Maps between Lie Groups

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Let $G \subset \mathrm{GL_m}(\mathbb{R})$ and $H \subset \mathrm{GL_n}(\mathbb{R})$ be Lie groups. Let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Let $\phi : G \to H$ be both

(1) a group homomorphism, meaning $\phi(\mathrm{Id_m}) = \mathrm{Id_n}$ and $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.

(2) a smooth map, meaning there is an open set $U \supseteq G$ and a C^∞ function $\widetilde{\phi} : U \to H$ such that $\widetilde{\phi}$ restricts to ϕ on G.

Problem 33. For $X \in \mathfrak{g}$, show that $\exp((D\phi)X) = \phi(\exp X)$.

Solution. Fix $X \in \mathfrak{g}$. Let γ be a smooth curve such that $\gamma(0) = Id_m$ and $\gamma'(0) = X$
From **Problem 23**, we know

$$\exp(X) = \lim_{n\to\infty} \gamma(\frac{1}{n})^n$$

Next we compose ϕ and γ to find:

$$\phi \circ \gamma(0) = \phi(Id_m) = Id_n$$

and

$$(\phi \circ \gamma)'(0) = \phi' \circ \gamma(0) * \gamma'(0) = \phi'(Id)X = (D\phi)X$$

Now, from **Problem 23**, we get

$$\exp((D\phi)X) = \lim_{n\to\infty} (\phi \circ \gamma(1/n)^n)$$

Since ϕ is smooth, we can take it out of the limit

$$= \phi(\lim_{n\to\infty} (\gamma(1/n)^n))$$

As noted earlier, $\exp(X) = \lim_{n\to\infty} \gamma(\frac{1}{n})^n$, so

$$\exp((D\phi)X) = \phi(\lim_{n\to\infty} (\gamma(1/n)^n)) = \phi(\exp X)\square$$

Problem 34. For X and Y in $\mathfrak{g}$, show that $\big(D\phi\big)\big([X, Y]\big) = \Big[(D\phi)(X),\ (D\phi)(Y)\Big]$.

Solution. Consider the following useful formula from **Problem 25.**:

$$\frac{d^2}{dt^2}\Big|_{t=0} e^{tx}e^{ty}e^{-tx}e^{-ty} = [X, Y]$$

Now, note that we have proved for functions where $f'(0) = 0$, then

$$\frac{d}{dt}\Big|_{t=0} f(\sqrt{t}) = \frac{d^2}{dt^2}\Big|_{t=0} f(t)$$

Let $f = e^{tx}e^{ty}e^{-tx}e^{-ty}$. Evidently, $f'(0) = 0$. Thus, we may assume that

$$\frac{d}{dt}\Big|_{t=0} e^{\sqrt{t}x}e^{\sqrt{t}y}e^{-\sqrt{t}x}e^{-\sqrt{t}y} = [X, Y]$$

Then, we have $\Big[(D\phi)(X),\ (D\phi)(Y)\Big] = \frac{d}{dt}e^{\sqrt{t}D\phi x}e^{\sqrt{t}D\phi y}e^{-\sqrt{t}D\phi x}e^{-\sqrt{t}D\phi y}$.

From **Problem 33** we know that $\exp((D\phi)X) = \phi(\exp X)$. Thus our expression

$$= \frac{d}{dt}\phi(e^{\sqrt{t}x})\phi(e^{\sqrt{t}y})\phi(e^{-\sqrt{t}x})\phi(e^{-\sqrt{t}y})$$

Since ϕ is a group homomorphism, this equals

$$\frac{d}{dt}\phi(e^{\sqrt{t}x}e^{\sqrt{t}y}e^{-\sqrt{t}x}e^{-\sqrt{t}y}) = D\phi(e^{\sqrt{t}x}e^{\sqrt{t}y}e^{-\sqrt{t}x}e^{-\sqrt{t}y})$$

Applying our beginning note, we get this equals $(D\phi)\big([X, Y]\big)$. Yay!