Four Definitions of Manifold

Let $X \subset \mathbb{R}^n$ and let d be a nonnegative integer. We consider the following conditions:

Condition 1: For every $x \in X$, there exists an open set $U \ni x$, an open set $P \subset \mathbb{R}^d$ and a C^1 immersion $f : P \to \mathbb{R}^n$ with continuous inverse $f^{-1} : f(P) \to P$ such that $X \cap U = f(P)$. (This one was stated wrongly the first time.)

Recall that an immersion is a map for which (Df) is injective at every point.

Condition 2: For every $x \in X$, there exists an open set $U \ni x$ and a C^1 submersion $g : U \to \mathbb{R}^{n-d}$ such that $X \cap U = g^{-1}(g(x))$.

Condition 3: For every $x \in X$, there exists an open set $U \ni x$, an open set $V \subset \mathbb{R}^n$ and an invertible C^1 map $h : U \to V$ such that $X \cap U = h^{-1}(V \cap \big(\mathbb{R}^d \times \{0\}\big)\big)$.

Condition 4: For every $x \in X$, we can reorder the coordinates on $\mathbb{R}^n$ into the order $(x_1, x_2, \dots, x_d, y_1, y_2, \dots, y_{n-d})$ so that we can find an open set $U' \times U'' \subset \mathbb{R}^d \times \mathbb{R}^{n-d}$, and a C^1 function $q : U' \to U''$ such that $X \cap (U' \times U'') = \big\{(\vec{x}, q(\vec{x})) : \vec{x} \in U'\big\}$.

Today's main result will be that these conditions are equivalent! Such a set is called a d***-dimensional submanifold of*** $\mathbb{R}^n$, or a d***-fold in*** $\mathbb{R}^n$ for short.

Remark: Next time I teach this, condition (1) will read: For every $x \in X$, there exists an open set $U \ni x$, an open set $P \subset \mathbb{R}^d$ and C^1 maps $f : P \to U$ and $\ell : U \to P$ such that $\ell \circ f = \mathrm{Id}$ and $f(P) = U \cap X$. The existence of ℓ easily implies that f is an immersion, that it is injective and that it has continuous inverse and, quite nontrivially, it also follows from this laundry list of conditions. We never seemed to have any good way to work with condition 1 without the left inverse ℓ.