Identities Involving the Lie Bracket

Connor Puritz, Matthew Polgar, William Newman

November 11, 2017

(29) We define ad_X to be the linear map $Y \mapsto [X, Y]$. So, if we going to write this as a matrix, it would be an $n^2 \times n^2$ matrix. Show that

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y)}{n!}.$$

We recall the formula

$$(D\exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{X^j Y X^{n-1-j}}{n!}.$$

Proof: First, we will show through induction that

$$\mathrm{ad}_X^n(Y) = \sum_{j=0}^{n} \binom{n}{j} X^j Y (-X)^{n-j}.$$

For $n = 0$, we have

$$\mathrm{ad}_X^0(Y) = Y$$

by definition. So the base case is true. Now suppose the hypothesis is true for some $n \in \mathbb{N}$.

$$\begin{aligned}
\mathrm{ad}_X^{n+1}(Y) = [X, \mathrm{ad}_X^n(Y)] &= X\left(\sum_{j=0}^{n} \binom{n}{j} X^j Y (-X)^{n-j}\right) - \left(\sum_{j=0}^{n} \binom{n}{j} X^j Y (-X)^{n-j}\right) X \\
&= \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} X^{j+1} Y X^{n-j}\right) + \left(\sum_{j=0}^{n} (-1)^{n-j+1} \binom{n}{j} X^j Y X^{n-j+1}\right) \\
&= \left(\sum_{j=1}^{n+1} (-1)^{n-j+1} \binom{n}{j-1} X^j Y X^{n-j+1}\right) + \left(\sum_{j=0}^{n} (-1)^{n-j+1} \binom{n}{j} X^j Y X^{n-j+1}\right) \\
&= X^{n+1} Y + \left(\sum_{j=1}^{n} (-1)^{n-j+1} \left(\binom{n}{j-1} + \binom{n}{j}\right) X^j Y X^{n-j+1}\right) + (-1)^{n+1} Y X^{n+1} \\
&= X^{n+1} Y + \left(\sum_{j=1}^{n} (-1)^{n-j+1} \binom{n+1}{j} X^j Y X^{n-j+1}\right) + (-1)^{n+1} Y X^{n+1} \\
&= \sum_{j=0}^{n+1} (-1)^{n-j+1} \binom{n+1}{j} X^j Y X^{n-j+1}
\end{aligned}$$

So the hypothesis holds for $n + 1$, so we're good. Using this, we see that

$$\sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y)}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{X^j Y (-X)^{n-j}}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{X^j Y (-X)^{n-j}}{j!(n-j)!}.$$

The right hand side is a Cauchy product, with

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{X^jY(-X)^{n-j}}{j!(n-j)!} = \left(\sum_{j=0}^{\infty}\frac{X^j}{j!}\right)Y\left(\sum_{k=0}^{\infty}\frac{(-X)^k}{k!}\right) = e^XYe^{-X}. \quad \square$$

(30) Let X be the diagonal matrix with entries $x_1, x_2, \ldots, x_n$. Let $i \neq j$ and let Y be the matrix with 1 in position (i,j) and 0 everywhere else. Show that

$$(D\exp)_X(Y) = \sum_{n=0}^{\infty}\frac{\mathrm{ad}_X^n(Y)}{(n+1)!}e^X = e^X\sum_{n=0}^{\infty}\frac{(-1)^n\mathrm{ad}_X^n(Y)}{(n+1)!}.$$

Proof: First, note that when we compute XY, the only nonzero term is x_i in position (i,j). Since X is diagonal, by induction, X^nY is zero except in position (i,j) with x_i^n as the entry. Similarly, YX^n is zero except in the position (i,j) with x_j^n as the entry. So we can write $X^nY = x_i^nY$ and $YX^n = x_j^nY$. Using this, we compute $\mathrm{ad}_X(Y) = XY - YX = (x_i - x_j)Y$. So Y is an eigenvector of ad_X, meaning $\mathrm{ad}_X^n(Y) = (x_i - x_j)^nY$. Note that if $x_i = x_j$, then all sums in the problem vanish. So suppose $x_i \neq x_j$.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\mathrm{ad}_X^n(Y)}{(n+1)!}e^X &= \sum_{n=0}^{\infty}\frac{(x_i-x_j)^n}{(n+1)!}Ye^x = \sum_{n=0}^{\infty}\frac{(x_i-x_j)^{n+1}}{(n+1)!(x_i-x_j)}Ye^x \\
&= \frac{Ye^X}{(x_i-x_j)}\left(-1+\sum_{n=0}^{\infty}\frac{(x_i-x_j)^n}{n!}\right) \\
&= \frac{e^{x_j}Y}{(x_i-x_j)}\left(-1+e^{x_i-x_j}\right) \\
&= \frac{e^{x_i}-e^{x_j}}{x_i-x_j}Y
\end{aligned}$$

$$\begin{aligned}
e^X\sum_{n=0}^{\infty}\frac{(-1)^n\mathrm{ad}_X^n(Y)}{(n+1)!} &= e^X\sum_{n=0}^{\infty}\frac{(-1)^n(x_i-x_j)^nY}{(n+1)!} \\
&= -\frac{e^XY}{x_i-x_j}\left(-1+\sum_{n=0}^{\infty}(-1)^n\frac{(x_i-x_j)^n}{n!}\right) \\
&= \frac{e^{x_i}Y}{x_i-x_j}\left(1-e^{x_j-x_i}\right) \\
&= \frac{e^{x_i}-e^{x_j}}{x_i-x_j}Y
\end{aligned}$$

So the second equal in the problem statement is true. We now just need to show the first equality. Consider the matrix component of each term in the sum for $(D\exp)_X(Y)$. It is of the form

$$X^kYX^{n-1-k} = x_i^kYX^{n-1-k} = x_i^kx_j^{n-1-k}Y.$$

So

$$\begin{aligned}
(D\exp)_X(Y) &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{X^kYX^{n-1-k}}{n!} \\
&= \left(\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{x_i^kx_j^{n-1-k}}{n!}\right)Y \\
&= \sum_{n=1}^{\infty}\frac{x_i^n-x_j^n}{n!(x_i-x_j)}Y \\
&= \frac{e^{x_i}-e^{x_j}}{x_i-x_j}Y.
\end{aligned}$$

This completes the proof. $\square$