Let $A \subset \mathbb{R}^n$ be an open set. We define an ***exhaustion*** of A to be a sequence

$$U_1 \subseteq K_1 \subseteq U_2 \subseteq K_2 \subseteq \cdots \subseteq A \quad (*)$$

such that

(1) All the U_i are open.
(2) All the K_i are compact.
(3) $A = \bigcup U_i = \bigcup K_i$.

The equality of $\bigcup U_i$ and $\bigcup K_i$ simply follows from the containments in $(*)$, the point is to make sure that every $x \in A$ lies in K_i once K_i is large enough.

The point of this note is to prove that every open set A has an exhaustion. Moreover, we can choose the exhaustion so that the ∂K_i and ∂U_i are measurable. In fact, K_i will be a union of finitely many closed rectangles and U_i will be its interior.

Let N be a positive integer. We'll define an ***N-box*** to be a closed rectangle of the form

$$[a_1/N, (a_1+1)/N] \times [a_2/N, (a_2+1)/N] \times \cdots \times [a_n/N, (a_n+1)/N]$$

with a_1, a_2, …, a_n integers. So the N-boxes tile $\mathbb{R}^n$.

Let $Z(N)$ be the union of all N-boxes which lie completely within $A \cap B_N(0)$, where $B_N(0)$ is the ball of radius N around the origin. Note that $Z(N)$ is compact, since it is a union of finitely many closed rectangles, and its boundary clearly has measure zero.

Claim For any $a \in A$, the point a is in every sufficiently large $Z(N)$.

Proof Once N is large enough, the distance from a to 0 will be far less than N. Since A is open, A contains a small ball around a and, once N is large enough, this will contain the N-box (or boxes) containing a. So any a lies in $Z(N)$ for N sufficiently large. $\square$

If we replace N by a multiple dN, then each N-box β breaks into d^n smaller dN-boxes. If $\beta \subset Z(N)$, then each of the smaller boxes is as well, as the condition that each of them lies in $A \cap B_{dN}(0)$ is only weaker. So $Z(N) \subseteq Z(dN)$.

Claim: If we choose d large enough, then $Z(N)$ lies in the interior of $Z(dN)$. In other words, we need to show that we can choose d large enough that, for all $z \in Z(N)$, all of the dN boxes containing z lie in $A \cap B_{dN}(0)$.

Proof: The union of all the dN-boxes touching $Z(N)$ will lie in the ball of radius $N +$ (small amount) (more precisely, the small amount is $\sqrt{n}/(dN)$, the diagonal of a dN box). So the condition to stay in the ball of radius dN is easily satisfied, and the issue is to stay in A. Suppose for the sake of contradiction that, for all $d > 0$, there is a $z_d \in Z(N)$ such that one of the dN-boxes touching z_d contains a point $y_d \notin A$. Since $Z(N)$ is compact, we can extract a convergent subsequence of z_{d_i}, approaching some $z_\infty \in Z(N)$. Since y_{d_i} and z_{d_i} lie in the same d_iN-box, we have $|y_{d_i} - z_{d_i}| \to 0$ as $i \to \infty$, so y_{d_i} also approaches z_∞. We have $z_\infty \in Z(N) \subset A$, so there is some open ball around z_∞ in A. Eventually, y_{d_i} moves into that ball. But then y_{d_i} lies in A, contradicting our construction. $\square$.

Once we have this, the rest is easy. Choose some N_1, say 1. Let $K_1 = Z(1)$ and let U_1 be the interior of K_1.

Choose d_2 large enough that $Z(N_1)$ lies in the interior of $Z(d_2N_1)$. We put $N_2 = d_2N_1$, define $K_2 = Z(N_2)$ and let U_2 be the interior of K_2.

Continuing in this manner, we have

$$U_1 \subseteq K_1 \subseteq U_2 \subseteq K_2 \subseteq \cdots \subseteq A \quad (*)$$

Assuming we always take $d_1 > 1$, the N_i approach ∞. We know that any point a of A is in $Z(N)$ for all sufficiently large N, so $\bigcup Z(N_i) = A$. **QED**