_{\alpha'}^2 A_1\|_{L^2}.\end{aligned} \tag{4.150}$$ eq:239

so

$$\|\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\|_{L^2} \le C(\mathfrak{E}, E_2) E_3^{1/4} + C(\mathfrak{E}, E_2), \tag{4.151}$$ eq:240

therefore

$$\|\partial_{\alpha'}^2(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\partial_{\alpha'}\overline{Z}_{tt}\|_{L^2}^2 \le C(\mathfrak{E}, E_2) E_3 + C(\mathfrak{E}, E_2). \tag{4.152}$$ eq:241

Now similar to (4.92) and (4.93), we compute $Z_{,\alpha'} U_h^{-1}[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), \frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^3\overline{z}_t$ by (B.28) and have

$$\int \left| Z_{,\alpha'} U_h^{-1}[(\partial_t^2 + i\mathfrak{a}\partial_\alpha), \frac{h_\alpha}{z_\alpha}](\frac{\partial_\alpha}{h_\alpha})^3\overline{z}_t \right|^2 d\alpha' \le C(\mathfrak{E}) E_3. \tag{4.153}$$ eq:243

Now we begining with (4.12) for $k=3$. After expansion, commuting and precomposing with h^{-1}, and using the above estimates, we arrive at

$$U_h^{-1}(\partial_t^2+i\mathfrak{a}\partial_\alpha)U_h\partial_{\alpha'}^3\overline{Z}_t=(\overline{Z}_{tt}-i)\partial_{\alpha'}^3(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})+e_1 \tag{4.154}$$ eq:244

with

$$\int |e_1|^2\,d\alpha'\le C(\mathfrak{E},E_2)E_3+C(\mathfrak{E},E_2). \tag{4.155}$$ eq:245

Going through similar calculations as in (4.95) to (4.98), then applying (4.106) to $U=\partial_{\alpha'}^3\overline{Z}_t$, we obtain

$$\|(\overline{Z}_{tt}-i)\partial_{\alpha'}^3(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1})\|_{L^2}^2\le C(\mathfrak{E},E_2)E_3+C(\mathfrak{E},E_2). \tag{4.156}$$ eq:246

This finishes the proof for Proposition 4.4.

□

complete1

4.3. **Completing the proof for Theorem 3.1.**

Proof. Let $s\ge 4$. Let the initial interface $Z(\cdot,0)=Z(0)$, the initial velocity $Z_t(\cdot,0)=Z_t(0)$ be given and satisfy (2.8) and $\overline{Z}_t(0)=\mathbb{H}\overline{Z}_t(0)$; let $A_1(0)$ satisfy (2.19) and the initial acceleration $Z_{tt}(0)$ satisfy (2.10). Assume $Z_{,\alpha'}(0)-1\in L^\infty(\mathbb{R})$, $Z_t(0)\in H^{s+1/2}(\mathbb{R})$, and $Z_{tt}(0)\in H^s(\mathbb{R})$. It is clear that $E_2(0)+E_3(0)<\infty$. Assume $Z=Z(\cdot,t)$, for $t\in[0,T^*)$ is a solution of (2.9)-(2.8), such that $(Z_t,Z_{tt},Z_{,\alpha'}-1)\in C([0,T^*),H^{s+1/2}(\mathbb{R})\times H^s(\mathbb{R})\times H^s(\mathbb{R}))$, and T^* is the maximum existence time as defined in Theorem 3.1. Assume $T^*<\infty$, for otherwise we are done; and assume $\sup_{t\in[0,T^*)}\mathfrak{E}(t):=M<\infty$. We want to show $\sup_{t\in[0,T^*)}(\|Z_{tt}(t)\|_{H^3}+\|Z_t(t)\|_{H^{3+1/2}})<\infty$.

Step 1. Controlling $\|Z_{tt}(t)\|_{L^2}$ and $\|Z_t(t)\|_{L^2}$ by $\mathfrak{E}$ and the initial data. We start with $\|Z_{tt}(t)\|_{L^2}$. By a change of the variables,

$$\frac{d}{dt}\|Z_{tt}(t)\|_{L^2}^2=\frac{d}{dt}\int|z_{tt}|^2h_\alpha\,d\alpha=2\,\mathrm{Re}\int z_{tt}\overline{z}_{ttt}h_\alpha\,d\alpha+2\int|z_{tt}|^2\frac{h_{t\alpha}}{h_\alpha}h_\alpha\,d\alpha; \tag{4.157}$$ eq:200

we estimate

$$\int|z_{tt}|^2\frac{h_{t\alpha}}{h_\alpha}h_\alpha\,d\alpha\le\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty}\|Z_{tt}(t)\|_{L^2}^2. \tag{4.158}$$ eq:25

Switching back to the Riemann mapping variable and using (4.5) gives

$$\begin{aligned}\int z_{tt}\overline{z}_{ttt}h_\alpha\,d\alpha&=\int Z_{tt}\overline{Z}_{ttt}\,d\alpha'\\&=-i\int Z_{tt}\mathcal{A}\overline{Z}_{t,\alpha'}\,d\alpha'+\int Z_{tt}\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}(\overline{Z}_{tt}-i)\,d\alpha'=I+II\end{aligned} \tag{4.159}$$ eq:21

Replacing $\mathcal{A}:=\frac{A_1}{|Z_{,\alpha'}|^2}$, we estimate I by

$$|I|\le\|A_1\|_{L^\infty}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}^2\|Z_{tt}\|_{L^2}\|Z_{t,\alpha'}\|_{L^2} \tag{4.160}$$ eq:22

In II we estimate $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^2}$ by (4.6), where we rewrite $D_{\alpha'}\overline{Z}_t:=\frac{1}{Z_{,\alpha'}}\overline{Z}_{t,\alpha'}$. Using (A.12), (A.13) and (A.16) yields

$$\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}\right\|_{L^2}\lesssim\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{L^\infty}+\|Z_{t,\alpha'}\|_{L^2}^3\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}, \tag{4.161}$$ eq:23

so

$$|II|\lesssim(\|Z_{t,\alpha'}\|_{L^2}\|Z_{tt}\|_{L^\infty}+\|Z_{t,\alpha'}\|_{L^2}^3\left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty})\|Z_{tt}\|_{L^2}(\|Z_{tt}\|_{L^\infty}+1). \tag{4.162}$$ eq:24

Sum up the above estimates and apply Appendix C, we arrive at

$$\frac{d}{dt}\|Z_{tt}(t)\|_{L^2}^2 \le c(\mathfrak{E}(t))\|Z_{tt}(t)\|_{L^2}^2 + c(\mathfrak{E}(t)).$$

Consequently by Gronwall,

$$\sup_{[0,T^*)} \|Z_{tt}(t)\|_{L^2} \le c(\|Z_{tt}(0)\|_{L^2}, M) < \infty. \tag{4.163}$$

eq:249

Changing to the Lagrangian coordinate, we have

$$\int |Z_t(\alpha',t)|^2\,d\alpha' = \int |z_t(\alpha,t)|^2 h_\alpha(\alpha,t)\,d\alpha,$$

so

$$\frac{d}{dt}\int |z_t|^2 h_\alpha\,d\alpha = 2\operatorname{Re}\int z_t\overline{z}_{tt}h_\alpha\,d\alpha + \int |z_t|^2 h_\alpha \frac{h_{t\alpha}}{h_\alpha}\,d\alpha. \tag{4.164}$$

eq:247

Using Cauchy-Schwarz and changing back to the Riemann mapping variable,

$$\frac{d}{dt}\int |z_t|^2 h_\alpha\,d\alpha \le 2\|Z_t(t)\|_{L^2}\|Z_{tt}(t)\|_{L^2} + \|(h_t\circ h^{-1})_{\alpha'}\|_{L^\infty}\|Z_t(t)\|_{L^2}^2, \tag{4.165}$$

eq:248

therefore

$$\frac{d}{dt}\|Z_t(t)\|_{L^2}^2 \le C(\mathfrak{E}(t))\|Z_t(t)\|_{L^2}^2 + \|Z_{tt}(t)\|_{L^2}^2, \tag{4.166}$$

eq:250

by Appendix C. Consequently by Gronwall's inequality and (4.163),

$$\sup_{t\in[0,T^*)} \|Z_t(t)\|_{L^2}^2 \le C(\|Z_t(0)\|_{L^2}, \|Z_{tt}(0)\|_{L^2}, M) < \infty. \tag{4.167}$$

eq:251

Step 2. Controlling $\|Z_{,\alpha'}\|_{L^\infty}$. We know

$$Z_{,\alpha'}\circ h = \frac{z_\alpha}{h_\alpha},$$

and

$$\frac{d}{dt}\left|\frac{z_\alpha}{h_\alpha}\right|^2 = 2\left|\frac{z_\alpha}{h_\alpha}\right|^2 \operatorname{Re}(D_\alpha z_t - \frac{h_{t\alpha}}{h_\alpha}),$$

so by Appendix C,

$$\frac{d}{dt}\left|\frac{z_\alpha}{h_\alpha}\right|^2 \le C(\mathfrak{E})\left|\frac{z_\alpha}{h_\alpha}\right|^2$$

therefore

$$\sup_{t\in[0,T^*)} \|Z_{,\alpha'}(t)\|_{L^\infty}^2 \le \|Z_{,\alpha'}(0)\|_{L^\infty}^2 e^{C(M)T^*} < \infty. \tag{4.168}$$

eq:252

Step 3. Controlling $\|Z_t(t)\|_{H^{3+1/2}} + \|Z_{tt}(t)\|_{H^3}$. Taking sup over $[0,T^*)$ on (4.17) gives

$$\begin{aligned} &\sup_{t\in[0,T^*)} E_2(t) \le E_2(0)e^{p_1(M)T^*} := M_2 < \infty;\\ &\sup_{t\in[0,T^*)} E_3(t) \le (E_3(0) + p_3(M,M_2)T^*)e^{p_2(M,M_2)T^*} := M_3 < \infty, \end{aligned} \tag{4.169}$$

By (4.135), (4.163),

$$\sup_{[0,T^*)} \|Z_{tt}(t)\|_{H^3} \lesssim \sup_{[0,T^*)} (\|\partial_{\alpha'}^3 Z_{tt}(t)\|_{L^2} + \|Z_{tt}(t)\|_{L^2}) < \infty. \tag{4.170}$$

eq:253

Now by (A.6),

$$\|\partial_{\alpha'}^3\overline{Z}_t\|_{\dot H^{1/2}} \lesssim \|Z_{,\alpha'}\|_{L^\infty}(\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t\|_{\dot H^{1/2}} + \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2}).$$

We know by (4.116) and Appendix C,

$$\|\partial_{\alpha'}^3\overline{Z}_t\|_{L^2},\quad \|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\overline{Z}_t\|_{\dot H^{1/2}} \le C(\mathfrak{E})E_3, \qquad \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2} \le C(\mathfrak{E});$$

so using (4.168) we have

$$\sup_{[0,T^*)} \|\partial_{\alpha'}^3 \overline{Z}_t\|_{\dot{H}^{1/2}} \le \|Z_{,\alpha'}(0)\|_{L^\infty}^2 e^{C(M)T^*} C(M) M_3 < \infty \tag{4.171}$$

Combine with (4.167), we have

$$\sup_{[0,T^*)} \|Z_t(t)\|_{H^{3+1/2}} < \infty. \tag{4.172}$$

By Proposition 2.3 this brings us a contradiction. This finishes the proof for Theorem 3.1. □

proof2

5. The proof of Theorem 3.4

We prove Theorem 3.4 by mollifying the initial data by the Poisson Kernel and approximating. We denote $z' = x' + iy'$, where $x', y' \in \mathbb{R}$. $f * g$ is the convolution in the spatial variable.

ID

5.1. **The initial data.**

Let $F(z', 0)$ be the initial fluid velocity in the Riemann mapping coordinate, $\Psi(z', 0) : P_- \to \Omega(0)$ be the Riemann mapping as given in §3.1 with $Z(\alpha', 0) = \Psi(\alpha', 0)$ the initial interface. We note that by the assumption

$$\sup_{y'<0} \|\partial_{z'}(\frac{1}{\Psi_{z'}(z',0)})\|_{L^2(\mathbb{R},dx')} \le \mathcal{E}_1(0) < \infty, \quad \sup_{y'<0} \|\frac{1}{\Psi_{z'}(z',0)} - 1\|_{L^2(\mathbb{R},dx')} \le c_0 < \infty;$$
$$\sup_{y'<0} \|F_{z'}(z',0)\|_{L^2(\mathbb{R},dx')} \le \mathcal{E}_1(0) < \infty, \quad \sup_{y'<0} \|F(z',0)\|_{L^2(\mathbb{R},dx')} \le c_0 < \infty,$$

$\frac{1}{\Psi_{z'}}(\cdot, 0)$, $F(\cdot, 0)$ can be extended continuously onto $\overline{P}_-$. We denote their boundary values by $\frac{1}{\Psi_{z'}}(\alpha', 0)$ and $F(\alpha', 0)$. So $Z(\cdot, 0) = \Psi(\cdot, 0)$ is continuous differentiable on the open set where $\frac{1}{\Psi_{z'}}(\alpha', 0) \neq 0$, and $\frac{1}{\Psi_{z'}}(\alpha', 0) = \frac{1}{Z_{,\alpha'}(\alpha',0)}$ where $\frac{1}{\Psi_{z'}}(\alpha', 0) \neq 0$. By $\frac{1}{\Psi_{z'}}(\cdot, 0) - 1 \in H^1(\mathbb{R})$ and Sobolev embedding, there is $N > 0$ sufficiently large, such that for $|\alpha'| \ge N$, $|\frac{1}{\Psi_{z'}}(\alpha', 0) - 1| \le 1/2$, so $Z = Z(\cdot, 0)$ is continuous differentiable on $(-\infty, -N) \cup (N, \infty)$, with $|Z_{,\alpha'}(\alpha', 0)| \le 2$, for all $|\alpha'| \ge N$. Moreover, $Z_{,\alpha'}(\cdot, 0) - 1 \in H^1\{(-\infty, -N) \cup (N, \infty)\}$.

mo-ap

5.2. **The mollified data and the approximate solutions.**

Let $\epsilon > 0$. We take

$$\begin{aligned} Z^\epsilon(\alpha', 0) = \Psi(\alpha' - \epsilon i, 0), \quad \overline{Z}_t^\epsilon(\alpha', 0) = F(\alpha' - \epsilon i, 0), \quad h^\epsilon(\alpha, 0) = \alpha, \\ F^\epsilon(z', 0) = F(z' - \epsilon i, 0), \quad \Psi^\epsilon(z', 0) = \Psi(z' - \epsilon i, 0). \end{aligned} \tag{5.1}$$ m-id

Notice that $F^\epsilon(\cdot, 0)$, $\Psi^\epsilon(\cdot, 0)$ are holomorphic on P_-, $Z^\epsilon(0)$ satisfies (2.8) and $\overline{Z}_t^\epsilon(0) = \mathbb{H}\overline{Z}_t^\epsilon(0)$. Let $Z_{tt}^\epsilon(0)$ be given by (2.10). It is clear $Z^\epsilon(0)$, $Z_t^\epsilon(0)$ and $Z_{tt}^\epsilon(0)$ satisfy the assumption of Theorem 3.1. Let $Z^\epsilon(t) := Z^\epsilon(\cdot, t)$ be the solution as given by Theorem 3.1, with the homeomorphism $h^\epsilon(t) = h^\epsilon(\cdot, t) : \mathbb{R} \to \mathbb{R}$, and $z^\epsilon(\alpha, t) = Z^\epsilon(h^\epsilon(\alpha, t), t)$. We know $z_t^\epsilon(\alpha, t) = Z_t^\epsilon(h^\epsilon(\alpha, t), t)$. Let

$$F^\epsilon(x' + iy', t) = K_{y'} * \overline{Z}_t^\epsilon(x', t), \quad \Psi_{z'}^\epsilon(x' + iy', t) = K_{y'} * Z_{,\alpha'}^\epsilon(x', t), \quad \Psi^\epsilon(\cdot, t)$$

be the holomorphic functions on P_- with boundary values $\overline{Z}_t^\epsilon(t)$, $Z_{,\alpha'}^\epsilon(t)$ and $Z^\epsilon(t)$;

$$\frac{1}{\Psi_{z'}^\epsilon}(x' + iy', t) = K_{y'} * \frac{1}{Z_{,\alpha'}^\epsilon}(x', t)$$

by uniqueness.[21] We denote the energy functional $\mathcal{E}$ for $Z^\epsilon(t)$, $\overline{Z}_t^\epsilon(t)$ by $\mathcal{E}^\epsilon(t)$ and the energy functional $\mathcal{E}_1$ for $F^\epsilon(t)$, $\Psi^\epsilon(t)$ by $\mathcal{E}_1^\epsilon(t)$. It is clear $\mathcal{E}^\epsilon(0) = \mathcal{E}_1^\epsilon(0) \le \mathcal{E}_1(0)$. By Theorem 3.1, Theorem 2.4 and Proposition 2.5, there exists $T_0 > 0$, T_0 depends only on $\mathcal{E}_1(0)$, such that

[21]By the maximum principle, $(K_{y'} * \frac{1}{Z_{,\alpha'}^\epsilon})(K_{y'} * Z_{,\alpha'}^\epsilon) \equiv 1$ on P_-.

on $[0, T_0]$, the system (2.9)-(2.8)-(2.18)-(2.19) has a unique solution $Z^\epsilon = Z^\epsilon(\cdot, t)$, satisfying $(Z_t^\epsilon, Z_{tt}^\epsilon, \frac{1}{Z_{,\alpha'}^\epsilon} - 1) \in C([0, T_0], H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ for $s > 4$, and

$$\sup_{[0,T_0]} \mathcal{E}_1^\epsilon(t) = \sup_{[0,T_0]} \mathcal{E}^\epsilon(t) \le M(\mathcal{E}_1(0)) < \infty. \tag{5.2}$$

eq:400

Moreover by (2.10), (4.163) and (4.167),

$$\sup_{[0,T_0]} (\|Z_t^\epsilon(t)\|_{L^2} + \|Z_{tt}^\epsilon(t)\|_{L^2} + \|\frac{1}{Z_{,\alpha'}^\epsilon(t)} - 1\|_{L^2}) \le c(c_0, \mathcal{E}_1(0)), \tag{5.3}$$

eq:401

so there is a constant $C_0 := C(c_0, \mathcal{E}_1(0)) > 0$, such that

$$\sup_{[0,T_0]} \{\sup_{y'<0} \|F^\epsilon(x' + iy', t)\|_{L^2(\mathbb{R}, dx')} + \sup_{y'<0} \|\frac{1}{\Psi_{z'}^\epsilon(x' + iy', t)} - 1\|_{L^2(\mathbb{R}, dx')}\} < C_0 < \infty. \tag{5.4}$$

eq:402

ubound

5.3. Uniformly bounded quantities. We would like to apply some compactness results to pass to the limits of the various quantities for the water waves. It is necessary to understand the boundedness properties of these quantities.

Let $b^\epsilon := h_t^\epsilon \circ (h^\epsilon)^{-1} = 2 \operatorname{Re} Z_t^\epsilon + \operatorname{Re}[Z_t^\epsilon, \mathbb{H}](\frac{1}{Z_{,\alpha'}^\epsilon} - 1)$ be as given by (2.18). By (A.18),

$$\|b^\epsilon(t)\|_{L^\infty} = \|h_t^\epsilon(t)\|_{L^\infty} \lesssim \|Z_t^\epsilon(t)\|_{L^\infty} + \|Z_{t,\alpha'}^\epsilon(t)\|_{L^2} \|\frac{1}{Z_{,\alpha'}^\epsilon}(t) - 1\|_{L^2}. \tag{5.5}$$

eq:408

Using (4.2) to rewrite $b^\epsilon = \operatorname{Re}(I - \mathbb{H})(Z_t^\epsilon \frac{1}{Z_{,\alpha'}^\epsilon})$, differentiating to get

$$\|b_{\alpha'}^\epsilon(t)\|_{L^2} \lesssim \|Z_{t,\alpha'}^\epsilon(t)\|_{L^2} \|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^\infty} + \|Z_t^\epsilon(t)\|_{L^\infty} \|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^2}. \tag{5.6}$$

eq:412

We know h^ϵ satisfies

$$\begin{cases} \frac{d}{dt} h^\epsilon = b^\epsilon(h^\epsilon, t); \\ h^\epsilon(\alpha, 0) = \alpha. \end{cases} \tag{5.7}$$

eq:405

Differentiating (5.7) gives

$$\begin{cases} \frac{d}{dt} h_\alpha^\epsilon = b_{\alpha'}^\epsilon(h^\epsilon, t) h_\alpha^\epsilon; \\ h_\alpha^\epsilon(\alpha, 0) = 1 \end{cases} \tag{5.8}$$

eq:406

therefore

$$e^{-t \sup_{[0,t]} \|b_{\alpha'}^\epsilon(s)\|_{L^\infty}} \le h_\alpha^\epsilon(\alpha, t) = e^{\int_0^t b_{\alpha'}^\epsilon(h^\epsilon, s)\, ds} \le e^{t \sup_{[0,t]} \|b_{\alpha'}^\epsilon(s)\|_{L^\infty}}. \tag{5.9}$$

eq:407

Now by (2.18), (B.24), with an application of (A.18) and (A.17),

$$\begin{aligned} &\|U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} b^\epsilon(t)\|_{L^\infty} \lesssim \|Z_{tt}^\epsilon(t)\|_{L^\infty} + \|Z_{tt,\alpha'}^\epsilon(t)\|_{L^2} \|\frac{1}{Z_{,\alpha'}^\epsilon}(t) - 1\|_{L^2} \\ &+ \|Z_{t,\alpha'}^\epsilon(t)\|_{L^2} (\|U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} \frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^2} + \|b_{\alpha'}^\epsilon(t)\|_{L^\infty} \|\frac{1}{Z_{,\alpha'}^\epsilon}(t) - 1\|_{L^2}); \end{aligned} \tag{5.10}$$

eq:409

where $U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} \frac{1}{Z_{,\alpha'}^\epsilon} = \frac{1}{Z_{,\alpha'}^\epsilon}((h_t^\epsilon \circ (h^\epsilon)^{-1})_{\alpha'} - D_{\alpha'} Z_t^\epsilon)$ gives

$$\begin{aligned} &\|U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} \frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^2} \le \|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^\infty} (\|b_{\alpha'}^\epsilon(t)\|_{L^2} + \|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^\infty} \|Z_{t,\alpha'}^\epsilon(t)\|_{L^2}) \\ &\|U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} \frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^\infty} \le \|\frac{1}{Z_{,\alpha'}^\epsilon}(t)\|_{L^\infty} (\|b_{\alpha'}^\epsilon(t)\|_{L^\infty} + \|D_{\alpha'} Z_t^\epsilon(t)\|_{L^\infty}) \end{aligned} \tag{5.11}$$

eq:411

and $U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} = \partial_t + b^\epsilon \partial_{\alpha'}$ gives

$$\|\partial_t b^\epsilon(t)\|_{L^\infty} \le \|U_{h^\epsilon}^{-1} \partial_t U_{h^\epsilon} b^\epsilon(t)\|_{L^\infty} + \|b^\epsilon(t)\|_{L^\infty} \|b_{\alpha'}^\epsilon(t)\|_{L^\infty}. \tag{5.12}$$

eq:410

Finally, differentiating (4.3) gives $z^\epsilon_{ttt}=(z^\epsilon_{tt}+i)(D_\alpha z^\epsilon_t+\frac{\mathfrak{a}^\epsilon_t}{\mathfrak{a}^\epsilon})$, so

$$\|z^\epsilon_{ttt}(t)\|_{L^\infty}\le\|z^\epsilon_{tt}(t)+i\|_{L^\infty}(\|D_\alpha z^\epsilon_t(t)\|_{L^\infty}+\|\frac{\mathfrak{a}^\epsilon_t}{\mathfrak{a}^\epsilon}(t)\|_{L^\infty}). \tag{5.13}$$ eq:4400

Let $M(\mathcal{E}_1(0))$, $c(c_0,\mathcal{E}_1(0))$, C_0 be the bounds in (5.2), (5.3) and (5.4). By Proposition 2.5, Sobolev embedding, Appendix C and (5.11), the following quantities are uniformly bounded with bounds depending only on $M(\mathcal{E}_1(0))$, $c(c_0,\mathcal{E}_1(0))$, C_0:

$$\begin{aligned}&\sup_{[0,T_0]}\|Z^\epsilon_t(t)\|_{L^\infty},\ \sup_{[0,T_0]}\|Z^\epsilon_{t,\alpha'}(t)\|_{L^2},\ \sup_{[0,T_0]}\|Z^\epsilon_{tt}(t)\|_{L^\infty},\ \sup_{[0,T_0]}\|Z^\epsilon_{tt,\alpha'}(t)\|_{L^2},\\&\sup_{[0,T_0]}\|\frac{1}{Z^\epsilon_{,\alpha'}}(t)\|_{L^\infty},\ \sup_{[0,T_0]}\|\partial_{\alpha'}(\frac{1}{Z^\epsilon_{,\alpha'}})(t)\|_{L^2},\ \sup_{[0,T_0]}\|U^{-1}_{h^\epsilon}\partial_tU_{h^\epsilon}\frac{1}{Z^\epsilon_{,\alpha'}}(t)\|_{L^\infty};\end{aligned}\tag{5.14}$$ eq:404

and with a change of the variables and (5.9), (5.13) and Appendix C,

$$\begin{aligned}&\sup_{[0,T_0]}\|z^\epsilon_t(t)\|_{L^\infty}+\sup_{[0,T_0]}\|z^\epsilon_{t\alpha}(t)\|_{L^2}+\sup_{[0,T_0]}\|z^\epsilon_{tt}(t)\|_{L^\infty}\le C(c_0,\mathcal{E}_1(0)),\\&\sup_{[0,T_0]}\|\frac{h^\epsilon_\alpha}{z^\epsilon_\alpha}(t)\|_{L^\infty}+\sup_{[0,T_0]}\|\partial_\alpha(\frac{h^\epsilon_\alpha}{z^\epsilon_\alpha})(t)\|_{L^2}+\sup_{[0,T_0]}\|\partial_t\frac{h^\epsilon_\alpha}{z^\epsilon_\alpha}(t)\|_{L^\infty}\le C(c_0,\mathcal{E}_1(0)),\\&\sup_{[0,T_0]}\|z^\epsilon_{tt}(t)\|_{L^\infty}+\sup_{[0,T_0]}\|z^\epsilon_{tt\alpha}(t)\|_{L^2}+\sup_{[0,T_0]}\|z^\epsilon_{ttt}(t)\|_{L^\infty}\le C(c_0,\mathcal{E}_1(0)).\end{aligned}\tag{5.15}$$ eq:414

Furthermore, by the estimates in (5.5)–(5.12), using (5.3) (5.14) and Appendix C, the following quantities are uniformly bounded:

$$\begin{aligned}&\sup_{[0,T_0]}\|b^\epsilon(t)\|_{L^\infty}+\sup_{[0,T_0]}\|b^\epsilon_{\alpha'}(t)\|_{L^\infty}+\sup_{[0,T_0]}\|b^\epsilon_t(t)\|_{L^\infty}\le C(c_0,\mathcal{E}_1(0))\\&\sup_{[0,T_0]}\|h^\epsilon_\alpha(t)\|_{L^\infty}+\sup_{[0,T_0]}\|h^\epsilon_t(t)\|_{L^\infty}\le C(c_0,\mathcal{E}_1(0))\end{aligned}\tag{5.16}$$ eq:415

In particular, by (5.9) and Appendix C, there are $c_1,c_2>0$, depending only on c_0 and $\mathcal{E}_1(0)$, such that

$$0<c_1\le\frac{h^\epsilon(\alpha,t)-h^\epsilon(\beta,t)}{\alpha-\beta}\le c_2<\infty,\qquad\forall\alpha,\beta\in\mathbb{R},\ t\in[0,T_0].\tag{5.17}$$ eq:416

5.4. **Some useful compactness results.** Here we give two compactness results that we will use to pass to the limits.

lemma1 **Lemma 5.1.** *Let $\{f_n\}$ be a sequence of smooth functions on $\mathbb{R}\times[0,T]$. Let $1<p\le\infty$. Assume that there is a constant C, independent of n, such that*

$$\sup_{[0,T]}\|f_n(t)\|_{L^\infty}+\sup_{[0,T]}\|\partial_xf_n(t)\|_{L^p}+\sup_{[0,T]}\|\partial_tf_n(t)\|_{L^\infty}\le C.\tag{5.18}$$

Then there is a function f, continuous and bounded on $\mathbb{R}\times[0,T]$, and a subsequence $\{f_{n_j}\}$, such that $f_{n_j}\to f$ uniformly on compact subsets of $\mathbb{R}\times[0,T]$.

Lemma 5.1 is an easy consequence of Arzela-Ascoli Theorem, we omit the proof.

lemma2 **Lemma 5.2.** *Assume that $f_n\to f$ uniformly on compact subsets of $\mathbb{R}\times[0,T]$, and assume there is a constant C, such that $\sup_n\|f_n\|_{L^\infty(\mathbb{R}\times[0,T])}\le C$. Then $K_{y'}*f_n$ converges uniformly to $K_{y'}*f$ on compact subsets of $\overline{P}_-\times[0,T]$.*

The proof follows easily by considering the convolution on two sets $|x'|<N$, and $|x'|\ge N$. We omit the proof.

Definition 5.3. We write

$$f_n\Rightarrow f\qquad\text{on } E\tag{5.19}$$ unif-notation

if f_n converge uniformly to f on compact subsets of E.

5.5. **Passing to the limit.** Notice that $h^\epsilon(\alpha,t)-\alpha=\int_0^t h_t^\epsilon(\alpha,s)\,ds$, so

$$\sup_{\mathbb{R}\times[0,T_0]}|h^\epsilon(\alpha,t)-\alpha|\le T_0\sup_{[0,T_0]}\|h_t^\epsilon(t)\|_{L^\infty}\le T_0C(c_0,\mathcal{E}_1(0))<\infty. \tag{5.20}$$ eq:425

By Lemma 5.1, there is a subsequence $\epsilon_j\to 0$, which we still write as ϵ instead of ϵ_j, and functions b, $h-\alpha$, w, u, $q:=w_t$, continuous and bounded on $\mathbb{R}\times[0,T_0]$, such that

$$b^\epsilon\Rightarrow b,\quad h^\epsilon\Rightarrow h,\quad z_t^\epsilon\Rightarrow w,\quad \frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}\Rightarrow u,\quad z_{tt}^\epsilon\Rightarrow q,\qquad \text{on } \mathbb{R}\times[0,T_0], \tag{5.21}$$ eq:417

as $\epsilon=\epsilon_j\to 0$. Moreover by (5.17),

$$0<c_1\le\frac{h(\alpha,t)-h(\beta,t)}{\alpha-\beta}\le c_2<\infty,\qquad \forall\alpha,\beta\in\mathbb{R},\ t\in[0,T_0]; \tag{5.22}$$ eq:420

hence $h(\cdot,t):\mathbb{R}\to\mathbb{R}$ is a homeomorphism, and

$$(h^\epsilon)^{-1}\Rightarrow h^{-1}\qquad \text{on } \mathbb{R}\times[0,T_0],\quad \text{as } \epsilon=\epsilon_j\to 0. \tag{5.23}$$ eq:418

This gives

$$\overline{Z}_t^\epsilon\Rightarrow w\circ h^{-1},\qquad \frac{1}{Z_{,\alpha'}^\epsilon}\Rightarrow u\circ h^{-1},\quad \overline{Z}_{tt}^\epsilon\Rightarrow w_t\circ h^{-1},\qquad \text{on } \mathbb{R}\times[0,T_0] \tag{5.24}$$ eq:419

as $\epsilon=\epsilon_j\to 0$. Now

$$F^\epsilon(z',t)=K_{y'}*\overline{Z}_t^\epsilon,\qquad \frac{1}{\Psi_{z'}^\epsilon}(z',t)=K_{y'}*\frac{1}{Z_{,\alpha'}^\epsilon}. \tag{5.25}$$ eq:421

Let $F(z',t)=K_{y'}*(w\circ h^{-1})(x',t)$, $\Lambda(z',t)=K_{y'}*(u\circ h^{-1})(x',t)$. By Lemma 5.2,

$$F^\epsilon(z',t)\Rightarrow F(z',t),\qquad \frac{1}{\Psi_{z'}^\epsilon}(z',t)\Rightarrow\Lambda(z',t)\qquad \text{on } \overline{P}_-\times[0,T_0]; \tag{5.26}$$ eq:422

as $\epsilon=\epsilon_j\to 0$. Moreover $F(\cdot,t)$, $\Lambda(\cdot,t)$ are holomorphic on P_- for each $t\in[0,T_0]$, and continuous on $\overline{P}_-\times[0,T]$. Furthermore applying the Cauchy integral formula to the first limit in (5.26) yields

$$F_{z'}^\epsilon(z',t)\Rightarrow F_{z'}(z',t)\qquad \text{on } P_-\times[0,T_0]. \tag{5.27}$$ eq:430

as $\epsilon=\epsilon_j\to 0$.

step4.1

Step 1. The limit of Ψ^ϵ. We consider the limit of Ψ^ϵ, as $\epsilon=\epsilon_j\to 0$. We know

$$\begin{aligned} z^\epsilon(\alpha,t)&=z^\epsilon(\alpha,0)+\int_0^t z_t^\epsilon(\alpha,s)\,ds\\ &=\Psi(\alpha-\epsilon i,0)+\int_0^t z_t^\epsilon(\alpha,s)\,ds, \end{aligned} \tag{5.28}$$ eq:423

therefore

$$\begin{aligned} Z^\epsilon(\alpha',t)-Z^\epsilon(\alpha',0)&=\Psi((h^\epsilon)^{-1}(\alpha',t)-\epsilon i,0)-\Psi(\alpha'-\epsilon i,0)\\ &\quad+\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha',t),s)\,ds. \end{aligned} \tag{5.29}$$ eq:424

Let

$$W^\epsilon(\alpha',t):=\Psi((h^\epsilon)^{-1}(\alpha',t)-\epsilon i,0)-\Psi(\alpha'-\epsilon i,0)+\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha',t),s)\,ds. \tag{5.30}$$ eq:431

Observe $Z^\epsilon(\alpha',t)-Z^\epsilon(\alpha',0)$ is the boundary value of the holomorphic function $\Psi^\epsilon(z',t)-\Psi^\epsilon(z',0)$. By (5.21) and (5.23), $\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha',t),s)\,ds\to\int_0^t w(h^{-1}(\alpha',t),s)\,ds$ uniformly on compact subsets of $\mathbb{R}\times[0,T_0]$, and by (5.15), $\int_0^t z_t^\epsilon((h^\epsilon)^{-1}(\alpha',t),s)\,ds$ is continuous and uniformly bounded in $L^\infty(\mathbb{R}\times[0,T_0])$. By the assumptions $\lim_{z'\to 0}\Psi_{z'}(z',0)=1$, $\Psi(\cdot,0)$ is continuous on $\overline{P}_-$ and (5.20), (5.23),

$$\Psi((h^\epsilon)^{-1}(\alpha',t)-\epsilon i,0)-\Psi(\alpha'-\epsilon i,0)$$

is continuous and uniformly bounded in $L^\infty(\mathbb{R}\times[0,T_0])$ for $0<\epsilon<1$, and converges uniformly on compact subsets of $\mathbb{R}\times[0,T_0]$, as $\epsilon=\epsilon_j\to 0$. This gives[22]

$$\Psi^\epsilon(z',t)-\Psi^\epsilon(z',0)=K_{y'}*W^\epsilon(x',t) \tag{5.31}$$ eq:426

and by Lemma 5.2, $\Psi^\epsilon(z',t)-\Psi^\epsilon(z',0)$ converges uniformly on compact subsets of $\overline{P}_-\times[0,T_0]$ to a function that is holomorphic on P_- for every $t\in[0,T_0]$ and continuous on $\overline{P}_-\times[0,T_0]$. Therefore there is a function $\Psi(\cdot,t)$, holomorphic on P_- for every $t\in[0,T_0]$ and continuous on $\overline{P}_-\times[0,T_0]$, such that

$$\Psi^\epsilon(z',t)\rightrightarrows\Psi(z',t)\qquad\text{on }\overline{P}_-\times[0,T_0] \tag{5.32}$$ eq:427

as $\epsilon=\epsilon_j\to 0$; as a consequence of the Cauchy integral formula,

$$\Psi^\epsilon_{z'}(z',t)\rightrightarrows\Psi_{z'}(z',t)\qquad\text{on }P_-\times[0,T_0] \tag{5.33}$$ eq:428

as $\epsilon=\epsilon_j\to 0$. Combining with (5.26), we have $\Lambda(z',t)=\frac{1}{\Psi_{z'}(z',t)}$, so $\Psi_{z'}(z',t)\neq 0$ for all $(z',t)\in P_-\times[0,T_0]$ and

$$\frac{1}{\Psi^\epsilon_{z'}(z',t)}\rightrightarrows\frac{1}{\Psi_{z'}(z',t)}\qquad\text{on }\overline{P}_-\times[0,T_0] \tag{5.34}$$ eq:429

as $\epsilon=\epsilon_j\to 0$.

Denote $Z(\alpha',t):=\Psi(\alpha',t)$, $\alpha'\in\mathbb{R}$, and $z(\alpha,t)=Z(h(\alpha,t),t)$. (5.32) gives $Z^\epsilon(\alpha',t)\rightrightarrows Z(\alpha',t)$, and with (5.21) it gives $z^\epsilon(\alpha,t)\rightrightarrows z(\alpha,t)$ on $\mathbb{R}\times[0,T_0]$, as $\epsilon=\epsilon_j\to 0$. Moreover by (5.28),

$$z(\alpha',t)=z(\alpha',0)+\int_0^t w(\alpha,s)\,ds,$$

so $w=z_t$. We denote $Z_t=z_t\circ h^{-1}$.

Step 2. The limits of Ψ^ϵ_t *and* F^ϵ_t. Observe that by (5.30), for fixed $\epsilon>0$, $\partial_t W^\epsilon(\cdot,t)$ is a bounded function on $\mathbb{R}\times[0,T_0]$, so by (5.31), $\Psi^\epsilon_t=K_{y'}*\partial_t W^\epsilon$ is bounded on $P_-\times[0,T_0]$. However we will not use this to pass to the limit for Ψ^ϵ_t, instead, we use (B.4).

By (B.4) and the above observation, since $\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}}$ is bounded and holomorphic on P_-,

$$\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}}=K_{y'}*\Big(\frac{Z^\epsilon_t}{Z^\epsilon_{,\alpha'}}-b^\epsilon\Big). \tag{5.35}$$ eq:432

By (5.21), (5.24) and Lemma 5.2, $\frac{\Psi^\epsilon_t}{\Psi^\epsilon_{z'}}$ converges uniformly on compact subsets of $\overline{P}_-\times[0,T_0]$ to a function that is holomorphic on P_- for each $t\in[0,T_0]$ and continuous on $\overline{P}_-\times[0,T_0]$. By (5.32), (5.33), we can conclude that Ψ is continuously differentiable and

$$\Psi^\epsilon_t\rightrightarrows\Psi_t\qquad\text{on }P_-\times[0,T_0] \tag{5.36}$$ eq:433

as $\epsilon=\epsilon_j\to 0$.

Now we consider the limit of F^ϵ_t as $\epsilon=\epsilon_j\to 0$. Since for fixed $\epsilon>0$, $\partial_t Z^\epsilon_t=Z^\epsilon_{tt}-b^\epsilon Z^\epsilon_{t,\alpha'}$ is in $L^\infty(\mathbb{R}\times[0,T_0])$, by (5.25),

$$F^\epsilon_t(z',t)=K_{y'}*\partial_t\overline{Z}^\epsilon_t=K_{y'}*(\overline{Z}^\epsilon_{tt}-b^\epsilon\overline{Z}^\epsilon_{t,\alpha'}). \tag{5.37}$$ eq:434

By Lemma 5.2, $K_{y'}*\overline{Z}^\epsilon_{tt}$ converges uniformly on compact subsets of $\overline{P}_-\times[0,T_0]$. With a change of variables

$$K_{y'}*(b^\epsilon\overline{Z}^\epsilon_{t,\alpha'})=\frac{-1}{\pi}\int\frac{y'}{(x'-h^\epsilon(\alpha,t))^2+y'^2}b^\epsilon\circ h^\epsilon(\alpha,t)\overline{z}^\epsilon_{t\alpha}(\alpha,t)\,d\alpha. \tag{5.38}$$ eq:436

Because (5.21): $z^\epsilon_t\to z_t$, $z^\epsilon_{tt}\to z_{tt}$ uniform on compact subsets of $\mathbb{R}\times[0,T_0]$, and (5.14): $\sup_{[0,T_0]}\|\overline{z}^\epsilon_{t\alpha}(t)\|_{L^2}\le C(c_0,\mathcal{E}_1(0))$, $\sup_{[0,T_0]}\|\overline{z}^\epsilon_{tt\alpha}(t)\|_{L^2}\le C(c_0,\mathcal{E}_1(0))$, $\overline{z}_{t\alpha}$, $\overline{z}_{tt\alpha}$ exist in

[22]Because $W^\epsilon(\cdot,t)$ and $\partial_{\alpha'}W^\epsilon(\cdot,t):=Z^\epsilon_{,\alpha'}(\alpha',t)-Z^\epsilon_{,\alpha'}(\alpha',0)$ are continuous and bounded on $\mathbb{R}$, $\Psi^\epsilon_{z'}(z',t)-\Psi^\epsilon_{z'}(z',0)=K_{y'}*(\partial_{\alpha'}W^\epsilon)(x',t)=\partial_{z'}K_{y'}*W^\epsilon(x',t)$.

$L^2(\mathbb{R})$ for each $t \in [0, T_0]$, with $\sup_{[0,T_0]} \|\overline{z}_{t\alpha}(t)\|_{L^2} \le C(c_0, \mathcal{E}_1(0))$, $\sup_{[0,T_0]} \|\overline{z}_{tt\alpha}(t)\|_{L^2} \le C(c_0, \mathcal{E}_1(0))$, and by (5.21), (5.20) and $K_{y'} * (b^\epsilon \overline{Z}^\epsilon_{t,\alpha'})$ converges point-wise on $P_- \times [0, T_0]$ to the continuous function

$$\frac{-1}{\pi} \int \frac{y'}{(x' - h(\alpha,t))^2 + y'^2} b \circ h(\alpha,t) \overline{z}_{t\alpha}(\alpha,t)\, d\alpha$$

as $\epsilon = \epsilon_j \to 0$ and by (5.14), (5.16),

$$\sup_{[0,T_0]} \|F_t^\epsilon(z',t)\|_{L^\infty(\mathbb{R},dx')} \le \Big(1 + \frac{1}{|y'|^{1/2}}\Big) C(c_0, \mathcal{E}_1(0)).$$

Therefore F is continuously differentiable with respect to t, with $\sup_{[0,T_0]} \|F_t(z',t)\|_{L^\infty(\mathbb{R},dx')} \le (1 + \frac{1}{|y'|^{1/2}}) C(c_0, \mathcal{E}_1(0))$ and

$$F_t^\epsilon(z',t) \to F_t(z',t), \qquad \text{as } \epsilon = \epsilon_j \to 0 \tag{5.39}$$ eq:435

point-wise on $P_- \times [0, T_0]$.

Step 3. The limit of $\mathfrak{P}^\epsilon$. By the calculation in §2.3, we know $Z^\epsilon_{,\alpha}(\overline{Z}^\epsilon_{tt} - i)$ is the boundary value of the function $\Psi^\epsilon_{z'} F^\epsilon_t - \Psi^\epsilon_t F^\epsilon_{z'} + \overline{F}^\epsilon F^\epsilon_{z'} - i\Psi^\epsilon_{z'}$ on ∂P_-. Since $\Psi^\epsilon_{z'} F^\epsilon_t - \Psi^\epsilon_t F^\epsilon_{z'} - i\Psi^\epsilon_{z'}$ is holomorphic and $\overline{F}^\epsilon F^\epsilon_{z'} = \partial_{z'}(\overline{F}^\epsilon F^\epsilon)$, where $\partial_{z'} = \frac{1}{2}(\partial_{x'} - i\partial_{y'})$, there is a real valued function $\mathfrak{P}^\epsilon$, such that

$$\Psi^\epsilon_{z'} F^\epsilon_t - \Psi^\epsilon_t F^\epsilon_{z'} + \overline{F}^\epsilon F^\epsilon_{z'} - i\Psi^\epsilon_{z'} = -(\partial_{x'} - i\partial_{y'})\mathfrak{P}^\epsilon, \qquad \text{in } P_-; \tag{5.40}$$ eq:437

and by $Z^\epsilon_{,\alpha}(\overline{Z}^\epsilon_{tt} - i) = iA^\epsilon_1$, which is pure imaginary, we know

$$\mathfrak{P}^\epsilon = \mathit{constant}, \qquad \text{on } \partial P_-. \tag{5.41}$$ eq:438

Without loss of generality we take the *constant* $= 0$. We now explore a few other properties of $\mathfrak{P}^\epsilon$. Moving $\overline{F}^\epsilon F^\epsilon_{z'} = \partial_{z'}(\overline{F}^\epsilon F^\epsilon)$ to the right of (5.40) gives

$$\Psi^\epsilon_{z'} F^\epsilon_t - \Psi^\epsilon_t F^\epsilon_{z'} - i\Psi^\epsilon_{z'} = -(\partial_{x'} - i\partial_{y'})(\mathfrak{P}^\epsilon + \frac{1}{2}|F^\epsilon|^2), \qquad \text{in } P_-; \tag{5.42}$$ eq:440

Applying $(\partial_{x'} + i\partial_{y'}) = 2\overline{\partial}_{z'}$ to (5.42) yields

$$-\Delta(\mathfrak{P}^\epsilon + \frac{1}{2}|F^\epsilon|^2) = 0, \qquad \text{in } P_-. \tag{5.43}$$ eq:439

So $\mathfrak{P}^\epsilon + \frac{1}{2}|F^\epsilon|^2$ is a harmonic function on P_- with boundary value $\frac{1}{2}|\overline{Z}^\epsilon_t|^2$. On the other hand, it is easy to check that $\lim_{y' \to -\infty}(\Psi^\epsilon_{z'} F^\epsilon_t - \Psi^\epsilon_t F^\epsilon_{z'} - i\Psi^\epsilon_{z'}) = -i$. Therefore

$$\mathfrak{P}^\epsilon(z',t) = -\frac{1}{2}|F^\epsilon(z',t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}^\epsilon_t|^2)(x',t). \tag{5.44}$$ eq:441

By (5.26), (5.24) and Lemma 5.2,

$$\mathfrak{P}^\epsilon(z',t) \Rightarrow -\frac{1}{2}|F(z',t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}_t|^2)(x',t), \qquad \text{on } \overline{P}_- \times [0, T_0] \tag{5.45}$$ eq:442

as $\epsilon = \epsilon_j \to 0$. We write

$$\mathfrak{P} := -\frac{1}{2}|F(z',t)|^2 - y + \frac{1}{2}K_{y'} * (|\overline{Z}_t|^2)(x',t).$$

$\mathfrak{P}$ is continuous on $\overline{P}_- \times [0, T_0]$ with $\mathfrak{P} \in C([0, T_0], C^\infty(P_-))$, and

$$\mathfrak{P} = 0, \qquad \text{on } \partial P_-. \tag{5.46}$$ eq:443

Moreover, since $K_{y'} * (|\overline{Z}^\epsilon_t|^2)(x',t)$ is harmonic on P_-, by interior derivative estimate for harmonic functions and by (5.26),

$$(\partial_{x'} - i\partial_{y'})\mathfrak{P}^\epsilon \Rightarrow (\partial_{x'} - i\partial_{y'})\mathfrak{P} \qquad \text{on } P_- \times [0, T_0] \tag{5.47}$$ eq:4450

as $\epsilon = \epsilon_j \to 0$.

Step 4. Conclusion. We now sum up Steps 1-3. We have shown that there are functions $\Psi(\cdot,t)$ and $F(\cdot,t)$, holomorphic on P_- for each fixed $t\in[0,T_0]$, continuous on $\overline{P}_-\times[0,T_0]$, and continuous differentiable on $P_-\times[0,T_0]$, with $\frac{1}{\Psi_{z'}}$ continuous on $\overline{P}_-\times[0,T_0]$, such that $\Psi^\epsilon\to\Psi$, $\frac{1}{\Psi^\epsilon_{z'}}\to\frac{1}{\Psi_{z'}}$, $F^\epsilon\to F$ uniform on compact subsets of $\overline{P}_-\times[0,T_0]$, $\Psi^\epsilon_t\to\Psi_t$, $\Psi^\epsilon_{z'}\to\Psi_{z'}$, $F^\epsilon_{z'}\to F_{z'}$ uniform on compact subsets of $P_-\times[0,T_0]$, and $F^\epsilon_t\to F_t$ pointwise on $P_-\times[0,T_0]$, as $\epsilon=\epsilon_j\to0$. We have also shown there is $\mathfrak{P}$, continuous on $\overline{P}_-\times[0,T_0]$ with $\mathfrak{P}=0$ on ∂P_- and $(\partial_{x'}-i\partial_{y'})\mathfrak{P}$ continuous on $P_-\times[0,T_0]$, such that $(\partial_{x'}-i\partial_{y'})\mathfrak{P}^\epsilon\to(\partial_{x'}-i\partial_{y'})\mathfrak{P}$ uniformly on compact subsets of $P_-\times[0,T_0]$, as $\epsilon=\epsilon_j\to0$. Let $\epsilon=\epsilon_j\to0$ in equation (5.40), we have

$$\Psi_{z'}F_t-\Psi_tF_{z'}+\overline{F}F_{z'}-i\Psi_{z'}=-(\partial_{x'}-i\partial_{y'})\mathfrak{P},\qquad\text{on } P_-\times[0,T_0].\tag{5.48}$$ eq:444

This shows Ψ, F is a generalized solution of the water wave equation in the sense given in §2.3. Furthermore because of (5.2), (5.4), letting $\epsilon=\epsilon_j\to0$ gives

$$\sup_{[0,T_0]}\mathcal{E}_1(t)\le M(\mathcal{E}_1(0))<\infty.\tag{5.49}$$

and

$$\sup_{[0,T_0]}\{\sup_{y'<0}\|F(x'+iy',t)\|_{L^2(\mathbb{R},dx')}+\sup_{y'<0}\|\frac{1}{\Psi_{z'}(x'+iy',t)}-1\|_{L^2(\mathbb{R},dx')}\}<C_0<\infty.\tag{5.50}$$

5.6. **The invertability of $\Psi(\cdot,t)$.** If in addition $\Sigma(t)=\{Z=\Psi(\alpha',t):=Z(\alpha',t)\mid\alpha'\in\mathbb{R}\}$ is a Jordan curve, then because $\lim_{|\alpha'|\to\infty}\Psi_{z'}(\alpha',t)=1$,[23] the domain $\Omega(t)$ bounded above by $\Sigma(t)$ is winded by $\Sigma(t)$ exactly once. By the argument principle, $\Psi(\cdot,t):\overline{P}_-\to\Omega(t)$ is one-to-one and onto, $\Psi^{-1}(\cdot,t):\Omega(t)\to P_-$ exists and is holomorphic. By the chain rule, it is easy to check (5.48) is equivalent to

$$(F\circ\Psi^{-1})_t+\overline{F}\circ\Psi^{-1}(F\circ\Psi^{-1})_z-i=-(\partial_x-i\partial_y)(\mathfrak{P}\circ\Psi^{-1}),\qquad\text{on }\Omega(t).\tag{5.51}$$ eq:445

This is the Euler equation, the first equation of (1.1) in complex form. Let $\overline{\mathbf{v}}=F\circ\Psi^{-1}$, $P=\mathfrak{P}\circ\Psi^{-1}$. Then $(\mathbf{v},P)$ is a solution of the water wave equation (1.1) in $\Omega(t)$, with fluid interface $\Sigma(t):Z=Z(\alpha',t)$, $\alpha'\in\mathbb{R}$.

5.7. **The chord-arc interfaces.** Now assume at time $t=0$, the interface $Z=\Psi(\alpha',0):=Z(\alpha',0)$, $\alpha'\in\mathbb{R}$ is chord-arc, that is, there is $0<\delta<1$, such that

$$\delta\int_{\alpha'}^{\beta'}|Z_{,\alpha'}(\gamma,0)|\,d\gamma\le|Z(\alpha',0)-Z(\beta',0)|\le\int_{\alpha'}^{\beta'}|Z_{,\alpha'}(\gamma,0)|\,d\gamma,\quad\forall-\infty<\alpha'<\beta'<\infty.$$

We want to show there is $T_1>0$, depending only on $\mathcal{E}_1(0)$, such that for $t\in[0,\min\{T_0,\frac{\delta}{T_1}\}]$, the interface $Z=Z(\alpha',t):=\Psi(\alpha',t)$ remains chord-arc. We begin with

$$-z^\epsilon(\alpha,t)+z^\epsilon(\beta,t)+z^\epsilon(\alpha,0)-z^\epsilon(\beta,0)=\int_0^t\int_\alpha^\beta z^\epsilon_{t\alpha}(\gamma,s)\,d\gamma\,ds\tag{5.52}$$ eq:446

for $\alpha<\beta$. Because

$$\frac{d}{dt}|z^\epsilon_\alpha|^2=2|z^\epsilon_\alpha|^2\operatorname{Re}D_\alpha z^\epsilon_t\tag{5.53}$$ eq:447

by Gronwall, for $t\in[0,T_0]$,

$$|z^\epsilon_\alpha(\alpha,t)|^2\le|z^\epsilon_\alpha(\alpha,0)|^2e^{2\int_0^t|D_\alpha z^\epsilon_t(\alpha,\tau)|\,d\tau};\tag{5.54}$$ eq:448

so

$$|z^\epsilon_{t\alpha}(\alpha,t)|\le|z^\epsilon_\alpha(\alpha,0)||D_\alpha z^\epsilon_t(\alpha,t)|e^{\int_0^t|D_\alpha z^\epsilon_t(\alpha,\tau)|\,d\tau};\tag{5.55}$$ eq:449

by Appendix C, (5.2) and Proposition 2.5,

$$\sup_{[0,T_0]}|z^\epsilon_{t\alpha}(\alpha,t)|\le|z^\epsilon_\alpha(\alpha,0)|C(\mathcal{E}_1(0)).\tag{5.56}$$ eq:450

[23]By a similar argument as in §5.1.

therefore for $t \in [0, T_0]$,

$$\int_0^t \int_\alpha^\beta |z_{t\alpha}^\epsilon(\gamma, s)| \, d\gamma \, ds \le tC(\mathcal{E}_1(0)) \int_\alpha^\beta |z_\alpha^\epsilon(\gamma, 0)| \, d\gamma \tag{5.57}$$

eq:451

Now $z^\epsilon(\alpha, 0) = Z^\epsilon(\alpha, 0) = \Psi(\alpha - \epsilon i, 0)$. Because $Z_{,\alpha'}(\cdot, 0) \in L^1_{loc}(\mathbb{R})$, and $Z_{,\alpha'}(\cdot, 0) - 1 \in H^1(\mathbb{R} \setminus [-N, N])$ for some large N,

$$\overline{\lim_{\epsilon \to 0}} \int_\alpha^\beta |\Psi_{z'}(\gamma - \epsilon i, 0)| \, d\gamma \le \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma \tag{5.58}$$

eq:452

Let $\epsilon = \epsilon_j \to 0$ in (5.52). We get, for $t \in [0, T_0]$,

$$||z(\alpha, t) - z(\beta, t)| - |Z(\alpha, 0) - Z(\beta, 0)|| \le tC(\mathcal{E}_1(0)) \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma \tag{5.59}$$

eq:453

hence for all $\alpha < \beta$ and $0 \le t \le \min\{T_0, \frac{\delta}{2C(\mathcal{E}_1(0))}\}$,

$$\frac{1}{2}\delta \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma \le |z(\alpha, t) - z(\beta, t)| \le 2 \int_\alpha^\beta |Z_{,\alpha'}(\gamma, 0)| \, d\gamma \tag{5.60}$$

eq:454

This show that for $\le t \le \min\{T_0, \frac{\delta}{2C(\mathcal{E}_1(0))}\}$, $z = z(\cdot, t)$ is absolute continuous on compact intervals of $\mathbb{R}$, with $z_\alpha(\cdot, t) \in L^1_{loc}(\mathbb{R})$, and is chord-arc. So $\Sigma(t) = \{z(\alpha, t) \mid \alpha \in \mathbb{R}\}$ is Jordan. This finishes the proof of Theorem 3.4.

ineq

Appendix A. Basic analysis preparations

Let $\Omega \subset \mathbb{C}$ be a domain with boundary $\Sigma : z = z(\alpha)$, $\alpha \in I$, oriented clockwise. Let $\mathfrak{H}$ be the Hilbert transform associated to Ω:

$$\mathfrak{H} f(\alpha) = \frac{1}{\pi i} \text{pv.} \int \frac{z_\beta(\beta)}{z(\alpha) - z(\beta)} f(\beta) \, d\beta \tag{A.1}$$

hilbert-t

We have the following characterization of the trace of a holomorphic function on Ω.

prop:hilbe

Proposition A.1. [19] *a. Let $g \in L^p$ for some $1 < p < \infty$. Then g is the boundary value of a holomorphic function G on Ω with $G(z) \to 0$ at infinity if and only if*

$$(I - \mathfrak{H})g = 0. \tag{A.2}$$

eq:1571

b. Let $f \in L^p$ for some $1 < p < \infty$. Then $\mathbb{P}_H f := \frac{1}{2}(I + \mathfrak{H})f$ is the boundary value of a holomorphic function $\mathfrak{G}$ on Ω, with $\mathfrak{G}(z) \to 0$ as $|z| \to \infty$.
c. $\mathfrak{H}1 = 0$.

Observe that Proposition A.1 gives $\mathfrak{H}^2 = I$ in L^p.

We next present the basic estimates we will rely on for this paper. We start with the Sobolev inequality.

sobolev

Proposition A.2 (Sobolev inequality)**.** *Let $f \in C_0^1(\mathbb{R})$. Then*

$$\|f\|_{L^\infty}^2 \le 2\|f\|_{L^2}\|f'\|_{L^2} \tag{A.3}$$

eq:sobolev

hardy-inequality

Proposition A.3 (Hardy's Inequality)**.** *Let $f \in C^1(\mathbb{R})$, with $f' \in L^2(\mathbb{R})$. Then there exists $C > 0$ independent of f such that for any $x \in \mathbb{R}$,*

$$\left| \int \frac{(f(x) - f(y))^2}{(x - y)^2} dy \right| \le C \, \|f'\|_{L^2}^2 . \tag{A.4}$$

eq:77

Let

$$\mathbb{H} f(x) = \frac{1}{\pi i} \text{pv.} \int \frac{1}{x - y} f(y) \, dy.$$

be the Hilbert transform associated with P_-. Let $f:\mathbb{R}\to\mathbb{C}$ be a function in $\dot H^{1/2}$, we note that

$$\|f\|_{\dot H^{1/2}}^2=\int i\mathbb{H}\partial_x f(x)\overline{f}(x)\,dx=\frac{1}{2\pi}\iint\frac{|f(x)-f(y)|^2}{(x-y)^2}\,dx\,dy. \tag{A.5}$$ def-hhalf

We have the following result on $\dot H^{1/2}$ functions.

prop:Hhalf **Proposition A.4.** *Let $f,\ g\in C^1(\mathbb{R})$. Then*

$$\|g\|_{\dot H^{1/2}}\lesssim\|f^{-1}\|_{L^\infty}(\|fg\|_{\dot H^{1/2}}+\|f'\|_{L^2}\|g\|_{L^2}). \tag{A.6}$$ Hhalf

The proof is straightforward from the definition of $\dot H^{1/2}$ and the Hardy's inequality. We omit the details.

Let $A_i\in C^1(\mathbb{R})$, $i=1,\dots m$. Define

$$C_1(A_1,\dots,A_m,f)(x)=\text{pv.}\int\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^{m+1}}f(y)\,dy. \tag{A.7}$$ 3.15

B1 **Proposition A.5.** *There exist constants $c_1>0$, $c_2>0$, such that*

1. *For any $f\in L^2$, $A_i'\in L^\infty$, $1\le i\le m$,*

$$\|C_1(A_1,\dots,A_m,f)\|_{L^2}\le c_1\|A_1'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^2}. \tag{A.8}$$ 3.16

2. *For any $f\in L^\infty$, $A_i'\in L^\infty$, $2\le i\le m$, $A_1'\in L^2$,*

$$\|C_1(A_1,\dots,A_m,f)\|_{L^2}\le c_2\|A_1'\|_{L^2}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^\infty}. \tag{A.9}$$ 3.17

(A.8) is a result of Coifman, McIntosh and Meyer [10]. (A.9) is a consequence of the Tb Theorem, a proof is given in [32].

Let A_i satisfies the same assumptions as in (A.7). Define

$$C_2(A,f)(x)=\int\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^m}\partial_y f(y)\,dy. \tag{A.10}$$ 3.19

We have the following inequalities.

B2 **Proposition A.6.** *There exist constants c_3, c_4 and c_5, such that*

1. *For any $f\in L^2$, $A_i'\in L^\infty$, $1\le i\le m$,*

$$\|C_2(A,f)\|_{L^2}\le c_3\|A_1'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^2}. \tag{A.11}$$ 3.20

2. *For any $f\in L^\infty$, $A_i'\in L^\infty$, $2\le i\le m$, $A_1'\in L^2$,*

$$\|C_2(A,f)\|_{L^2}\le c_4\|A_1'\|_{L^2}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f\|_{L^\infty}. \tag{A.12}$$ 3.21

3. *For any $f'\in L^2$, $A_1\in L^\infty$, $A_i'\in L^\infty$, $2\le i\le m$,*

$$\|C_2(A,f)\|_{L^2}\le c_5\|A_1\|_{L^\infty}\|A_2'\|_{L^\infty}\dots\|A_m'\|_{L^\infty}\|f'\|_{L^2}. \tag{A.13}$$ 3.22

Using integration by parts, the operator $C_2(A,f)$ can be easily converted into a sum of operators of the form $C_1(A,f)$. (A.11) and (A.12) follow from (A.8) and (A.9). To get (A.13), we rewrite $C_2(A,f)$ as the difference of the two terms $A_1C_1(A_2,\dots,A_m,f')$ and $C_1(A_2,\dots,A_m,A_1f')$ and apply (A.8) to each term.

prop:half-dir **Proposition A.7.** *There exists a constant $C>0$ such that for any $f,g\in C^1(\mathbb{R})$ with $f'\in L^2$ and $g'\in L^2$,*

$$\|[f,\mathbb{H}]g\|_{L^2}\le C\,\|f\|_{\dot H^{1/2}}\,\|g\|_{L^2} \tag{A.14}$$ eq:b10

$$\|[f,\mathbb{H}]\partial_{\alpha'}g\|_{L^2}\le C\,\|f'\|_{L^2}\,\|g\|_{\dot H^{1/2}} \tag{A.15}$$ eq:b11

(A.14) is straightforward by Cauchy-Schwarz and the definition of $\dot H^{1/2}$. (A.15) follows from integration by parts, then Cauchy-Schwarz, Hardy's inequality, the definition of $\dot H^{1/2}$ and (A.14).

Recall $[f,g;h]$ as given in (2.1).

Proposition A.8. *There exists a constant $C > 0$ such that for any $f, g \in C^1(\mathbb{R})$ with $f', g' \in L^2$ and $h \in L^2$,*

$$\|[f,g;h]\|_{L^2} \le C \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}\,; \tag{A.16}$$
eq:b12

$$\|[f,g;h]\|_{L^\infty} \le C \|f'\|_{L^2} \|g'\|_{L^\infty} \|h\|_{L^2}\,. \tag{A.17}$$
eq:b15

(A.16) follows directly from Cauchy-Schwarz, Hardy's inequality and Fubini Theorem; (A.17) follows from Cauchy-Schwarz, Hardy's inequality and the mean value Theorem.

Proposition A.9. *There exists a constant $C > 0$ such that for any $f \in C^1(\mathbb{R})$ with $f' \in L^2$, $g \in L^2$,*

$$\|[f,\mathbb{H}]g\|_{L^\infty} \le C \|f'\|_{L^2} \|g\|_{L^2}\,. \tag{A.18}$$
eq:b13

(A.18) is straightforward from Cauchy-Schwarz and Hardy's inequality.

Proposition A.10. *There exists a constant $C > 0$ such that for any $f, g \in C^1(\mathbb{R})$ with $f', g' \in L^2$, and $h \in L^2$,*

$$\|\partial_{\alpha'}[f,[g,\mathbb{H}]]h\|_{L^2} \lesssim \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}\,. \tag{A.19}$$
eq:b14

Taking derivative under the integral $[f,[g,\mathbb{H}]]h$, (A.19) directly follows from (A.16) and (A.18).

basic-iden

Appendix B. Identities

B.1. **Basic identities.** Here we derive a few basic identities from the system (2.9)-(2.8), without assuming $Z = Z(\cdot,t)$ being non-self-intersecting. Theses identities provide an alternative way of deriving the quasi-linearization of the system (2.9)-(2.8) in this more general context, they also show that the argument in [21] can be modified, so that the a priori estimate of [21] and the characterization of the energy in §10 of [21] hold for solutions of the system (2.9)-(2.8) without the non-self-intersecting requirement.

Let $Z = Z(\cdot,t)$ be sufficiently regular[24] and satisfy (2.9)-(2.8):

$$\begin{cases} Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}, \\ \overline{Z}_t = \mathbb{H}\overline{Z}_t, \\ Z_{,\alpha'} - 1 = \mathbb{H}(Z_{,\alpha'} - 1), \qquad \frac{1}{Z_{,\alpha'}} - 1 = \mathbb{H}(\frac{1}{Z_{,\alpha'}} - 1); \end{cases} \tag{B.1}$$
c1

where Z and Z_t are related through (2.6), (2.7):

$$z(\alpha,t) = Z(h(\alpha,t),t), \qquad z_t(\alpha,t) = Z_t(h(\alpha,t),t) \tag{B.2}$$
c2

for some (sufficiently regular) homeomorphism $h(\cdot,t) : \mathbb{R} \to \mathbb{R}$. Let $\mathfrak{a}h_\alpha := \mathcal{A} \circ h$, $A_1 := \mathcal{A}|Z_{,\alpha'}|^2$. Precomposing the first equation of (B.1) with h gives (2.3):

$$z_{tt} + i = i\mathfrak{a}z_\alpha \tag{B.3}$$
c9

We first show that (2.18) can be derived from (B.1) and (B.2). Let Ψ be a holomorphic function on P_-, continuously differentiable on $\overline{P}_-$, such that

$$\Psi(\alpha',t) = Z(\alpha',t), \qquad \Psi_{z'}(\alpha',t) = Z_{,\alpha'}(\alpha',t).$$

Therefore $z(\alpha,t) = \Psi(h(\alpha,t),t)$ and by the chain rule, $z_t = \Psi_t \circ h + h_t \Psi_{z'} \circ h$. Precomposing with h^{-1} then gives

$$Z_t = \Psi_t + Z_{,\alpha'} h_t \circ h^{-1};$$

dividing by $Z_{,\alpha'}$ yields

$$h_t \circ h^{-1}(\alpha',t) = \frac{Z_t(\alpha',t)}{Z_{,\alpha'}(\alpha',t)} - \frac{\Psi_t}{\Psi_{z'}}(\alpha',t). \tag{B.4}$$
c3

[24]Here we do not specify what precisely "sufficiently regular" means, but assume it is enough so that the calculations make sense.

Notice that $\frac{\Psi_t}{\Psi_{z'}}$ is a holomorphic function on P_-. By Proposition A.1, applying $(I-\mathbb{H})$ to (B.4) then taking the real parts and using the second and third equations of(B.1) to rewrite into the commutator gives (2.18). Conversely, if h satisfies (2.18) for a function Z satisfying the second and third equations of (B.1), expanding the commutator gives

$$h_t\circ h^{-1}=\operatorname{Re}(I-\mathbb{H})(\frac{Z_t}{Z_{,\alpha'}})=\frac{Z_t}{Z_{,\alpha'}}+\frac{1}{2}(I+\mathbb{H})(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\frac{Z_t}{Z_{,\alpha'}}). \tag{B.5}$$ c4

By Proposition A.1, $\frac{1}{2}(I+\mathbb{H})(\frac{\overline{Z}_t}{\overline{Z}_{,\alpha'}}-\frac{Z_t}{Z_{,\alpha'}})$ is the boundary value of a holomorphic function on P_-, tending to zero at the spatial infinity.

In what follows we use the following notations. We write $U_1\equiv U_2$, if $(I-\mathbb{H})(U_1-U_2)=0$; that is if U_1-U_2 is the boundary value of a holomorphic function on P_- that tends to zero at infinity.

Assume Z satisfies the second and third equations of (B.1) and h satisfies (2.18), so (B.5) holds.

prop:basic-iden **Proposition B.1.** *Let $U(\cdot,t):\mathbb{R}\to\mathbb{C}$ be sufficiently regular, and $u=U\circ h$. Assume $U\equiv 0$. We have 1.*

$$u_t\circ h^{-1}\equiv Z_tD_{\alpha'}U; \tag{B.6}$$ c5

2.

$$u_{tt}\circ h^{-1}\equiv Z_{tt}D_{\alpha'}U+2Z_tD_{\alpha'}(u_t\circ h^{-1}-Z_tD_{\alpha'}U)+Z_t^2D_{\alpha'}^2U. \tag{B.7}$$ c6

3.

$$U_h^{-1}(u_{tt}+i\mathfrak{a}\partial_\alpha u)\equiv 2Z_{tt}D_{\alpha'}U+2Z_tD_{\alpha'}(u_t\circ h^{-1}-Z_tD_{\alpha'}U)+Z_t^2D_{\alpha'}^2U. \tag{B.8}$$ c8

Proof. Applying the chain rule to $u=U\circ h$ and precompose with h^{-1} gives

$$u_t\circ h^{-1}=\partial_tU+\partial_{\alpha'}Uh_t\circ h^{-1}.$$

Observe that $U\equiv 0$ gives $\partial_tU\equiv 0$ and $\partial_{\alpha'}U\equiv 0$. (B.6) follows from (B.5) and the fact that product of holomorphic functions is holomorphic.

Now we apply (B.6) to $u_t\circ h^{-1}-Z_tD_{\alpha'}U$. This gives

$$U_h^{-1}\partial_t(u_t-z_tD_\alpha u)\equiv Z_tD_{\alpha'}(u_t\circ h^{-1}-Z_tD_{\alpha'}U). \tag{B.9}$$ c7

Expanding the left hand side by the product rule, and observe that $\partial_tD_\alpha u=D_\alpha(u_t-z_tD_\alpha u)+z_tD_\alpha^2u$, so

$$\begin{aligned}\partial_t(u_t-z_tD_\alpha u)&=u_{tt}-z_{tt}D_\alpha u-z_t\partial_tD_\alpha u\\&=u_{tt}-z_{tt}D_\alpha u-z_tD_\alpha(u_t-z_tD_\alpha u)-z_t^2D_\alpha^2u.\end{aligned}$$

Precomposing with h^{-1} and substituting in (B.9) gives (B.7).

(B.8) follows from (B.7) and the fact that $i\mathfrak{a}\partial_\alpha u=(z_{tt}+i)D_\alpha u$ and $D_{\alpha'}U\equiv 0$.

□

Now assume Z satisfies (B.1)[25]. Applying (B.6) to $\overline{Z}_t$ gives $\overline{Z}_{tt}\equiv Z_tD_{\alpha'}\overline{Z}_t$. Following the rest of the argument in section 2.2.1 of [35] gives (2.19). Similarly, applying (B.8) to $\overline{Z}_t$ and following the rest of the argument in section 2.2.3 of [35] gives

$$\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ h^{-1}=\frac{-\operatorname{Im}(2[Z_t,\mathbb{H}]\overline{Z}_{tt,\alpha'}+2[Z_{tt},\mathbb{H}]\partial_{\alpha'}\overline{Z}_t-[Z_t,Z_t;D_{\alpha'}\overline{Z}_t])}{A_1} \tag{B.10}$$ c10

where

$$[Z_t,Z_t;D_{\alpha'}\overline{Z}_t]:=\frac{1}{\pi i}\int\frac{(Z_t(\alpha',t)-Z_t(\beta',t))^2}{(\alpha'-\beta')^2}D_{\beta'}\overline{Z}_t(\beta',t)\,d\beta' \tag{B.11}$$ c11

For the periodic case studied in [21], the same computations above and Proposition B.1 hold, and the corresponding equations for (2.18), (2.19), (B.10) can be derived without

[25]Here $Z=Z(\cdot,t)$ need not be non-self-intersecting.

the non-self-intersecting assumption. The periodic version of Proposition B.1 shows that the argument in [21] can be modified so that the a priori estimate, Theorem 2 of [21] and the characterization of the energy in §10 of [21] hold more generally without the non-self-intersecting assumption. Proposition B.1 and a small modification of the argument in [21] show that a similar a priori estimate and a similar characterization of the energy as in [21] hold in the whole line case for solutions of (2.9)-(2.8).

comm-iden

B.2. **Commutator identities.** We include here for reference the various commutator identities that are necessary. The first set: (B.12)-(B.15) has already appeared in [21].

$$[\partial_t, D_\alpha] = -(D_\alpha z_t)D_\alpha; \tag{B.12}$$ eq:c1

$$\left[\partial_t, D_\alpha^2\right] = -2(D_\alpha z_t)D_\alpha^2 - (D_\alpha^2 z_t)D_\alpha; \tag{B.13}$$ eq:c2

$$\left[\partial_t^2, D_\alpha\right] = (-D_\alpha z_{tt})D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t)D_\alpha\partial_t; \tag{B.14}$$ eq:c3

$$\left[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha\right] = (-2D_\alpha z_{tt})D_\alpha - 2(D_\alpha z_t)\partial_t D_\alpha. \tag{B.15}$$ eq:c5

We need some additional commutator identities. In general for operators A, B and C,

$$[A, BC^k] = [A,B]C^k + B[A, C^k] = [A,B]C^k + \sum_{i=1}^k BC^{i-1}[A,C]C^{k-i}. \tag{B.16}$$ eq:c12

We note that for $f = f(\cdot, t)$, $U_h\partial_{\alpha'}U_{h^{-1}}f = \frac{\partial_\alpha}{h_\alpha}f$. So

$$[\partial_t, \frac{\partial_\alpha}{h_\alpha}]f = -\frac{h_{t\alpha}}{h_\alpha}\frac{1}{h_\alpha}\partial_\alpha f = -U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f\}; \tag{B.17}$$ eq:c7

$$[U_h^{-1}\partial_t U_h, \partial_{\alpha'}]g = U_h^{-1}[\partial_t, \frac{\partial_\alpha}{h_\alpha}]U_h g = -(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}g. \tag{B.18}$$ eq:20

Applying (B.16) yields

$$\begin{aligned}\left[\partial_t, (\frac{\partial_\alpha}{h_\alpha})^2\right] f &= \frac{\partial_\alpha}{h_\alpha}[\partial_t, \frac{\partial_\alpha}{h_\alpha}]f + [\partial_t, \frac{\partial_\alpha}{h_\alpha}]\frac{\partial_\alpha}{h_\alpha}f \\ &= -2U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}^2 U_{h^{-1}}f\} - U_h\{\partial_{\alpha'}^2(h_t\circ h^{-1})\partial_{\alpha'}U_{h^{-1}}f\};\end{aligned} \tag{B.19}$$ eq:c11

$$\begin{aligned}\left[\partial_t^2, \frac{\partial_\alpha}{h_\alpha}\right] f &= \partial_t[\partial_t, \frac{\partial_\alpha}{h_\alpha}]f + [\partial_t, \frac{\partial_\alpha}{h_\alpha}]\partial_t f \\ &= -\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f\} - U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f_t\}.\end{aligned} \tag{B.20}$$ eq:c8

To calculate $[i\mathfrak{a}\partial_\alpha, \frac{\partial_\alpha}{h_\alpha}]f$, we use the definition $\mathcal{A}\circ h := \mathfrak{a}h_\alpha$, and $i\mathfrak{a}\partial_\alpha := i\mathcal{A}\circ h\frac{\partial_\alpha}{h_\alpha}$. We have

$$[i\mathfrak{a}\partial_\alpha, \frac{\partial_\alpha}{h_\alpha}]f = [i\mathcal{A}\circ h\frac{\partial_\alpha}{h_\alpha}, \frac{\partial_\alpha}{h_\alpha}]f = -iU_h\{\mathcal{A}_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f\}. \tag{B.21}$$ eq:c9

Adding (B.20) and (B.21), we conclude that

$$\begin{aligned}\left[\partial_t^2 + i\mathfrak{a}\partial_\alpha, \frac{\partial_\alpha}{h_\alpha}\right] f = -\partial_t U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f\} - U_h\{(h_t\circ h^{-1})_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f_t\} \\ - iU_h\{\mathcal{A}_{\alpha'}\partial_{\alpha'}U_{h^{-1}}f\}.\end{aligned} \tag{B.22}$$ eq:c10

We note that $U_h^{-1}\partial_t U_h = \partial_t + b\partial_{\alpha'}$ where $b := h_t\circ h^{-1}$. Therefore

$$[U_h^{-1}\partial_t U_h, \mathbb{H}] = [h_t\circ h^{-1}, \mathbb{H}]\partial_{\alpha'} \tag{B.23}$$ eq:c21

A straightforward differentiation gives

$$\begin{aligned}U_h^{-1}\partial_t U_h[f, \mathbb{H}]g &= [U_h^{-1}\partial_t U_h f, \mathbb{H}]g \\ &\quad + [f, \mathbb{H}](U_h^{-1}\partial_t U_h g + (h_t\circ h^{-1})_{\alpha'}g) - [f, h_t\circ h^{-1}; g];\end{aligned} \tag{B.24}$$ eq:c14'

with an application of (B.18) yields

$$U_h^{-1}\partial_t U_h[f,\mathbb{H}]\partial_{\alpha'}g = [U_h^{-1}\partial_t U_h f,\mathbb{H}]\partial_{\alpha'}g + [f,\mathbb{H}]\partial_{\alpha'}U_h^{-1}\partial_t U_h g - [f, h_t\circ h^{-1};\partial_{\alpha'}g]. \tag{B.25}$$

eq:c14

The following commutators are straightforward from the product rule. We have

$$[Z_{,\alpha'}, U_h^{-1}\partial_t U_h]f = [U_h^{-1}\frac{z_\alpha}{h_\alpha}, U_h^{-1}\partial_t U_h]f = -\{U_h^{-1}\partial_t\big(\frac{z_\alpha}{h_\alpha}\big)\}f = -Z_{,\alpha'}(D_{\alpha'}Z_t - (h_t\circ h^{-1})_{\alpha'})f; \tag{B.26}$$

eq:c13

$$[\partial_t, \frac{h_\alpha}{z_\alpha}]f = \partial_t\big(\frac{h_\alpha}{z_\alpha}\big)f = \frac{h_\alpha}{z_\alpha}(U_h(h_t\circ h^{-1})_{\alpha'} - D_\alpha z_t)f; \tag{B.27}$$

eq:c15

by $i\mathfrak{a}z_\alpha = z_{tt}+i$ (2.3),

$$[i\mathfrak{a}\partial_\alpha, \frac{h_\alpha}{z_\alpha}]f = [(z_{tt}+i)D_\alpha, \frac{h_\alpha}{z_\alpha}]f = (z_{tt}+i)D_\alpha\big(\frac{h_\alpha}{z_\alpha}\big)f; \tag{B.28}$$

eq:c17

by (B.16), (B.27), (B.28) and the product rule,

$$[\partial_t^2 + i\mathfrak{a}\partial_\alpha, \frac{h_\alpha}{z_\alpha}]f = 2\frac{h_\alpha}{z_\alpha}(U_h(h_t\circ h^{-1})_{\alpha'} - D_\alpha z_t)f_t + \frac{h_\alpha}{z_\alpha}(U_h(h_t\circ h^{-1})_{\alpha'} - D_\alpha z_t)^2 f + \frac{h_\alpha}{z_\alpha}(\partial_t U_h(h_t\circ h^{-1})_{\alpha'} - \partial_t D_\alpha z_t)f + (z_{tt}+i)D_\alpha\big(\frac{h_\alpha}{z_\alpha}\big)f. \tag{B.29}$$

eq:c16

quantities

Appendix C. Main quantities controlled by $\mathfrak{E}$

We list here the various quantities that we have shown in [21] are controlled by polypromials of $\mathfrak{E}(t)$. [26]

$$\left\|D_{\alpha'}^2\overline{Z}_{tt}\right\|_{L^2}, \left\|D_{\alpha'}^2 Z_{tt}\right\|_{L^2}, \left\|D_{\alpha'}^2\overline{Z}_t\right\|_{L^2}, \left\|D_{\alpha'}^2 Z_t\right\|_{L^2}, \|D_\alpha\partial_t D_\alpha\overline{z}_t\|_{L^2(h_\alpha d\alpha)},$$
$$\left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\overline{Z}_t\right\|_{\dot H^{1/2}}, \left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{L^\infty}, \|D_{\alpha'}Z_{tt}\|_{L^\infty}, \left\|D_{\alpha'}\overline{Z}_t\right\|_{L^\infty}, \|D_{\alpha'}Z_t\|_{L^\infty},$$
$$\left\|\overline{Z}_{tt,\alpha'}\right\|_{L^2}, \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}, \int|D_\alpha\overline{z}_t|^2\frac{d\alpha}{\mathfrak{a}}, \int|D_\alpha\overline{z}_{tt}|^2\frac{d\alpha}{\mathfrak{a}}, \left\|\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}, \|Z_{tt}+i\|_{L^\infty}, \|A_1\|_{L^\infty}; \tag{C.1}$$

eq:1550

- $\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\right\|_{L^\infty} = \left\|\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^\infty}$;
- $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^2}$;
- $\left\|\frac{h_{t\alpha}}{h_\alpha}\right\|_{L^\infty}$;
- $\|(I+\mathbb{H})D_{\alpha'}Z_t\|_{L^\infty}$;
- $\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$, $\left\|(Z_{tt}+i)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty}$;
- $\left\|\partial_{\alpha'}\mathbb{P}_A\frac{Z_t}{Z_{,\alpha'}}\right\|_{L^\infty}$, $\left\|\mathbb{P}_A\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{L^\infty}$.

In addition from (179), (186) of [21],

$$\left\|D_{\alpha'}(h_t\circ h^{-1})_{\alpha'}\right\|_{L^2} \lesssim C(\mathfrak{E}).$$

[26] The same proof for the symmetric periodic setting in [21] applies to the whole line setting. We leave it to the reader to check the details.

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