Math 395: Continuity and Differentiability of the Exponential

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8. Show that

$$|AB| \leq |A| \cdot |B|$$

Proof: We have that

$$|AB|^2 = \sum_{i,j} {(AB)_{ij}}^2 = \sum_{i,j}\sum_{k} (A_{ik}B_{kj})^2 \leq \sum_{i,j}\Big(\sum_{k} A_{ik}^2\Big)\Big(\sum_{l} {B_{lj}}^2\Big) = \Big(\sum_{i,k} A_{i,k}^2\Big)\Big(\sum_{j,l} B_{l,j}^2\Big) = |A|^2|B|^2$$

The first inequality comes from the Cauchy Schwarz Inequality.

□

9. Let R be a positive real number and let $B(R) = \{X \in Mat_{n\times n}(\mathbb{R})\colon |X| \leq R$. Show that exp: $B(R) \to GL_n(\mathbb{R})$. Deduce that exp is continuous.

Proof: We will show that the function is continuous using the Weierstrass M-Test. Let $M_n = R^n/n!$. Then we have that

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{R^n}{n!} = e^R$$

Observe that, since $|A| \leq R$,

$$\left|\frac{A^n}{n!}\right| = \frac{|A^n|}{n!} \leq \frac{|A|^n}{n!} \leq \frac{R^n}{n!}$$

Then, $\sum_{n=0}^{\infty} A^n/n!$ is convergent and continuous on $B(R)$ by the Weierstrass M-test. Note now that for any $X \in Mat_{n\times n}(\mathbb{R})$ there exists R such that $X \in B(R)$. Thus exp is continuous on $Mat_{n\times n}(\mathbb{R})$.

□

10. Show that if X is an $n \times n$ real matrix then $\det \circ \exp(X) > 0$.

Proof: We will give two proofs. The first slick proof is from class. Note that $e^A = e^{A/2}e^{A/2}$. Thus

$$\det(e^A) = \det(e^{A/2}e^{A/2}) = \det(e^{A/2})\det(e^{A/2}) > 0$$

Now for the second proof: Note that both exp and det are continuous functions, and so $\det \circ \exp$ is also continuous. We have also shown that $\exp(X) \in GL_n(\mathbb{R})$. Thus $\det(X) \neq 0$.

Now note that $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is connected. Since $\det \circ \exp$ is continuous, it maps connected sets to connected sets, so $\det(X) > 0$ or $\det(X) < 0$ for all X. Finally, recall that $\det(e^0) = \det(\mathrm{Id}) = 1$, and we conclude that $\det(X) > 0$ for all X.

□

11. For $X \in \mathrm{Mat}_{k\times k}(\mathbb{R})$, ket $g(X) = X^n$. For any $k \times k$ matrix Y, we want to show that $D(g)_X(Y) = \sum_{j=0}^{n-1} X^j Y X^{n-1-j}$. That is, this expression is the derivative of the function g at the matrix X evaluated at the matrix Y.

Proof: Well, we want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|Y| < \delta$, we have that:

$$\frac{|g(X+Y) - g(X) - D(g)_X(Y)|}{|Y|} < \varepsilon$$

Fix $\varepsilon > 0$. Let $\delta = \min\{1, \varepsilon / \left(\sum_{j=2}^n \binom{n}{j} |Y||X|^{n-j}\right)\}$. Fix $Y \in \mathrm{Mat}_{k\times k}(\mathbb{R})$ such that $|Y| < \delta$. Note that:

$$\begin{aligned}
\frac{\left|g(X+Y) - g(X) - D(g)_X(Y)\right|}{|Y|} &= \frac{\left|(X+Y)^n - X^n - \sum_{j=0}^{n-1} X^j Y X^{n-1-j}\right|}{|Y|} \\
&\leq \frac{\sum_{j=2}^n \binom{n}{j} |Y|^j |X|^{n-j}}{|Y|} = \sum_{j=2}^n \binom{n}{j} |Y|^{j-1} |X|^{n-j} \\
&< \sum_{j=2}^n \binom{n}{j} |Y||X|^{n-j} = |Y| \sum_{j=2}^n \binom{n}{j} |Y||X|^{n-j} \\
&< \frac{\varepsilon}{\sum_{j=2}^n \binom{n}{j} |Y||X|^{n-j}} \sum_{j=2}^n \binom{n}{j} |Y||X|^{n-j} = \varepsilon
\end{aligned}$$

The inequality from the first to the second line comes from the noncommutative expansion of $(X+Y)^n$. Every term in the expression with H^0 or H^1 is cancelled out by X^n or a term in $\sum_{j=0}^{n-1} X^j Y X^{n-1-j}$ respectively. Then, via triangle inequality and the inequality in Problem 8, all of the remaining matrices may be replaced with their magnitude—effectively turning $(X+Y)^n$ into $(|X|+|Y|)^n$. Since the norm of a matrix is a real number and real numbers are commutative, this new expansion can be expressed via the binomial theorem (minus the first two terms, since the first two terms in the binomial theorem sum expansion exactly correspond with the parts of the non-commutative sum which cancelled out). This final expression is what is seen in the numerator on the second line.

The inequality from the second to the third line is true because $|Y| < \delta \leq 1$, and likewise the inequality from the third to fourth line follows from how we chose delta as well. Thus, we have that $D(g)_X(Y) = \sum_{j=0}^{n-1} X^j Y X^{n-1-j}$.

$\square$