Syllabus for Math 297
Winter Term 2019
Classes MWF 1-2:30, 3866 East Hall

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Office hours: DeBacker's: M: 10:45 – 11:45am; M: 8:30 – 9:30pm; T: 2 – 3pm ; θ: 11 pm – midnight
Annie's: W: 7 – 8pm (EH3866)
Problem Session: T: 4 – 5pm (EH3866)

Text: Abbott, Stephen. *Understanding Analysis*, Second edition. Undergraduate Texts in Mathematics. Springer, New York, 2015. ISBN: 978-1-4939-2711-1.[1]

Philosophy: "He was not fast. Speed means nothing. Math doesn't depend on speed. It is about deep." (Yuriu Burago commenting on Fields Medal winner Grigory Perelman in Sylvia Nasar and David Gruber, *Manifold Destiny*, The New Yorker, August 28, 2006, pp. 44–57.)

An Overview: It is assumed that you have acquired a solid foundation in the theoretical aspects of linear algebra (as in Math 217). This includes familiarity with how to read and write proofs; the notions of linear independence, basis, spanning sets; linear transformations; eigenvalues and eigenvectors; the spectral theorem; and basic results about inner product spaces including the Gram-Schmidt process.

This is a course in analysis, the study of how/why calculus works. Over the course of the semester, we will develop an appreciation for the importance of the completeness and ordering of $\mathbb{R}$. We will also learn how to read, write, and understand ε–δ style arguments as we cover topics including basic topology, uniform continuity, and the properties of derivatives, definite integrals, and infinite sequences and series. To the extent possible, we will carry out our investigations in the setting of inner product spaces.

Next year's Math 395-6 sequence will develop the theory behind differentiation and integration in several real variables. It will also generalize the entirety of multivariable calculus to the geometric setting of abstract manifolds (where one proves a general Stokes' Theorem that includes the "vector calculus" incarnations due to Green, Gauss, and Stokes as special cases).

Although it will almost never be discussed in class, you will continue your study of complex numbers, linear algebra, and relations (specifically, equivalence relations). Each *Monday Homework* set will usually include one problem about complex numbers, one about relations, and one about linear algebra; it is important that you both attempt and understand each of these problems.

Pedagogy: This course will be taught in an Inquiry Based Learning style, a teaching method that "emphasizes discovery, analysis and investigation to deepen students' understanding of the material and its applications."[2] If you took Math 217, then you are familiar with this pedagogical approach to teaching mathematics. There will be very little lecture; instead, you will work through carefully crafted worksheets that guide you as you learn and internalize the material.

Grade: There will be homework counting 40%, two midterm exams counting 15% each, and a final counting 30%. The midterms will probably occur on February 20 and April 3. It appears that the registrar has scheduled the final for Tuesday, May 1 at 1:30 pm. Please verify this on your own. All exams are modified take home exams. Additionally, there will be four Gateways, which will run March 11 to March 23. These will cover differentiation, integration, basic matrix manipulations, and diagonalization. Failure to pass a Gateway exam will lower your course grade by half a letter grade. So, if you fail two of the four gateways and your grade would have been a B, then your grade will be a C.

Homework Zero has two problems:

(1) Complete the **Student Data Form** at `http://instruct.math.lsa.umich.edu/`.

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[1] A free electronic edition is available at: `https://link.springer.com/book/10.1007%2F978-1-4939-2712-8`
[2] `https://lsa.umich.edu/math/centers-outreach/ibl-center-for-inquiry-based-learning.html`

(2) After having read and understood the handouts *Joy of Sets* and *Mathematical Hygiene*, complete the **Ma297: WebHW** at http://instruct.math.lsa.umich.edu/

Homework: Homework will usually be assigned every Friday. The main part will be due at the beginning of class on the following Friday. The linear algebra/relations part (Monday Homework) will usually be due on the Monday that falls ten days after it is assigned, and the complex numbers/concept review part (Wednesday Homework) on the Wednesday twelve days after it is assigned. The two lowest homework grades (of each type) will be dropped.

To facilitate the grading of homework: do the problems in order, write on only one side of the paper, and use standard sized paper. No credit will be given if you misstate a problem, so pay attention. You are encouraged to discuss the problems with other students, but you must write up your solutions independently. Warning: It is unbelievable easy to detect plagiarism in mathematics; if you are caught, you will fail. Your solutions should be understandable to your peers; that is, your solutions should be correct, complete, and justified. Late homework will not be accepted (it receives a zero).

Waiting to begin your homework until the evening before it is due is an extremely bad idea.

Exams: There are no alternate or makeup exams (except in cases of extreme human tragedy). The exams will be based entirely on homework, class work (notes/worksheets), and True/False questions. In other words: do your homework and *understand* it; review your notes/worksheets and *understand* them; and complete the True/False questions and *understand* them!

Accommodations for a disability: If you think you need an accommodation for a disability, please let me know as soon as possible. In particular, a Verified Individualized Services and Accommodations (VISA) form must be provided to me at least two weeks prior to the need for a test/quiz accommodation. The Services for Students with Disabilities (SSD) Office (G664 Haven Hall; http://ssd.umich.edu/) issues VISA forms.

Student Responsibilities: College students are often advised to spend 2 to 3 hours outside of class for every hour of time spent in class. This rule of thumb is for the typical college class; math and science classes usually fall more towards the 3 hour end, and it is not unusual for students to spend about 12 hours per week, on average, outside of class for a course like this. If you are regularly spending more than 15 hours per week outside of class, you should speak with me about the work load and your studying strategies since this is more time than is intended.

People who take EECS 280 quickly learn that it is best to begin projects the moment they are assigned, and it is strongly recommended that you take the same approach to your Math 297 homework. Experience shows that students who attempt a substantial portion of their written homework the weekend it is assigned do quite well in their math courses.

Climate: Each of you deserves to learn in an environment where you feel safe and where you are respected.

The climate in the mathematical community at Michigan is heavily influenced by the students in it; this now includes you. Our choices directly affect, for good and/or bad, the health of the communities in which we live and work, please choose to build and maintain a welcoming, thriving mathematical community.[1]

Please politely call out your peers and professors on inappropriate behavior. Examples of inappropriate behavior include parading one's knowledge of advanced mathematics, dismissing others' questions as trivial, and any disparaging comments about other students' abilities, grades, appearance, course selection, likelihood to date, likelihood to go to grad school, mathematical tastes, etc. If you don't feel comfortable doing this, then talk to me or some university official that you trust (e.g., your general advisor, the university ombudsperson, ...).

Finally, mathematics requires the courage to take risks – intellectually, socially, emotionally, ... In particular, many of us find it unnerving to share our mathematical ideas, especially those we are not sure are correct, with others. In order to develop our capacity for taking these risks, we must respect and support each other.

[1]In case you think this is all abstraction with no real world implications, I include here comments of two alumns. A recent UofM Math undergraduate described their experience here in this way: "I found that, in general, the honors math majors are overly competitive and unwilling to help each other. ... I think the students that I took classes with were generally competitive rather than collaborative. From what I can tell, the current underclassmen seem to be much friendlier with each other, which goes a long way to having a strong community." Another student from a few years back wrote that "the worst part of my experience at Michigan was the general attitude that 'honors-sequence majors' have towards the rest of their 'honors major' and 'non-honors major' peers. While working in the Nesbitt room, I've been told by 'honors-sequence majors' that I'm not a 'real' math major because I did not participate in the honors sequence and that my opinions about math do not matter."

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Math 297: Tentative day-by-day syllabus			
Week	Monday	Wednesday	Friday
Jan 7 – Jan 11 §§1.1–1.2	No class	Introduction. Discussion of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\sqrt{2}$; small discussion of $\mathbb{R}$, and enough discussion about $\mathbb{C}$ to establish the polar form of a complex number	Establish basic notation for class: functions, more set theory notation (especially quantifiers)
Jan 14 – Jan 18 §§1.3–1.5	$\mathbb{R}$: Axiom of Completeness, sups and infs	$\mathbb{R}$: $\mathbb{N}$ as smallest inductive subset of $\mathbb{R}$, induction and Archimedean property	$\mathbb{R}$: nested interval property, existence of square roots, light on cardinality
Jan 21 – Jan 25 §2.2	Please attend the annual Marjorie Lee Browne Colloquium and other University of Michigan Martin Luther King, Jr. Day Symposia. No class.	$\mathbb{R}$ Recap ($\mathbb{R}$ecap)	Sequences: definition and limits
Jan 28 – Feb 1 §§2.3–2.5	Sequences: basic limit properities	Sequences: monotone convergence & peek at series	Sequences: subsequences and Bolzano-Weierstrass
Feb 4 – Feb 8 §§2.6–2.7	Sequences: Cauchy criterion and completeness	Series: basic results & tests	Sequences and Series Recap
Feb 11 – Feb 15 §§3.2–3.4	Topology: open and closed sets in $\mathbb{R}^n$	Topology: sequential compactness for $\mathbb{R}^n$	Topology: connected sets in $\mathbb{R}^n$
Feb 18 – Feb 22 §4.2	Topology Recap	Midterm I	Continuity: limit and sequence definitions (in $\mathbb{R}^n$ – to the extent possible)
Feb 25 – Feb 29 §§4.3–4.4	Continuity: basic results	Continuity: continuous functions carry compact sets to compact sets; Extreme Value Theorem	Continuity: continuous functions on compact sets are uniformly continuous
Mar 4– Mar 8	Vacation	Vacation	Vacation
Mar 11 – Mar 15 §§4.5 & 7.2	Continuity: Continuous functions carry connected sets to connected sets; Intermediate Value Theorem	Continuity Recap	Integrability: definition and discussion of existence of integrable functions
Mar 18 – Mar 22 §7.4	Integrability: basic properties including the Darboux integrability criterion	Integrability: definition of $\ln$ and $\exp$	Integrability Recap
Mar 25 – Mar 29 §§5.2 – 5.3	Differentiation: definitions and basic results	Differentiation: Darboux's intermediate value theorem for derivatives; Fermat's theorem about the behavior of the derivative at an extremum	Differentiation: Various versions of the Mean Value Theorem
Apr 1 – Apr 5 §7.5	Differentiation Recap	Midterm II	Fundamental Theorem(s) of Calculus
Apr 8 – Apr 12 §§6.2–6.4	Sequences of Functions: pointwise and uniform convergence	Sequences of Functions: continuity, integration, and differentiability of limits; introduction of $C^p(\mathbb{R})$ notation	Series of Functions: Weierstrass M-test, uniform Cauchy Criterion

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Math 297: Tentative day-by-day syllabus continued			
Week	Monday	Wednesday	Friday
Apr 15 – Apr 19 §§6.5–6.6	Series of Functions: Power series including Hadamard's Theorem and a brief discussion of analyticity	Series of Functions: Taylor Series, including a discussion of e^{-1/x^2}	Sequences and Series of Functions Recap
Apr 22 – Apr 26	Semester Recap	No class	No class

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Worksheet for 9 Jan 2019
Introduction and Notation
§1.1

Vocabulary: natural numbers, integers, rational numbers, real numbers, Cartesian product, commutative, associative, field, order, (binary) operations, power sets

Why do we need the real numbers? Why not just do calculus on the rational numbers? What are the real numbers? For that matter, what are the rational numbers? We have a long way to travel, so let's get started.

Notation: The set of *natural numbers*, denoted $\mathbb{N}$, is the set $\{1, 2, 3, 4, 5, \ldots\}$. The set of *integers*, denoted $\mathbb{Z}$, is the set $\{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7 \ldots\}$. The set of *rational numbers*, denoted $\mathbb{Q}$, may be thought of[1] as the set $\{a/b \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{N}\}$.

The plan: The course will start by isolating a defect of $\mathbb{Q}$ that causes calculus to fail there. We will then define $\mathbb{R}$ to be a field that, by definition, doesn't have this defect. How do we know that $\mathbb{R}$ actually exists? We don't, but any discussion of calculus must begin somewhere, and our discussion will start with the assumption that $\mathbb{R}$ exists. [At the end of the term, we may discuss how to push the starting point of our discussions back to the Peano axioms.]

(1) What is the footnote above about $2/3 = 94/141$ trying to express?

Before tackling the nuances of calculus, we need to spend some time developing/recalling basic mathematical words.

(2) Suppose S and T are sets, what is a function $f\colon S \to T$? What do we call S? What do we call T?

(3) Provide five nontrivial examples of functions.

(4) Choose your favorite function from among the five above. If $g\colon X \to Y$ is your function, then for $B \subset Y$ define
$$g^{-1}[B] := \{x \in X \mid g(x) \in B\}.$$
Does this define a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$? [If V is a set, then $\mathcal{P}(V)$ denotes the *power set* of V; that is, $\mathcal{P}(V)$ denotes the set of all subsets of V.]

You are familiar with the Cartesian plane, often denoted $\mathbb{R}^2$. As a set, it is just all ordered pairs (x, y) with $x, y \in \mathbb{R}$. More generally,

Definition. For sets C and D we define the *Cartesian product* of C and D as
$$C \times D := \{(c, d) \mid c \in C \text{ and } d \in D\}.$$

In this notation, the familiar $\mathbb{R}^2$ is just $\mathbb{R} \times \mathbb{R}$, and you can draw pictures of $C \times D$ just as you do for $\mathbb{R} \times \mathbb{R}$.

(5) What does $\{a, b, c, d\} \times \{☾, \bowtie, ☺\}$ look like?

(6) If C and D are finite sets, how many elements does $C \times D$ have? How many elements in $\mathcal{P}(D)$?

(7) *Bonus.* A function $f\colon S \to T$ may be thought of as the subset of $S \times T$ by identifying f with its graph:
$$f := \{(s, f(s)) \mid s \in S\}.$$
Which subsets of $S \times T$ correspond to functions?

Definition. Suppose S is a set. A *binary operation* on S is a function from $S \times S$ to S. If $*$ is a binary operation on S, then it is called *commutative* provided that $a * b = b * a$ for all $a, b \in S$, and it is called *associative* provided that $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

(8) What sort of (natural) binary operations can you define on each of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$?

(9) Which of the operations that you came up with are commutative? which are associative?

Definition. A binary operation $*$ on a set S is said to have an *identity*, e, provided that $s * e = e * s = s$ for all $s \in S$.

(10) Which of your binary operations on $\mathbb{N}$ have an identity in $\mathbb{N}$? Which of your binary operations on $\mathbb{Z}$ have an identity in $\mathbb{Z}$? Which of your binary operations on $\mathbb{Q}$ have an identity in $\mathbb{Q}$?

[1] We understand that $2/3 = 94/141$ in $\mathbb{Q}$, but encoding this equality into the definition of $\mathbb{Q}$ requires some sophisticated words that we won't have under control until the end of the semester.

(11) Show that a binary operation $*$ on a set S has at most one identity. [Hint: Suppose it has two e and e'; show that $e = e'$.]

Definition. Suppose $*$ is a binary operation on a set S. Also suppose that there exists a $*$-identity e. An element $s \in S$ is said to have a $*$-*inverse* provided that there is an $s' \in S$ for which $s * s' = s' * s = e$.

(12) For each of your binary operations on $\mathbb{N}$, discuss which elements of $\mathbb{N}$ have inverses. For each of your binary operations on $\mathbb{Z}$, discuss which elements of $\mathbb{Z}$ have inverses. For each of your binary operations on $\mathbb{Q}$, discuss which elements of $\mathbb{Q}$ have inverses.

(13) Suppose $*$ is an associative binary operation on a set S. Also suppose that there exists a $*$-identity e. Show that an element $s \in S$ has at most one $*$-inverse.

(14) *Bonus.* Can you cook up an example of a non-associative binary operation for which inverses are not unique?

Definition. A *field* is a set F equipped with two commutative, associative binary operations, $+$ and $\times$, such that

- there is a $+$-identity $0 \in F$;
- there is $\times$-identity $1 \in F$;
- each $x \in F$ has a $+$-inverse, usually denoted $-x$;
- each $x \in F \setminus \{0\}$ has a $\times$-inverse, usually denoted x^{-1};
- for all $a, b, c \in F$ we have $a \times (b + c) = a \times b + a \times c$;
- $0 \neq 1$.

We will almost always write ab for $a \times b$. We refer to $+$ as addition and $\times$ as multiplication.

(15) Which of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are fields?

(16) *Bonus.* Abbott says that "The finite set $\{0, 1, 2, 3, 4\}$ is a field when addition and multiplication are computed modulo 5." Prove this. This field is often denoted $\mathbb{F}_5$.

Definition. A field is said to be *ordered* provided that there is subset $P \subset F$ satisfying the following

- (closure under addition and multiplication) for all $a, b \in P$ we have $a + b \in P$ and $ab \in P$.
- (trichotomy) for all $f \in F$ exactly one of the following is true
  - we have $f \in P$ (in which case we write $f > 0$)
  - we have $f = 0$
  - we have $-f \in P$ (in which case we write $f < 0$)

(17) Is $\mathbb{Q}$ ordered? What about $\mathbb{R}$?

(18) *Bonus.* Is $\mathbb{C}$ ordered? Is $\mathbb{F}_5$ ordered?

There are many interesting questions one can ask about fields. For example: if F is a field, is it true that $ab = 0$ implies that one of a or b must be zero? is it possible for a field to be ordered in more than one way? We may return to these (and other) questions later, for now we take up one of the critical differences between $\mathbb{Q}$ and $\mathbb{R}$.

(19) Recall that the intermediate value theorem says:

If $f \colon [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < 0 < f(b)$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Consider the function $g \colon \mathbb{Q} \to \mathbb{Q}$ defined by $g(q) = q^2 - 2$; this function is continuous (trust me). Use this function to show that the intermediate value theorem is not valid for $\mathbb{Q}$.

(20) Prove that the square root of two is not a rational number.

Notation: We will be using quantifiers ("for all" and "there exists") quite often in this class. For this reason, we adopt the following shorthand: The symbol $\forall$ mean *for all* and the symbol $\exists$ means *there exists*.

Something to Think About

Given a function $g \colon X \to Y$, we defined a function $g^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by setting $g^{-1}[B] = \{x \in X \mid g(x) \in B\}$ for $B \in \mathcal{P}(Y)$. For $A \subset X$ we define

$$g[A] := \{g(a) \mid a \in A\}.$$

Does this define a function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$? Is this an inverse of $g^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$?

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Worksheet for 11 Jan 2019
Ordered fields
§1.2

Vocabulary: upper bound, lower bound, absolute value function, infinite, triangle inequality

We begin by checking that if F is an ordered field, then some of our favorite facts are true. Checking these facts is often difficult for students; I think this is because these facts are all, in some sense, too familiar and so seem obvious. Consequently, I have given plenty of hints in the hopes that we can move through these quickly.

(21) Suppose $r \in F$. Show that $r0 = 0$. [Hint: $0 = 0 + 0$ and the additive identity is unique.]
(22) Suppose $a, b \in F$. If $ab = 0$, then at least one of a or b is zero. [Hint: WOLOG, $a \neq 0$. Note that $1 = aa^{-1}$.]
(23) Suppose $a, b \in F$. We have $-ab = (-a)b$. [Hint: additive inverses are unique.]
(24) Show that for all $a, b \in F$ we have $(-a)(-b) = ab$. [Hint: additive inverses are unique.] Conclude that $a^2 = (-a)^2$.
(25) If $a, b \in F$ with $a < 0$ and $b > 0$, then $ab < 0$. [Hint: Use Prompt 23 and trichotomy.]
(26) If $x \in F \setminus \{0\}$, then $x^2 > 0$. [Hint: Use Prompt 24.] Conclude that $1 > 0$.
(27) Suppose $a \in F_{>0}$. Show that $a^{-1} > 0$.
(28) Suppose $c, d \in F_{\geq 0}$. Show that $c^2 < d^2$ if and only if $c < d$. (Note: for $x, y \in F$ we write $x < y$ provided that $y - x > 0$.)
(29) Suppose $c, d \in F_{\geq 0}$. Show that $c^2 = d^2$ if and only if $c = d$.
(30) Suppose $c, d \in F_{\geq 0}$. Show that $c^2 \leq d^2$ if and only if $c \leq d$. (Note: for $x, y \in F$ we write $x \leq y$ provided that $y - x > 0$ or $x = y$.)

From now on, we will assume that our favorite facts about ordered fields are true. This may include some favorite facts that are not listed above, but we'll try to be careful.

(31) Show that $\sqrt{2}$ is not a rational number; how many different proofs can you give?
(32) Consider the set $A = \{q \in \mathbb{Q} \,|\, q^2 < 2\}$. Show that if $p \in A$ and $p > 0$, then $p' = \frac{2p+2}{p+2}$ is also in A. Which is larger, p or p'? What does this say about the set A? [Note: $2 := 1 + 1 > 0$.]
(33) Consider the set $A = \{q \in \mathbb{Q} \,|\, q^2 < 2\}$. Show that if $t \in \mathbb{Q} \setminus A$ and $t > 0$, then $t > q$ for all $q \in A$. The element t is called an *upper bound* of A.

Definition. Suppose F is an ordered field and $C \subset F$. An element $x \in F$ is called an *upper bound* for C provided that $x \geq c$ for all $c \in C$. If C has an upper bound, then C is said to be *bounded above*. An element $y \in F$ is called a *lower bound* for C provided that $y \leq c$ for all $c \in C$. If C has a lower bound, then C is said to be *bounded below*.

(34) Is the emptyset bounded above? bounded below?
(35) Consider the set $A = \{q \in \mathbb{Q} \,|\, q^2 < 2\}$. Show that if $t \in \mathbb{Q} \setminus A$ and $t > 0$, then $t' = \frac{2t+2}{t+2} > 0$ is also in $\mathbb{Q} \setminus A$. Which is larger, t or t'? What does this say about the set of upper bounds of A in $\mathbb{Q}$ (i.e. the set $\{v \in \mathbb{Q} \,|\, v \text{ is an upper bound of } A\}$)?

The subset A of $\mathbb{Q}$ is interesting. On the one hand, from Prompt 32 we can conclude that A has no largest element. On the other hand, from Prompts 33 and 35 we know that there is no least rational upper bound of A. While it is not at all obvious at this point, the fact that some nonempty subsets of $\mathbb{Q}$, like A, do not have least upper bounds in $\mathbb{Q}$ is a big part of the reason that calculus doesn't work for $\mathbb{Q}$.

Sets that are not bounded (above or below) can behave poorly. We consider one cautionary example. For each $m \in \mathbb{N}$ set $\mathbb{N}_{>m} = \{n \in \mathbb{N} \,|\, n > m\}$. Note that the sets $\mathbb{N}_{>m}$ are nested; indeed: $\mathbb{N} \supsetneq \mathbb{N}_{>1} \supsetneq \mathbb{N}_{>2} \supsetneq \mathbb{N}_{>3} \cdots$.

Definition. A set A is said to be *infinite* provided that there is a proper subset $B \subsetneq A$ and an injective map from A to B.

(36) Show that for each $m \in \mathbb{N}$ the set $\mathbb{N}_{>m}$ is infinite.
(37) Show that $\bigcap_m \mathbb{N}_{>m}$ is empty.
(38) *Bonus.* What if we replaced $\mathbb{N}_{>m}$ by $\mathbb{Q}_{>m} = \{q \in \mathbb{Q} \,|\, q > m\}$ or by $\mathbb{R}_{>m} = \{r \in \mathbb{R} \,|\, r > m\}$?

The triangle inequality is one of the more important inequalities in mathematics. The proof outlined in Exercise 1.2.6 of your book is not very conceptual; here we present a different approach.

Definition. Suppose F is an ordered field. The *absolute value function* $|\ |: F \to F$ is defined by

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

(39) Show that for all $x \in F$ we have $x \leq |x|$ and $x^2 = |x|^2$.

(40) Let F be an ordered field, $a, b \in F$. Show that $|ab| = |a|\,|b|$. [Hint: Thanks to Prompt 29 it is enough to show $|ab|^2 = (|a|\,|b|)^2$.]

(41) Let F be an ordered field, $a, b \in F$. The triangle inequality says

$$|a+b| \leq |a| + |b| .$$

Prove it. [Hint: Thanks to Prompt 30 it is enough to show $|a+b|^2 \leq (|a| + |b|)^2$. Also, $2 = 1 + 1 > 0$.]

Some Things to Think About

[A] Can an ordered field have a finite number of elements or must it be infinite?

[B] Can -1 be a square in an ordered field? Why or why not?

Worksheet for 14 Jan 2019
Axiom of Completeness, supremums, and infimums
§1.3

Vocabulary: axiom of completeness, supremum, infimum, least upper bound, greatest lower bound, inductive sets

Recall that if F is an ordered field and $B \subset F$, then we say that $\beta \in F$ is a lower bound for B provided that $\beta \leq b$ for all $b \in B$. We define upper bounds of B in a similar fashion.

Definition. Suppose F is an ordered field and $A \subset F$. An element $\alpha \in F$ is a *least upper bound* for A provided that

- α is an upper bound for A;
- if x is any upper bound for A, then $\alpha \leq x$.

(42) Suppose F is an ordered field and $A \subset F$. Show that A has at most one least upper bound.

Thanks to Prompt 42, the following definition makes sense.

Definition. Suppose F is an ordered field and $A \subset F$ has a least upper bound. The least upper bound of A is also often called the *supremum* of A and it is denoted $\mathrm{lub}(A)$ or $\sup(A)$.

(43) Suppose F is an ordered field. Show that $F_{<0} = \{x \in F \mid x < 0\}$ has a least upper bound.
(44) Suppose F is an ordered field. Show that $F_{\leq 2} = \{x \in F \mid x \leq 2\}$ has a least upper bound.
(45) Provide an example of a field F and a bounded above subset of F that does not have a supremum in F. [Hint: You have already done this.]
(46) Is the empty set bounded above? Does it have a least upper bound?
(47) Suppose F is an ordered field and $A \subset F$. Define what it means to be a *greatest lower bound* (or *infimum*) of A. If the greatest lower bound of A exists, is it unique?

Definition. Suppose F is an ordered field and $A \subset F$. An element $M \in F$ is called a *maximum* of A provided that

- $M \in A$ and
- $M \geq a$ for all $a \in A$.

(48) Suppose F is an ordered field and $A \subset F$. Show that A has at most one maximal element. [If A has a maximal element, then we denote it by $\max(A)$.]
(49) Provide an example of a field F and a bounded above subset A of F for which $\sup(A)$ exists in F but A has no maximum.
(50) Suppose F is an ordered field and $A \subset F$ is bounded above. Show that $\max(A)$ exists if and only if $\sup(A)$ exists and $\sup(A) \in A$.
(51) Suppose F is an ordered field and $A \subset F$. Define what it means to be a *minimum* of A. If a minimum of A exists, is it unique? What is the relationship between $\min(A)$ and $\inf(A)$?

Definition. The ordered field F is said to be *complete* provided that every nonempty bounded above subset of F has a supremum in F.

(52) Show that $\mathbb{Q}$ is not complete. [Hint: You have already done this.]

It turns out that there is a unique[1] ordered field that is also complete. However, as Abbott points out, at some point we need to make a decision about where we are going to begin our study of calculus – should we establish the existence and uniqueness of a complete ordered field (a nonenlightening (at this stage) and somewhat tedious process), or should we assume that such a field exists and start thinking about calculus. We will do the latter.

Axiom of Completion. The field of real numbers, denoted $\mathbb{R}$, is the complete ordered field.

(53) Suppose $A \subset \mathbb{R}$ is a bounded above nonempty subset and $b \in \mathbb{R}$. Show that $\sup(b + A) = b + \sup(A)$.

[1] up to order-preserving isomorphism

(54) Suppose A is a bounded above subset of $\mathbb{R}$ and α is an upper bound for A. Show that $\alpha = \sup(A)$ if and only if for all $\varepsilon > 0$ there is an $a \in A$ such that
$$\alpha - \varepsilon < a \leq \alpha.$$

(55) Show that if $B \subset \mathbb{R}$ is a nonempty bounded below subset of $\mathbb{R}$, then $\inf(B)$ exists. [Hint: You may want to consider $L = \{\ell \in \mathbb{R} \mid \ell \text{ is a lower bound for } B\}$.]

(56) *Bonus.* Suppose $A, B \subset \mathbb{R}$ are nonempty subsets of $\mathbb{R}$. If for all $a \in A$ and $b \in B$ we have $a \leq b$, then (i) $\inf(B)$ and $\sup(A)$ exist and (ii) $\sup(A) \leq \inf(B)$. Moreover, we have $\sup(A) = \inf(B)$ if and only if for all $\varepsilon > 0$ there exists $a \in A$ and $b \in B$ such that $b - a < \varepsilon$.

One difficulty with using the Axiom of Completion as our starting point is that we no longer have a handle on how $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are related to $\mathbb{R}$. We would like to know that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. One way to approach this difficulty is to ask: what properties does $\mathbb{N}$ have that are important? After coming up with a reasonable list of properties, we then try to find a minimal subset of $\mathbb{R}$ that satisfies the properties on our list. Let's do this:

Definition. A subset I of $\mathbb{R}$ is said to be *inductive* provided that
- $1 \in I$ and
- if $x \in I$, then $(x+1) \in I$

(57) Show that $\mathbb{R}$ and $\mathbb{R}_{>0}$ are inductive.

(58) Let $X = \{I \subset \mathbb{R} \mid I \text{ is inductive}\}$. Note that X is not empty. Show that
$$\mathbb{N} := \bigcap_{J \in X} J$$
is inductive. Moreover, if I is any inductive subset of $\mathbb{R}$, then $\mathbb{N} \subset I$. That is, $\mathbb{N}$ is the (unique) smallest inductive subset of $\mathbb{R}$.

(59) Show that
$$1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$$
for all $k \in \mathbb{N}$. [Hint: Let $S \subset \mathbb{N}$ be the set
$$S = \left\{ m \in \mathbb{N} : \ 1 + 2 + 3 + \cdots + (m-1) + m = \frac{m(m+1)}{2} \right\}.$$
Show that S is inductive. Conclude that $S \subset \mathbb{N}$ and so $\mathbb{N} = S$.]

We can now define the integers in $\mathbb{R}$ by
$$\mathbb{Z} = \{n \in \mathbb{R} : \ n = 0 \text{ or } |n| \in \mathbb{N}\}$$
and the rational numbers in $\mathbb{R}$ by
$$\mathbb{Q} = \{x \in \mathbb{R} \mid \text{ there exists } a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \text{ such that } x = ab^{-1}\}.$$

Something to Think About

Now that we have $\mathbb{R}$, we need to make sure that our intuition about how $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ play with $\mathbb{R}$ is correct. For example, is $\mathbb{N}$ unbounded in $\mathbb{R}$? Is it true that between every two distinct real numbers there is a rational number? If $x \in \mathbb{R}$ is there an $m \in \mathbb{Z}$ so that $m \leq x < m + 1$? How would you prove these things?

A Brown Sharpie comic by Courtney Gibbons (http://brownsharpie.courtneygibbons.org)

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Worksheet for 16 Jan 2019
Nested interval property, Archimedean property, density of $\mathbb{Q}$ in $\mathbb{R}$
§1.4

Vocabulary: Archimedean Property, Nested Interval Property, Density of $\mathbb{Q}$ in $\mathbb{R}$

This point of this worksheet is to verify some properties of $\mathbb{R}$ that will be used repeatedly throughout the course.

Archimedean Property. While it may seem "obvious" that the natural numbers are unbounded in $\mathbb{R}$, the fact is that there are plenty of fields in which the natural numbers occur as a bounded set. For this reason, we begin by showing that the natural numbers are not bounded in $\mathbb{R}$.

(60) Suppose α is an upper bound for $\mathbb{N}$. Show that for all $n \in \mathbb{N}$ we have $n \leq \alpha$ and $n+1 \leq \alpha$.

(61) Show that $\mathbb{N}$ is unbounded in $\mathbb{R}$. [Hint: Suppose that $\mathbb{N}$ is a bounded subset of $\mathbb{R}$. Use completeness to find a supremum, and the show that one less than this supremum is also an upper bound of $\mathbb{N}$.]

(62) Prove the Archimedean property:

For all $\varepsilon \in \mathbb{R}_{>0}$ there is an $n \in \mathbb{N}$ such that $1/n < \varepsilon$.

(63) What is $\inf(S)$ where $S = \{\frac{1}{m} \,|\, m \in \mathbb{N}\}$? Prove it.

(64) *Bonus.* Consider the field

$$\mathbb{R}(x) := \left\{ \frac{p(x)}{q(x)} \,|\, \text{where } p(x) \text{ and } q(x) \text{ are polynomials with coefficients in } \mathbb{R} \right\}.$$

If $f, g \in \mathbb{R}(x)$, then we write $f < g$ provided that there exists an $\varepsilon > 0$ such that $f(t) < g(t)$ for all $t \in (0, \varepsilon)$. Note that $\mathbb{N}$ naturally sits inside $\mathbb{R}(x)$. Show that $\mathbb{R}(x)$ is an ordered field, but $\mathbb{N}$ is not bounded in $\mathbb{R}(x)$. [Careful: which is larger, $1/x$ or x?]

Nested Interval Property. Suppose we have a sequence[1] of intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

We will investigate conditions on the I_j that guarantee that the intersection of the intervals is nonempty.

Definition. An interval is a subset I of $\mathbb{R}$ with the property that whenever $x, y, z \in \mathbb{R}$ with $x < y < z$ and $x, z \in I$, then $y \in I$.

(65) There are eleven different kinds of intervals in $\mathbb{R}$. Can you enumerate them? [Note: there is a very big difference between finite intervals and intervals with an infinite number of points.]

(66) Suppose $I_j = (0, 1/j)$. What's true about $\bigcap_j I_j$?

(67) Suppose $I_j = [j, \infty)$. What's true about $\bigcap_j I_j$?

(68) Suppose each $I_j = \emptyset$. What's true about $\bigcap_j I_j$?

(69) Suppose each I_j is closed, bounded, and nonempty; that is $I_j = [a_j, b_j]$ for some $a_j \leq b_j$. For $k, \ell \in \mathbb{N}$, what is the relationship between a_k and b_ℓ? If $A = \{a_k \,|\, k \in \mathbb{N}\}$ and $B = \{b_\ell \,|\, \ell \in \mathbb{N}\}$, show that $\sup(A) \leq \inf(B)$ and $\sup(A) \in \bigcap_j I_j$. You have just proved the Nested Interval Property:

The intersection of a sequence of nonempty, closed, bounded, and nested intervals in $\mathbb{R}$ is nonempty.

Note that each of the adjectives – nonempty, closed, bounded – is required.

Density of $\mathbb{Q}$ in $\mathbb{R}$. Our next task is to show that between any two real numbers there is a rational number. Suppose $a, b \in \mathbb{R}$ with $a < b$

(70) Show that there is an $n \in \mathbb{N}$ so that $1/n < b - a$.

(71) Our goal is to find $m \in \mathbb{Z}$ such that $a < m/n < b$. This is equivalent to finding an $m \in \mathbb{Z}$ such that $an < m < bn$. As a *first step* in this direction, show that there is a smallest integer m such that $an < m$.[2]

(72) For the n and m you chose above, note that $m - 1 \leq an$, so $m \leq an + 1$. Now use $1/n < b - a$ to conclude that $m < nb$.

[1] The word *sequence* signals that we will be indexing objects by $\mathbb{N}$.

[2] Hint: You may want to start by showing that if $S \subset \mathbb{Z}$ is nonempty and bounded below, then $\ell = \inf(S) \in S$. To show this, you will probably need to know that the distance between distinct integers is greater than or equal to one. That is, if $i, j \in \mathbb{Z}$ with $i < j$, then $i + 1 \leq j$ – see the *Handout on Induction* for a proof of this fact for $\mathbb{N}$. Note also that if $\ell \notin S$, then for all $\varepsilon > 0$ there exists $s \in S$ such that $\ell < s < \ell + \varepsilon$. Why?

If we knew that $\sqrt{2}$ was a real number (we will show this shortly), then we could use the density of $\mathbb{Q}$ in $\mathbb{R}$ to show that between every two distinct real numbers a and b there is an irrational number as follows: By density, there is a rational number q between $a - \sqrt{2}$ and $b - \sqrt{2}$. Then the irrational number $q + \sqrt{2}$ is between a and b.

Every positive element of $\mathbb{R}$ has a square root. Our final task is to show that for each $a \in \mathbb{R}_{>0}$ the equation $x^2 = a$ has a solution in $\mathbb{R}$. In fact, if d is a solution, then so is $-d$. We will denote by $\sqrt{a}$ the unique positive solution to $x^2 = a$.

(73) Fix $a \in \mathbb{R}_{>0}$ and set $S_a = \{r \in \mathbb{R} \mid r^2 < a\}$. Show that S_a is nonempty and bounded above. [Hint: Show that $a < (a+1)^2$.]

(74) Let $d = \sup(S_a)$. We want to show that $d^2 = a$. By trichotomy, it is enough to rule out the cases $d^2 < a$ and $d^2 > a$. Given enough time on a rainy Sunday, you can come up with the mathematics below; I hope that it is correct.

  (a) If $d^2 > a$, consider $c = d - (\frac{d^2-a}{2d})$. Show that $0 < c < d$ and c is an upper bound of S_a. Why is this a problem? [Hint: compute c^2.]

  (b) If $d^2 < a$, consider $c = \min(d+1, d + \frac{a-d^2}{2(d+1)})$. Show that $c > d \geq 0$ and $c \in S_a$. Why is this a problem? [Hint: Show: $d + 1 \leq d + \frac{a-d^2}{2(d+1)} \Rightarrow 2(d+1) \leq a - d^2$ and $d + \frac{a-d^2}{2(d+1)} < d + 1 \Rightarrow \left(\frac{a-d^2}{2(d+1)}\right)^2 < \frac{a-d^2}{2(d+1)}$. In both cases, compute $c^2 - a$.]

One consequence of this result is that $\mathbb{R}_{\geq 0}$ can be described as the set of squares in $\mathbb{R}$. That is, $\mathbb{R}_{\geq 0} = \{d^2 \mid d \in \mathbb{R}\}$.

Something to Think About

Given that between any two distinct real numbers there is a rational number, it is natural to ask questions like: Are there more rational numbers than irrational numbers? How many rational numbers are there? How many irrationals? How many reals?

What do you think?

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Worksheet for 18 Jan 2019
Cardinality
§§1.5 & 1.6

Vocabulary: Cardinality, countably infinite, uncountable

Cardinality is how mathematicians refer to the size of a set. The cardinality of any finite set is the number of elements in the set. So, for example, the cardinality of the set of Michigan Math T-shirts that Professor DeBacker owns is 92. For infinite sets, we need to be careful. After many years of bickering, mathematicians settled on the following definition.

Definition. Suppose A and B are two sets. We say that A and B *have the same cardinality* provided that there is a bijective function from A to B.

Since the composition of bijective functions is again bijective, if A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

(75) Show that $\mathbb{N}$ and the set of positive even integers have the same cardinality.
(76) Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.
(77) Show that $(-1, 1)$ has the same cardinality as $\mathbb{R}$. [Hint: $x/(x^2 - 1)$.]
(78) *Bonus.* Suppose $a, b \in \mathbb{R}$ with $a < b$. Show that (a, b) has the same cardinality as $\mathbb{R}$.
(79) Consider the map $f \colon \mathbb{Q} \to \mathbb{N}$ defined as follows. Suppose $p/q \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ is in lowest terms (by definition, 0 is $0/1$ in lowest terms). We set

$$f(p/q) = \begin{cases} 2^p \cdot 3^q & \text{if } p/q \geq 0 \\ 2^{-p} \cdot 3^q \cdot 47 & \text{if } p/q < 0 \end{cases}$$

Show that f is injective. What can we conclude about the cardinality of $\mathbb{Q}$? [Warning: In this problem we have assumed that every natural number has a unique prime factorization. This is a fact that we really should verify at some point.]

The Schröder–Bernstein Theorem (see Exercise 1.5.11 in your book) says that if X and Y are sets and there exist injective functions from X to Y and from Y to X, then the sets X and Y have the same cardinality. You will prove this theorem later in the semester, so we will assume it for now. Thanks to this theorem, the following definition makes sense.

Definition. Suppose A and B are two sets. If there is an injective function from A to B, then we say that the cardinality of A is less than or equal to that of B and write $|A| \leq |B|$.

Note that since there is a natural injection from $\mathbb{N}$ to $\mathbb{Q}$, thanks to Prompt 79 we conclude that $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality, that is $|\mathbb{N}| = |\mathbb{Q}|$.

Definition. A set A is said to be *countably infinite* (or sometimes just *countable*) if it has the same cardinality as $\mathbb{N}$. An infinite set that is not countably infinite is said to be *uncountable*.

Since $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite and since $\mathbb{Q}$ is dense in $\mathbb{R}$, one might think that $\mathbb{R}$ is also countably infinite. However, this is very far from the truth.

(80) Suppose that $\mathbb{R}$ is countably infinite. Then there is a bijective function $f \colon \mathbb{N} \to \mathbb{R}$. An interval with more than one point must have infinitely many points; thus, in this class it is called an iWimp™. Let I_1 be a nonempty, closed, bounded iWimp™ that does not contain $f(1)$. Can you find a nonempty, closed, bounded iWimp™ I_2 such that
  - $I_2 \subset I_1$ and
  - $f(2) \notin I_2$?

(81) Continuing in the fashion we can construct a sequence of nonempty, closed, bounded, nested iWimp™s

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

such that for each $n \in \mathbb{N}$ we have $f(n) \notin I_n$. Show that $\bigcap_j I_j$ cannot contain $f(m)$ for all $m \in \mathbb{N}$. Is this a problem? What can we conclude about the cardinality of $\mathbb{R}$?

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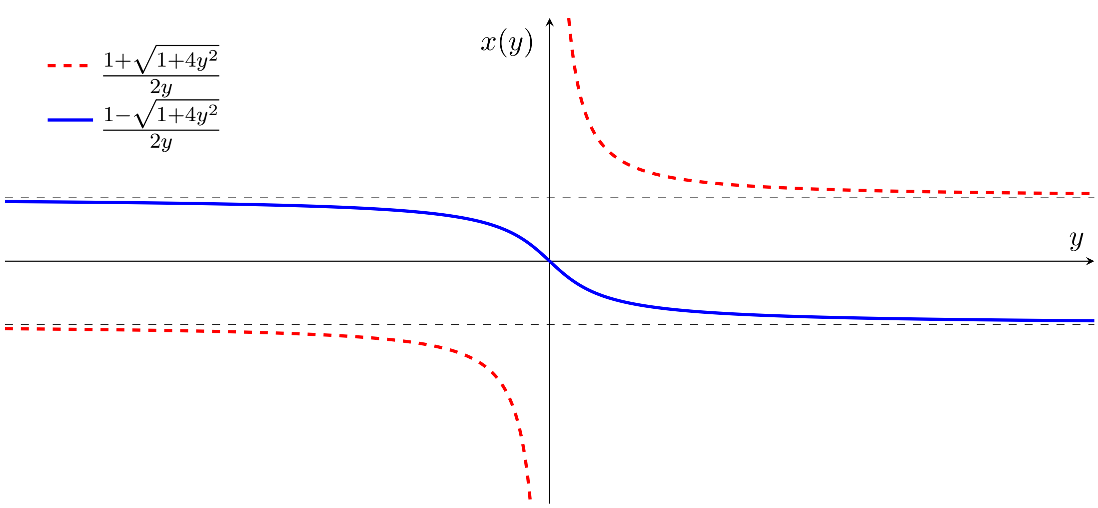
Something to Think About

One can spend considerable time thinking about cardinality. For example, one can ask: is the Cartesian product of countable sets countable? Is the cardinality of the power set of a countable set again countable? Is the union of a countable number of countable sets always countable? Etc. These are all interesting and worthwhile questions to explore, but they are somewhat tangential to our program at present.

(82) Is it possible to have an uncountable collection of nonempty disjoint open intervals?

(83) *Bonus.* Suppose $C \subset [0, 1]$. If C is uncountable, show that there exists $a \in (0, 1)$ so that $C \cap [a, 1]$ is uncountable. If C is just infinite, is there an $a \in (0, 1)$ so that $C \cap [a, 1]$ is infinite?



Perhaps useful for Prompt 77; neither function is defined at zero

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Worksheet[1] for 25 Jan 2019
Sequences: Definition and Limits
§2.2

Vocabulary: sequence, convergence, limit, divergence, ε-neighborhood.

In your calculus courses you learned that for a sequence of real numbers to have a limit it must get "closer and closer" to some number. Conceptually, this is a reasonable definition, but it is extremely difficult to work with. For example, the sequence $(1, 1/2, 1/3, 1/4, 1/5, \ldots)$ is getting closer and closer to the number 0, but it is also getting closer to the numbers -23 and $-\pi$. As another example, the sequence $(\cos(n^3)/n)$, which, when rounded to four decimal places, looks like $(.5403, -.0728, -.0793, .0979, .1575, -.1197, -.1025, -.1246, \ldots)$ converges to zero, but the terms are often moving farther away from zero rather than closer to it! We need to develop a mathematical way to express this notion of "closer and closer." We begin by exploring the idea of a sequence.

Definition. A *sequence* of real numbers is a function $s\colon \mathbb{N} \to \mathbb{R}$ from the natural numbers to the real numbers.

By setting $s_n = s(n)$, we can think of a sequence of real numbers s as a list $(s_1, s_2, s_3, \ldots)$. We will use the notation (s_n) or $(s_n)_{n=1}^{\infty}$ for such a sequence. There is nothing special about limiting our attention to sequences with values in $\mathbb{R}$, so let's explore what sequences may look like in other contexts.

(84) Suppose V is a real vector space. What would it mean to have a sequence of vectors in V?

(85) Give an example of a sequence of vectors in $\mathbb{R}^2$ that converges[2] to $[5, 4\pi]^T$. Give an example of a sequence of vectors in $\mathbb{R}^2 \setminus \{[5, 4\pi]^T\}$ that converges to $[5, 4\pi]^T$.

(86) Give an example of a sequence of vectors in $C^0(\mathbb{R})$ that converges[3] to sin. Give an example of a sequence of vectors in $C^0(\mathbb{R}) \setminus \{\sin\}$ that converges to sin.

(87) Give an example of a sequence $(\vec{v}_n)$ of mutually distinct vectors in $\mathbb{R}^2$ possessing both of the following two properties:
- $\|\vec{v}_n\| < 3$ for all $n \in \mathbb{N}$. (Recall that $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ for $\vec{x} \in \mathbb{R}^m$.)
- The sequence $(\vec{v}_n)$ does not converge.

Do you notice anything (mathematically) interesting about your example?

We now mathematicate™ the notion of "closer and closer."

Definition. A sequence of real numbers (s_n) is said to *converge* to $s \in \mathbb{R}$ provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $|s_n - s| < \varepsilon$.

If a sequence of real numbers (b_n) converges to $b \in \mathbb{R}$, then we will write $\lim_{n\to\infty} b_n = b$ or $b_n \to b$. We often call b the *limit* of (b_n).

Warning: Note that we are using the symbol s in two very different ways: on the one hand it is a function $s\colon \mathbb{N} \to \mathbb{R}$ and on the other it is the limit $s \in \mathbb{R}$ of (s_n). This is awful, but standard. It will be clear from context which we mean, and when when we need both uses, as in Prompt 96, we call the limit by some other name.

(88) Show that the sequence $(1/n) = (1, 1/2, 1/3, 1/4, \ldots)$ converges to 0. [Hint: In the definition of the term *converge*, the number N is a function of ε. That is, for each ε greater than 0 you need to produce an $N = N(\varepsilon)$ so that $n \geq N \Rightarrow |1/n - 0| < \varepsilon$. Thus, your proof should begin something like this "Fix $\varepsilon > 0$. Let $N = \ldots$" and end with something like "Thus, if $n \geq N$, then $|1/n| < \varepsilon$. Consequently, we have shown that $1/n \to 0$."]

(89) *Bonus.* Show that $(\cos(n^3)/n)$ converges to 0.

(90) Discuss in standard, non-mathematical English what it means to say that the sequence (b_n) converges to b.

(91) Suppose $(V, \langle\ ,\ \rangle)$ is an inner product space. As usual, for $\vec{v} \in V$ define $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. Define what it means for a sequence $(\vec{v}_n)$ in V to converge to $\vec{v} \in V$.

[1] Partially based on worksheets developed at The University of Chicago and by Annalisa Crannell at Franklin and Marshall College.

[2] That is, we want a sequence of vectors $(\vec{v}_n)$ in $\mathbb{R}^2$ that get "closer and closer" to $[5, 4\pi]^T$.

[3] Wait! What does this mean? As we shall discover later in the term, there are multiple ways to define convergence in spaces like $C^0(\mathbb{R})$. For now, we will go with the naïve notion of point-wise convergence: A sequence (f_n) of functions in $C^0(\mathbb{R})$ converges to $f \in C^0(\mathbb{R})$ provided that for all $y \in \mathbb{R}$ we have that the sequence of real numbers $(f_n(y))$ is getting "closer and closer" to $f(y)$.

(92) *Important idea.* Suppose $\vec{v}_n = [a_n, b_n, c_n]^T \in \mathbb{R}^3$. Show that $(\vec{v}_n)$ converges to $\vec{v} = [a, b, c]^T \in \mathbb{R}^3$ if and only if (a_n) converges to a, (b_n) converges to b, and (c_n) converges to c in $\mathbb{R}$.

(93) Generalize the statement in the prompt above to the inner product space $(\mathbb{R}^m, \cdot)$. Sketch why the statement is true.

(94) *Bonus.* Now generalize to the setting of a finite dimensional inner product space $(V, \langle\, , \,\rangle)$.

Later, it will be convenient to have a more geometric/topological interpretation of convergence; hence:

Definition. Given a real number $a \in \mathbb{R}$ and an $\varepsilon > 0$, define the *ball of radius ε centered at a* by

$$B_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}.$$

Note that the book uses the notation $V_\varepsilon(a)$ instead of $B_\varepsilon(a)$. Sets of the form $B_\varepsilon(a)$ are often called 'ε-neighborhoods' or 'ε-neighborhoods of a.'

(95) Draw pictures of $B_{\frac{1}{n(n+1)}}(\frac{1}{n})$ for $n \in \{1, 2, 3, 4\}$. *Bonus.* Which $n \in \mathbb{N}$ must be used if we want to cover[1] all of $(0, 1)$ with sets of the form $B_{\frac{1}{n(n+1)}}(\frac{1}{n})$?

(96) Show that if a sequence of real numbers (t_n) converges to $\tau \in \mathbb{R}$, then for all $\varepsilon > 0$ we have that $(\mathbb{R}\backslash B_\varepsilon(\tau)) \cap t[\mathbb{N}]$ is finite. [Hint: First think about what the statement "$(\mathbb{R} \backslash B_\varepsilon(\tau)) \cap t(\mathbb{N})$ is finite" means.] *Bonus.* Is the converse true? If not, under what conditions on (t_n) might the converse be true? [Hint: See Prompt 100.]

(97) Suppose $\vec{v}_n \in \mathbb{R}^3$. Show that $(\vec{v}_n)$ converges to $\vec{v} \in \mathbb{R}^3$ if and only if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that for all $n \geq N$ we have $\vec{v}_n \in B_\varepsilon(\vec{v}) = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x} - \vec{v}\| < \varepsilon\}$.

(98) Generalize the statement in the above prompt to the inner product space $(\mathbb{R}^m, \cdot)$. Sketch why the statement is true.

(99) *Bonus.* Now generalize to the setting of an inner product space $(V, \langle\, , \,\rangle)$.

Alas, many sequences do not converge.

Definition. A sequence of real numbers (s_n) that does not converge is said to *diverge*.

(100) Prove that the sequence $(-1, 1, -1, 1, -1, \ldots) = ((-1)^n)$ in $\mathbb{R}$ diverges. [Hint: Prompt 96 may be useful here.]

(101) Does the sequence $\left(\left[(-1)^n, \dfrac{n+1}{n}\right]^T\right)$ in $\mathbb{R}^2$ converge or diverge? Justify your answer.

Remark: In Prompt 101 and elsewhere it is implicitly understood that we are considering $\mathbb{R}^2$, or more generally $\mathbb{R}^m$, as an inner product space with respect to the dot product.

(102) *Important example*. Draw pictures of the functions $f_n \in C^0([0, 1])$ defined by

$$f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n] \\ 0 & x \in (1/n, 1] \end{cases}$$

Does (f_n) converge (point-wise) in $C^0([0, 1])$?

(103) *Limits are unique.* The above exercise shows that sequences can fail to converge because the "limit" isn't in the same space as the sequence. However, when limits do exist (that is, when our sequence converges), our intuition is that there should only be one limit. We now verify this intuition. Suppose $(\vec{v}_n)$ is a sequence in $\mathbb{R}^m$ and $\vec{v}, \vec{w} \in \mathbb{R}^m$. If $\vec{v}_n \to \vec{v}$ and $\vec{v}_n \to \vec{w}$, then $\vec{v} = \vec{w}$. [Hint: Draw pictures to get a feeling for how to proceed.]

(104) *Bonus.* Generalize the above exercise to the setting of an inner product space $(V, \langle\, , \,\rangle)$.

A convention: We also use the term *sequence* to refer to any function s defined on a set of the form $\mathbb{Z}_{\geq n_0} = \{n \in \mathbb{Z} \mid n \geq n_0\}$ for some $n_0 \in \mathbb{Z}$. We denote such a sequence by $(s_n)_{n=n_0}^{\infty}$. The definition of convergence for such a sequence is pretty much the same – just replace "$N \in \mathbb{N}$" with "$N \in \mathbb{Z}_{\geq n_0}$."

Something to Think About

If (a_n) is a sequence of positive real numbers that converges to $a \in \mathbb{R}$, what can you conclude about a? For example, can a be negative? Must it be positive?

[1] What do you think the word "cover" means in this context? Check with me before proceeding.

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Worksheet for 28 Jan 2019
Sequences: Basic limit properties
§2.3

Vocabulary: bounded, eventually

In today's worksheet we verify that convergent sequences have basic properties that agree with our intuition. For example, we would expect that convergent sequences are "not too far away from the origin" and that they play well with basic algebraic manipulations – addition, multiplication, quotients, etc. Because we are trying to prove that life **always** works as we want, our approach is necessarily very abstract. We begin by recalling some basic definitions. Suppose that $(V, \langle\, ,\ \rangle)$ is an inner product space.[1] For $\vec{x} \in V$, let $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x}\rangle}$.

Definition. A *sequence* of vectors in V is a function from $\mathbb{N}$ to V.

Definition. A sequence of vectors $(\vec{v}_n)$ is said to *converge* to $\vec{v} \in V$ provided that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $\|\vec{v}_n - \vec{v}\| < \varepsilon$.

Recall that the statement '$(\vec{v}_n)$ converges to $\vec{v}$ ' has the following geometric interpretation:

(*) for all $\varepsilon > 0$ the set $\{m \in \mathbb{N} \mid \vec{v}_m \notin B_\varepsilon(\vec{v})\}$ is finite

where $B_\varepsilon(\vec{v}) = \{\vec{x} \in V \mid \|\vec{v} - \vec{x}\| < \varepsilon\}$.
Let us mathematicate™ what "not too far away from the origin" means.

Definition. A sequence $(\vec{v}_n)$ in V is *bounded* provided that there exists a number $M > 0$ such that $\|\vec{v}_n\| < M$ for all $n \in \mathbb{N}$.

Note that $\|\vec{v}_n\| < M$ if and only if $\vec{v}_n \in B_M(\vec{0})$. The book definition of the term bounded allows equality in the statement $\|\vec{v}_n\| < M$; can a sequence be bounded under one definition but not the other?

(105) Suppose $\vec{v} \in V$ and $r > 0$. Show $B_r(\vec{v}) \subset B_{r+\|\vec{v}\|}(\vec{0})$.

(106) If $(\vec{w}_n)$ is a convergent sequence in V, then must $(\vec{w}_n)$ be bounded? [Hint: The geometric interpretation of limits may be useful. (See (*) above.)]

(107) If $(\vec{x}_n)$ is a bounded sequence in V, then must $(\vec{x}_n)$ converge?

Hopefully your response to Prompt 107 is consistent with your response to Prompt 87. If not, do not proceed without discussing things with me.

We now explore how limits interact with the vector space operations of scalar multiplication and vector addition. Suppose (α_n) is a sequence of real numbers that converges to $\alpha \in \mathbb{R}$. Let $(\vec{v}_n)$ and $(\vec{w}_n)$ be sequences in V that converge to $\vec{v}$ and $\vec{w}$, respectively.

(108) Show that the sequence of vectors $(\alpha_n \vec{v}_n)$ converges to $\alpha\vec{v}$ in V. Note: this implies that for all $\vec{x} \in V$ we have $\alpha_n \vec{x} \to \alpha\vec{x}$.

(109) Is the converse true? That is, suppose (β_n) is a sequence of real numbers and $(\vec{x}_n)$ is a sequence in V. If $(\beta_n \vec{x}_n)$ converges in V, then does (β_n) converge in $\mathbb{R}$ and does $(\vec{x}_n)$ converge in V?

(110) Show that the sequence of vectors $(\vec{v}_n + \vec{w}_n)$ converges to $\vec{v} + \vec{w}$.

(111) Is the converse true? That is, suppose $(\vec{x}_n)$ and $(\vec{y}_n)$ are sequences in V. with the property that $(\vec{x}_n + \vec{y}_n)$ converges in V, then do $(\vec{x}_n)$ and $(\vec{y}_n)$ converge in V?

(112) If $\alpha \neq 0$ and $\alpha_n \neq 0$ for all $n \in \mathbb{N}$, is it true that $\dfrac{1}{\alpha_n}\vec{v}_n \to \dfrac{1}{\alpha}\vec{v}$? [Hint: use one of the earlier prompts to reduce this to a question about sequences in $\mathbb{R}$, then use a result from your reading.]

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[1]**Important later.** Most everything that we will say and prove about inner-product spaces holds in the larger context of *normed vector spaces*. A normed vector space is an $\mathbb{R}$-vector space W together with a function $\|\ \|\colon W \to \mathbb{R}$ with the following properties:

- $\|\vec{v} + \vec{w}\| \leq \|\vec{x}\| + \|\vec{w}\|$ for all $\vec{x}, \vec{w} \in W$
- for all $\vec{x} \in W$ we have $\|\vec{x}\| \geq 0$ with equality if and only if $\vec{x} = 0$
- $\|c\vec{x}\| = |c|\,\|\vec{x}\|$ for all $c \in \mathbb{R}$ and $\vec{x} \in W$

The function $\|\ \|\colon W \to \mathbb{R}$ is called a norm. Note that every inner product space is a normed vector space.

Our inner product space has structure beyond scalar multiplication and vector addition, and we now explore how that additional structure, the inner product, interacts with the formation of limits. This is a bit delicate,[1] so we now assume that our inner product space $(V, \langle\ ,\ \rangle)$ is finite dimensional. Suppose (α_n) and (β_n) are sequences of real numbers that converge to α and β, respectively, in $\mathbb{R}$. Let $(\vec{v}_n)$ and $(\vec{w}_n)$ be sequences in V that converge to $\vec{v}$ and $\vec{w}$, respectively.

(113) Is it true that for all $\vec{x}, \vec{y} \in V$ we have $\langle \alpha_n \vec{x}, \vec{y} \rangle \to \alpha \langle \vec{x}, \vec{y} \rangle$? [Hint: What sort of objects are $\langle \alpha_n \vec{x}, \vec{y} \rangle$ and $\alpha \langle \vec{x}, \vec{y} \rangle$? Use Prompt 108 with $V = \mathbb{R}$.]

(114) Is it true that for all $\vec{x} \in V$ we have $\langle \vec{v}_n, \vec{x} \rangle \to \langle \vec{v}, \vec{x} \rangle$? [Hint: Cauchy-Schwarz is a very useful tool, as are your responses to Prompts 97 through 99.]

(115) Is it true that $\langle \vec{v}_n, \vec{w}_n \rangle \to \langle \vec{v}, \vec{w} \rangle$?

(116) *Bonus.* Is it true that $\langle \alpha_n \vec{v}_n, \beta_n \vec{w}_n \rangle \to \alpha\beta \langle \vec{v}, \vec{w} \rangle$?

A convention: Many results, like those found in Prompt 112, hold under more relaxed hypotheses. For example, in Prompt 112 we do not really need that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$. Instead, it would be enough to require that *eventually* we have $\alpha_n \neq 0$; that is, it is OK for a finite number of α_n to be zero – we would disregard such n since we are only interested in the behavior of $\dfrac{1}{\alpha_n}$ in the limit.

Some Things to Think About

[A] Consider the sequence $(s_n) = (\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots)$. Since $\sqrt{2} < 2$, an induction argument shows that $s_1 < s_2 < s_3 < \cdots$. Similarly, since $\sqrt{2} < 2$, an induction argument shows that $s_n < 2$ for all $n \in \mathbb{N}$. Does the sequence need to converge? Justify your answer.

[B] Which of the definitions & results presented on this worksheet do not make sense in the context of normed vector spaces? Which of those that make sense are true?

[1]Please take a course in functional analysis to see how delicate.

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Worksheet for 30 Jan 2019
Monotonicity and an Introduction to Series
§2.4

Vocabulary: nonincreasing, nondecreasing, strictly decreasing, strictly increasing, increasing, decreasing, monotone, series, sequence of partial sums, harmonic series

In this worksheet, we restrict our attention to sequences in $\mathbb{R}$ and discuss properties of limits that result from the fact that the real numbers are ordered.[1] Most of the results we explore in this worksheet do not make sense for general inner product spaces. However, thanks to Gram-Schmidt[2] we will be able to use many of these results about sequences in $\mathbb{R}$ to say something about sequences in an inner product space.

Suppose that (a_n) and (c_n) are sequences of real numbers that converge to a and c, respectively.

(117) *The Squeeze Lemma.* Suppose (b_n) is a sequence of real numbers such that we eventually have $a_n \leq b_n \leq c_n$. What can we conclude about (b_n)? If $a = c$, what can we conclude about (b_n)?

(118) If we eventually have $a_n < 47$, what can we conclude about a?

(119) If $a_n < c_n$ eventually, what can we conclude about a and c?

Definition. A sequence of real numbers (a_n) is said to be *nondecreasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *nonincreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either nonincreasing or nondecreasing.

Definition. A sequence of real numbers (a_n) is said to be *strictly increasing* if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$ and *strictly decreasing* if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *strictly monotone* if it is either strictly increasing or strictly decreasing.

(120) If possible, provide an example of
- a sequence of real numbers that is bounded but not strictly monotonic.
- a sequence of real numbers that is monotonic but not bounded.
- a sequence of real numbers that is both monotonic and bounded.
- a sequence of real numbers that is monotonic but does not converge.
- a sequence of real numbers that is bounded but does not converge.
- a sequence of real numbers that is bounded and monotonic but does not converge.

(121) Show that if (t_n) is a nonincreasing bounded sequence of real numbers, then it converges. [Hint: Your book shows that every nondecreasing bounded sequence converges.]

(122) For $\rho \in [0, 1)$ conclude that the sequence (ρ^n) converges by showing that it is nonincreasing and bounded below by zero.

(123) Define $s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)}$. Show that (s_m) converges by showing that it is an increasing bounded sequence. [Hint: "partial fraction decomposition."]

Prompt 123 provides an example of a sequence defined by sums, such sequences give rise to the notion of a series:

Definition. Let (a_n) be a sequence of real numbers. A *series* (sometimes called an *infinite series*) is a formal expression of the form

$$\sum_{n=1}^{\infty} a_n$$

(often written $\sum a_n$ to conserve ink).

[1] Recall that this means that the following are true:
- if $x, y \in \mathbb{R}_{>0}$, then $x + y \in \mathbb{R}_{>0}$ and $xy \in \mathbb{R}_{>0}$
- if $x \in \mathbb{R}$ then exactly one of the following is true: $x \in \mathbb{R}_{>0}$, $x = 0$, or $-x \in \mathbb{R}_{>0}$.

[2] Why is this thanks to Gram-Schmidt? This is key, so make sure you can answer this question.

A note on words. The word *formal* is often used as a synonym for *rigorous*; however in the above definition and elsewhere in math it is used to convey that an expression is given little meaning beyond the symbols of which it is comprised. We have to accept that it is "just something we write" (at least for the moment).

Definition (continued). The *sequence of partial sums* for the series $\sum a_n$ is the sequence (ps_m) defined by

$$ps_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + \cdots + a_m.$$

The series $\sum a_n$ is said to *converge* to $\alpha \in \mathbb{R}$ provided that the sequence (ps_m) converges to α. In this case we write $\sum a_n = \alpha$. If the sequence of partial sums diverges, then we say that $\sum a_n$ *diverges*.

(124) If possible, provide an example of
- a (non-trivial) series that converges
- a series that diverges.

(125) For $\rho \in [0, 1)$, show that the $\sum \rho^n$ converges by using the definition.

(126) Suppose (a_n) is a sequence. Show that if $\sum a_n$ converges, then $a_n \to 0$. As we shall see in Prompt 127, the converse of this statement is false. [Hint: What is $ps_{n+1} - ps_n$?]

(127) The *harmonic series* is the series $\sum 1/n$. Your book gives one of the older proofs that the harmonic series diverges (due to Nicole Oresme circa 1350), and uses the proof to motivate its presentation of the *Cauchy Condensation Test*. You will provide two more proofs that the harmonic series diverges.
 (a) A proof by contradiction. Suppose that the harmonic series converges, to say S. Derive a contradiction by showing that $0 = S - S > 1/2$. [Hint: Since $\frac{1}{2k-1} + \frac{1}{2k} = \frac{1}{k} + \frac{1}{(2k)(2k-1)}$, we have $ps_{2m} = ps_m + \sum_{k=1}^{m} \frac{1}{(2k)(2k-1)}$.]
 (b) Recall that e^x is an everywhere increasing function, and from calculus we know $e^x > 1 + x$ for all $x \in \mathbb{R}$. (If you can't prove this, don't worry – we will do it later.) Show that e^{ps_m} is not bounded, and conclude that ps_m must therefore diverge.

In Prompt 127a we considered the sequence $(ps_{2m}) = (ps_2, ps_4, ps_6, \ldots)$; this is an example of a subsequence of the sequence $(ps_1, ps_2, ps_3, \ldots)$. Subsequences, which will be formally defined soon, play a very important role in this course.

A convention: For most people, including Abbott, the word *increasing* is synonymous with the word *nondecreasing* and the word *decreasing* is synonymous with the word *nonincreasing*.

Some Things to Think About

[A] What about series where not all of the terms are of the same sign? An example of such a sequence is the alternating harmonic series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}.$$

Can you develop a general technique for evaluating such a series? If $\sum a_n$ converges, what can we conclude about $\sum |a_n|$? If $\sum |a_n|$ converges, what can we conclude about $\sum a_n$?

[B] Above we said that a series converges if and only if the sequence of partial sums converges. There are other reasonable choices for a definition of series convergence. For example, we could instead require that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} ps_m \quad \text{or} \quad \lim_{r\to 1^-} \lim_{m\to\infty} \sum_{n=1}^{m} r^n a_n$$

converge. In the first case we say that $\sum a_n$ is *Cesàro summable* and in the latter we say it is *Abel summable*. Both definitions (and others) are useful (e.g., in Fourier analysis). One can show that (i) if a series converges, then it is Cesàro summable with the same limit and (ii) if it is Cesàro summable, then it is Abel summable with the same limit. What about the converses of statements (i) and (ii)?

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Worksheet for 1 Feb 2019
Subsequences and Bolzano-Weierstrass
§2.5

Vocabulary: subsequences, Bolzano-Weierstrass Theorem

In the introduction to the worksheet for §2.4 we noted that Gram-Schmidt would play an outsized role in our exploration of analysis in inner product spaces. The following warm-up exercise reinforces this point.

(128) Let $(V, \langle\, ,\, \rangle)$ be a finite dimensional inner product space. Thanks to Gram-Schmidt we can choose an orthonormal basis $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ for V. Let $(\vec{x}_n)$ be a sequence in V and for each $1 \leq j \leq m$ let $x_n^j = \langle \vec{x}_n, \vec{v}_j \rangle$. Verify each of the following statements.
  (a) The sequence $(\vec{x}_n)$ converges in V if and only if for each $1 \leq j \leq m$ we have that (x_n^j) converges in $\mathbb{R}$.
  (b) The sequence $(\vec{x}_n)$ is bounded in V if and only if for each $1 \leq j \leq m$ we have that (x_n^j) is bounded in $\mathbb{R}$.

Definition. Let $(\vec{v}_n)$ be a sequence of vectors in an inner product space $(V, \langle\, ,\, \rangle)$, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$(\vec{v}_{n_1}, \vec{v}_{n_2}, \vec{v}_{n_3}, \vec{v}_{n_4}, \vec{v}_{n_5}, \vec{v}_{n_6}, \ldots)$$

is called a *subsequence* of $(\vec{v}_n)$ and is denoted by $(\vec{v}_{n_j})$ where $j \in \mathbb{N}$ indexes the subsequence.

Note that a subsequence is a sequence. Indeed, using the notation of the definition, a subsequence is a function $\mathbb{N} \to V$ that carries k to $\vec{v}_{n_k}$.

(129) *Increasing Index Inequality.* Suppose (n_k) is an increasing sequence of natural numbers. Show that $k \leq n_k$.

(130) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space and $(\vec{x}_n)$ is a convergent sequence in V that converges to $\vec{x} \in V$. Show that every subsequence of $(\vec{x}_n)$ converges to $\vec{x}$.

(131) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space, $\vec{a}, \vec{b} \in V$, and $(\vec{x}_n)$ is a sequence in V. Suppose $(\vec{x}_{n_j})$ and $(\vec{x}_{n_k})$ are two convergent subsequences in $(\vec{x}_n)$ with $(\vec{x}_{n_j}) \to \vec{a}$ and $(\vec{x}_{n_k}) \to \vec{b}$. Show that if $\vec{a} \neq \vec{b}$, then $(\vec{x}_n)$ diverges. Is the converse true? [Careful – what is the converse?]

(132) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space and $(\vec{v}_n)$ is a sequence in V for which $\|\vec{v}_n\| > n$. Does $(\vec{v}_n)$ have a convergent subsequence?

The following exercise provides a proof of the Bolzano-Weierstrass Theorem for $\mathbb{R}$ that is different from the one in your text.

(133) *Bonus.* Let (a_n) be a bounded sequence in $\mathbb{R}$. Define

$$S = \{x \in \mathbb{R} \mid \text{ the set } \{n \in \mathbb{N} \mid x < a_n\} \text{ is an infinite subset of } \mathbb{N}\ \}.$$

Show that there exists a subsequence (a_{n_j}) of (a_n) such that $a_{n_j} \to \sup(S)$. [Hint: Be sure that your proof works for the sequence $((-1)^n/n)$]

(134) (Bolzano-Weierstrass) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space. Every bounded sequence in V has a convergent subsequence.

(135) Suppose that (b_n) is a sequence with values in (e, π). By the Bolzano-Weierstrass Theorem, the sequence (b_n) has a convergent subsequence, call it (b_{n_j}). Must the subsequence (b_{n_j}) converge to an element of (e, π)? If not, to which values outside of (e, π) do you think (b_{n_j}) can converge?

The Bolzano-Weierstrass Theorem will play a key role in our development of analysis; so invest some time in understanding it well.

(136) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space and $(\vec{x}_n)$ is a convergent sequence in V. Show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}_{\geq N}$ we have $\|\vec{x}_n - \vec{x}_m\| < \varepsilon$.

Something to Think About

It is not very difficult to manufacture sequences that appear to converge, but for which we don't know the limit. For example, from calculus we know that

$$\gamma_n = \sum_{j=1}^{n} \frac{1}{j} - \ln(n)$$

is a bounded decreasing sequence of real numbers. How could we show that (γ_n) converges? (You will not be able to find the limit – it is not even known if the limit is rational or not; Prompt 136 may help.)

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Worksheet for 4 Feb 2019
The Cauchy Criterion
§2.6

Vocabulary: Cauchy sequence, Cauchy Criterion

Definition. Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space. A sequence $(\vec{v}_n)$ is called a *Cauchy sequence* provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that whenever $m, \ell \in \mathbb{N}_{\geq N}$ it follows that $\|\vec{v}_\ell - \vec{v}_m\| < \varepsilon$.

(137) Provide an example of each of the following.
- An unbounded sequence that contains a Cauchy subsequence.
- A Cauchy sequence that is not monotone.
- A monotone sequence that is not Cauchy.

(138) Suppose (a_n) is a sequence in $\mathbb{R}$ with the property that $|a_{j+1} - a_j| < 1/j$ for each $j \in \mathbb{N}$. Do you think (a_n) is Cauchy (it is OK to guess)? Does (a_n) need to converge?

(139) Suppose (c_n) is a sequence in $\mathbb{R}$ with the property that $|c_{j+1} - c_j| < 1/2^j$ for each $j \in \mathbb{N}$. Do you think (c_n) is Cauchy? Justify your answer. Does (c_n) need to converge (it is OK to guess)?

(140) Show that every subsequence of a Cauchy sequence is also a Cauchy sequence.[1]

(141) Show that a Cauchy sequence with a convergent subsequence converges.

(142) Show that every Cauchy sequence in a finite dimensional inner product space is bounded.

From prompt 136 we know that every convergent sequence is Cauchy. The proof that every Cauchy sequence converges is considerably deeper.

(143) Let $(V, \langle\, ,\, \rangle)$ be a finite dimensional inner product space. Thanks to Gram-Schmidt we can choose an orthonormal basis $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ for V. Let $(\vec{x}_n)$ be a sequence in V and for each $1 \leq j \leq m$ let $x_n^j = \langle \vec{x}_n, \vec{v}_j \rangle$. Show that $(\vec{x}_n)$ is Cauchy in V if and only if for each $1 \leq j \leq m$ we have that (x_n^j) is Cauchy in $\mathbb{R}$.

(144) (Cauchy Criterion). Show that every Cauchy sequence in a finite dimensional inner product space converges.

(145) Suppose that (b_n) is a Cauchy sequence with values in $(-\ln(2), \sqrt{2})$. Must the sequence (b_n) converge to an element of $(-\ln(2), \sqrt{2})$? If not, to which values outside of $(-\ln(2), \sqrt{2})$ do you think (b_n) can converge?

A convention: In the definition of a Cauchy sequence, we choose $\ell, m \in \mathbb{N}_{\geq N}$. Some authors will require $N \in \mathbb{N}$ and some will require that $\ell > m$. These are cosmetic differences that do not alter our understanding of what a Cauchy sequence is.

Something to Think About

The proof that every Cauchy sequence converges relies heavily on the axiom of completeness for $\mathbb{R}$. Since it is straightforward to produce Cauchy sequences in $\mathbb{Q}$ that do not converge to elements of $\mathbb{Q}$, one might suspect that the link between the two ideas, Cauchy Criterion and the axiom of completeness, is quite strong. Might they be equivalent notions on $\mathbb{R}$?

[1] Warning: You may not assume that a sequence converges if and only if it is Cauchy until after Prompt 144.

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Worksheet for 6 Feb 2019
Series: Basic Results
§2.7

Vocabulary: series, convergence, divergence, sequence of partial sums, sum, geometric sequence, absolute convergence, conditional convergence, Cauchy Criterion for Series, alternating harmonic series, rearrangement

On the Worksheet for §2.5 we introduced the notion of a series in $\mathbb{R}$. Today we extend the definition to inner product spaces and establish some properties for series.

Definition. Suppose $(V, \langle\, ,\, \rangle)$ is an inner product space. Let $(\vec{x}_n)$ be a sequence in V. A *series* (sometimes called an *infinite series*) is a formal expression of the form

$$\sum_{n=1}^{\infty} \vec{x}_n$$

(often written $\sum \vec{x}_n$ to conserve ink). The *sequence of partial sums* for the series $\sum \vec{x}_n$ is the sequence $(\vec{ps}_m)$ in V defined by

$$\vec{ps}_m = \sum_{n=1}^{m} \vec{x}_n = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_m.$$

The series $\sum \vec{x}_n$ is said to *converge* to the *sum* $\vec{x} \in V$ provided that the sequence $(\vec{ps}_m)$ converges to $\vec{x}$. In this case we write $\sum \vec{x}_n = \vec{x}$ and say $\sum \vec{x}_n$ *sums* to $\vec{x}$. If the sequence of partial sums diverges, then we say that $\sum \vec{x}_n$ *diverges*.

(146) Suppose $(\vec{z}_n)$ is a sequence in an inner product space. In the middle of the night, someone changes just under forty-seven trillion of the terms in the sequence $(\vec{z}_n)$. Does this affect the convergence/divergence of $\sum \vec{z}_n$?

(147) Let $(V, \langle\, ,\, \rangle)$ be a finite dimensional inner product space. Thanks to Gram-Schmidt we can choose an orthonormal basis $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ for V. Let $(\vec{x}_n)$ be a sequence in V and for each $1 \le j \le m$ let $x_n^j = \langle \vec{x}_n, \vec{v}_j \rangle$. Show that the series $\sum \vec{x}_n$ converges in V if and only if for each $1 \le j \le m$ we have that the series $\sum x_n^j$ converges in $\mathbb{R}$.

(148) (Linearity for series) Suppose $(V, \langle\, ,\, \rangle)$ is an inner product space. Let $(\vec{x}_n)$ and $(\vec{y}_n)$ be sequences in V. Suppose $\sum \vec{x}_k = \vec{x}$ while $\sum \vec{y}_j = \vec{y}$ with $\vec{x}, \vec{y} \in V$.

(a)

$$\sum_{k=1}^{\infty} \alpha \vec{x}_k = \alpha \vec{x}.$$

(b)

$$\sum_{\ell=1}^{\infty} (\vec{x}_\ell + \vec{y}_\ell) = \vec{x} + \vec{y}.$$

(149) (Cauchy Criterion for Series) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space. Let $(\vec{x}_n)$ be a sequence in V. Show that $\sum \vec{x}_k$ converges if and only if, given $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ such that whenever $\ell > m \in \mathbb{N}_{\ge N}$ it follows that

$$\|\vec{x}_{m+1} + \vec{x}_{m+2} + \vec{x}_{m+3} + \vec{x}_{m+4} + \cdots + \vec{x}_{\ell-1} + \vec{x}_\ell\| < \varepsilon.$$

(150) (Nth term test) Suppose $(V, \langle\, ,\, \rangle)$ is a finite dimensional inner product space. Let $(\vec{y}_n)$ be a sequence in V. Show that if $\sum \vec{y}_j$ converges, then $(\vec{y}_j)$ converges to $\vec{0}$.

(151) For each of the following series in $\mathbb{R}^3$ explain why the series converges/diverges. You may use results you recall from your prior study of caclulus.

-
$$\sum_{n=1}^{\infty} (5/n, 5/n, 5/n)$$

-
$$\sum_{n=47}^{\infty} ((-1)^n 5/n, (-1)^n 5/n, (-1)^{n+1} 5/n)$$

- $$\sum_{t=0}^{\infty}((-1/2)^t,(1/3)^t,(1/7)^t)$$
- $$\sum_{s=0}^{\infty}((-1)^s/2^s,(-1)^s/3^s,(-1)^{s+1}/7^s)$$
- $$\sum_{m=5}^{\infty}((-1/m)^2,(-1)^m/m^2,m^5/(m^{18}\ln(m)))$$

Definition. Suppose $(V,\langle\ ,\ \rangle)$ is an inner product space. Let $(\vec{y}_n)$ be a sequence in V. The *series* $\sum_{n=1}^{\infty}\vec{y}_n$ is said to *converge absolutely* provided that $\sum_{n=1}^{\infty}\|\vec{y}_n\|$ converges to an element of $\mathbb{R}$.

(152) Let $(V,\langle\ ,\ \rangle)$ be a finite dimensional inner product space. Show that if a series converges absolutely, then it satisfies the Cauchy Criterion for Series and thus converges.

Definition. Suppose $(V,\langle\ ,\ \rangle)$ is an Let $(\vec{z}_n)$ be a sequence in V. A converging series $\sum_{n=1}^{\infty}\vec{z}_n$ is said to *converge conditionally* provided that it is not absolutely convergent.

(153) For each of the series that appears in Prompt 151 indicate if the series diverges, converges absolutely, or converges conditionally. (Is this an exhaustive list of options?)

(154) *Bonus.* Let $(V,\langle\ ,\ \rangle)$ be a finite dimensional inner product space. Thanks to Gram-Schmidt we can choose an orthonormal basis $(\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m)$ for V. Let $(\vec{x}_n)$ be a sequence in V and for each $1\leq j\leq m$ let $x_n^j=\langle\vec{x}_n,\vec{v}_j\rangle$. Decide if the following statements are true or false. If true, give a proof. If false, provide a counterexample. [Hint: is it true that $|x_n^1|+|x_n^2|+\cdots+|x_n^m|\geq\|\vec{x}_n\|\geq\max\{|x_n^1|,|x_n^2|,\ldots,|x_n^m|\}$?]
   (a) The series $\sum\vec{x}_n$ converges absolutely in V if and only if for each $1\leq j\leq m$ we have that the series $\sum x_n^j$ converges absolutely in $\mathbb{R}$.
   (b) Suppose $\sum\vec{x}_n$ converges. The series $\sum\vec{x}_n$ converges conditionally in V if and only if there exists j with $1\leq j\leq m$ such that $\sum x_n^j$ converges conditionally in $\mathbb{R}$.

Something to Think About

At the end of §2.7 Abbott demonstrates that any rearrangement of an absolutely convergent sequence converges to the same sum as the original series does. Recall the definition of a rearrangement of a series:

Definition. Suppose $(V,\langle\ ,\ \rangle)$ is a finite dimensional inner product space. Let $\sum\vec{v}_k$ be a series in V. A series $\sum\vec{w}_k$ is called a *rearrangement* of $\sum\vec{v}_k$ if there exists a one-to-one, onto function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\vec{w}_{f(k)}=\vec{v}_k$ for all $k\in\mathbb{N}$.

It is natural to wonder about rearrangements of conditionally convergent sequences. Indeed, in §2.1 Abbott shows that there is a rearrangement of the conditionally convergent alternating harmonic series whose sum is three-halves of its original sum. Riemann showed that, in fact, if $\sum a_n$ is a conditionally convergent series of real numbers, then for all $\beta\in\mathbb{R}$ there is a bijection $f\colon\mathbb{N}\to\mathbb{N}$ so that $\sum a_{f(n)}=\beta$. One way to think about this is as follows:

> The set of all possible sums of rearrangments of a given series of real numbers is either empty, a single point, or the entire real line.

Suppose V is a finite dimensional inner product space. What is the set of all possible sums of rearrangments of a given series of vectors in V?

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Worksheet for 11 Feb 2019
Topology: Open and Closed Sets in V
§3.2

Vocabulary: topology, open, closed, topological space, closure

The basic notions of topology (open, closed, compact, connected, continuous, ...) are everywhere in mathematics. In this worksheet we will explore the Euclidean topology, τ_{E}, on an inner product space $(V, \langle\, , \,\rangle)$. To make it clear that this is part of a much bigger picture, we give the most general definition of a *topological space*:

Definition. A *topological space* is a pair (X, τ) where X is a set and τ is a set of subsets of X satisfying
(1) $\emptyset, X \in \tau$,
(2) the union of any collection of elements of τ is again in τ, and
(3) the intersection of a finite collection of elements of τ is again in τ.
The elements of τ are called the *open* sets in X, and a subset C of X is said to be *closed* provided that $X \setminus C$ is open.

The set τ is often referred to as a *topology* on X. Property 2 says that if I is an indexing set[1] and $\{\mathcal{O}_i \mid i \in I\} \subset \tau$, then

$$\bigcup_{i \in I} \mathcal{O}_i \in \tau.$$

Property 3 says that if $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n \in \tau$, then

$$\bigcap_{i=1}^{n} \mathcal{O}_i = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \cdots \cap \mathcal{O}_n \in \tau.$$

Math 490 is a good place to learn all about general topological spaces.
For us X will usually be V, a finite dimensional inner product space. What will the open sets be? Recall that for $\varepsilon > 0$ and $\vec{v} \in V$ the ball of radius ε around $\vec{v}$ is the set $B_\varepsilon(\vec{v}) = \{\vec{x} \in V \mid \|\vec{x} - \vec{v}\| < \varepsilon\}$.

Definition. A subset $\mathcal{O}$ of V belongs to τ_{E} provided that for each $\vec{a} \in \mathcal{O}$ there exists $\varepsilon > 0$ so that $B_\varepsilon(\vec{a}) \subset \mathcal{O}$.

(155) Give two examples of each of the following in $\mathbb{R}^3$: a set that is open; a set that is closed; a set that is neither open nor closed, a set that is both open and closed.

(156) Decide whether or not the following sets in $\mathbb{R}$ are open, closed, neither, or both.
- $\mathbb{Q}$
- $\{1/n \mid n \in \mathbb{N}\}$
- $\mathbb{N}$
- $(-\pi, e)$
- $[-\pi, e)$
- $[-\pi, e]$

(157) Suppose $A \subset V$ is not open. Show that this means that there is a point $\vec{a} \in A$ such that for all $n \in \mathbb{N}$ we have $(V \setminus A) \cap B_{1/n}(\vec{a}) \neq \emptyset$.

(158) Suppose $\delta > 0$ and $\vec{y} \in V$. Show that $B_\delta(\vec{y})$ is open in V. (That is, show $B_\delta(\vec{y}) \in \tau_{\mathrm{E}}$.)

(159) Show that the x-axis in $\mathbb{R}^2$ is closed in $\mathbb{R}^2$.

(160) Show that any finite collection of points in V is closed in V. Is any countable (i.e., listable) collection of points in V closed?

(161) Show that (V, τ_{E}) is a topological space.

One important property of finite dimensional inner product spaces is that Cauchy sequences can be used to characterize their closed subsets; the next few problems explore this topic.

(162) Suppose C is a subset of V. If C is closed, then every Cauchy sequence in C converges to an element of C.

[1] It is important to keep in mind that I can be any set (e.g. $\{2, \pi, 3\}$, $\mathbb{N}$, $\mathbb{R}$, ...) – its only function is to index a collection of elements in τ.

(163) Suppose C is a subset of V. If every Cauchy sequence in C converges to a point in C, then C is closed. [Hint: Consider proving the contrapostive.]

(164) **Very important.** A subset C of V is closed if and only if every Cauchy sequence in C converges to a point in C.

Definition. Suppose (X, τ) is a topological space and $B \subset X$. The closure of B, denoted $\bar{B}$, is the smallest closed subset of X that contains B.

How do we even know that there is a "smallest" closed subset of X that contains B? Such a set would have the following property: if C is any closed subset of X such that $B \subset C$, then $\bar{B} \subset C$. Well, since the intersection of any collection of closed subsets of X is again a closed subset of X (see today's "Something to Think About"), we see that the intersection of all closed sets that contain B is, in fact, the smallest closed subset of X that contains B. Hence $\bar{B}$ exists.

(165) Find the closure of each of the following subsets of $\mathbb{R}$.
- (a) $(-\pi, e)$
- (b) $[\ln(2), \sqrt{2}]$
- (c) $\mathbb{Z}$
- (d) $\mathbb{Q}$

(166) Suppose (X, τ) is a topological space and $A, C, D, E \subset X$. Show
- If $A \subset C$, then $\bar{A} \subset \bar{C}$.
- $\overline{D \cap E} \subset \bar{D} \cap \bar{E}$.

Definition. Suppose $(V, \langle\, , \rangle)$ is a finite dimensional inner product space and $B \subset V$. The *sequential closure* of B is the set

$$\{x \in V \mid \text{there exists a Cauchy sequence in } B \text{ that converges to } x\}.$$

The next few prompts establish that the sequential closure of a subset B of V is equal to the closure, $\bar{B}$, of B.

(167) Suppose $(V, \langle\, , \rangle)$ is a finite dimensional inner product space and $D \subset V$. Show that D is contained in the sequential closure of D.

(168) Suppose $(V, \langle\, , \rangle)$ is a finite dimensional inner product space and $C \subset V$. Show that the sequential closure of C is closed. [Hint: Let $\hat{C}$ denote the sequential closure of C. Let (x_n) be a Cauchy sequence in $\hat{C}$ that converges to $x \in V$. Show that $x \in \hat{C}$.]

(169) Suppose $(V, \langle\, , \rangle)$ is a finite dimensional inner product space and $E \subset V$. Show that $\bar{E}$ is equal to the sequential closure of E.

Summary. Suppose $(V, \langle\, , \rangle)$ is a finite dimensional inner product space and $A \subset V$.
- The closure, $\bar{A}$, of A is equal to the sequential closure of A.
- A is closed if and only if $A = \bar{A}$.
- A is closed if and only if every Cauchy sequence in A converges to an element of A.

Wait! Why is A closed if and only if $A = \bar{A}$? Can you prove that?

Some Things to Think About

[A] DeMorgan's Laws tell us (a) the complement of a union is the intersection of the complements and (b) the complement of an intersection is the union of the complements. Using these rules and their generalizations, the notion of a topological space can be recast in terms of closed sets. Try to carry this out.

[B] The *contraction mapping theorem* is an extremely useful result that is used in many branches of mathematics from differential equations to multivariable calculus. It was first proved in the PhD thesis of Stefan Banach in the early 1920s. It states the following: Suppose $K \subset V$ is nonempty and closed. If $f\colon K \to K$ is Lipschitzian,[1] then f has a unique fixed-point. How might you show this?

[1] That is, there is some $c \in [0, 1)$ such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for all $x, y \in K$.

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Worksheet for 13 Feb 2019
Topology: Sequential Compactness
§3.3

Vocabulary: compact, sequentially compact, open cover, finite subcover, Heine-Borel Theorem, bounded, unbounded

In this worksheet we explore one of the fundamental notions in topology: compactness. To make it clear that this is part of a much bigger picture, we first give the most general definition of the term *compact*.

Definition. Suppose (X, τ) is a topological space and $A \subset X$. An *open cover* for A is a collection of open sets $\{\mathcal{O}_\lambda \,|\, \lambda \in \Lambda\} \subset \tau$ whose union contains the set A; that is

$$A \subset \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda.$$

(Here Λ is an indexing set.) Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A. A subset A of X is said to be *compact* provided that every open cover for A has a finite subcover.

Most people find this definition difficult to understand. Thankfully, for a finite dimensional inner product space there are several equivalent ways to understand compactness, and you may find some of these equivalent notions to be more approachable. Before exploring these equivalent ways to think about compactness, we should first check that we understand the definitions above.

(170) Let $S = \{1/n \,|\, n \in \mathbb{N}\}$
  (a) Show that the collection of open sets $\mathcal{C} = \{B_{1/m(m+1)}(1/m) \,|\, m \in \mathbb{N}\}$ is an open cover for S.
  (b) Does $\mathcal{C}$ contain a finite subcover of S? [Hint: which elements of $\mathcal{C}$ contain $1/47$?]

(171) Show that any finite subset of $\mathbb{R}$ is compact in $\mathbb{R}$.

(172) Show that $\mathbb{N}$ is not a compact subset of $\mathbb{R}$.

Definition. Suppose$(V, \langle\, ,\, \rangle)$ is an inner product space. A subset D of V is said to be *bounded* provided that there exists an $n \in \mathbb{N}$ for which $D \subset B_n(\vec{0})$. A subset of V is said to be *unbounded* provided that it is not bounded.

(173) Let A be an unbounded subset of an inner product space V. By considering the open cover $\mathcal{C} = \{B_\ell(\vec{0}) \,|\, \ell \in \mathbb{N}\}$, show that A cannot be compact. Conclude that every compact subset of V must be bounded. (Sadly, as Prompt 170 shows, the converse is not true.)

(174) Suppose D is a subset of an inner product space V that is not closed. Suppose $\vec{v} \in \bar{D} \setminus D$. By considering the open cover $\mathcal{C} = \{V \setminus \bar{B}_{1/n}(\vec{v}) \,|\, n \in \mathbb{N}\}$ of D (why is it an open cover?), show that D cannot be compact. Conclude that every compact subset of V must be closed. (Sadly, as Prompt 172 shows, the converse is not true.)

(175) Show that if K is a compact subset of an inner product space, then K is closed and bounded.

Definition. Suppose$(V, \langle\, ,\, \rangle)$ is an inner product space. A subset A of V is said to be *sequentially compact* provided that every sequence in A has a subsequence that converges to an element of A.

(176) Some exercise in using the definition of sequentially compact.
  (a) Show that the set S in Prompt 170 is not sequentially compact.
  (b) Show that any finite subset of $\mathbb{R}$ is sequentially compact in $\mathbb{R}$.
  (c) Show that $\mathbb{N}$ is not a sequentially compact subset of $\mathbb{R}$.
  (d) Let A be an unbounded subset of an inner product space V. For each $n \in \mathbb{N}$, choose $\vec{a}_n \in A \setminus B_n(\vec{0})$ (why must such an $\vec{a}_n$ exist?). Show that $(\vec{a}_n)$ does not have a convergent subsequence and so A cannot be sequentially compact. Conclude that every sequentially compact subset of V must be bounded. (Sadly, as Prompt 176a shows, the converse is not true.)
  (e) Show that every bounded and closed interval in $\mathbb{R}$ is sequentially compact. [Hint: Bolzano-Weierstrass may be useful.]

(177) Recall that in Prompt 164 you showed that a subset of a finite dimensional inner product space is closed if and only if every Cauchy sequence in the subset converges to an element of the subset. Use this result to show that a sequentially compact subset of an inner product space is closed.

(178) Prove the Heine-Borel theorem: A subset C of a finite dimensional inner product space $(V, \langle\ ,\ \rangle)$ is sequentially compact if and only if it is both closed and bounded. [Hint: Bolzano-Weierstrass may be useful.]

(179) Show that a nonempty sequentially compact subset of $\mathbb{R}$ has both a maximum and a minimum.

The definition of compact presented at the start of this worksheet was not stated until relatively late – some eighty years after Bolzano understood that bounded sequences of real numbers have convergent subsequences. This gives you some idea of how people struggled with it. Thanks to this worksheet and upcoming homework problems, the following theorem is valid.

Theorem. Suppose K is a subset of a finite dimensional inner product space $(V, \langle\ ,\ \rangle)$. The following statements are equivalent.

- K is compact.
- K is sequentially compact.
- K is closed and bounded.

Something to Think About

Which of the equivalent statements appearing in the theorem above one chooses to use depends on the situation at hand. It is kind of like choosing a basis in linear algebra – the basis that one chooses to work with is heavily influenced by the problem that one is trying to solve. Can you think of a situation where the first statement may be the better one to use?

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Worksheet for 15 Feb 2019
Topology: Connected Sets
§3.4

Vocabulary: separated, disconnected, connected, subspace topology, interval

In this worksheet we explore another of the fundamental notions in topology: connectedness. To make it clear that this is part of a much bigger picture, we explore the most general definition of the term *connected*.

Definition. Suppose (X, τ) is a topological space. Two nonempty subsets A, B of X are said to be *separated* provided that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Note the requirement that both A and B be **nonempty**; many people neglect this condition to their detriment.

(180) Are any of the following pairs of subsets of $\mathbb{R}$ separated? Justify your answer.
  (a) $\{\pi\}$, $\mathbb{N}$
  (b) $\mathbb{N}$, $(-5, 5) \setminus \{1\}$
  (c) $\{\pi\}$, $(-5, 5) \setminus \{1\}$

Definition. Suppose (X, τ) is a topological space. A subset E of X is said to be *disconnected* provided that it can be written as the union of two separated sets. A set that is not disconnected is said to be *connected*.

(181) Are any of the following subsets of $\mathbb{R}$ disconnected? Justify your answer.
  (a) $\{22/7\}$
  (b) $\mathbb{Q}$
  (c) $(-17, 8) \setminus \{-1\}$
  (d) $S = \{1/n \,|\, n \in \mathbb{N}\}$
  (e) $\emptyset$

As you may have noticed, we have assiduously avoided asking about the connectedness of non-trivial sets. For example, it would be nice to know that every interval in $\mathbb{R}$ is connected. We now work to correct this shortcoming.

(182) Suppose $(V, \langle\ ,\ \rangle)$ is a finite dimensional inner product space.
  (a) Suppose that V is disconnected. Then there exist nonempty $A, B \subset V$ with $A \cap \bar{B} = \bar{A} \cap B = \emptyset$ and $V = A \cup B$. We are going to construct a Cauchy sequence $(\vec{a}_n)$ in A and a Cauchy sequence $(\vec{b}_n)$ in B such that $(\vec{a}_n)$ and $(\vec{b}_n)$ both converge to the same element of V. Wait! Can such a sequence exist? Why or why not?
  (b) Continue with the notation of Prompt 182a. Since A and B are nonempty, we can choose $\vec{a}_1 \in A$ and $\vec{b}_1 \in B$. Let $\vec{c}_1 = (\vec{a}_1 + \vec{b}_1)/2 \in V$. Since $\vec{c}_1 \in V = A \cup B$, we know that either $\vec{c}_1 \in A$ or $\vec{c}_1 \in B$. If $\vec{c}_1$ belongs to A, then define $\vec{a}_2 = \vec{c}_1$ and $\vec{b}_2 = \vec{b}_1$. Otherwise, set $\vec{a}_2 = \vec{a}_1$ and $\vec{b}_2 = \vec{c}_1$. How should we define $\vec{c}_2$? $\vec{a}_3$? $\vec{b}_3$? $\vec{c}_3$? $\vec{a}_4$? $\vec{b}_4$? ...
  (c) Show that both $(\vec{a}_n)$ and $(\vec{b}_n)$ are Cauchy. [Hint: How far can $\vec{c}_\ell$ be from $\vec{c}_k$ if $\ell > k$?[1]]
  (d) Show there is a $\vec{v} \in V$ so that $\vec{a}_n \to \vec{v}$ if and only if $\vec{b}_n \to \vec{v}$.
  (e) Conclude that every finite dimensional inner product space is connected.

(183) The above problem suggests a general approach for determining if a subset of a finite dimensional inner product space is connected. Suppose $(V, \langle\ ,\ \rangle)$ is a finite dimensional inner product space. Show that a subset E of V is connected iff for all nonempty disjoint subsets $A, B \subset V$ for which $E = A \cup B$ there exists a Cauchy sequence contained in one of A or B that converges to an element of the other.

Definition. A subset I of $\mathbb{R}$ is called an interval provided that whenever $x, y, z \in \mathbb{R}$ with $x < y < z$ and $x, z \in I$ we have $y \in I$.

(184) Are any of the following subsets of $\mathbb{R}$ intervals? Justify your answer.

[1] More precisely, perhaps you may want to show that $\|\vec{c}_k - \vec{c}_{k+1}\| \leq \frac{\|\vec{a}_1 - \vec{b}_1\|}{2^{k+1}}$, $\|\vec{a}_k - \vec{a}_{k+1}\| \leq \frac{\|\vec{a}_1 - \vec{b}_1\|}{2^k}$, $\|\vec{b}_k - \vec{b}_{k+1}\| \leq \frac{\|\vec{a}_1 - \vec{b}_1\|}{2^k}$, and $\|\vec{a}_k - \vec{b}_k\| \leq \frac{\|\vec{a}_1 - \vec{b}_1\|}{2^{k-1}}$.

(a) $\{22/7\}$
(b) $\mathbb{Q}$
(c) $(-17, 8) \setminus \{-1\}$
(d) $S = \{1/n \,|\, n \in \mathbb{N}\}$
(e) $\emptyset$

(185) Show that if a subset D of $\mathbb{R}$ is connected, then it is an interval. [Hint: If D has fewer than two elements, this is obvious. So, assume it has two or more elements. Fix $a, b \in D$ with $a < b$. For c between a and b, look at $A = (-\infty, c) \cap D$ and $B = (c, \infty) \cap D$. Remember: you want to show that $c \in D$. Prompt 166 may be useful.]

(186) Show that if a subset D of $\mathbb{R}$ is an interval, then it is connected.

(187) Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

Something to Think About

Suppose (X, τ) is a topological space. If $Y \subset X$, then we could try to endow Y with a topology by defining $\tau_Y = \{\mathcal{O} \cap Y \,|\, \mathcal{O} \in \tau\}$. It is an **excellent** exercise to show that (Y, τ_Y) is a topological space. We call τ_Y the *subspace topology*.

This notion of a subspace topology allows us to write down a more "standard" definition of connected: The space X is connected provided that it is not disconnected. The space X is disconnected if there exists nonempty, disjoint $A, B \in \tau$ such that $X = A \sqcup B$. A subset Y of X is connected provided that Y is connected in the subspace topology.

Can you show that this definition of connected is equivalent to the one given above?

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Worksheet for 22 Feb 2019
Limits
§4.2

Vocabulary: limit points, limits

In your calculus course(s) you evaluated limits like $\lim_{x\to 3}(x^2+2x-15)/(x-3)$. Note that 3 is **not** in the domain of the function $x\mapsto(x^2+2x-15)/(x-3)$, yet the limit as x approaches 3 of the function makes sense. On the other hand, if you were given the function $g\colon \mathbb{N}\to\mathbb{R}$ defined by $g(n)=3^n$, the expression $\lim_{x\to 3} g(x)$ makes no sense. These two examples show that we need to be a little careful about the points at which we attempt to evaluate limits.

(188) Explain why taking the limit at 3 makes sense in the first example, but not in the second.

The points where 'taking a limit makes sense' will be called *limit points*. Here is a more formal definition.

Definition. Suppose $(V,\langle\,,\,\rangle)$ is a finite dimensional inner product space and $A\subset V$. A point $\vec{v}\in V$ is called a *limit point of* A provided that for all $\delta>0$ we have that $B_\delta(\vec{v})$ contains an element of A other than $\vec{v}$, that is $A\cap(B_\delta(\vec{v})\backslash\{\vec{v}\})\neq\emptyset$.

(189) Show that 3 is a limit point of both $\mathbb{R}$ and $\mathbb{R}\setminus\{3\}$, but 3 is not a limit point of $\mathbb{N}$.

(190) Suppose $(V,\langle\,,\,\rangle)$ is a finite dimensional inner product space and $A\subset V$. Show that $\vec{v}\in V$ is a limit point of A if and only if there is a sequence in $A\setminus\{\vec{v}\}$ that converges to $\vec{v}$.

(191) Suppose $(V,\langle\,,\,\rangle)$ is a finite dimensional inner product space and $B\subset V$. Is every point in B a limit point of B? Is every limit point of B an element of B?

(192) Suppose $(V,\langle\,,\,\rangle)$ is a finite dimensional inner product space and $C\subset V$. Show that C is closed if and only if every limit point of C belongs to C.

We are now in a position to define what it means to evaluate a limit. As usual, there will be several equivalent ways to think about this concept.

Definition. Suppose $(V,\langle\,,\,\rangle_V)$ and $(W,\langle\,,\,\rangle_W)$ are finite dimensional inner product spaces. Suppose $A\subset V$, $\vec{v}$ is a limit point of A, $\vec{w}\in W$, and $f\colon A\to W$ is a function. We say $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}$ provided that for all $\varepsilon>0$ there is $\delta>0$ such that

$$\text{whenever } \vec{x}\in A \text{ and } 0<\|\vec{x}-\vec{v}\|_V<\delta \text{ it follows that } \|f(\vec{x})-\vec{w}\|_W<\varepsilon.$$

(193) With the notation of the definition, what does it mean to say $\lim_{\vec{x}\to\vec{v}} f(\vec{x})\neq\vec{w}$?

(194) Suppose $(V,\langle\,,\,\rangle_V)$ and $(W,\langle\,,\,\rangle_W)$ are finite dimensional inner product spaces. Fix a linear transformation $T\colon V\to W$ and an ONB $\{\vec{e}_i\,|\,1\le i\le n=\dim(V)\}$ for V. Set $c=\sqrt{n}\cdot\max\{\|T(\vec{e}_i)\|_W\,|\,1\le i\le n\}$.

(a) Show that $\|T(\vec{v})\|_W\le c\cdot\|\vec{v}\|_V$ for all $\vec{v}$ in V. [Hint: Apply the Cauchy-Schwarz inequality to the vectors $[a_1,a_2,\ldots,a_n]^T$ and $[1,1,1,1,\ldots,1]^T$ in $\mathbb{R}^n$.]

(b) Conclude that $\lim_{\vec{x}\mapsto\vec{0}} T(\vec{x})=\vec{0}$.

Note that the definition of limit can be recast as follows (same notation as in the definition): We say $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}$ provided that for all $\varepsilon>0$ there is $\delta>0$ such that

$$\text{whenever } x\in A\cap B_\delta(\vec{v})\backslash\{\vec{v}\} \text{ it follows that } f(\vec{x})\in B_\varepsilon(\vec{w}).$$

The definition can also be recast in terms of sequences:

(195) Show: if $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}$, then for all sequences $(\vec{a}_n)$ in $A\backslash\{\vec{v}\}$ that converge to $\vec{v}$ we have that the sequence $(f(\vec{a}_n))$ converges to $\vec{w}$.

(196) Show: If $\lim_{\vec{x}\to\vec{v}} f(\vec{x})\neq\vec{w}$, then there is a sequence $(\vec{a}_n)$ in $A\setminus\{\vec{v}\}$ that converge to $\vec{v}$ yet $(f(\vec{a}_n))$ does not converge to $\vec{w}$.

(197) Conclude: $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}$ if and only if for all sequences $(\vec{a}_n)$ in $A\backslash\{\vec{v}\}$ that converge to $\vec{v}$ we have that the sequence $(f(\vec{a}_n))$ converges to $\vec{w}$.

Q: Should one use the $\varepsilon-\delta$ or sequence defintion of limit when attacking a problem? **A:** Use the one that is best suited to the problem at hand.

(198) Show that limits are unique; that is, if $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}$ and $\lim_{\vec{x}\to\vec{v}} f(\vec{x})=\vec{w}'$, then $\vec{w}=\vec{w}'$

Definition. Suppose $(V, \langle\, , \rangle_V)$ and $(W, \langle\, , \rangle_W)$ are finite dimensional inner product spaces. Suppose $A \subset V$, $\vec{v}$ is a limit point of A, and $f\colon A \to W$ is a function. We say $\lim_{\vec{x}\to\vec{v}} f(\vec{x})$ does not exist if there is no $\vec{w} \in W$ for which $\lim_{\vec{x}\to\vec{v}} f(\vec{x}) = \vec{w}$.

(199) Evaluate $\lim_{x\to -3} 5x + 2$.
(200) Evaluate $\lim_{x\to 3}(x^2 + 2x - 15)/(x - 3)$.
(201) Let $A = \{[s, t]^T \in \mathbb{R}^2 \mid t \neq 3\}$. Define $h\colon A \to \mathbb{R}^2$ by $h([x, y]^T) = [5x + 2, (y^2 + 2y - 15)/(y - 3)]^T$. Evaluate $\lim_{\vec{x}\to[-3,3]^T} h(\vec{x})$.
(202) Define $j\colon \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} x/\,|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Evaluate $\lim_{z\to 0} j(z)$.
(203) Define $\ell\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\ell([x, y]^T) = \begin{cases} xy/(x^2 + y^2) & [x, y]^T \neq \vec{0} \\ 0 & [x, y]^T = \vec{0} \end{cases}$$

Evaluate $\lim_{\vec{z}\to\vec{0}} \ell(\vec{z})$.

To put your mind at ease: polynomials, exponential, logarithms, etc. are all continuous where they are supposed to be. This will be taken up elsewhere.

Alternate notation. Some people refer to limit points as *accumulation points* or *cluster points*.

Something to Think About

Recasting the notion of taking a limit in terms of sequences (see Prompt 197) allows us to apply previous results to (easily) obtain new results for limits. Here are some examples.
Suppose $(V, \langle\, , \rangle_V)$ and $(W, \langle\, , \rangle_W)$ are finite dimensional inner product spaces. Suppose $A \subset V$, $\vec{v}$ is a limit point of A, $f, g, h\colon A \to W$ are functions, $k \in \mathbb{R}$, and $\vec{w}, \vec{w}' \in W$. If $\lim_{\vec{x}\to\vec{v}} f(\vec{x}) = \vec{w}$ and $\lim_{\vec{x}\to\vec{v}} g(\vec{x}) = \vec{w}'$, then

- $\lim_{\vec{x}\to\vec{v}} kf(\vec{x}) = k\vec{w}$
- $\lim_{\vec{x}\to\vec{v}}(f + g)(\vec{x}) = \vec{w} + \vec{w}'$
- $\lim_{\vec{x}\to\vec{v}}\langle f(\vec{x}), g(\vec{x})\rangle = \langle \vec{w}, \vec{w}'\rangle$

Moreover, we have a pretty straightforward *divergence criterion*: If there are two sequences $(\vec{y}_n)$ and $(\vec{z}_n)$ in $A\backslash\{\vec{v}\}$ that both converge to $\vec{v}$ but for which $(h(\vec{y}_n))$ and $(h(\vec{z}_n))$ do not converge to the same limit, then $\lim_{\vec{x}\to\vec{v}} h(\vec{x})$ does not exist.
Can you prove all of the above in your head? What other results follow easily from our previous work?

Epsilon, an SMBC comic by Zach Weinersmith (www.smbc-comics.com)

Worksheet for 25 Feb 2019
Continuity
§4.3

Vocabulary: continuous, continuous at a point

In this worksheet we will explore the the notion of continuity for functions between (finite dimensional) inner product spaces. To make it clear that this is part of a much bigger picture, we give the most general definition of *continuous*:

Definition. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. A function from X to Y is *continuous* provided that the inverse image of every open set is open. That is, $f: X \to Y$ is continuous provided that $f^{-1}[\mathcal{O}] \in \tau_X$ for all $\mathcal{O} \in \tau_Y$.

Over the next few worksheets we will explore how continuity plays with the notions of compactness and connectedness. In this worksheet, we will work to better understand the notion of continuity. However, before doing this we do a little work in the general setting.

(204) Show: If $A \subset Y$, then $f^{-1}[Y \setminus A] = X \setminus f^{-1}[A]$. [Recall that $f^{-1}[A] = \{x \in X \mid f(x) \in A\}$.]

(205) Show that $f: X \to Y$ is continuous if and only if the inverse image of every closed set is closed.

(206) Suppose (Z, τ_Z) is another topological space and $g: Y \to Z$ is continuous. Show that $g \circ f: X \to Z$ is continuous. That is, show that the composition of continuous functions is continuous.

Definition. Suppose $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ are inner product spaces. Suppose $A \subset V$, $\vec{a} \in A$, and $f: A \to W$ is a function. We say that f is *continuous at* $\vec{a}$ provided that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{whenever } \vec{x} \in A \text{ and } \|\vec{x} - \vec{a}\|_V < \delta \text{ it follows that } \|f(\vec{x}) - f(\vec{a})\|_W < \varepsilon.$$

(207) With the notation of the definition, what does it mean to say f is not continuous at $\vec{a}$?

(208) Suppose $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ are finite dimensional inner product spaces. Fix a linear transformation $T: V \to W$.

  (a) Show that T is continuous at $\vec{0}$. [Hint: See Prompt 194a.]

  (b) Use the linearity of T to show that it is continuous at each $\vec{v} \in V$.

Note that the definition of continuity at a point can be recast as follows (same notation as in the definition): We say $f: A \to W$ is continuous at $\vec{a}$ provided that for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{whenever } \vec{x} \in A \cap B_\delta(\vec{a}) \text{ it follows that } f(\vec{x}) \in B_\varepsilon(f(\vec{a})).$$

The definition of continuity at a point can also be recast in terms of sequences:

(209) Show: If $f: A \to W$ is continuous at $\vec{a}$, then for all sequences $(\vec{a}_n)$ in A that converge to $\vec{a}$ we have that the sequence $(f(\vec{a}_n))$ converges to $f(\vec{a})$.

(210) Show: If $f: A \to W$ is not continuous at $\vec{a}$, then there is a sequence $(\vec{a}_n)$ in A that converge to $\vec{a}$ yet $(f(\vec{a}_n))$ does not converge to $f(\vec{a})$.

(211) Conclude: The function $f: A \to W$ is continuous at $\vec{a}$ if and only if for all sequences $(\vec{a}_n)$ in A that converge to $\vec{a}$ we have that the sequence $(f(\vec{a}_n))$ converges to $f(\vec{a})$.

(212) Define $h: \mathbb{R}^2 \to \mathbb{R}^2$ by $h([x, y]^T) = [5xy + 2, y + x - 5]^T$. Is h continuous at $[5, 8]^T$?

(213) Is the function $g: \mathbb{N} \to \mathbb{R}$ defined by $g(n) = 3^n$ continuous at 3?

(214) Prompt 213 highlights a key difference between the 'calculus definition of continuity' in terms of limits, and the definition given here; can you identify the difference? [Theorem 4.3.2 in Abbott discusses the equivalence of the two notions for limit points of A.]

For the next few problems, we suppose $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ are inner product spaces, $A \subset V$, and $h: A \to W$ is a function. The goal of this sequence of problems is to show that $h: A \to W$ is continuous if and only if it is continuous at each $\vec{a} \in A$.

(215) To say that h is continuous means that $h^{-1}[\mathcal{O}]$ is open in A for all open $\mathcal{O}$ in W. What does it mean for a subset $\mathcal{O}'$ of A to be open? It means that $\mathcal{O}'$ is open in the subspace topology on A (see the worksheet for §3.4); that is,

$$\text{a subset } \mathcal{O}' \text{ of } A \text{ is open in } A \text{ if and only if } \mathcal{O}' = U \cap A \text{ for some open } U \text{ in } V.$$

Show that a subset $\mathcal{O}'$ of A is open if and only if for every $\vec{v} \in \mathcal{O}'$ there is a $\delta_{\vec{v}} > 0$ so that $B_{\delta_{\vec{v}}}(\vec{v}) \cap A \subset \mathcal{O}'$.

(216) Show: If $h \colon A \to W$ is continuous, then it is continuous at each $\vec{a} \in A$.

(217) Show: If $h \colon A \to W$ is continuous at each $a \in A$, then for each open $\mathcal{O} \subset W$ we have that $h^{-1}[\mathcal{O}]$ is open in A.

(218) **Important**. Conclude that $h \colon A \to W$ is continuous if and only if it is continuous at each $\vec{a} \in A$.

We close with some examples.

(219) Show that every linear transformation between two finite dimensional inner product spaces is continuous.

(220) Suppose $(V, \langle \, , \, \rangle_V)$ is a finite dimensional inner product space. Fix $\vec{x} \in V$. Is the function $\vec{v} \mapsto \langle \vec{x}, \vec{v} \rangle$ from V to $\mathbb{R}$ continuous?

(221) Suppose $(V, \langle \, , \, \rangle_V)$ is a finite dimensional inner product space. Is the function $\vec{v} \mapsto \|\vec{v}\|$ from V to $\mathbb{R}$ continuous?

Something to Think About

Recasting the notion of taking a limit in terms of sequences (see Prompt 211) allows us to apply previous results to (easily) obtain new results for continuous functions. Here are some examples.

Suppose $(V, \langle \, , \, \rangle_V)$ and $(W, \langle \, , \, \rangle_W)$ are finite dimensional inner product spaces. Suppose $A \subset V$, $\vec{v} \in A$, $f, g, h \colon A \to W$ are functions, and $k \in \mathbb{R}$. If $f \colon A \to W$ and $g \colon A \to W$ are continuous at $\vec{v}$, then

- $kf \colon A \to W$ is continuous at $\vec{v}$.
- $f + g \colon A \to W$ is continuous at $\vec{v}$.
- The function $\vec{x} \mapsto \langle f(\vec{x}), g(\vec{x}) \rangle_W$ from A to $\mathbb{R}$ is continuous.[1]

Moreover, we have a pretty straightforward *criterion for discontinuity*: If there is a sequence $(\vec{y}_n)$ in A that converges to $\vec{v}$ but for which $(f(\vec{y}_n))$ does not converge to $f(\vec{v})$, then $f \colon A \to W$ is not continuous.

Can you prove all of the above in your head? What other results follow easily from our previous work?

[1] Warning: this function is really a composition of functions $A \to W \times W$ and $W \times W \to \mathbb{R}$. The first function sends $\vec{a} \in A$ to $(f(\vec{a}), g(\vec{a}))$ and the second sends $(\vec{w}_1, \vec{w}_2) \in W \times W$ to $\langle \vec{w}_1, \vec{w}_2 \rangle$. Thus, we may need to think about how to define $\tau_{W \times W}$.

Worksheet for 27 & 29 Feb 2019
Continuity and Compactness
§4.4

Vocabulary: uniform continuity, maximum, minimum, local maximum, local minimum, extremum, local extremum, extreme value theorem

In the next few problems we prove one of the more important relations between continuity and compactness: continuous functions take compact sets to compact sets.

Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. Let $f \colon X \to Y$ be a continuous function. Recall that this means that the inverse image of every open set is open; that is, $f^{-1}[\mathcal{O}] \in \tau_X$ for all $\mathcal{O} \in \tau_Y$.

(222) Suppose $A \subset X$. Suppose $\{S_j \mid j \in J\}$ is a collection of subsets of X, indexed by a set J. Show: If $\{S_j \mid j \in J\}$ covers A, then $\{f[S_j] \mid j \in J\}$ covers $f[A]$. That is, show that if $A \subset \bigcup_{j\in J} S_j$, then $f[A] \subset \bigcup_{j\in J} f[S_j]$.

(223) Suppose $A \subset X$. Suppose $\{\mathcal{O}_i \mid i \in I\}$ is an open cover of $f[A]$ that is indexed by a set I. Show that $\{f^{-1}[\mathcal{O}_i] \mid i \in I\}$ is an open cover of A. [Recall that saying '$\{\mathcal{O}_i \mid i \in I\}$ is an open cover of $f[A]$' means (a) $f[A] \subset \bigcup_{i\in I} \mathcal{O}_i$ and (b) $\mathcal{O}_i \in \tau_Y$ for all $i \in I$.]

(224) Show: If $B \subset Y$, then $f[f^{-1}[B]] \subset B$.

(225) Conclude: if $K \subset X$ is compact, then $f[K]$ is compact. [That is, show that every open cover of $f[K]$ has a finite subcover.]

The result of Prompt 225 has consequences throughout mathematics. For example, the Extreme Value Theorem follows immediately from this statement (see Prompt 228).

Definition. Suppose $(V, \langle\, , \rangle)$ is an inner product space and $A \subset V$ Suppose $f \colon A \to \mathbb{R}$ is a function and $\vec{a} \in A$. We say that f attains a *maximum* at $\vec{a}$ provided that $f(\vec{a}) \geq f(\vec{x})$ for all $\vec{x} \in A$. We say that f attains a *local maximum* at a provided that there exists a $\delta > 0$ such that $f(\vec{a}) \geq f(\vec{x})$ for all $\vec{x} \in B_\delta(\vec{a}) \cap A$.

We define the terms *minimum* and *local minimum* in a similar fashion. The function f is said to attain an *extremum* (resp. *local extremum*) at $\vec{a}$ provided that it attains a maximum or minimum (resp. local maximum or a local minimum) at $\vec{a}$. Note that every extremum is a local extremum.

(226) Suppose $K \subset X$ is compact. If $g \colon K \to Y$ is continuous, then $g[K]$ is compact. [Another exercise in using the subspace topology: $U' \subset K$ is open in K if and only if there exists open U in X such that $U' = U \cap K$.]

(227) Suppose $(V, \langle\, , \rangle)$ is an inner product space. If $K \subset V$ is compact and $h \colon K \to \mathbb{R}$ is continuous, then h attains both its minimum and maximum values. That is, there exist $k_0, k_1 \in K$ such that $h(k_0) \leq h(k) \leq h(k_1)$ for all $k \in K$. (Hint: Prompt 175.)

(228) **Extreme Value Theorem**. If $g \colon [a, b] \to \mathbb{R}$ is continuous, then g must obtain both its maximum and minimum.

Some functions have the property that when establishing continuity at a point x via the $\varepsilon - \delta$ definition, the δ can be chosen independently of the point x. For example, for the function $\ell \colon \mathbb{R} \to \mathbb{R}$ that maps x to $47x + 34$, when your challenger gives you an $\varepsilon > 0$, you calmly respond 'Let $\delta = \varepsilon/47$.' A function for which the choice of δ may be made independently of x is called *uniformly continuous*.

Definition. Suppose $(V, \langle\, , \rangle_V)$ and $(W, \langle\, , \rangle_W)$ are finite dimensional inner product spaces. Suppose $A \subset V$. A function $f \colon A \to W$ is *uniformly continuous* provided that for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{whenever } \vec{x}, \vec{y} \in A \text{ and } \|\vec{x} - \vec{y}\|_V < \delta \text{ it follows that } \|f(\vec{x}) - f(\vec{y})\|_W < \varepsilon.$$

Sadly, not all continuous functions are uniformly continuous. In Example 4.4.4 (ii) of Abbott, the continuous function $x \mapsto x^2$ is shown to not be uniformly continuous and in Example 4.4.7 of Abbott, the continuous function $x \mapsto \sin(1/x)$ for $x \in (0, 1)$ is also shown to fail to be uniformly continuous.

(229) Show: Every uniformly continuous function is continuous.

(230) Show that the continuous function $h \colon (0, \infty) \to \mathbb{R}$ given by $h(t) = 1/t$ is not uniformly continuous.

Many important results in integration theory require uniform continuity. For this reason, it is important to have a large class of functions that are uniformly continuous. Suppose $(V, \langle\, , \rangle_V)$ and $(W, \langle\, , \rangle_W)$ are finite dimensional inner product spaces.

(231) Suppose $A \subset V$. What does it mean to say that a function $f\colon A \to W$ is not uniformly continuous?

(232) Suppose $A \subset V$. Show that a function $f\colon A \to W$ fails to be uniformly continuous if and only if there exists a particular $\varepsilon > 0$ and two sequences $(\vec{x}_n)$ and $(\vec{y}_n)$ in A satisfying

$$\|\vec{x}_n - \vec{y}_n\|_V \to 0 \quad \text{but} \quad \|f(\vec{x}_n) - f(\vec{y}_n)\|_W \geq \varepsilon \ \forall n \in \mathbb{N}.$$

(233) Suppose $K \subset V$ is compact. If $g\colon K \to W$ is continuous, then it is uniformly continuous. [Hint: Argue by contradiction and use both sequenital-compactness and Prompt 232.]

Something to Think About

Suppose $(V, \langle\, , \rangle_V)$ and $(W, \langle\, , \rangle_W)$ are finite dimensional inner product spaces. Suppose also that $A \subset V$ and $f\colon A \to W$ is continuous. The example of Prompt 230 shows that we cannot hope that if $(\vec{a}_n)$ is a Cauchy sequence in A, then $(f(\vec{a}_n))$ a Cauchy sequence in $f[A]$. (Do you see why?). However, if we assume that $f\colon A \to W$ is uniformly continuous, then the image of every Cauchy sequence is again a Cauchy sequence. Can you show this?

This small observation leads us to the following remarkable claim: Suppose $A \subset V$ is bounded.

Claim. A function $g\colon A \to W$ is uniformly continuous if and only if there is a continuous function $\bar{g}\colon \bar{A} \to W$ for which $g(\vec{a}) = \bar{g}(\vec{a})$ for all $\vec{a} \in A$.

Moreover, if such a $\bar{g}$ exists, then it is the unique continuous extension of g. Given g, how would you define $\bar{g}$? Is the condition that A be bounded necessary? Can you establish this result?

Please report any errors you find to smdbackr@umich.edu

Worksheet for 11 Mar 2019
Continuity and Connectedness
§4.5

Vocabulary: Intermediate Value Theorem

In the next few problems we prove one of the more important relations between continuity and connectedness: continuous functions take connected sets to connected sets.

Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. Let $g\colon X \to Y$ be a continuous function. Recall that this means that the inverse image of every open set is open; that is, $g^{-1}[\mathcal{O}] \in \tau_X$ for all $\mathcal{O} \in \tau_Y$.

(234) Show that if A, B are subsets of Y, then $g^{-1}[A \cup B] = g^{-1}[A] \cup g^{-1}[B]$.

(235) Show that if A is a subset of Y, then $\overline{g^{-1}[A]} \subset g^{-1}[\overline{A}]$.

Recall that two nonempty subsets A, B of X are said to be *separated* provided that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

(236) If the sets C and D are separated in Y and both intersect $g[X]$ nontrivially, then the sets $g^{-1}[C]$ and $g^{-1}[D]$ are separated in X.

Recall that a subset E of X is connected provided that it is not disconnected. Moreover, the set E is disconnected provided that it can be written as $E = A \cup B$ where A and B are nonempty separated sets.

(237) Show: if $F \subset X$ is connected, then $g[F]$ is connected.

The result of Prompt 237 has consequences throughout mathematics. For example, the Intermediate Value Theorem follows immediately from this statement (see Prompt 238).

(238) **Intermediate Value Theorem**. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. If $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ so that $f(c) = 0$. [Hint: Recall from Prompt 187 that a subset of $\mathbb{R}$ is connected if and only if it is an interval.]

(239) The version of the Intermediate Value Theorem appearing in the book states:

> If $f\colon [a, b] \to \mathbb{R}$ is continuous, and L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.

Show that the book version of the IVT follows from the version of the IVT stated in Prompt 238.

(240) Show that the IVT is false for $\mathbb{Q}$: Note that the function $g\colon \mathbb{Q} \to \mathbb{Q}$ that maps $x \in \mathbb{Q}$ to $x^2 - 2$ is negative at 0 and positive at 2. Therefore, if the IVT were true for $\mathbb{Q}$, then there would exist $c \in \mathbb{Q}$ between 0 and 2 such that $g(c) = 0$. What would c be? Is this a problem?

(241) Some True/False problems to test your understanding of some words that start with the letter C.

- T/F There is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ whose image is $\mathbb{Q}$.
- T/F Continuous functions take bounded open intervals to bounded open intervals.
- T/F Continuous functions take bounded open intervals to bounded open sets.
- T/F Continuous functions take bounded open intervals to bounded sets.
- T/F Continuous functions take bounded closed intervals to bounded closed intervals.

Something to Think About

Exercise 4.5.7 in the book introduces the one-dimensional version of Brouwer's fixed-point theorem: If $f\colon [0, 1] \to [0, 1]$ is continuous, then there is an $x \in [0, 1]$ for which $f(x) = x$. Can you prove this? Can you generalize it to other subsets of $\mathbb{R}$? Can you formulate what Brouwer's fixed-point theorem might say in $\mathbb{R}^n$?

Brouwer's fixed-point theorem and its relatives play an important role in many branches of mathematics, from analysis to differential equations to game theory. Its namesake, Luitzen Egbertus Jan Brouwer, was a mathematician and philosopher (as a philosopher, he is responsible for intuitionism which stood in opposition to the formalism of Hilbert in the early 20th century).

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Worksheet for 15 Mar 2019
Integrability: the basics
§7.2

Vocabulary: bounded function, partition, Darboux upper sum, Darboux lower sum, refinement, Darboux integrable

As discussed in §7.1 of Abbott, from the time of Newton until the mid 19th century, integration was understood as the 'inverse operation' to differentiation; that is, the integral of a function f was a function F for which $F' = f$. This understanding of integration was eventually rejected because it is too limiting – for example, functions with 'jump discontinuities' cannot be integrated with respect to this understanding. Around the mid-1800s the notion of integration was completely separated from the notion of differentiation and instead understood from the point of view of 'area under a curve.' To emphasize this point of view, in this class we have chosen to present integration prior to differentiation. It will probably go easier for you if you draw many pictures while working through the material on integration.

(242) Abbott notes that for the function

$$h(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 2 & \text{for } x \in [1,2] \end{cases}$$

there is no $H\colon [0,2] \to \mathbb{R}$ for which $H' = h$. Using ideas you knew prior to taking this class, justify Abbott's statement.

The area of a rectangle of base b and height h is bh. The basic idea of integration is to approximate the (signed) area under a curve using rectangles and then take a limit. We make this precise.

Definition. A *partition* P of $[a,b]$ is a finite, ordered set

$$P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}.$$

(243) Write down three distinct partitions of the interval $[0,2]$. At least one of these partitions should have n points.

Definition. A function $f\colon [a,b] \to \mathbb{R}$ is said to be *bounded* provided that the set $\{f(x) \mid x \in [a,b]\}$ is bounded.

(244) Give an example of a function $g\colon [0,1] \to \mathbb{R}$ that is not bounded.

Definition. Suppose $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ is a partition of $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is a bounded function. For each subinterval $[t_{k-1}, t_k]$ of P, let

$$m_k = \inf\{f(t) \mid t \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(t) \mid t \in [t_{k-1}, t_k]\}.$$

The *lower Darboux sum* of f with respect to P is given by

$$L(f,P) = \sum_{j=1}^{n} m_j (t_j - t_{j-1})$$

and the *upper Darboux sum* of f with respect to P is given by

$$U(f,P) = \sum_{j=1}^{n} M_j (t_j - t_{j-1}).$$

(245) For each of the three distinct partitions you gave in Prompt 243, compute the lower and upper Darboux sums of the function $h\colon [0,2] \to \mathbb{R}$ given in Prompt 242.

(246) If P is a partition of $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is a bounded function, then $L(f,P) \le U(f,P)$.

Definition. A partition Q of $[a,b]$ is a *refinement* of a partition P of $[a,b]$ provided that every point in P is contained in Q. In this case, we write $P \subset Q$.

(247) Find a partition Q which is a simultaneous refinement of each of the three distinct partitions you gave in Prompt 243.

(248) If P and Q are partitions of $[a,b]$, $f\colon [a,b] \to \mathbb{R}$ is bounded, $P \subset Q$, and $|Q| = |P| + 1$, then $U(f,Q) \le U(f,P)$.

(249) If P and Q are partitions of $[a,b]$, $f\colon [a,b] \to \mathbb{R}$ is bounded, and $P \subset Q$, then $U(f,Q) \leq U(f,P)$. [The inequality $L(f,P) \leq L(f,Q)$ is established in the book.]
(250) If R and S are partitions of $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is bounded, then $L(f,R) \leq U(f,S)$.

Definition. Let $\mathcal{P}$ denote the set of all partitions of $[a,b]$. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function. The *upper Darboux integral* of f is

$$U(f) = \inf\{U(f,P) \mid P \in \mathcal{P}\}$$

and the *lower Darboux integral* of f is

$$L(f) = \sup\{L(f,P) \mid P \in \mathcal{P}\}$$

(251) If $f\colon [a,b] \to \mathbb{R}$ is a bounded function, then $L(f) \leq U(f)$. [Hint: See Prompt 56 and Homework 12.]

Definition. A function $f\colon [a,b] \to \mathbb{R}$ is said to be *Darboux integrable* provided that

(1) f is bounded and
(2) $L(f) = U(f)$.

When these conditions hold, we define $\int_a^b f = U(f)$.

Note, functions like $g\colon [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1/x^{1/2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

are automatically disqualified from being integrable because they fail to be bounded.

(252) Show that the function $h\colon [0,2] \to \mathbb{R}$ given in Prompt 242 is Darboux integrable. What is $\int_0^2 h$?
(253) Prove that the modified Dirichlet function $g\colon [-1,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 3 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not integrable.

Conventions: Some people write $\int_a^b f(s)\,ds$ for $\int_a^b f$; either notation is fine. The book says that a function for which $U(f) = L(f)$ is *Riemann integrable*; while true, this is **not** the standard definition of Riemann integrable. While we will not take it up here, to put your mind at ease: A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. Moreover, if a function $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then it is bounded.

Something to Think About

Our approach to integration has been to partition the x-axis and then create a bunch of rectangles whose base is determined by the partition and whose height is determined by the function. This approach works well, but, as it turns out, is not quite general enough for many applications. For example, the modified Dirichlet function $g\colon [-1,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 3 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not integrable. However, we really would like for $\int_{-1}^1 g$ to be 6. What might happen if we instead partitioned the y-axis? Draw some pictures.

Worksheet for 18 Mar 2019
Integration: a criterion for integrability
§7.2

Vocabulary: Darboux integrability criterion

Deciding if a given function is integrable[1] can be a bit tricky. In this worksheet we will develop a convenient integration criterion and show that continuous functions are integrable.

(254) Suppose that A, B are nonempty subsets of $\mathbb{R}$. Suppose that for every $a \in A$ and every $b \in B$ we have $a \leq b$.
  (a) Show that $\sup(A) \leq \inf(B)$.
  (b) Show that $\sup(A) = \inf(B)$ if and only if for all $\varepsilon > 0$ there exist an $a \in A$ and a $b \in B$ such that $b - a < \varepsilon$.

(255) **Darboux Integrability Criterion.** A bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that
$$U(f,P) - L(f,P) < \varepsilon.$$

(256) *Bonus.* Use the Darboux integrability Criterion to show that a bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ of $[a,b]$ satisfying
$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

Recall that every continuous function on a compact subset of $\mathbb{R}$ is uniformly continuous. We now use this to demonstrate that every continuous function $f\colon [a,b] \to \mathbb{R}$ is integrable. WOLOG, $a < b$.

(257) Every continuous function $f\colon [a,b] \to \mathbb{R}$ is bounded.

(258) Every continuous function $f\colon [a,b] \to \mathbb{R}$ is uniformly continuous.

(259) Fix $\varepsilon > 0$. Show that there is a $\delta > 0$ so that if $x, y \in [a,b]$ with $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon/(b-a)$.

(260) Fix $\varepsilon > 0$. Show that there is a partition $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ of $[a,b]$ so that $M_k - m_k < \varepsilon/(b-a)$ for $1 \leq k \leq n$. [Hint: if you don't use the EVT in your proof, then it is almost certainly wrong.]

(261) Use the Darboux integrability Criterion to conclude that every continuous function $f\colon [a,b] \to \mathbb{R}$ is integrable.

If V is a finite dimensional inner product space, $K \subset V$ is compact, and $f\colon K \to \mathbb{R}$ is a bounded function, then one can develop a theory of integrability for f as follows. Let $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ be an orthonormal basis for V. Since K is compact, it is closed and bounded. Hence, there exists $a_j, b_j \in \mathbb{R}$ for $1 \leq j \leq n$ such that K is a subset of the 'rectangle'
$$R = \left\{ \sum x_i \vec{e}_i \mid x_i \in [a_i, b_i] \right\}.$$

We can extend $f\colon K \to \mathbb{R}$ to a function $\tilde{f}\colon R \to \mathbb{R}$ by setting
$$\tilde{f}(y) = \begin{cases} f(y) & \text{if } y \in K \\ 0 & \text{if } y \in R \setminus K. \end{cases}$$

Let P_i be a partition of $[a_i, b_i]$. Then the n-tuple of partitions $P = (P_1, P_2, \ldots, P_n)$ partitions R into subrectangles. On each subrectangle we can take take the infimum and supremum of $\tilde{f}$ to form lower and upper Darboux sums $L(f,P)$ and $U(f,P)$ (the 'base' is the volume of the subrectangle). The function $f\colon K \to \mathbb{R}$ is integrable provided that
$$\sup_{P \in \mathcal{P}} L(\tilde{f}, P) = \inf_{Q \in \mathcal{P}} U(\tilde{f}, Q)$$
where $\mathcal{P}$ is the set of partitions of R. If f is integrable, we set $\int_K f = \int_R \tilde{f} = \inf_Q U(\tilde{f}, Q)$. Darboux's integrability criterion holds, continuous functions are integrable, etc. Since the theory is notationally cumbersome and nearly identical to the one-variable theory, we will stick to the one-variable case.

(262) In defining $\int_K f$ above, we made a choice of an orthonormal basis and a rectangle R with respect to this basis that contains K. Discuss why $\int_K f$ is independent of these choices. [I just want a discussion; an actual proof is pretty involved and not a good use of our time at present.]

[1]We will say "integrable" rather than "Darboux integrable" because we are only considering one type of integration in this class. In practice, there are all kinds of adjectives to place in front of the word "integrable." Some types of integration one may encounter include Riemann integration, Lebesgue integration, Henstock-Kurzweil integration, Riemann-Stieltjes integration, . . .

(263) *Bonus.* Given two finite dimensional inner product spaces V and W and $K \subset V$ compact one can define a notion of what it means to integrate a bounded function $f\colon K \to W$. What is the expected output? How would you proceed?

Something to Think About

Suppose that $f\colon [a,b] \to \mathbb{R}$ is an integrable function. While you are away, a jolly prankster changes the function f to a function g. The functions f and g agree except at one million points. Is g integrable? If so, what is the value of $\int_a^b g$? Explain.

Suppose $f, h\colon [a,b] \to \mathbb{R}$ are integrable functions. Is $fg\colon [a,b] \to \mathbb{R}$ integrable?

Worksheet for 20 Mar 2019
Properties of the Integral
§7.4

Vocabulary: Int, ln, exp

Suppose $a, b \in \mathbb{R}$ with $a < b$. Much of this worksheet is given over to verifying and/or using properties of integration with which you are very familiar. For example, although it is not verified on this worksheet, it does follow from the results used and/or proved below that if f is integrable on $[a, b]$ and $c, d, e \in [a, b]$, then

$$\int_c^e f = \int_c^d f + \int_d^e f.$$

Note: We did **not** specify that $c \leq d \leq e$.

(264) Let $\mathrm{Int}([a, b])$ denote the set of functions on $[a, b]$ that are integrable.
- (a) Show that $\mathrm{Int}([a, b])$ is a vector space. [Hint: You may assume, from Math 217, that $\mathrm{Fun}([a, b])$, the set of functions with source $[a, b]$ and target $\mathbb{R}$, is a vector space.]
- (b) Show that the function $I\colon \mathrm{Int}([a, b]) \to \mathbb{R}$ defined by $I(f) = \int_a^b f$ is linear.

(265) Suppose $f \in C^0([a, b])$ is a non-negative function with the property that $f(t) > 0$ for some $t \in [a, b]$. Show that $\int_a^b f > 0$.

> **Definition**. Suppose $f \in \mathrm{Int}([a, b])$. For all $c \in [a, b]$ we set
> $$\int_c^c f = 0,$$
> and we define
> $$\int_b^a f = -\int_a^b f$$

(266) Since the function $h\colon (0, \infty) \to (0, \infty)$ defined by $h(t) = 1/t$ is continuous, we can define a function $\mathrm{ellen}\colon (0, \infty) \to \mathbb{R}$ by

$$\mathrm{ellen}(x) = \int_1^x h(t)\, dt.$$

The function ellen plays a central role in mathematics; here we show it is strictly increasing.
- (a) Show that $\mathrm{ellen}(1) = 0$.
- (b) Use Theorem 7.4.1 of your book to show that ellen is strictly increasing on $[1, \infty)$.
- (c) Use Theorem 7.4.1 of your book to show that ellen is strictly increasing on $(0, 1]$.
- (d) Conclude: ellen is a strictly increasing function that is positive on $(1, \infty)$ and negative on $(0, 1)$.

(267) Show that $\mathrm{ellen}\colon (0, \infty) \to \mathbb{R}$ is continuous. [Hint: use Theorem 7.4.1 of your book.[1]] Conclude that $\mathrm{ellen}[(0, \infty)]$ is an interval.

(268) What is $\mathrm{ellen}[(0, \infty)]$?
- (a) Show
$$\lim_{x\to\infty} \mathrm{ellen}(x) = \infty \quad \text{and} \quad \lim_{x\to 0^+} \mathrm{ellen}(x) = -\infty.$$
[Hint: Observe that $1/2 \geq 1/2$, $1/3 + 1/4 \geq 1/2$, and $1/5 + 1/6 + 1/7 + 1/8 \geq 1/2$.]
- (b) Conclude that $\mathrm{ellen}[(0, \infty)] = \mathbb{R}$.

(269) *Bonus*. Show that for all $a, b \in (0, \infty)$, we have $\mathrm{ellen}(ab) = \mathrm{ellen}(a) + \mathrm{ellen}(b)$. [Hint: Use the definition of ellen and the definition of what it means to be integrable.]

(270) If $h \in \mathrm{Int}([a, b])$ is bounded below by m and above by M, then

$$m(b - a) \leq \int_a^b h \leq M(b - a).$$

(271) Suppose $f, g \in \mathrm{Int}([a, b])$ with $f \leq g$ (that is, $f(x) \leq g(x)$ for all $x \in [a, b]$). Show that $\int_a^b f \leq \int_a^b g$.

[1]Or, you could follow N.J. and show ellen is continuous at x_0 by setting $\delta = \varepsilon x_0/(2 + \varepsilon) \ldots$

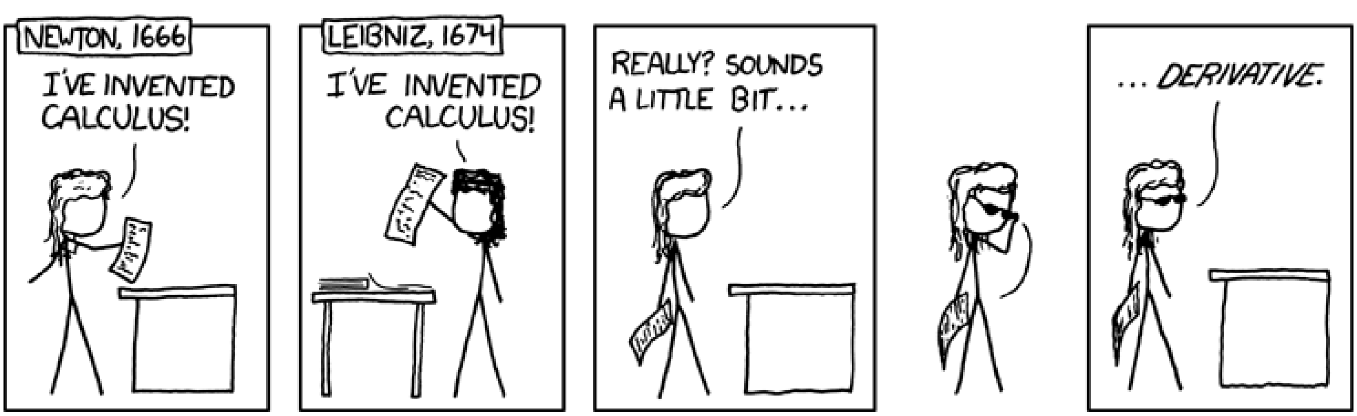
(272) Suppose $h \in \text{Int}([a,b])$.
   (a) Show[1] that $|h| \in \text{Int}([a,b])$
   (b) Show that the 'integral version of the triangle inequality' holds:

$$\left| \int_a^b h \right| \leq \int_a^b |h| \, .$$

Convention: Most people denote the function $\text{ellen}\colon (0,\infty) \to \mathbb{R}$ by $\ln\colon (0,\infty) \to \mathbb{R}$ and call it the *natural log function*. We will follow this convention on all subsequent worksheets.

Something to Think About

Since $\ln$ is a continuous, strictly increasing function from $(0,\infty)$ to $\mathbb{R}$, it is bijective. In particular, $\ln$ has an inverse which is usually called $\exp$. Is $\exp\colon \mathbb{R} \to (0,\infty)$ continuous? What is $\exp(0)$? Is $\exp$ increasing? What is $\exp(a+b)$?



Newton and Leibniz, an xkcd comic by Randall Munroe (xkcd.com)

[1][Hint: For $\emptyset \neq A \subset \mathbb{R}$ compare $\sup(A) - \inf(A)$ and $\sup(|A|) - \inf(|A|)$ where $|A| = \{|a| : a \in A\}$. You can break this into cases, or be extremely clever like Z.Q. and do the following: fix $\varepsilon > 0$; there are $a, a' \in A$ such that $\sup(|A|) - |a| < \varepsilon/2$ and $|a'| - \inf(|A|) < \varepsilon/2$. So $\sup(|A|) - \inf(|A|) - \varepsilon < ||a| - |a'||$; now use the reverse triangle inequality ....]

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Worksheet for 25 Mar 2019
Differentiation: definitions and basic results
§5.2

Vocabulary: best linear approximation, derivative, differentiation, differentiable, total derivative, degree one Taylor polynomial

For the next little bit, we will throw rigor to the wind as we try to develop some intuition about differentiation in higher dimensional spaces.

When you took calculus, you were told (correctly) to think of the derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ at the point a in $\mathbb{R}$ as the slope of the tangent line to the graph of $y = f(x)$ at the point a. That is, the tangent line had equation $P_{f,a,1}(a+h) = f(a) + f'(a)(h)$ or, equivalently, $P_{f,a,1}(x) = f(a) + f'(a)(x-a)$. In a strong sense, the *degree one Taylor polynomial* of f at a, $P_{f,a,1}$, is the *best linear approximation* of f near a.

(273) Suppose $a \in \mathbb{R}$. Compute $P_{f,a,1}(a+h)$ for the functions $3x+5$, $7x^2-8x+9$, e^x, and $\ln(1+x)$.

Suppose $(V, \langle\, , \,\rangle_V)$ and $(W, \langle\, , \,\rangle_W)$ are finite dimensional inner product spaces. What might the *best linear approximation* of a function $g\colon V \to W$ at a point $\vec{a} \in V$ be? Being a linear approximation, it would need to have the form

$$P_{g,\vec{a},1}(\vec{x}) = g(\vec{a}) + T_{\vec{a}}(\vec{x}-\vec{a}) \text{ or, equivalently, } P_{g,\vec{a},1}(\vec{a}+\vec{h}) = g(\vec{a}) + T_{\vec{a}}(\vec{h})$$

where $T_{\vec{a}} \in \operatorname{Hom}(V,W)$. What properties should $T_{\vec{a}}$ have?

Here is a way to think about $T_{\vec{a}}$: Since $P_{g,\vec{a},1}$ is supposed to be the best linear approximation of g at $\vec{a}$, then we expect that $\operatorname{err}_{g,\vec{a}}(\vec{h}) := g(\vec{a}+\vec{h}) - g(\vec{a}) - T_{\vec{a}}(\vec{h})$ should be "quadratic or worse" in $\vec{h}$. This would mean that $\|\operatorname{err}_{g,\vec{a}}(\vec{h})\|_W$ is quadratic (or worse) in $\|\vec{h}\|_V$. But, if $\|\operatorname{err}_{g,\vec{a}}(\vec{h})\|_W$ is quadratic in $\|\vec{h}\|_V$, then

$$\lim_{\vec{h}\to\vec{0}} \frac{\|\operatorname{err}_{g,\vec{a}}(\vec{h})\|_W}{\|\vec{h}\|_V} = 0.$$

In other words, we expect

$$(\smiley) \qquad \lim_{\vec{h}\to\vec{0}} \frac{\|g(\vec{a}+\vec{h}) - g(\vec{a}) - T_{\vec{a}}(\vec{h})\|_W}{\|\vec{h}\|_V} = 0.$$

We now return to being rigorous.

Definition. Suppose I is an open subset of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$. If $a \in I$ and

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists, then we say that f is *differentiable* at a. In this case we let $f'(a)$ denote the value of this limit and say that the derivative of f at a is $f'(a)$.
If f is differentiable at each $a \in I$, then the function $f'\colon I \to \mathbb{R}$ that sends $a \in I$ to $f'(a)$ is called the *derivative* of f and f is said to be *differentiable* on I.

(274) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$. Show that $\smiley$ holds, that is:

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - f'(a)h|}{|h|} = 0.$$

(275) Suppose $(V, \langle\, , \,\rangle_V)$ and $(W, \langle\, , \,\rangle_W)$ are finite dimensional inner product spaces and $\mathcal{O} \subset V$ is open. Suppose $g\colon \mathcal{O} \to W$ and $\vec{a} \in \mathcal{O}$. Show that if there exists $T_{\vec{a}} \in \operatorname{Hom}(V,W)$ satisfying Equation ($\smiley$), then $T_{\vec{a}}$ is unique. [Hint: For $\vec{h}_0 \neq \vec{0}$, look at $\|(T_{\vec{a}} - T'_{\vec{a}})(\vec{h}_0)\|_W / \|\vec{h}_0\|_V$ and note that we can replace $\vec{h}_0$ by $t\vec{h}_0$ for $t \neq 0$ without changing the value of the quotient. Now add zero in a creative way and take a limit as $t \to 0$ to arrive at a contradiction.]

Thanks to Prompt 275, the following definition makes sense.

Definition. Suppose $(V, \langle\,,\,\rangle_V)$ and $(W, \langle\,,\,\rangle_W)$ are finite dimensional inner product spaces and $\mathcal{O} \subset V$ is open. Suppose $g\colon \mathcal{O} \to W$ and $\vec{a} \in \mathcal{O}$. If there exists $T_{\vec{a}} \in \operatorname{Hom}(V, W)$ satisfying

$$\lim_{\vec{h}\to\vec{0}} \frac{\|g(\vec{a}+\vec{h}) - g(\vec{a}) - T_{\vec{a}}(\vec{h})\|_W}{\|\vec{h}\|_V} = 0,$$

then g is said to be *differentiable* at $\vec{a}$, and the *derivative* of g at $\vec{a}$ is $T_{\vec{a}}$.
If g is differentiable at each $\vec{v} \in \mathcal{O}$, then the function

$$Dg\colon \mathcal{O} \to \operatorname{Hom}(V, W)$$

that maps $\vec{v} \in \mathcal{O}$ to $T_{\vec{v}}$ is called the *total derivative* of g, and g is said to be *differentiable* on $\mathcal{O}$.

Notation. Because the value of Dg at $\vec{v}$ is a function that will take inputs of its own, we often write $D_{\vec{v}}g$ for the total derivative of g at $\vec{v}$ rather than $Dg(\vec{v})$. So

$$P_{g,\vec{a},1}(\vec{a}+\vec{h}) = g(\vec{a}) + D_{\vec{a}}g(\vec{h}) \quad \text{or, equivalently,} \quad P_{g,\vec{a},1}(\vec{x}) = g(\vec{a}) + D_{\vec{a}}g(\vec{x}-\vec{a}).$$

(276) If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\vec{a} \in \mathbb{R}^2$, describe the graph of $P_{f,\vec{a},1}(\vec{x}) = f(\vec{a}) + D_{\vec{a}}f(\vec{x}-\vec{a})$.

(277) Explain how the two definitions of differentiation given above coincide when $V = W = \mathbb{R}$.

(278) Suppose $L \in \operatorname{Hom}(V, W)$. Show that $D_{\vec{v}}L = L$ for all $\vec{v} \in V$. That is, the best linear approximation of a linear map is the linear map itself.

(279) Basic differentiation facts. You will prove at least one of the problems below — which one? the one DeBacker tells you to prove. [Hint: You will probably need that if $T \in \operatorname{Hom}(V, W)$, then there exists $c \in \mathbb{R}_{>0}$ such that $\|T(\vec{x})\|_W \leq c\|\vec{x}\|_V$ for all $\vec{x} \in V$. Why is this true?]

  (a) Show that differentiation is linear. That is, with notation as above, if $f, \ell\colon \mathcal{O} \to W$ are differentiable, then for all $c \in \mathbb{R}$ and all $\vec{v} \in \mathcal{O}$ we have
  - $D_{\vec{v}}(cf) = cD_{\vec{v}}f$
  - $D_{\vec{v}}(f+\ell) = D_{\vec{v}}f + D_{\vec{v}}\ell$

  [Hint: use uniqueness.]

  (b) With notation as above, if $g\colon \mathcal{O} \to W$ is differentiable, then g is continuous.

  (c) Suppose $f, \ell\colon \mathcal{O} \to \mathbb{R}$ are differentiable. Show that $D_{\vec{v}}(f\ell) = f(\vec{v})D_{\vec{v}}\ell + \ell(\vec{v})D_{\vec{v}}f$ for all $\vec{v} \in \mathcal{O}$. [Hint: use uniqueness.] Can you formulate what the quotient rule will say in this context?

  (d) *Bonus.* Suppose $\mathcal{V} \subset W$ is open and U is a finite dimensional inner product space. If $h\colon \mathcal{V} \to U$ is differentiable and $f\colon \mathcal{O} \to \mathcal{V}$ is differentiable, then $h \circ f\colon \mathcal{O} \to U$ is differentiable and $D_{h(f(\vec{a}))}(h \circ f) = D_{f(\vec{a})}h \circ D_{\vec{a}}f$ for all $\vec{a} \in \mathcal{O}$

Some Things to Think About

[A] Because we are so used to thinking of the derivative as a scalar rather than as a linear map from $\mathbb{R}$ to $\mathbb{R}$, the notion of differentiation in multiple dimensions can cause one's brain to hurt a bit. A tried-and-true method to get used to a new idea is to think about how/if familiar concepts make sense in this new context. For example, think about the chain rule, critical points, the derivative of an inverse, higher order derivatives, the mean value theorem(s), how one might compute anything non-trivial, ...

[B] In verifying various expected properties of the derivative, uniqueness was invoked repeatedly to great effect. This is a very powerful technique, where else might you be able to use it? Much later in your mathematical journey you may encounter an idea that feels very similar and is extremely useful: in category theory *universal properties* serve to define objects uniquely;[1] when you get there, embrace the idea – it will save you much grief.

[1] up to unique isomorphism

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Worksheet for 27 Mar 2019
Differentiation: Mean Value Theorems
§5.3

Vocabulary: Fermat's Theorem, Interior Extremum Theorem, Rolle's Theorem, Cauchy Mean Value Theorem, Mean Value Theorem, Darboux's Theorem

Today we reap some of the rewards for the effort that you've put in over the last few months. Enjoy.
There are a number of results in the theory of differentiation in one variable that do not have exact analogues for functions of multiple variables. The mean value theorem, which, as Abbott notes, is the most important result in all of calculus, is one such result.

(280) Show that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if and only if for all sequences (x_n) in $\mathbb{R} \setminus \{a\}$ that converge to a we have $\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n-a}$ exists. [Hint: Recall Prompt 197.]

(281) (Fermat's Theorem (AKA the Interior Extremum Theorem)). Suppose $I \subset \mathbb{R}$ is open and $f\colon I \to \mathbb{R}$ is differentiable at $c \in I$. If f attains a local extremum at $c \in I$, then $f'(c) = 0$. [Hint: sequences – one from the left and one from the right.]

(282) *Bonus.* Suppose $(V, \langle\, ,\rangle)$ is a finite dimensional inner product space and $\mathcal{O} \subset V$ is open. If $f\colon \mathcal{O} \to \mathbb{R}$ attains a local extremum at $\vec{c} \in \mathcal{O}$, then $D_{\vec{c}} f = 0$. [Hint: Let $I = (-1, 1)$. For each $\vec{v} \in V$ construct a differentiable $\gamma\colon I \to \mathcal{O}$ with $\gamma(0) = \vec{c}$ and $D_0\gamma = \vec{v}$. Consider the function $h = f \circ \gamma\colon I \to \mathbb{R}$. Apply the chain rule to h to show that $D_{\vec{c}} f(\vec{v}) = 0$.]

You should draw some pictures to make sure that you have the correct visual interpretation of Fermat's Theorem.
Fermat's Theorem is particularly useful for solving optimization problems. However, one must not forget the boundary – Fermat's theorem often fails to detect extrema on the boundary. For example, consider the function $f(x) = x^2$ on $[-4, 2]$. The extrema occur at -4 and 0, but Fermat's Theorem only detects the minimum at 0.

(283) (Rolle's Theorem). Suppose $a < b$, $f\colon [a, b] \to \mathbb{R}$ is continuous, and f is differentiable at each $x \in (a, b)$. If $f(a) = f(b)$, then there is a $c \in (a, b)$ for which $f'(c) = 0$. [Hint: $[a, b]$ is compact and f is continuous.]

Lest you begin to think that all of calculus is a triviality, remember how hard we had to work to establish results like the extreme value theorem and the intermediate value theorem.

(284) (Cauchy's Mean Value Theorem). Suppose $a < b$, $f, g\colon [a, b] \to \mathbb{R}$ are continuous, and f, g are differentiable at each $x \in (a, b)$. Then there is a $c \in (a, b)$ for which

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

[Hint: Examine the function $\Omega(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$.]

(285) (Mean Value Theorem). Suppose $a < b$, $f\colon [a, b] \to \mathbb{R}$ is continuous, and f is differentiable at each $x \in (a, b)$. Then there is a $c \in (a, b)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$.

You should draw some pictures to make sure that you have the correct visual interpretation of the Mean Value Theorem.

(286) Suppose I is an open iWimp$^{\text{TM}}$. If $f\colon I \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in I$, then f is a constant function.

(287) Suppose $a < b$ and $f, g\colon [a, b] \to \mathbb{R}$ are continuous. If $f, g\colon (a, b) \to \mathbb{R}$ are differentiable and $f'(t) = g'(t)$ for all $t \in (a, b)$, then there exists $C \in \mathbb{R}$ for which $f(x) = g(x) + C$ for all $x \in [a, b]$

At this point it is traditional to present one or another of L'Hospital's Rules. We choose to look at the following $0/0$ version:

(288) Suppose I is an open iWimp$^{\text{TM}}$ and $a \in I$. Suppose f and g are both differentiable on $I \setminus \{a\}$, $f(a) = g(a) = 0$, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, and $\lim_{x\to a^-} \frac{f'(x)}{g'(x)} = L$.
   (a) Show that there is a $\delta > 0$ for which g' and g are nonzero on $(a - \delta, a)$.
   (b) Choose a sequence (x_n) in $(a - \delta, a)$ that converges to a. Use the Cauchy Mean Value Theorem to produce a sequence (y_n) in $(a - \delta, a)$ that converges to a and has the property that $\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)}$ for all n.
   (c) Show that $\lim_{x\to a^-} \frac{f(x)}{g(x)} = L$.

Suppose that I is an iWimp™ and $g\colon I \to \mathbb{R}$ is differentiable. What can we say about the function $g'\colon I \to \mathbb{R}$? In principle, it could be the case that, for example, g' has a jump discontinuity, a 'removeable' discontinuity, an oscillation discontinuity, an 'infinite' discontinuity, ...

(289) Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Sketch a graph of g. Be sure to indicate the asymptotic behaviour of g as $x \to \infty$ and $x \to -\infty$.
(b) Show that g is differentiable.
(c) Show that $g'\colon \mathbb{R} \to \mathbb{R}$ is not continuous.

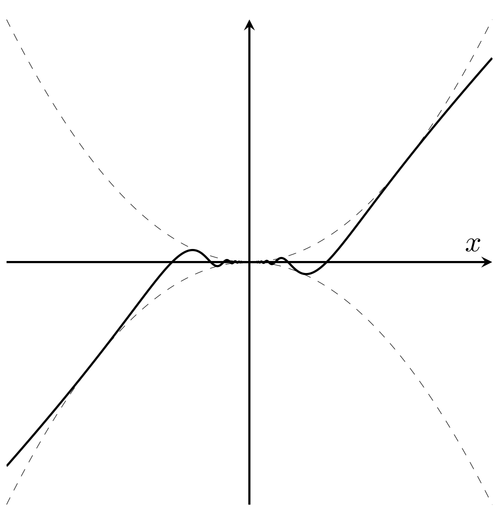
(290) Suppose that I is an open iWimp™ and $f\colon I \to \mathbb{R}$ is differentiable. Suppose $a, b \in I$ with $a < b$. Suppose $z \in \mathbb{R}$ is strictly between $f'(a)$ and $f'(b)$.

(a) Show that the function $f_a\colon [a,b] \to \mathbb{R}$ defined by $f_a(a) = f'(a)$ and $f_a(x) = \dfrac{f(x) - f(a)}{x - a}$ for $x \in (a,b]$ is continuous.
(b) Show that the function $f_b\colon [a,b] \to \mathbb{R}$ defined by $f_b(b) = f'(b)$ and $f_b(x) = \dfrac{f(x) - f(b)}{x - b}$ for $x \in [a,b)$ is continuous.
(c) Show that $z \in \operatorname{im}(f_a)$ or $z \in \operatorname{im}(f_b)$. [Hint: See Homework 96.]
(d) If $z \in \operatorname{im}(f_a)$, use the MVT to show that there is a $t \in (a,b)$ so that $f'(t) = z$.
(e) If $z \in \operatorname{im}(f_b)$, use the MVT to show that there is a $t \in (a,b)$ so that $f'(t) = z$.
(f) (Darboux's Theorem). Conclude: There exists $c \in (a,b)$ for which $f'(c) = z$.

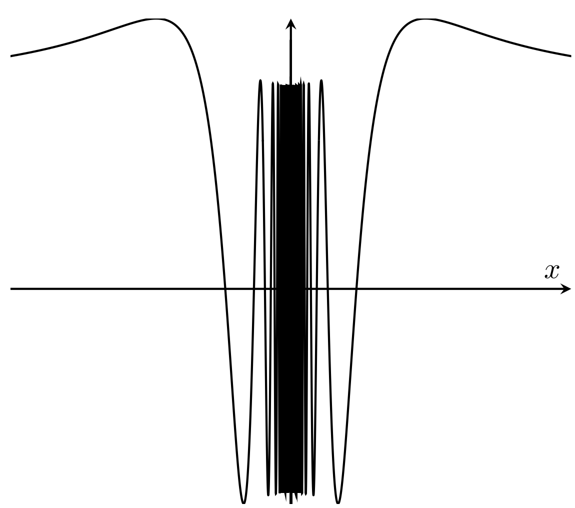
Darboux's Theorem is an analogue of the IVT, but for derivatives. It shows that derivatives cannot have, for example, jump or removeable discontinuities. The proof you provided in Prompt 290 is quite different from the one given in your book; both require the introduction of an auxiliary function.

Something to Think About

How might you generalize the mean value theorem to the finite dimensional inner product situation? For example, for a function $g\colon V \to \mathbb{R}$ or to a function $f\colon \mathbb{R} \to V$? It turns out that in the former case there is an analogue, can you formulate it? In the latter case, one must be content with the 'mean value inequality' which says that there exists $c \in (a,b)$ so that $\|D_c f\|_V \geq \frac{\|f(b) - f(a)\|_V}{(b-a)}$. How would you prove such a statement?



$x^2 \sin(1/x)$ on $[-1,1] \setminus \{0\}$



$x \sin(1/x) - \cos(1/x)$ on $[-1,1] \setminus \{0\}$

Worksheet for 5 Apr 2019
Fundamental Theorem(s) of Calculus
§7.5

Vocabulary: Fundamental Theorem of Calculus (i), Fundamental Theorem of Calculus (ii), Antiderivative

The Fundamental Theorem of Calculus is fundamental both because it links integration and differentiation, but also because nearly all of the major results in calculus from this point forward use it or the Mean Value Theorem. For example, it is used in the remainder theorems for Taylor polynomials, in the proof of the equality of mixed partials, in the proof of the inverse function theorem, when looking at convergence properties for sequences of differential functions, ...

(291) (Warmup) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $\ell\colon [a, b] \to \mathbb{R}$ is continuous. Show that ℓ is integrable. [Hint: See the Worksheet for §7.2.]

We will now create a proof of the Fundamental Theorem of Calculus (ii). Suppose $g\colon [a, b] \to \mathbb{R}$ is integrable.

(292) Define $G\colon [a, b] \to \mathbb{R}$ by $G(x) = \int_a^x g$. Suppose $t, h \in \mathbb{R}$ with $t, t + h \in [a, b]$. Interpret

$$\frac{G(t + h) - G(t)}{h}$$

in terms of the area of an appropriate rectangle. [Hint: Draw some pictures.] What is the relationship between $\frac{G(t+h)-G(t)}{h}$ and $g(t)$?

(293) Show that if g is continuous at a, then G is differentiable at a and $G'(a) = g(a)$. [Hint: You need to show that $\lim_{h\to 0^+} \frac{G(a+h)-G(a)}{h} = g(a)$, so fix $\varepsilon > 0$ and get started. You will probably need to use the result at Prompt 270.]

(294) Prove the Fundamental Theorem of Calculus (ii):

If $c \in [a, b]$ and g is continuous at c, then G is differentiable at c and $G'(c) = g(c)$.

[Hint: The proof is nearly identical to the one you just gave in Prompt 293; it is also in your book.]

(295) Show that if we define $\tilde{G}(x) = \int_x^b g$ and g is continuous at $c \in [a, b]$, then $\tilde{G}'(c) = -g(c)$.

(296) Show that $\ln\colon (0, \infty) \to \mathbb{R}$ is differentiable and $\ln'(x) = 1/x$ for all $x \in (0, \infty)$.

(297) Show that $\ln\colon \mathbb{R}_{>0} \to \mathbb{R}$ is a group homomorphism. That is, show that for all $a, b \in \mathbb{R}_{>0}$ we have $\ln(ab) = \ln(a) + \ln(b)$. [Hint: Take the derivative of both sides with respect to a, plug in $a = 1$, and use the Mean Value Theorem.]

We will now create a proof of the Fundamental Theorem of Calculus (i). Suppose $f\colon [a, b] \to \mathbb{R}$ is integrable and $F\colon [a, b] \to \mathbb{R}$ is differentiable with $F' = f$. The function F is called an *antiderivative* of f.

(298) If $\tilde{F}$ is another antiderivative of f, then how are F and $\tilde{F}$ related?

(299) Suppose $t_{47}, t_{48} \in [a, b]$ with $t_{47} < t_{48}$. Show that there is an $x^*_{48} \in [t_{47}, t_{48}]$ for which

$$F(t_{48}) - F(t_{47}) = f(x^*_{48})(t_{48} - t_{47}).$$

(300) Suppose $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ is a partition of $[a, b]$. For each $i \in \mathbb{N}_n$ choose $x^*_i \in [t_{i-1}, t_i]$ for which $F(t_i) - F(t_{i-1}) = f(x^*_i)(t_i - t_{i-1})$. Show that $L(f, P) \leq \sum f(x^*_i)(t_i - t_{i-1}) \leq U(f, P)$.

(301) (Fundamental Theorem of Calculus (i)) Conclude that

$$\int_a^b f = F(b) - F(a).$$

(302) Suppose $m \in \mathbb{N}$. Show:

$$\int_a^b t^m\, dt = \frac{b^{m+1} - a^{m+1}}{m + 1}.$$

Something to Think About

You encountered a few generalizations of the Fundamental Theorem of Calculus (i) in your multivariable class; here are the wikipedia versions of them:

- Green's Theorem: Let C be a positively oriented, piece-wise smooth, simple closed curve in $\mathbb{R}^2$, and let D be the region bounded by C (often indicated with $\partial D = C$). Suppose $\mathcal{O} \subset \mathbb{R}^2$ is open and contains D. If $L, M\colon \mathcal{O} \to \mathbb{R}$ have continuous partial derivatives, then
$$\iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\, dy = \oint_{\partial D = C} (L\, dx + M\, dy)$$
where the path of integration along C is anticlockwise.
- Stokes' Theorem: Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a piecewise smooth Jordan plane curve. Let D denote the compact part of $\mathbb{R}^2$ that is bounded by γ and suppose $\psi\colon D \to \mathbb{R}^3$ is smooth, with $S := \psi(D)$. If $\Gamma = \partial S$ is the space curve defined by $\Gamma(t) = \psi(\gamma(t))$ and $\mathbf{F}$ is a smooth vector field on $\mathbb{R}^3$, then:
$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S = \Gamma} \mathbf{F} \cdot d\mathbf{\Gamma}$$
- Divergence Theorem: Suppose $V \subset \mathbb{R}^3$ is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then we have:
$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iint_{\partial V = S} (\mathbf{F} \cdot \mathbf{n})\, dS$$
The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V and $\mathbf{n}$ is the outward pointing unit normal field of the boundary ∂V.

All of these theorems, including the Fundamental Theorem of Calculus (i)
$$\int_{[a,b]} F' = F(b) - F(a),$$
have the general beautiful form
$$\int_\Omega \partial\omega = \int_{\partial\Omega} \omega.$$
How general might one be able to take Ω to be? What happens when Ω doesn't have a boundary? What is the meaning of $\partial\omega$? All this and more will be discussed in Math 395 and Math 396.

An SMBC comic by Zach Weinersmith (www.smbc-comics.com)

Please report any errors you find to smdbackr@umich.edu

Worksheet for 8 Apr 2019
Sequences of Functions: pointwise vs. uniform convergence
§6.2

Vocabulary: point-wise convergence, uniform convergence

We have already mentioned that infinite dimensional vector spaces interact with analysis in more complicated & interesting ways than their finite dimensional siblings. For example, for a finite dimensional inner product space there is really only one notion of convergence that is compatible with having calculus work.[1] However, there are all kinds of interesting and useful ways for things to converge on function spaces (for example, do an Internet search for the term "L^p space").

Definition. Suppose V and W are inner product spaces and $A \subset V$. A sequence of functions $(h_n : A \to W)$ is said to *converge point-wise* to $h: A \to W$ provided that for all $\vec{a} \in A$ the sequence $(h_n(\vec{a}))$ converges to $h(\vec{a})$.

(303) This example is similar to the example discussed in Prompt 102. Define $f_n \in C^0([-1,1])$ by

$$f_n(x) = \begin{cases} 1 & x \in [-1,0] \\ 1 - nx & x \in (0, 1/n] \\ 0 & x \in (1/n, 1] \end{cases}$$

Does (f_n) converge point-wise in $C^0([-1,1])$?

(304) Recall that $C^0([-1,1])$ is an inner product space with respect to $\langle f, h\rangle = \int_{-1}^{1} f(t)\cdot h(t)\,dt$. Show that the sequence (f_n) from Prompt 303 is Cauchy with respect to this inner product. [Hint: Don't try to evaluate. Do note that for $m > n$ we have $\|f_n - f_m\|^2 < 1/n$.] Do you think that with respect to the inner product on $C^0([-1,1])$ that (f_n) has a limit in $C^0([-1,1])$?[2]

The two types of convergence discussed in the above Prompts fail to preserve continuity. To address this shortfall, we introduce the notion of *uniform convergence*.

Definition. Suppose V and W are inner product spaces and $A \subset V$. A sequence of functions $(h_n : A \to W)$ is said to *converge uniformly* to $h: A \to W$ provided that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\|h_n(\vec{a}) - h(\vec{a})\|_W < \varepsilon \text{ for all } \vec{a} \in A\,.$$

Nota bene: In the definition of uniform convergence the N does not depend on the point $\vec{a}$. Compare to the relationship between the concepts of uniform continuity and continuity.

(305) Draw some pictures[3] of a sequence of functions $(g_n : [-6,2] \to \mathbb{R})$ that converge uniformly to $g: [-6,2] \to \mathbb{R}$.

(306) Show that if $(h_n : A \to W)$ converges uniformly, then it converges pointwise.

(307) Show that the sequence $(f_n : [-1,1] \to \mathbb{R})$ defined in Prompt 303 does not converge uniformly to a function $f: [-1,1] \to \mathbb{R}$. [Hint: Suppose it does. By Prompt 306 we have that (f_n) converges to f point-wise. Look at $f_m(1/2m)$ and $f(1/2m)$.]

(308) Suppose V and W are finite dimensional inner product spaces and $A \subset V$. Let $(h_n : A \to W)$ be a sequence of continuous functions that converges uniformly to $h: A \to W$. Show that h is continuous. [Hint: h_{N+1} is a continuous function and $\varepsilon = \varepsilon/3 + \varepsilon/3 + \varepsilon/3$.]

So, the uniform limit of continuous functions is continuous. Do you think the same can be said for differentiable functions? Integrable functions? We will take up these questions later. For now we close with the following extremely useful result.

(309) (Uniform Cauchy Criterion). Suppose V and W are finite dimensional inner product spaces and $A \subset V$. Let $(h_n : A \to W)$ be a sequence of functions. The following statements are equivalent.
- There exists $h: A \to W$ for which (h_n) converges uniformly to h.

[1] See today's Something to Think About.

[2] If we were to continue in this direction, we would want to expand beyond continuous functions to $L^2([0,1])$, the space of square-integrable functions on $[0,1]$. However, this direction requires measure theory, equivalence relations, and various other mathematical ideas with which we are not fully acquainted. This circle of ideas is mighty important and well worth studying in, for example, Math 597.

[3] I am fond of the expression 'ε-tube.'

- For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have
$$\|h_n(\vec{a}) - h_m(\vec{a})\|_W < \varepsilon \text{ for all } \vec{a} \in A.$$

[Hint: One direction is straightforward. For the other direction, you must first use what you know about Cauchy sequences to construct an $h\colon A \to W$. Then remember that for all $\vec{a} \in A$, $\|h_{N+1}(\vec{a}) - h_n(\vec{a})\| < \varepsilon'/47$ for $n > N$, so $\|h_{N+1}(\vec{a}) - h(\vec{a})\| < 2\varepsilon'/47$.]

Something to Think About

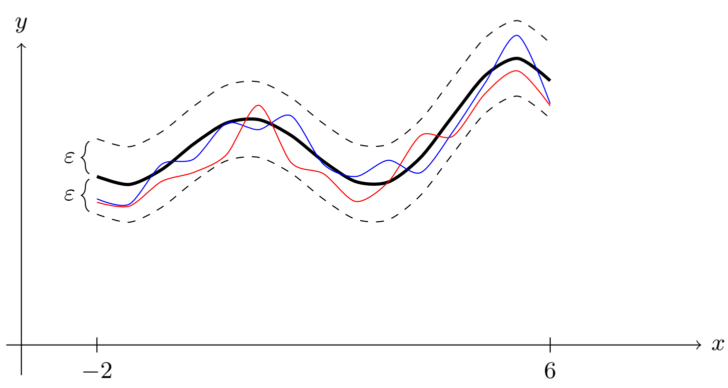
As alluded to above, in the finite-dimensional case, all norms[1] arising from inner products on finite dimensional real vector spaces are 'created equal.' More precisely, we say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a finite dimensional $\mathbb{R}$-vector space V are *equivalent* if there exist positive $C \geq c$ such that
$$c\|\cdot\| \leq \|\cdot\|' \leq C\|\cdot\|$$
on V. The condition of equivalence is a symmetric and transitive relation on the set of norms on V. Moreover, the concepts of *open set*, *Cauchy sequence*, *limit of a convergent sequence*, and *continuous map* between open sets in normed vector spaces are unaffected by replacing the given norms with equivalent ones. Hence, for the purposes of analysis there is no preference among any two equivalent norms.

We just need to convince ourselves that all norms are equivalent. Let $\|\cdot\|$ be a norm on V. Suppose $\mathbf{e} = (\vec{e}_1, \ldots, \vec{e}_d)$ is an ordered basis for V. Using arguments similar to ones we have used before, one can find a positive constant C such that
$$\|\cdot\| \leq C\|\cdot\|_{\max,\mathbf{e}}$$
where $\|\cdot\|_{\max,\mathbf{e}}\colon V \to \mathbb{R}$ is the norm defined by $\|\sum a_i\vec{e}_i\|_{\max,\mathbf{e}} = \max\{|a_i| \mid 1 \leq i \leq d\}$

To prove that there exists a positive c such that $\|\cdot\|_{\max,\mathbf{e}} \leq c\|\cdot\|$ is a bit trickier. Suppose no such constant exists. Thanks to the scaling-invariance of the inequality, one may find an infinite sequence of vectors $\vec{v}_n$ satisfying $\|\vec{v}_n\|_{\max,\mathbf{e}} = 1$ and $\|\vec{v}_n\| \leq 1/n$. Write $\vec{v}_n = \sum a_{j,n}\vec{e}_j$, and use sequential compactness of $[-1,1]^d$ to find a subsequence $\vec{v}_{n_i}$ with

- $a_{j,n_i} \to a_j \in [-1,1]$ as $n_i \to \infty$ and
- $\max(|a_1|, \ldots, |a_d|) = 1$.

Note that $\vec{v} = \sum a_j\vec{e}_j \neq 0$. The triangle inequality applied to $\|\cdot\|$ tells us that $\vec{v}_n \to \vec{v} = \sum a_j\vec{e}_j$ with respect to $\|\cdot\|$. We then obtain a contradiction from the nonvanishing of $\vec{v}$ and the upper bounds $\|\vec{v}_n\| \leq 1/n$. Do you see why?



An ε-tube around the graph of g, containing the graphs of possible g_n and g_m

[1]A norm does not need to arise from an inner product. A *norm* on a real vector space V is a function $\|\|\colon V \to \mathbb{R}$ that satisfies

- $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ for all $c \in \mathbb{R}$ and $\vec{v} \in V$,
- $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all $\vec{u}, \vec{v} \in V$,
- $\|v\| \geq 0$ for all $\vec{v} \in V$ and equality holds if and only if $\vec{v} = \vec{0}$.

Worksheet for 10 Apr 2019
Uniform convergence: integration and differentiation
§6.3

Vocabulary: C^p-type, $C^p(I)$, C^∞-type, $C^\infty(I)$

In Prompt 308 we showed that the uniform limit of continuous functions is continuous. In this worksheet we investigate how uniform convergence interacts with integration and differentiation.

To avoid getting lost in notation, we have often restricted our attention to functions from $\mathbb{R}$ to $\mathbb{R}$; we will do that again here. However, the results of this worksheet make sense in a more general setting; see the Something to Think About section below.

(310) Suppose $a, b \in \mathbb{R}$ with $a < b$. Let $(f_n : [a,b] \to \mathbb{R})$ be a sequence of integrable functions that converge uniformly to $f : [a,b] \to \mathbb{R}$.

(a) Show that f is integrable. [Hint: Use the Darboux Integrability Criterion (Prompt 255) and note that we can choose M so that $|f_M(x) - f(x)| < \varepsilon'/(47(b-a))$. Use the fact that f_M is integrable.]

(b) Show[1]

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

For continuity and integration, life could not be more swell. The story for differentiation is not as simple.

(311) Consider the sequence of functions $(h_n : \mathbb{R} \to \mathbb{R})$ defined by

$$h_n(x) = \begin{cases} |x| & \text{if } x \notin (-1/n, 1/n) \\ \frac{1}{n} - \frac{2\cos(n\pi x/2)}{n\pi} & \text{if } x \in (-1/n, 1/n) \end{cases}$$

Sketch the functions h_n and compute h_n'. Is each h_n' continuous? Show that h_n converges uniformly to the function $h(x) = |x|$.

A problem with the example in Prompt 311 is that, though they are continuous, the derivatives $(h_n' : \mathbb{R} \to \mathbb{R})$ do not converge uniformly. Because this type of situation arises so often, there is an entire suite of notation that has arisen around it.

Definition. Suppose $I \subset \mathbb{R}$ is an iWimp™. A function $f : I \to \mathbb{R}$ is said to be of C^0-*type* provided that f is continuous. The set of C^0-type functions on I is denoted $C^0(I)$.
A function $\ell : I \to \mathbb{R}$ is said to be of C^1-*type* provided that ℓ is differentiable and ℓ' is of C^0-type. The set of C^1-type functions on I is denoted $C^1(I)$.

(312) Show that $C^0(I)$ is a vector space and, since every differentiable function is continuous, $C^1(I)$ is a subspace of $C^0(I)$. Show that $C^1(I)$ is a proper subspace of $C^0(I)$. [Hint: While it is overkill, Prompt 289 may be worth looking at.]

Definition. More generally, for $p \in \mathbb{N}$ a function $h : I \to \mathbb{R}$ is said to be of C^p-*type* provided that h is differentiable and h' is of C^{p-1}-type. The vector space of functions of C^p-type on I is denoted $C^p(I)$.

We have

$$C^0(I) \supsetneq C^1(I) \supsetneq C^2(I) \supsetneq C^3(I) \supsetneq \cdots$$

(313) Produce a function that is in $C^1(\mathbb{R})$ but not $C^2(\mathbb{R})$. [Hint: Consider $x^\alpha \sin(1/x)$ for appropriate α.]

Definition. A function $j : I \to \mathbb{R}$ is said to be of C^∞-*type* provided that $j \in C^p(I)$ for all p. The vector space of functions of C^∞-type on I is denoted $C^\infty(I)$.

All polynomial functions and your favorite functions like $\ln$, $\sin$, $\cos$, and $\exp$ are of C^∞-type. Many people refer to C^∞-type functions as *smooth functions*.

[1] Hint: Prompt 270 may be useful.

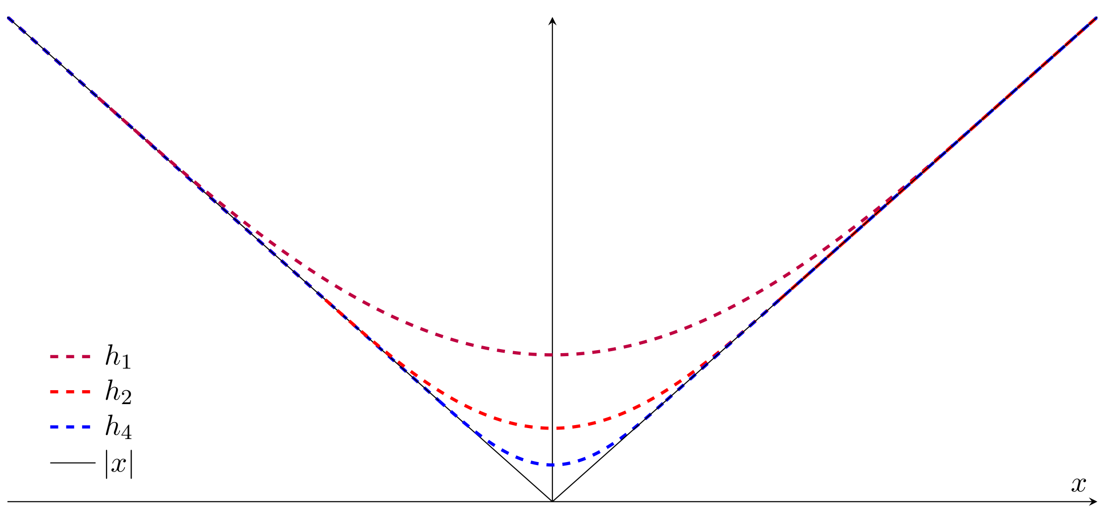
(314) Suppose $I \subset \mathbb{R}$ is an iWimp™. Suppose (f_n) is a sequence in $C^1(I)$ and both (f_n) and (f_n') satisfy the Uniform Cauchy Criterion.
  (a) Show there exist $f, g \in C^0(I)$ so that (f_n) converges uniformly to f and (f_n') converges uniformly to g.
  (b) Show that $f_m(x) - f_m(a) = \int_a^x f_m'$ for all $x \in I$. Use Prompt 310 to show that $f(x) = f(a) + \int_a^x g$
  (c) Conclude that $f \in C^1(I)$ and $f' = g$.

Abbott presents a different version of this result; see Theorem 6.3.3. Note that Theorem 6.3.3 has both a weaker hypothesis and a weaker conclusion – so it is neither stronger nor weaker than the result in Prompt 314.

(315) Generalize the statement of the result of Prompt 314 to a sequence (f_n) in $C^p(I)$.
(316) *Bonus.* Prove your generalization.

Something to Think About

Suppose V and W are finite dimensional inner product spaces.
The result of Prompt 310 can be generalized to the setting where $K \subset V$ is compact and $(f_n \colon K \to W)$ is a sequence of integrable functions that satisfies the Uniform Cauchy Criterion. Can you see how to proceed?
If $\mathcal{O} \subset V$ is open, then one can define $C^0(\mathcal{O}, W)$ to be the space of continuous functions $f \colon \mathcal{O} \to W$. One defines $C^p(\mathcal{O}, W)$ and $C^\infty(\mathcal{O}, W)$ similarly. Prompt 314 can be generalized to this context. Can you see how to proceed?



Perhaps useful for Prompt 311.

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Worksheet for 12 Apr 2019
Series of Functions: Weierstrass M-test & term-by-term results
§6.4

Vocabulary: Weierstrass M-test, Uniform Cauchy Criterion, term-by-term integration, term-by-term differentiation

We want to apply what we have learned about sequences of functions to power series. Before we can do this, we need to develop a general theory of series of functions.

Definition. Suppose that V and W are inner product spaces and $A \subset V$. Suppose $f\colon A \to W$ is a function and $(f_n \mid A \to W)$ is a sequence of functions. The infinite series

$$\sum_{n=1}^{\infty} f_n$$

converges pointwise (resp. *uniformly*) to f provided that the sequence of partial sums $(\vec{ps}_m : A \to W)$ converges pointwise (resp. uniformly) to f. Here $\vec{ps}_m = \sum_{n=1}^{m} f_n$. In either case we write

$$f = \sum_{n=1}^{\infty} f_n.$$

To save ink, we will often write $\sum f_n$ for $\sum_{n=1}^{\infty} f_n$

(317) (Uniform Cauchy Criterion). Suppose V and W are finite dimensional inner product spaces and $A \subset V$. Let $(h_n\colon A \to W)$ be a sequence of functions. The following statements are equivalent.
- There exists $h\colon A \to W$ for which $\sum h_n$ converges uniformly to h.
- For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > \ell > N$ we have

$$\|h_{\ell+1}(\vec{a}) + h_{\ell+2}(\vec{a}) + \cdots + h_m(\vec{a})\|_W < \varepsilon \text{ for all } \vec{a} \in A.$$

(318) (Weierstrass M-test). Suppose that V and W are finite dimensional inner product spaces and $A \subset V$. Suppose $(f_n \mid A \to W)$ is a sequence of functions. If (M_n) is a sequence for which
- there is an L with the property that for all $\ell > L$ we have $\|f_\ell(\vec{a})\|_W \le M_\ell$ for all $\vec{a} \in A$ and
- $\sum M_n$ converges,

then there exists $f\colon A \to W$ to which $\sum f_n$ converges uniformly and absolutely. [Hint: Apply the Uniform Cauchy Criterion to $\sum \|f_n\|_W$.]

Warning. With notation as in the Weierstrass M-test lemma: While both $\|\sum f_n\|_W$ and $\sum \|f_n\|_W$ converge, it will not usually happen that $\|\sum f_n\|_W = \sum \|f_n\|_W$.

(319) (Term-by-term Continuity). Suppose that V and W are inner product spaces and $A \subset V$. Suppose $g\colon A \to W$ is a function and let $(g_n \mid A \to W)$ be a sequence of functions in $C^0(A, W)$. If $\sum g_n$ converges uniformly to g, then $g \in C^0(A, W)$.

To avoid getting lost in notation, we now restrict our attention to functions from $\mathbb{R}$ to $\mathbb{R}$. You may want to think about how to recast the following results in a more general setting.

(320) (Term-by-term Integration). Suppose $[a, b] \subset \mathbb{R}$ with $a < b$. Suppose $(h_n\colon [a, b] \to \mathbb{R})$ is a sequence of integrable functions and $h\colon [a, b] \to \mathbb{R}$ is a function. If $\sum h_n$ converges uniformly to h, then h is integrable and

$$\int_a^b h = \sum (\int_a^b h_n).$$

(321) (Term-by-term Differentiation). Suppose $I \subset \mathbb{R}$ is an iWimp™. Suppose (f_n) is a sequence in $C^1(I)$. If both $\sum f_n$ and $\sum f_n'$ satisfy the Uniform Cauchy Criterion, then there exists $f \in C^1(I)$ such that $f = \sum f_n$ and $f' = \sum f_n'$.

Some Things to Think About

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[A] Since all polynomial functions are continuous, integrable, and differentiable, we expect the same will be true for power series. But, what is a power series? For example, when we write

$$\sum (-1)^{n+1} x^n$$

we know that this expression doesn't make sense at, for example, $x = 47$. How should we make sense of a general power series like

$$\sum c_n x^n \text{ ?}$$

[B] One of the more useful and beautiful areas of mathematics involves the study of trigonometric polynomials (this area is usually called *Fourier analysis*). While born in physics (solutions to the wave and heat equations), it is now used pretty much everywhere.[1] The basic idea is that you can study a complex valued 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ by approximating it with a *trigonometric polynomial* of the form

$$\vec{ps}_N = \sum_{n=-N}^{N} a_n e^{in\theta}$$

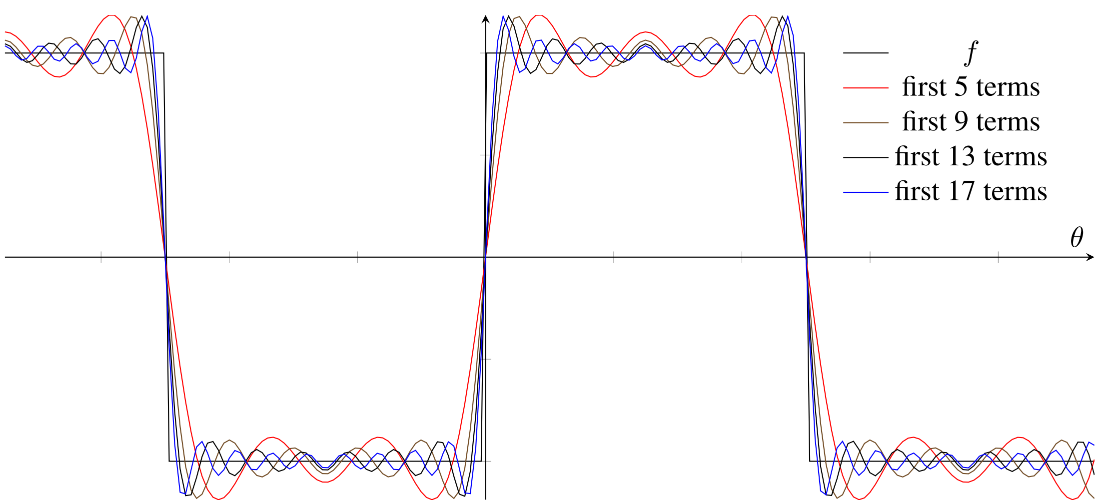
with $a_n \in \mathbb{C}$. If your original function is real-valued, then you can approximate it with a function of the form

$$\vec{ps}_N = \sum_{n=0}^{N} b_n \cos(n\theta) + c_n \sin(n\theta)$$

with $b_n, c_n \in \mathbb{R}$. The cool thing is that thanks to the trigonometric orthogonality relations, we have

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

(with similar equations for b_n, c_n). The sequence $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ is called the *Fourier series* of f. Which 2π-periodic functions on $\mathbb{R}$ can be approximated uniformly by their Fourier series? which can be approximated point-wise? Can we differentiate these series term by term to get the derivative? What about integration?



Graphs of some partial sums associated to the Fourier series of a function f.

[1] According to wikipedia, Fourier analysis is used in "physics, partial differential equations, number theory, combinatorics, signal processing, digital image processing, probability theory, statistics, forensics, option pricing, cryptography, numerical analysis, acoustics, oceanography, sonar, optics, diffraction, geometry, protein structure analysis, and other areas."

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Worksheet for 15 Apr 2019
Power Series
§6.5

Vocabulary: power series, real analytic functions, locally represented by a power series, $C^\omega(I)$, Hadamard's Theorem

A power series is **not** a function. It is important that you not confuse it for one. So then, what is a power series?

Definition. Suppose $a \in \mathbb{R}$ and (c_n) is a sequence of real numbers. The collection of symbols

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a *power series centered at* a.

In general, a power series will converge for some values of x (e.g., $x = a$) and diverge for others. We are not making any claims like '$\sum c_n(x-a)^n$ is a function.' We will often talk about a function being *represented by a power series* or, more accurately, *locally represented by a power series*. The following definition explains what we mean by this.

Definition. Suppose $I \subset \mathbb{R}$ is an iWimp$^{\text{TM}}$ and $f\colon I \to \mathbb{R}$. We say that f is *real analytic* or *locally represented by a power series* provided that for all $a \in I$ there exist $\varepsilon > 0$ and a sequence (c_n) such that for all $h \in B_\varepsilon(0)$ we have

$$\text{if } a+h \in I\text{, then } f(a+h) = \sum c_n h^n.$$

The vector space of real analytic functions on I is denoted $C^\omega(I)$.

(322) Show that the function $g\colon (0,\infty) \to \mathbb{R}$ defined by $g(x) = 1/x$ is real analytic. [Hint: Fix $a \in (0,\infty)$. Note that

$$g(x) = \frac{1}{x} = \frac{1}{a + (x-a)} = \frac{1}{a} \cdot \frac{1}{1 + (\frac{x-a}{a})}.$$

For $h \in (-a, a)$ we have $g(a+h) = \cdots$.]

This example demonstrates at least two important things: (a) the choice of coefficients (c_n) depends very much on the $a \in I$ under consideration; and (b) there may not be a 'single power series' that represents f on all of I, hence the need for the word *locally*.

(323) Verify that $C^\omega(I)$ is a vector space.

Not all functions belong to $C^\omega(I)$, and, in fact, we will end up showing that $C^\omega(I) \subsetneq C^\infty(I)$. We now turn to the question of determining for which inputs a given power series might converge.

(324) (Hadamard's Theorem). Suppose $\sum c_n x^n$ is a power series. Define $\rho \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$\rho = \frac{1}{\limsup(|c_n|^{1/n})}.$$

You will work on at least one of the following two prompts – ask DeBacker which one you should look at.

(a) Show: If $b \in (-\rho, \rho)$, then $\sum c_n b^n$ converges absolutely.[1]

(b) Show: If $d \notin [-\rho, \rho]$, then $\sum c_n d^n$ diverges.[2]

(325) Why does Hadamard's Theorem use $\limsup(|c_n|^{1/n})$ rather than $\lim |c_n|^{1/n}$?

Definition. ρ is called the *radius of convergence* for $\sum c_n x^n$.

Note that $\sum c_n x^n$ converges at every $b \in (-\rho, \rho)$. The question of convergence at $-\rho$ and ρ is often quite subtle and almost always important. However, the semester is nearly done, so we will let you study the endpoints on your own.

[1][Hint: WOLOG $b \neq 0$. Choose γ so that $|b| < \gamma < \rho$. Note that $1/|b| > 1/\gamma > 1/\rho$. Use the definition of $\limsup$ to conclude that there is an N such that for all $n > N$ we have $|c_n|^{1/n} < 1/\gamma < 1/|b|$. Transform this into something that can be compared to a geometric series.]

[2][Hint: Note that $1/|d| < 1/\rho$. Use the definition of $\limsup$ to conclude that for all N there is an $m > N$ such that $|c_m|^{1/m} > 1/|d|$. Transform this into something to which you can apply the nth term test.

(326) Suppose that $\sum d_n h^n$ is a power series for which $\lim_{m\to\infty} \frac{|d_{m+1}|}{|d_m|} = 47$. What is the radius of convergence for $\sum d_n h^n$?

To tie everything together, we now discuss a very important Corollary to the Weierstrass M-test.

(327) Let $\sum_{\ell=0}^{\infty} c_\ell x^\ell$ be a power series with radius of convergence ρ. Define $f\colon (-\rho,\rho) \to \mathbb{R}$ by setting $f(b) = \sum c_\ell b^\ell$ for $b \in (-\rho,\rho)$.
  (a) Fix $\delta \in (0,\rho)$. Use the Weierstrass M-test to show[1] that the series $\sum c_\ell x^\ell$ converges uniformly to f on $[-\delta,\delta]$.
  (b) Show that f is continuous.
  (c) Show that f is integrable. What is $\int_0^t f$ for $t \in (-\rho,\rho)$? Why?[2]
  (d) Use Hadamard's Theorem to show that $\sum_{\ell=1}^{\infty} (\ell c_\ell) x^{\ell-1}$ has radius of convergence ρ. Conclude that $f \in C^1(-\rho,\rho)$ and $f'(b) = \sum (\ell c_\ell) b^{\ell-1}$ for $b \in (-\rho,\rho)$.
  (e) Conclude that $f \in C^\infty((-\rho,\rho))$.
  (f) Conclude that for any iWimp™ $I \subset \mathbb{R}$ we have $C^\omega(I) \subset C^\infty(I)$.

Do you think f is also real analytic? It does have a power series representation at 0, but what about at $\rho/2$? It is **not** obvious that $f \in C^\omega((-\rho,\rho))$, but it is true – see handout.

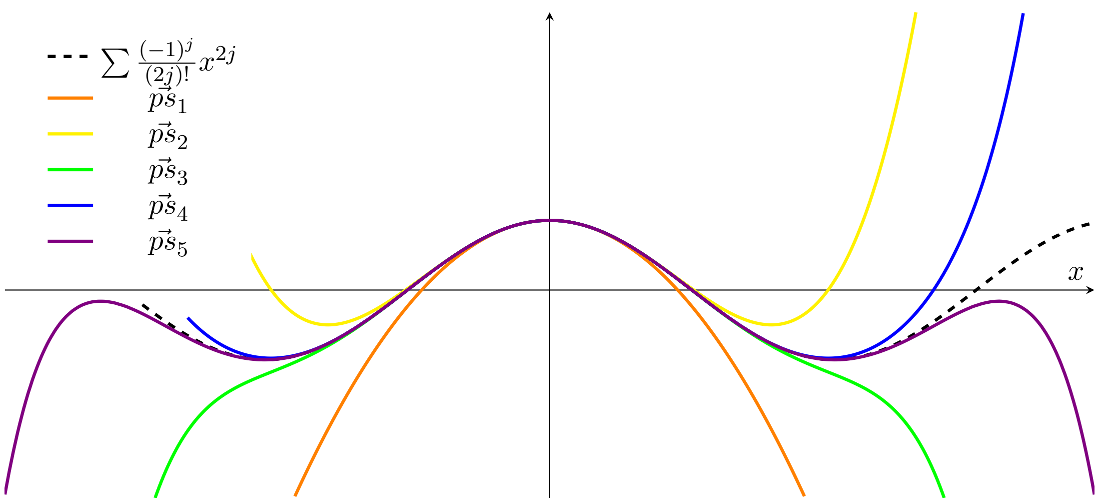
Remark. Thanks to Hadamard's Theorem and the other results of this worksheet, it is easy to check that your favorite functions (e.g. sin, cos, exp, ln, any polynomial function) are real analytic on their domains.

Something to Think About

In your calculus class you developed the theory of Taylor Series at a point a. For a real analytic function f what is the relationship between its Taylor Series expansion at a

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j$$

and the power series that locally represents f at a? Can you prove it?



Graphs of $\vec{ps}_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} x^{2j}$ and the graph of $\sum \frac{(-1)^j}{(2j)!} x^{2j}$.

[1][Hint: WOLOG $\rho \neq 0$. Use the definition of lim sup to conclude that there is an N such that for all $n > N$ we have $|c_n|^{1/n} < 1/\gamma$. Set $M_n = (\delta/\gamma)^n$.]

[2][Hint: There is a small subtlety here – the power of x and the index on c_ℓ do not match up as in the statement of Hadamard's Theorem. Here is a way around it. Suppose $t \in \mathbb{R}$. To check if $\sum a_k t^{k+2}$ converges at t, we look at $\sum a_k t^2 x^k$ and apply Hadamard's theorem. This new series has the same radius of convergence as $\sum a_k x^k$. Since t was arbitrary, we can conclude that the series $\sum a_k x^{k+2}$ has the same radius of convergence as $\sum a_k x^k$. Why?]

Please report any errors you find to smdbackr@umich.edu

Worksheet for 17 Apr 2019
The Principle of Analytic Continuation
§6.6

Vocabulary: Taylor series, Taylor polynomials, pliable, rigid, analytic continuation

Suppose f is smooth and $a \in \mathbb{R}$ belongs to the domain of f. The *Taylor series* for f centered at a is defined to be the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It is tempting to think of a Taylor Series at a as the limit of the *Taylor polynomials*

$$P_{f,a,m}(x) := \sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This should be done with some care. After all, the Taylor polynomials of f at a are functions on $\mathbb{R}$ and the Taylor series of f at a is **not** a function. However, the Taylor series of f at a does represent f on some interval of a (and that interval may be $\{a\}$); this is related to your homework on the Taylor remainder formulas.

(328) Suppose I is an iWimp™ and $f \in C^\omega(I)$. For every $a \in I$ show that the power series that locally represents f at a is the Taylor series of f at a.

(329) Show that $C^\omega(\mathbb{R}) \subsetneq C^\infty(\mathbb{R})$. [Hint: You've already done the hard work for this problem.]

Suppose I is an iWimp™. On your homework you demonstrated that smooth functions are *pliable* in the following sense: Suppose $f, g \in C^\infty(I)$ and $a \in I$. For all $\varepsilon > 0$ there exists $h \in C^\infty(I)$ with the property that

- $f(x) = h(x)$ for all $x \in I \cap (-\infty, a-\varepsilon)$ and
- $g(x) = h(x)$ for all $x \in I \cap (a+\varepsilon, \infty)$

In comparison, as the *Principle of Analytic Continuation* shows, real analytic functions are *rigid*:

Principle of Analytic Continuation. Suppose I is an iWimp™ and $\ell \in I$. Suppose (x_n) is a sequence in $I \setminus \{\ell\}$ that converges to ℓ. If $f, g \in C^\omega(I)$ have the property that $f(x_n) = g(x_n)$ for all n, then $f = g$ on I.

The Principle of Analytic Continuation plays an important role in many branches of mathematics. For example, in number theory it is used to define the Riemann zeta function.[1] You will now prove the $\mathbb{R}$ version of it.

(330) Motivating Example. Suppose $p, q \in \mathbb{R}[x]_{<k}$. If there exist distinct $t_1, t_2, \ldots, t_k$ in $\mathbb{R}$ for which $p(t_i) = q(t_i)$ for $1 \le i \le k$, then $p = q$. There are several ways to do this, I want you to use analysis. [Hint: WOLOG $t_1 < t_2 < \cdots < t_k$.]

(331) Easy Preliminary Lemma. Suppose J is an iWimp™, $j \in J$, and (j_n) a sequence in $J \setminus \{j\}$ that converges to j. If $h \in C^\omega(J)$ and $h(j_n) = 0$ for all n, then $h(j) = 0$.

(332) Define $h \in C^\omega(I)$ by $h = f - g$. Here are two ways to show that h is zero in a neighborhood of ℓ.
  (a) Show[2] that $h^{(j)}(\ell) = 0$ for all $j \ge 0$; the result will then follow from Prompt 328.
  (b) *Bonus*. Show[3] that if h is not zero in a neighborhood of ℓ, then $0 = 1$.

(333) Complete the proof of the Principle of Analytic Continuation by showing[4] that for all $x \in I$ we have $h(x) = 0$.

Something to Think About

[1]Check out this amazing video: https://www.youtube.com/watch?v=sD0NjbwqlYw

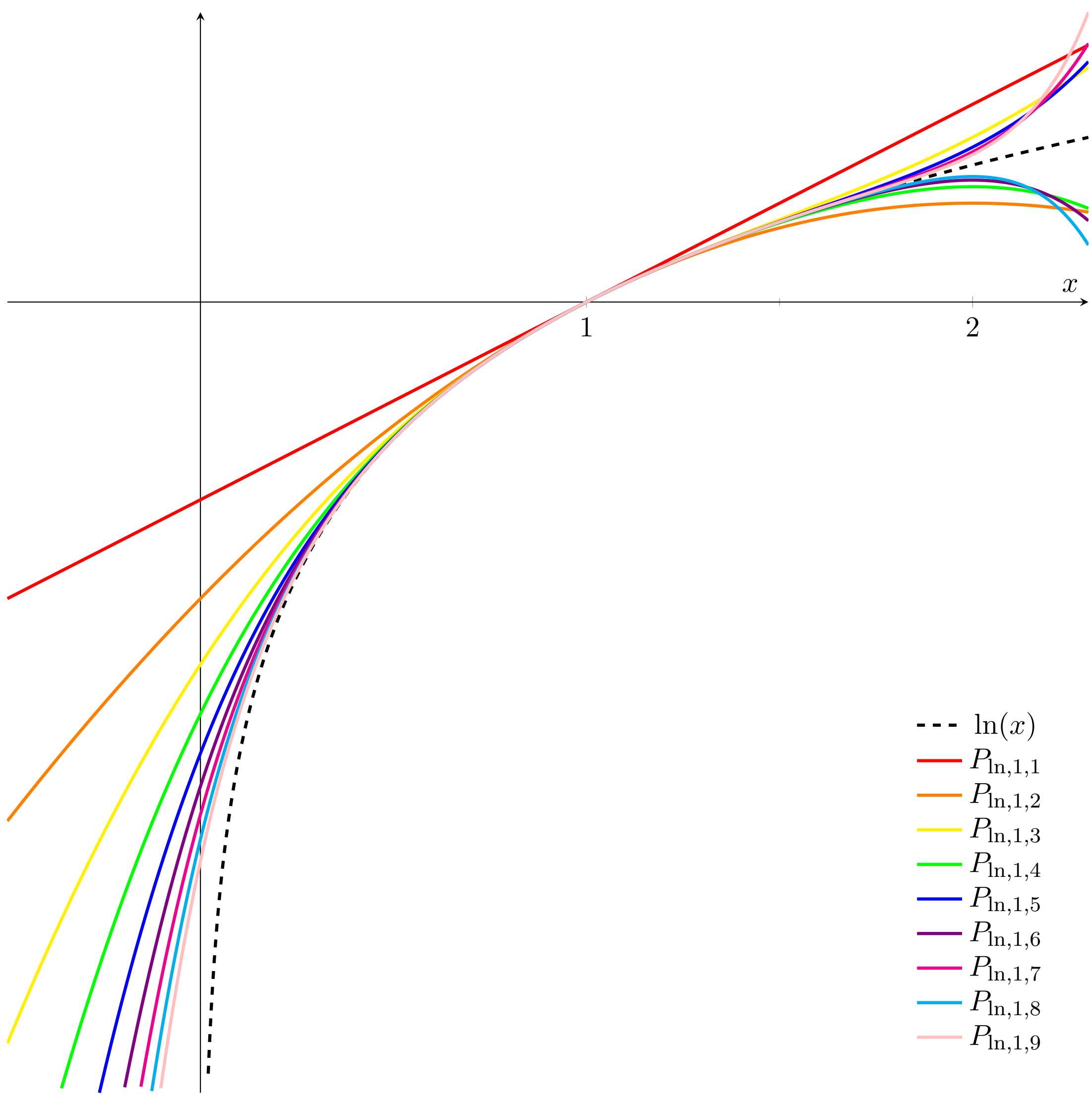
[2][Hint: WOLOG (x_n) is strictly increasing. Thanks to Prompt 331, we have $h^{(0)}(\ell) = h(\ell) = 0$. Thanks to Rolle's theorem, for each $n \in \mathbb{N}$ we can find $x_n^1 \in (x_n, x_{n+1})$ for which $h'(x_n^1) = 0$. Use Prompt 331 to show that $h^{(1)}(\ell) = 0$. ...]

[3][Hint: If h is not zero in a neighborhood of ℓ, then h is represented in some neighborhood of ℓ by a power series of the form $\sum_{m \ge k} a_m (x-\ell)^m$ with $a_k \neq 0$. Thus, in a neighborhood of ℓ we can write $h(x) = a_k(x-\ell)^k \tilde{g}(x)$ where $\tilde{g}$ is represented in a neighborhood of ℓ by $1 + \sum_{i \ge 1}(a_{i+k}/a_k)(x-\ell)^i$. Use Prompt 331 together with the facts that $h(x_n) = 0$ and $(x_n - \ell)^k \neq 0$ to conclude that $0 = 1$.]

[4][Hint: Fix $x \in I$. WOLOG, $x > \ell$. Set $\sigma = \sup\{t : \ell \le t \le x \text{ and } \operatorname{res}_{[\ell,t]} h = 0\}$. Construct an increasing sequence (y_n) in $[\ell, \sigma] \setminus \{\sigma\}$ with $h(y_n) = 0$. Use Prompt 332 to show h is zero in a neighborhood of σ.]

We veered a bit from Abbott's treatment of this material. Since you have already seen linear algebra, I chose to introduce vector spaces like $C^p(\mathbb{R})$ and explore how functions in these spaces behave (e.g. rigid vs. pliable).

A main takeaway from this section is that polynomial functions are simple, and we should stick to simple things. In your homework, you showed that every continuous function arises as the uniform limit of smooth functions. Do you think that on, for example, a compact set, the situation might be even better – for example, could it be that every continuous function on a compact set is the uniform limit of polynomials?



The graphs of some Taylor polynomials of $\ln(x)$ centered at $a = 1$.