IBL Notes September 29 2017

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Problem 12. Consider exp as a map from $n \times n$ matrices. Show that the derivative of exp at 0 is the $n^2 \times n^2$ identity.

Proof. By results of Week 2 problem,

$$(Dexp)_x(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{X^j Y X^{n-1-j}}{n!}$$

So $(Dexp)_0(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{0^j Y 0^{n-1-j}}{n!} = \sum_{j=0}^{0} \frac{0^j Y 0^{n-1-j}}{1!} + 0 = Y$

Thus, $(Dexp)_0$ is an identity map.

Since $exp : Mat_{n\times n}(\mathbb{R}) \to Mat_{n\times n}(\mathbb{R})$, so $(Dexp)_0 = Id_{n^2}$. □

Problem 13. Show that there are open neighborhoods U of 0 and V of Id_{n^2} such that exp is a bijection $U \to V$, with smooth inverse.

Proof. By previous problem,

$$(Dexp)_0 = Id_{n^2}$$

By problems of Week 2, we know that

$$(Dexp)_x(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{X^j Y X^{n-1-j}}{n!}$$

i.e. exp is differentiable and its derivative is a polynomial.

Thus, $exp \in C^{\infty}$, i.e. exp is smooth.

Choose an open ball A around 0 in $Mat_{n\times n}(\mathbb{R})$.

Then we have:

- an open set $A \subset Mat_{n\times n}(\mathbb{R})$

- a smooth map $exp : A \to Mat_{n\times n}(\mathbb{R})$
- a point $0 \in A$
- $(Dexp)_0 = Id_{n^2} \neq 0$, i.e. $(Dexp)_0$ is invertible.

By inverse function theorem, there exist open $U \subset A$ containing 0, open $V \subset Mat_{n\times n}(\mathbb{R})$ containing $exp(0) = Id_{n^2}$, s.t. $res_U(exp) : U \to V$ is invertible and its inverse is smooth. $\square$

Problem 14. Show that $exp(X^T) = exp(X)^T$. Show that, if Y is small enough for Y and Y^T to lie in V, then $\log(Y^T) = \log(Y)^T$.

Proof. By definition,

$$exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

And for matrices A and B, we know $(AB)^T = B^T A^T$, $(A+B)^T = A^T + B^T$.

So,

$$exp(X^T) = \sum_{n=0}^{\infty} \frac{(X^T)^n}{n!} = \sum_{n=0}^{\infty} \frac{(X^n)^T}{n!} = \sum_{n=0}^{\infty} \left(\frac{X^n}{n!}\right)^T = \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right)^T = (exp(X))^T$$

Note that

$$exp(log(Y^T)) = Y^T$$

Because exp and log are inverses.

So,

$$exp(log(Y)^T) = (exp(log(Y))^T = Y^T$$

Hence,

$$exp(log(Y^T)) = exp(log(Y)^T)$$

By Problem 13, $log : V \to U$ is a bijection and is smooth,

So $log(Y)$ and $log(Y^T) \in U$, since exp is a bijection on U,

Thus,

$$exp(log(Y^T)) = exp(log(Y)^T) \Rightarrow log(Y^T) = log(Y)^T$$

$\square$

Problem 15. Let $\mathfrak{so}(n)$ be the vector space of skew-symmetric $n \times n$ matrices. Show that $exp(\mathfrak{so}(n)) \subseteq O(n)$ and there is a neighborhood V of Id_n such that $V \cap O(n) \subseteq exp(\mathfrak{so}(n))$.

Proof. Choose $A \in \mathfrak{so}(n)$, want to show:

$$exp(A) \in O(n)$$

i.e. exp(A) is orthogonal.

Since A is skew-symmetric, $A = -A^T$.

So, we have:

$$exp(A) \cdot (exp(A))^T = exp(A) \cdot (exp(A^T)) = exp(A) \cdot exp(-A) = exp(A + (-A)) = exp(0) = Id_n$$

Hence, $exp(A)$ is orthogonal.

By problem 13, there exist open $U \ni 0$, open $V \ni Id_n$ s.t. $exp : U \to V$ is invertible with smooth inverse.

Further shrink U to U' so that $U' = U \cap U^T \cap -U$. Then, $U' \subset U$, and $0 \in U'$. Let $V' = exp(U')$, notice that $V' = (V')^T = (V')^{-1}$.

Choose $B \in V' \cap O(n)$, then we want to show that $B \in exp(\mathfrak{so}(n))$, i.e. $log(B) \in \mathfrak{so}(n)$ by bijectivity.

Since B is orthogonal, we have $B \cdot B^T = Id$.

Note that

$$exp(log(B)) \cdot exp(log(B)^T) = exp(log(B)) \cdot exp(log(B))^T = B \cdot B^T = Id_n$$

$$exp(log(B)) \cdot exp(-log(B)) = exp(log(B) + (-log(B))) = exp(0) = Id_n$$

So $exp(log(B))^T$ and $exp(-log(B))$ are both inverses of $exp(log(B)) = B$.

Since B^{-1} is unique, thus $exp(log(B))^T = exp(-log(B))$.

Finally, since $exp : U' \to V'$ is invertible, $log(B)^T = -log(B)$, i.e. $log(B) \in \mathfrak{so}(n)$, so $B \in exp(\mathfrak{so}(n))$.

$\square$