IBL Notes September 22 2017

Jeremy D'Silva and Charles Devlin VI

September 22 2017

<u>Recall and review:</u> We begin by giving an example to motivate this theorem.
Suppose $f : A \to \mathbb{R}^{\nvDash}; (x, y) \mapsto (\frac{1}{xy}, x^2 + y^2)$ where A is an open set chosen "sensibly" (e.g. excluding the x and y axes, such that f is well-defined).
<u>Questions:</u> We might want to know, is this map invertible at $(1, 2) \in A$? If so, is f^{-1} differentiable? And if so, what is it? The Inverse Function Theorem is the tool we need to answer these questions.
<u>Theorem:</u> (Inverse Function Theorem). Let V, W be finitely generated $\mathbb{R}$ vector spaces. Let $A \subset V$ be open, f a C^1 map $f : A \to W$, and $a \in A$. Then if $[Df]_a$ is invertible then f is invertible in a neighborhood of a, and f^{-1} is C^1. In particular, $\exists$ open neighborhood $A' \subset A$ containing a, and $\exists$ B', an open neighborhood containing $b = f(a)$, such that $\text{res}_{A'} f : A' \to B'$ is invertible (bijective) and, with this restriction, f^{-1} is C^1. (Also, note $[Df^{-1}]_b = ([Df]_a)^{-1}$).

1. Show that f is injective on A.
   Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$. By our previous work, A is shrunken such that there exists $c > 0$ such that $|f(x) - f(y)| \geq c|x - y|$ for all $x, y \in A$. Then
   $$\begin{aligned} 0 &= |f(x_1) - f(x_2)| \\ &\geq c|x_1 - x_2| \\ &\geq 0 \end{aligned}$$
   So we must have $c|x_1 - x_2| = 0$. As $c \neq 0$, we have $x_1 = x_2$ and f is injective.

2. Show that there is a neighborhood $B^{'} \ni b$ such that $B^{'} \subset f(A)$.
   From previous work, we know we can shrink A sufficiently small such that for all open $U \subset A$, we have $f(U)$ is open. Fix U with $a \in U$ and let $B^{'} = f(U)$. Since $a \in U$, we have $f(a) \in f(U) = B^{'}$. As $U \subset A$, $f(U) \subset f(A)$.

3. Show that there is a neighborhood $A^{'} \ni a$ in A such that f is a bijection $A^{'} \to B^{'}$.
   As f is continuous, the inverse image of open sets is open. Therefore, $f^{-1}(B^{'})$ is open in A. Let $A^{'} = f^{-1}(B^{'})$. From 1, f is injective on A and therefore is injective on $A^{'}$. Surjectivity follows from the definition of $A^{'}$. So f is a bijection on $A^{'}$.

4. Let y_1 and $y_2 \in B^{'}$ and let $x_i = g(y_i)$. Show that there is a constant c_2 such that $|x_1 - x_2| \leq c_2|y_1 - y_2|$. Deduce as a corollary that g is continuous.
   The result follows from previous work, in which it was shown that with $A^{'}$ sufficiently shrunk, there exists a constant $c > 0$ such that
   $|f(x_1) - f(x_2)| \geq c|x_1 - x_2|$ for all x_1, $x_2 \in A^{'}$. Choose $y_1, y_2 \in B^{'}$. Since $g = f^{-1}$, there exists $x_1, x_2 \in A^{'}$ such that $g(y_1) = x_1, g(y_2) = x_2$. So we have
   $$\begin{aligned} c|g(y_1) - g(y_2)| &= c|x_1 - x_2| \\ &\leq |f(x_1) - f(x_2)| \\ &= |y_1 - y_2| \end{aligned}$$
   As $c \geq 0$, we may divide both sides by c to obtain
   $$|x_1 - x_2| \leq c^{-1}|y_1 - y_2|$$

To see that g is continuous, we note that g is Lipschitz continuous on $B^{'}$ by the inequality above. As Lipschitz continuity implies continuity, g is also continuous on $B^{'}$.

For brevity, set $E = D(f)(a)$ and recall that E is assumed invertible.

5. For h small enough that $b + h \in B^{'}$, show that

$$\frac{|g(b+h) - g(b)|}{|h|} \leq c_2$$

From 4, if h is made small enough so that $b + h \in B^{'}$, then we have

$$\begin{aligned} |g(b+h) - g(b)| &\leq c_2 |b + h - b| \\ &= c_2 |h| \\ \implies \frac{|g(b+h) - g(b)|}{|h|} &\leq c_2 \end{aligned}$$

For h as above, put $k(h) = g(b+h) - g(b)$.

6. Show that, as $h \to 0$, the function $k(h)$ goes to 0 as well.
This follows from the continuity of g from 4. Since g is continuous at b, we have $g(b+h) - g(b) \to 0$ as $h \to 0$. So then $k(h) = g(b+h) - g(b) \to 0$ as $h \to 0$.

7. Show that

$$\begin{aligned} \frac{|g(b+h) - g(b) - E^{-1}(h)|}{|g(b+h) - g(b)|} &= \frac{|k(h) - E^{-1}(f(a + k(h)) - f(a))|}{|k(h)|} \\ &= \frac{|E^{-1}(E(k(h)) - f(a + k(h)) + f(a))|}{|k(h)|} \end{aligned}$$

First, observe that

$$\begin{aligned} f(a + k(h)) - f(a) &= f(a + g(b+h) - g(b)) - f(a) \\ &= f(a + g(b+h) - a) - f(a) \\ &= f(g(b+h)) - f(a) \\ &= b + h - b \\ &= h \end{aligned}$$

so

$$\begin{aligned} \frac{|g(b+h) - g(b) - E^{-1}(h)|}{|g(b+h) - g(b)|} &= \frac{|k(h) - E^{-1}(f(a + k(h)) - f(a))|}{|k(h)|} \\ &= \frac{|E^{-1}(E(k(h))) - E^{-1}(f(a + k(h)) - f(a))|}{|k(h)|} \\ &= \frac{|E^{-1}(E(k(h)) - f(a + k(h)) + f(a))|}{|k(h)|} \end{aligned}$$

8. Show that, as $h \to 0$, we have

$$\frac{|g(b+h) - g(b) - E^{-1}(h)|}{|g(b+h) - g(b)|} \to 0.$$

Proof: Well,

$$\frac{|g(b+h)-g(b)-E^{-1}(h)|}{|g(b+h)-g(b)|} = \frac{|E^{-1}(E(k(h))-f(a+k(h))+f(a))|}{|k(h)|} \leq |E^{-1}|\frac{|Ek(h)-f(a+k(h))-f(a)|}{k(h)}$$

Note that as $h \to 0, k(h) \to 0$ (by Problem 6). Thus, we note that as $k(h) \to 0$

$$|E^{-1}|\frac{|Ek(h)-f(a+k(h))-f(a)|}{k(h)} \to |E^{-1}|\frac{|E(h)-f(a+h)-f(a)|}{h} \to 0.$$

This is simply the definition of $E = (Df)_a$, multiplied by a constant, hence it goes to 0.

9. Prove the theorem by proving g is differentiable at b with derivative E^-1. In other words, as $h \to 0$, we have

$$\frac{|g(b+h)-g(b)-E^{-1}(h)|}{|h|} \to 0$$

Proof: Well, rearranging terms of the inequality proven in Problem 5, we note that

$$\frac{c_2|h|}{|g(b+h)-g(b)|} \geq 1$$

So then we see that if $|h|$ is small,

$$\frac{|g(b+h)-g(b)-E^{-1}(h)|}{|h|} \leq \frac{|g(b+h)-g(b)-E^{-1}(h)|}{|h|}\frac{c_2|h|}{|g(b+h)-g(b)|} = c_2\frac{|g(b+h)-g(b)-E^{-1}(h)|}{|g(b+h)-g(b)|}$$

But this is just a constant times the expression in 8, which goes to 0 as $|h|$ goes to 0, so this expression goes to 0 as well.

Recapitulation: Let us be clear about what we have proven. We begin with the hypotheses stated in the theorem statement. At $a \in A$, $(Df)_a$ is bijective. Applying results from Monday (9/18) and Wednesday (9/20) in class, we produce restrictions to make f bijective on a neighborhood around a mapping to a neighborhood around $f(a)$. This means that f^{-1} is defined on this neighborhood. We then showed that $g : B' \to A' = f^{-1} : B' \to A'$ is differentiable at $b = f(a)$, with inverse given by $E^{-1} = ((Df)_a)^{-1}$.